THE DICHOTOMY THEOREMS

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1. The G_0 dichotomy

A *digraph* (or directed graph) on a set X is a subset $G \subseteq X^2 \setminus \Delta$. Given a digraph G on a set X and a subset $A \subseteq X$, we say that A is *G-discrete* if for all $x, y \in A$ we have $(x, y) \notin G$.

Now let $s_n \in 2^n$ be chosen for every $n \in \mathbb{N}$ such that $\forall s \in 2^{<\mathbb{N}}\ \exists n\ s \sqsubseteq s_n$. Then we can define a digraph G_0 on $2^{\mathbb{N}}$ by

$$G_0 = \{(s_n 0x, s_n 1x) \in 2^{\mathbb{N}} \times 2^{\mathbb{N}} \bigm| n \in \mathbb{N}\ \&\ x \in 2^{\mathbb{N}}\}.$$

Lemma 1. *If $B \subseteq 2^{\mathbb{N}}$ has the Baire property and is non-meagre, then B is not G_0-discrete.*

Proof. By assumption on B, we can find some $s \in 2^{<\mathbb{N}}$ such that B is comeagre in N_s. Also, by choice of (s_n), we can find some n such that $s \sqsubseteq s_n$, whereby B is comeagre in N_{s_n}. By the characterisation of comeagre subsets of $2^{\mathbb{N}}$, we see that for some $x \in 2^{\mathbb{N}}$, we have $s_n 0x, s_n 1x \in B$, showing that B is not G_0-discrete. $\square$

Suppose G and H are digraphs on sets X and Y respectively. A *homomorphism* from G to H is a function $h\colon X \to Y$ such that for all $x, y \in X$,

$$(x, y) \in G \Rightarrow (h(x), h(y)) \in H.$$

Also, if Z is any set, a *Z-colouring* of a digraph G on X is a homomorphism from G to the digraph $\neq$ on Z, i.e., a function $h\colon X \to Z$ such that for all $x, y \in X$,

$$(x, y) \in G \Rightarrow h(x) \neq h(y).$$

Proposition 2. *There is no Baire measurable $\mathbb{N}$-colouring of G_0.*

Proof. Note that if $h\colon 2^{\mathbb{N}} \to \mathbb{N}$ is a Baire measurable function, then for some $n \in \mathbb{N}$, $B = h^{-1}(n)$ is non-meagre with the Baire property and hence not G_0-discrete. So h cannot be a homomorphism from G_0 to $\neq$ on $\mathbb{N}$. $\square$

Theorem 3 (The G_0 dichotomy)**.** *Suppose G is an analytic digraph on a Polish space X. Then exactly one of the following holds:*

- *there is a continuous homomorphism from G_0 to G,*
- *there is a Borel $\mathbb{N}$-colouring of G.*

Proof. If X is countable, the result is trivial. So if not, let $f\colon \mathbb{N}^{\mathbb{N}} \to P$ be a continuous bijection onto the perfect kernel P of X. By replacing G with $(f \times f)^{-1}[G]$, there is no loss of generality in assuming that $X = \mathbb{N}^{\mathbb{N}}$.

So suppose $F \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ is a closed set such that

$$(x, y) \in G \Leftrightarrow \exists z\ (x, y, z) \in F.$$

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In order to produce a continuous homomorphism h from G_0 to G it suffices to find monotone Lipschitz functions $u, v^m\colon 2^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$, $m \in \mathbb{N}$, such that for all $m < k$ and $t \in 2^{k-m-1}$,

$$\big(N_{u(s_m0t)} \times N_{u(s_m1t)} \times N_{v^m(t)}\big) \cap F \neq \emptyset.$$

In this case, we can define $h, \tilde{v}^m\colon 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ by $h(w) = \bigcup_n u(w|_n)$ and $\tilde{v}^m(w) = \bigcup_n v^m(w|_n)$. For then if $m \in \mathbb{N}$ and $w \in 2^{\mathbb{N}}$ are given, there are $x_k, y_k, z_k \in \mathbb{N}^{\mathbb{N}}$ such that $x_k \to h(s_m0w)$, $y_k \to h(s_m1w)$ and $z_k \to \tilde{v}^m(w)$ such that for all k, (x_k, y_k, z_k). So, as F is closed, also

$$(h(s_m0w), h(s_m1w), \tilde{v}^m(w)) \in F,$$

whence $(h(s_m0w), h(s_m1w)) \in G$, showing that h is a homomorphism from G_0 to G.

