Unconditional basic sequences

A series $\sum_{n=1}^{\infty} x_n$ in a Banach space X is said to be unconditionally convergent if $\sum_{n=1}^{\infty} x_{\pi(n)}$ converges for all permutations π of $\mathbb{N}$. It is absolutely convergent if $\sum_{n=1}^{\infty} \|x_n\|$ converges. Notice that by Riemann's theorem unconditional and absolute convergence coincide in $\mathbb{R}$ and hence in all finite-dimensional spaces.

Notice ~~that~~ if $\sum_{n=1}^{\infty} x_n$ is unconditionally convergent, ~~then~~ $\sum_{n=1}^{\infty} x_{\pi(n)}$ has the same limit for all π. For if $x^* \in X^*$ then $\sum_{n=1}^{\infty} x^*(x_n) = x^*\left(\sum_{n=1}^{\infty} x_n\right)$ is unconditionally and hence absolutely convergent and thus

$$x^*\left(\sum_{n=1}^{\infty} x_n\right) = x^*\left(\sum_{n=1}^{\infty} x_{\pi(n)}\right) \text{ for all } \pi.$$

Proposition (Orlicz) TFAE for $(x_n)_{n=1}^{\infty}$ in a Banach space X:

(i) $\sum_{n=1}^{\infty} x_n$ is unconditionally convergent

(ii) $\sum_{i=1}^{\infty} x_{n_i}$ converges for all $n_1 < n_2 < \dots$.

(iii) $\sum_{i=1}^{\infty} \theta_n x_n$ converges for all choices of signs $\theta_n \in \{-1, 1\}$.

Proof (iii) $\Rightarrow$ (ii) : Set $\theta_n = 1$ if $n = n_i$ for some i and otherwise $\theta_n = -1$. Then

$$\sum_{i=1}^{\infty} x_{n_i} = \frac{1}{2}\left(\sum_{n=1}^{\infty} x_n + \sum_{n=1}^{\infty} \theta_n x_n \right)$$ and hence

converges.

(ii) $\Rightarrow$ (i) : Suppose π is a permutation of $\mathbb{N}$ but $\sum_{n=1}^{\infty} x_{\pi(n)}$ diverges. Then Cauchy's criterion cannot be satisfied and we can therefore find intervals of integers $I_1 < I_2 < \dots$ such that for some $\varepsilon > 0$ and all i,

$$\Big\| \sum_{n \in I_i} x_{\pi(n)} \Big\| \geq \varepsilon .$$

But then by passing to a subsequence we can suppose that for $i<j$ and $m \in I_i$, $k \in I_j$, $\pi(m) < \pi(k)$. Thus if we enumerate

$$\{\pi(m) \mid m \in \bigcup_{i=1}^{\infty} I_i\} \quad \text{increasingly}$$

as $n_1 < n_2 < n_3 < \dots$ we see that $\{\pi(m) \mid m \in I_i\}$ forms an interval in $(n_j)_{j=1}^{\infty}$ and hence $\sum_{j=1}^{\infty} x_{n_j}$ diverges contrary to our assumption.

(i) $\Rightarrow$ (iii) : If $\sum_{n=1}^{\infty} \theta_n x_n$ does not converge then we can find intervals $I_1 < I_2 < \dots$ of integers and some $\varepsilon > 0$ such that

$$\Big\| \sum_{n \in I_i} \theta_n x_n \Big\| \geq 2\varepsilon ,$$

whereby either $\Big\| \sum_{\substack{n \in I_i \\ \theta_n = 1}} x_n \Big\| \geq \varepsilon$ or $\Big\| \sum_{\substack{n \in I_i \\ \theta_n = -1}} x_n \Big\| \geq \varepsilon$.

Let π be a permutation of $\mathbb{N}$ such that on I_i π first enumerates the n such that $\varepsilon_n = 1$. Then $\sum_{n=1}^{\infty} x_{\pi(n)}$ fails Cauchy's criterion, contradicting our assumption. $\square$

__Definition__ A basis $(e_n)_{n=1}^{\infty}$ is called __unconditional__ if for all $x \in X$, $\sum_{n=1}^{\infty} e_n^*(x) e_n$ converges unconditionally (to x).