An *n-approximation* is a pair (u, v) of functions $u\colon 2^n \to \mathbb{N}^n$ and $v\colon 2^{<n} \to \mathbb{N}^n$. Also, if (u, v) is an n-approximation and (u', v') is an $n+1$-approximation, we say that (u', v') *extends* (u, v) if $u(s) \sqsubseteq u'(si)$ and $v(t) \sqsubseteq v'(ti)$ for all $s \in 2^n$, $t \in 2^{<n}$ and $i = 0, 1$.

Suppose $A \subseteq X$ and (u, v) is an n-approximation. We define the set of A-realisations, $\mathbb{R}(A, u, v)$, to be the set of pairs of tuples $(x_s)_{s\in 2^n} \in \prod_{s\in 2^n} \big(A \cap N_{u(s)}\big)$ and $(z_t)_{t\in 2^{<n}} \in \prod_{t\in 2^{<n}} N_{v(t)}$ such that

$$(x_{s_m0t}, x_{s_m1t}, z_t) \in F$$

for all $s \in 2^n$, $m \in \mathbb{N}$ and $t \in 2^{n-m-1}$. So if (u_0, v_0) is the unique 0-approximation (i.e., $u(\emptyset) = \emptyset$ and v is the function with empty domain), we have $\mathbb{R}(A, u_0, v_0) = \{x_\emptyset \mid x_\emptyset \in A\} = A$. If (u, v) has no A-realised extension, we say that (u, v) is *A-terminal.*

Lemma 4. *Suppose (u, v) is an A-terminal n-approximation, then*

$$\mathbb{D}(A, u, v) = \{x_{s_n} \mid ((x_s)_{s\in 2^n}, (z_t)_{t\in 2^{<n}}) \in \mathbb{R}(A, u, v)\}$$

is G-discrete.

Proof. Suppose toward a contradiction that

$$((x^0_s)_{s\in 2^n}, (z^0_t)_{t\in 2^{<n}}),\ ((x^1_s)_{s\in 2^n}, (z^1_t)_{t\in 2^{<n}}) \in \mathbb{R}(A, u, v)$$

satisfy $(x^0_{s_n}, x^1_{s_n}) \in G$. Then for some $z_\emptyset \in \mathbb{N}^{\mathbb{N}}$, we have

$$(x^0_{s_n}, x^1_{s_n}, z_\emptyset) \in F,$$

and hence, setting $x_{si} = x^i_s$ and $z_{ti} = z^i_t$ for all $si \in 2^n$ and $ti \in 2^{<n+1} \setminus \{\emptyset\}$, we get an A-realisation $((x_s)_{s\in 2^{n+1}}, (z_t)_{t\in 2^{<n+1}})$ of an extension of (u, v), contradicting that (u, v) is A-terminal. □

Now define $\Phi \subseteq P(X)$ by

$$\Phi(A) \Leftrightarrow A \text{ is } G\text{-discrete}.$$

Since G is analytic, Φ is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$, and so, by the First Reflection Theorem, any G-discrete analytic set A is contained in a G-discrete Borel set A'. Using this, we can define a function D assigning to each Borel set $A \subseteq X$ a Borel subset given by

$$D(A) = A \setminus \bigcup\{\mathbb{D}(A, u, v)' \mid (u, v) \text{ is } A\text{-terminal }\}.$$

Note that, as there are only countably many approximations (u, v), the set $A \setminus D(A)$ is a countable union of G-discrete Borel sets.

Lemma 5. *Suppose (u,v) is an n-approximation all of whose extensions are A-terminal. Then (u,v) is $D(A)$-terminal.*

Proof. Note that if (u,v) is not $D(A)$-terminal, there is some extension (u',v') of (u,v) and some realisation $((x_s)_{s\in 2^{n+1}},(z_t)_{t\in 2^{<n+1}})\in \mathbb{R}(D(A),u',v')\subseteq \mathbb{R}(A,u',v')$. But since (u',v') is A-terminal, we have $\mathbb{D}(A,u',v')\cap D(A)=\emptyset$, contradicting that $\phi(x_{s_{n+1}})\in \mathbb{D}(A,u',v')\cap D(A)$. □

Now define, by transfinite induction, $D^0(X)=X$, $D^{\xi+1}(X)=D(D^\xi(X))$ and $D^\lambda(X)=\bigcap_{\xi<\lambda}D^\xi(X)$, whenever λ is a limit ordinal. Then $(D^\xi(X))_{\xi<\omega_1}$ is a well-ordered, decreasing sequence of Borel subsets of X, so the sets T_ξ of approximations (u,v) that are $D^\xi(X)$-terminal is an increasing sequence of subsets of the countable set of all approximations. It follows that for some $\xi<\omega_1$, we have $T_\xi=T_{\xi+1}$.

Now if $(u,v)\notin T_{\xi+1}$, then (u,v) is not $D(D^\xi(X))$-terminal and hence admits an extension (u',v') that is not $D^\xi(X)$-terminal either, whereby $(u',v')\notin T_\xi=T_{\xi+1}$. So if (u_0,v_0) denotes the unique 0-approximation and $(u_0,v_0)\notin T_{\xi+1}$, we can inductively construct $(u_n,v_n)\notin T_{\xi+1}$ extending each other. Setting

$$u=\bigcup_n u_n$$

and for $t\in 2^n$

$$v^m(t)=v_{n+m+1}(t),$$

we have the required monotone Lipschitz functions $u,v^m\colon 2^{<\mathbb{N}}\to\mathbb{N}^{<\mathbb{N}}$ to produce a continuous homomorphism from G_0 to G.

Conversely, if $(u_0,v_0)\in T_{\xi+1}$, then (u_0,v_0) is $D^{\xi+1}(X)$-terminal and hence $D^{\xi+2}(X)\subseteq D^{\xi+1}(X)\setminus \mathbb{D}(D^{\xi+1}(X),u_0,v_0)$. But, since (u_0,v_0) is the unique 0-approximation, we have

$$\mathbb{D}(D^{\xi+1}(X),u_0,v_0)=\mathbb{R}(D^{\xi+1}(X),u_0,v_0)=D^{\xi+1}(X),$$

whereby $D^{\xi+2}(X)=\emptyset$. It follows that

$$X=\bigcup_{\zeta<\xi+2}D^\zeta(X)\setminus D^{\zeta+1}(X)$$

is a countable union of G-discrete Borel sets. We can then define a Borel $\mathbb{N}$-colouring of G by letting $c(x)$ be a code for the discrete Borel subset of X to which x belongs. □

2. The Mycielski, Silver and Burgess dichotomies

Theorem 6 (Mycielski's Independence Theorem)**.** *Suppose X is a perfect Polish space and $R\subseteq X^2$ is a comeagre set. Then there is a continuous injection $\phi\colon 2^{\mathbb{N}}\to X$ such that for all distinct $x,y\in 2^{\mathbb{N}}$ we have $(\phi(x),\phi(y))\in R$.*

Proof. Let $d\le 1$ be a compatible complete metric on X and choose a decreasing sequence of dense open subsets $U_n\subseteq X^2$ such that $\bigcap_{n\in\mathbb{N}}U_n\subseteq R$. We construct a Cantor scheme $(C_s)_{s\in 2^{<\mathbb{N}}}$ of non-empty open subsets of X by induction on the length of s such that for all distinct $s,t\in 2^n$ and $i=0,1$, we have

$$\overline{C}_{si}\subseteq C_s,\quad \mathrm{diam}(C_s)\le\frac{1}{|s|+1},\quad\text{and}\quad C_s\times C_t\subseteq U_{n-1}.$$

To see how this is done, suppose that C_s has been defined for all $s\in 2^n$. Since X is perfect, we can find disjoint, non-empty open subsets D_{s0} and D_{s1} of C_s for every

$s \in 2^n$. Now, as U_n is dense, $U_n \cap (D_t \times D_{t'}) \neq \emptyset$ for all distinct $t, t' \in 2^{n+1}$ and so we can inductively shrink the D_t to open subsets C_t such that whenever $t, t' \in 2^{n+1}$ are distinct, we have $C_t \times C_{t'} \subseteq U_n$. By further shrinking the C_{si} if necessary, we can ensure that $\overline{C}_{si} \subseteq C_s$ and $\mathrm{diam}(C_s) \leq \frac{1}{|s|+1}$. Now letting $\phi \colon 2^{\mathbb{N}} \to X$ be defined by $\{\phi(x)\} = \bigcap_{n \in \mathbb{N}} C_{x|_n}$, we see that ϕ is continuous. Also, if $x, y \in 2^{\mathbb{N}}$ are distinct, then for all but finitely many n we have $(\phi(x), \phi(y)) \in C_{x|_n} \times C_{y|_n} \subseteq U_{n-1}$, so, since the U_n are decreasing, we have $(x, y) \in \bigcap_{n \in \mathbb{N}} U_n \subseteq R$. □