Notice we can then define for all $\Theta \in \{-1,1\}^{\mathbb{N}}$ an operator on X by

$$M_\Theta\left(\sum_{n=1}^{\infty} a_n e_n\right) = \sum_{n=1}^{\infty} \Theta_n a_n e_n .$$

By the closed graph theorem, M_Θ is continuous. For if $\sum_{n=1}^{\infty} a_n^{(k)} e_n \xrightarrow[k]{} \sum_{n=1}^{\infty} a_n e_n$ and

$\sum_{n=1}^{\infty} \Theta_n a_n^{(k)} e_n \xrightarrow[k]{} \sum_{n=1}^{\infty} b_n e_n$, then for every m,

$a_m^{(k)} \xrightarrow[k]{} a_m$ and $\Theta_m a_m^{(k)} \xrightarrow[k]{} b_m$, whereby $b_m = \Theta_m a_m$.

We claim also that $\{M_\Theta\}_{\Theta \in \{-1,1\}^{\mathbb{N}}}$ is uniformly bounded. If not, by the principle of uniform boundedness, it cannot be pointwise bounded and thus for some $x = \sum_{n=1}^{\infty} a_n e_n$ and $\Theta^{(k)} \in \{-1,1\}^{\mathbb{N}}$,

$\| M_{\Theta^{(k)}} x \| \to \infty$. Using the fact that $\{-1,1\}^N$ is finite for all N, this easily implies that we can find intervals $I_0 < I_1 < \dots$ of integers and $k_i \uparrow$ such that $\left\| \sum_{n \in I_{k_i}} \theta_n^{(k_i)} a_n e_n \right\| \to \infty$. Letting

$$\theta_n = \begin{cases} \theta_n^{(k_i)} & \text{if } n \in I_{k_i} \\ 1 & \text{otherwise} \end{cases}, \text{ we see that}$$

$\sum_{n=1}^{\infty} \theta_n a_n x_n$ diverges, which is a contradiction.

The number $K_u = \sup_\theta \|M_\theta\|$ is called the <u>unconditional basis</u> constant of (e_n).

<u>Proposition</u> For all $\bar{\lambda} = (\lambda_n)$ with $|\lambda_n| \leq 1$ and $x = \sum_{n=1}^{\infty} a_n e_n$ we have

$$\left\| \sum_{n=1}^{\infty} \lambda_n a_n e_n \right\| \leq K_u \left\| \sum_{n=1}^{\infty} a_n e_n \right\|.$$

<u>Proof</u> Define $M_{\bar{\lambda}} : \operatorname{span}\{e_n\}_{n=1}^{\infty} \to \operatorname{span}\{e_n\}_{n=1}^{\infty}$ by $M_{\bar{\lambda}}\left(\sum_{n=1}^{N} a_n e_n\right) = \sum_{n=1}^{N} \lambda_n a_n e_n$. It suffices to show that $\|M_{\bar{\lambda}}\| \leq K_u$. So suppose $x = \sum_{n=1}^{N} a_n e_n$ is given and find $x^* \in X$ of norm 1 such that $x^*\left(\sum_{n=1}^{N} \lambda_n a_n e_n\right) = \left\| \sum_{n=1}^{N} \lambda_n a_n e_n \right\|$. Now, let $\theta_n = 1$ if $a_n x^*(e_n) \geq 0$ and -1 otherwise.

Then,

$$\left\|\sum_{n=1}^{N}\lambda_n a_n e_n\right\| \leq \sum_{n=1}^{N}|\lambda_n|\,|a_n x^*(e_n)| \leq \sup_{1\leq n\leq N}|\lambda_n| \sum_{n=1}^{N}|a_n x^*(e_n)|$$

$$\leq \sum_{n=1}^{N}\theta_n a_n x^*(e_n) = x^*\left(\sum_{n=1}^{N}\theta_n a_n e_n\right)$$

$$\leq \|x^*\|\,\|M_\theta\|\,\left\|\sum_{n=1}^{N}a_n e_n\right\| \leq K_u \left\|\sum_{n=1}^{N}a_n e_n\right\| . \qquad \square$$

In particular if $A \subseteq \mathbb{N}$ we can set $\lambda_n = \begin{cases} 1 & \text{if } n\in A \\ 0 & \text{if not} \end{cases}$

and see that if $P_A : X \to X$ is the projection

$P_A\left(\sum_{n=1}^{\infty}a_n e_n\right) = \sum_{n\in A}a_n e_n$ then $\|P_A\| \leq K_u$.

The number $\sup_{A\subseteq\mathbb{N}}\|P_A\| = K_s$ is the <u>suppression constant</u> of (e_n) and (e_n) is <u>suppression unconditional</u> if $K_s = 1$.