Theorem 7 (The Silver Dichotomy)**.** *Suppose E is a conalytic equivalence relation on a Polish space X. Then exactly one of the following holds*

- *E has at most countably many classes,*
- *there is a continuous injection $\phi \colon 2^{\mathbb{N}} \to X$ such that for distinct $x, y \in 2^{\mathbb{N}}$, $\neg \phi(x) E \phi(y)$.*

Proof. We define an analytic digraph G on X by setting $G = X^2 \setminus E$. Notice first that if $c \colon X \to \mathbb{N}$ is a Borel $\mathbb{N}$-colouring of G, then for all $x, y \in X$,

$$\neg x E y \Rightarrow (x, y) \in G \Rightarrow c(x) \neq c(y).$$

So for any $n \in \mathbb{N}$, $c^{-1}(n)$ is contained in a single equivalence class of E. Moreover, as $X = \bigcup_{n \in \mathbb{N}} c^{-1}(n)$, this shows that X is covered by countably many E-equivalence classes.

So suppose instead that there is no Borel $\mathbb{N}$-colouring of G. Then by Theorem 3 there is a continuous homomorphism $h \colon 2^{\mathbb{N}} \to X$ from G_0 to G. Now let $F = \{(x, y) \in 2^{\mathbb{N}} \times 2^{\mathbb{N}} \mid h(x) E h(y)\}$. Then F is meagre. For otherwise, by the Kuratowski–Ulam Theorem, there is some $x \in 2^{\mathbb{N}}$ such that F_x is non-meagre and hence, by Lemma 1, there are $y, z \in F_x$ such that $(y, z) \in G_0$. As h is a homomorphism it follows that $(h(y), h(z)) \in G = X^2 \setminus E$, which contradicts that $h(y) E h(x) E h(z)$. Therefore, applying Mycielski's Theorem to the meagre set F, we get a continuous function $f \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ such that for distinct $x, y \in 2^{\mathbb{N}}$, $(f(x), f(y)) \notin F$, i.e., $\neg h \circ f(x) E h \circ f(y)$. Letting $\phi = h \circ f$, we have the result. □

Lemma 8. *Suppose E is an analytic equivalence relation on a Polish space X. Then there is a decreasing sequence $(E_\xi)_{\xi < \omega_1}$ of Borel equivalence relations on X whose intersection is E.*

Proof. We claim that if $C \subseteq X^2$ is an analytic set disjoint from E, there is a Borel equivalence relation F separating E from C. To see this, define the following property Φ of pairs of subsets of X^2

$$\begin{aligned}
\Phi(A, B) \Leftrightarrow & \forall x, y, z \; \big((x, y) \notin A \;\vee\; (y, z) \notin A \;\vee\; (x, z) \notin B\big) \\
& \& \; \forall x, y \; \big((x, y) \notin A \;\vee\; (y, x) \notin B\big) \\
& \& \; \forall x \; (x, x) \notin B \\
& \& \; \forall x, y \; \big((x, y) \notin A \;\vee\; (x, y) \notin C\big).
\end{aligned}$$

Clearly, Φ is $\boldsymbol{\Pi}^1_1$ on $\boldsymbol{\Sigma}^1_1$, hereditary, and continuous upward in the second variable. Moreover, $\Phi(E, \sim E)$, so by the Second Reflection Theorem there is a Borel set $F \supseteq E$ such that $\Phi(F, \sim F)$. By definition of Φ, F is then a Borel equivalence relation disjoint from C.

Now, by the Lusin–Sierpiński Theorem, there is a decreasing sequence $(H_\xi)_{\xi < \omega_1}$ of Borel sets with intersection E. Using the above claim, we can, by transfinite induction, choose a sequence $(F_\xi)_{\xi<\omega_1}$ of Borel equivalence relations containing E such that each F_ξ separates E from $\sim (H_\xi \cap \bigcap_{\zeta<\xi} F_\zeta)$. It follows that the sequence $(F_\xi)_{\xi<\omega_1}$ is decreasing and, by choice of H_ξ, that it has intersection E. □

Theorem 9 (The Burgess Dichotomy)**.** *Let E be an analytic equivalence relation on a Polish space X. Then one of the following holds*

- *E has at most $\aleph_1$ classes,*
- *there is a continuous injection $\phi\colon 2^{\mathbb{N}} \to X$ such that for distinct $x, y \in 2^{\mathbb{N}}$, $\neg\phi(x)E\phi(y)$.*

Proof. Using Lemma 8, we can find a decreasing sequence $(E_\xi)_{\xi<om_1}$ of Borel equivalence relations whose intersection is E. Note that if for any ξ there is a continuous injection $\phi\colon 2^{\mathbb{N}} \to X$ such that for distinct $x, y \in 2^{\mathbb{N}}$, $\neg\phi(x)E_\xi\phi(y)$, then the same holds for E. So suppose not. Then by Silver's Dichotomy, Theorem 7, each E_ξ has at most countably many classes $B_{\xi,n}$, $n \in \mathbb{N}$. Let $\{A_\xi\}_{\xi<\omega_1} = \{B_{\xi,n}\}_{\xi<\omega_1, n\in\mathbb{N}}$.

Suppose now that E has at least $\aleph_2$ classes. We say that $C \subseteq X$ is *large* if it intersects at least $\aleph_2$ classes of E. Note that if C is large, then for some ξ, both $C \cap A_\xi$ and $C \setminus A_\xi$ are large. For if not, we let for every $\xi < \omega_1$,

$$C_\xi = \begin{cases} C \cap A_\xi, & \text{if } C \cap A_\xi \text{ is not large;} \\ C \setminus A_\xi, & \text{otherwise.} \end{cases}$$

Then $\bigcup_{\xi<\omega_1} C_\xi$ will intersect at most $\aleph_1$ E-classes and so

$$C \setminus \bigcup_{\xi<\omega_1} C_\xi = \bigcap_{\xi<\omega_1} C \setminus C_\xi$$

will be large. But for all $x, y \in \bigcap_{\xi<\omega_1} C \setminus C_\xi$ and all $\xi < \omega_1$, we have $x \in A_\xi$ if and only if $y \in A_\xi$ and hence xEy, contradicting the largeness of C.

Now let $\mathcal{U}_0 = \{U_n\}_{n\in\mathbb{N}}$ be a countable basis for the topology on X. We define inductively countable families $\mathcal{U}_n$ and $\mathcal{A}_n$ of Borel subsets of X such that

- $\mathcal{U}_0 \subseteq \mathcal{A}_0 \subseteq \mathcal{U}_1 \subseteq \mathcal{A}_1 \subseteq \ldots$,
- each $\mathcal{U}_n$ is the basis for a Polish topology on X,
- each $\mathcal{A}_n$ is a Boolean algebra of subsets of X,
- if $C \in \mathcal{U}_n$ is large, then there is some $A_\xi \in \mathcal{A}_n$, such that both $C \cap A_\xi$ and $C \setminus A_\xi$ are large.

It follows that $\mathcal{A} = \bigcup_{n\in\mathbb{N}} \mathcal{A}_n$ is a Boolean algebra whose elements form the basis for a Polish topology on X and such that whenever $C \in \mathcal{A}$ is large there is another $A_\xi \in \mathcal{A}$ such that both $C \cap A_\xi$ and $C \setminus A_\xi$ are large. Let $d \leq 1$ be a complete metric on X compatible with the topology induced by $\mathcal{A}$. Now, using that if $\bigcup_{n\in\mathbb{N}} C_n$ is large, then some C_n is large, we can build a Cantor scheme $(C_s)_{s\in 2^{<\mathbb{N}}}$ of elements of $\mathcal{A}$ such that $C_\emptyset = X$, $\mathrm{diam}(C_s) \leq \frac{1}{|s|+1}$, each C_s is large and for every s there is some $\xi < \omega_1$ such that $C_{s0} \subseteq A_\xi$, while $C_{s1} \subseteq \sim A_\xi$. It follows that if $\phi\colon 2^{\mathbb{N}} \to X$ is defined by $\{\phi(x)\} = \bigcap_{n\in\mathbb{N}} C_{x|_n}$, then for distinct $x, y \in 2^{\mathbb{N}}$ we have $\neg\phi(x)E\phi(y)$. □

Now as the isomorphism relation between the countable models of an $L_{\omega_1\omega}$-sentence is an analytic equivalence relation, we have the following corollary, initially proved by analysing the space of complete types.

Corollary 10 (Morley's Theorem)**.** *Suppose L is a countable language and σ is a $L_{\omega_1\omega}$ sentence. Then there are either a continuum of non-isomorphic countable models of σ or at most $\aleph_1$ non-isomorphic models of σ.*