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Complexity and homogeneity in Banach spaces

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Abstract. We provide an overview of a number of results concerning the complexity of isomorphism between separable Banach spaces. We also include some new results on the lattice structure of the set of spreading models of a Banach space.

Key words. Isomorphism of Banach spaces, Complexity of analytic equivalence relations, Minimal Banach spaces, Spreading models.

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To Nigel Kalton on the occasion of his 60th birthday.

Contents

1 Complexity of equivalence relations

A topic much in vogue in descriptive set theory for at least the last twenty years is the study of the relative complexity of Borel and analytic equivalence relations on Polish, i.e., completely metrisable separable spaces. The motivation comes from the general mathematical problem of classifying one class of mathematical objects by another, that is, given some class $\mathcal{A}$ of mathematical objects, e.g., separable Banach spaces, and

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a corresponding notion of isomorphism one tries to find complete invariants for the objects in $\mathcal{A}$ up to isomorphism. More explicitly, one would like to assign to each object in $\mathcal{A}$ an object in another category $\mathcal{B}$, such that two objects in $\mathcal{A}$ are isomorphic if and only if their assignments in $\mathcal{B}$ are isomorphic. If this can be done, one purports to have *classified* the elements of $\mathcal{A}$ by the elements of $\mathcal{B}$ up to isomorphism. Thus, for example, the discrete spectrum theorem of Halmos–von Neumann is a classification of the ergodic measure preserving transformations with discrete spectrum by their sets of eigenvalues. Similarly, by Stone duality, boolean algebras can by classified by compact spaces up to homeomorphism.

In descriptive set theory there has been an effort to systematise the notion of classification itself and to determine which classes of objects can properly be said to be classifiable in terms of another. This is done by restricting attention to classes of objects that can readily be made into a standard Borel space (i.e., the measurable space of a Polish, or separable completely metrisable space) and the corresponding equivalence relation of isomorphism. For example, the class of separable Banach spaces can be identified with the set of closed linear subspaces of some isometrically universal space, e.g., $C(2^{\mathbb{N}})$, as we shall see later. However, as uncountable standard Borel spaces are all Borel isomorphic, the perspective changes from the set of objects themselves to the equivalence relation that really encodes the complexity of the objects up to isomorphism. The precise definition is as follows.

Definition 1. *Let E and F be equivalence relations on standard Borel spaces X and Y respectively. We say that E is* Borel reducible *to F if there is a Borel function $f : X \to Y$ such that*
$$xEy \leftrightarrow f(x)Ff(y)$$
for all $x, y \in X$. We denote this by $E \leqslant_B F$ and informally say that E less complex than F. If both $E \leqslant_B F$ and $F \leqslant_B E$, then E and F are called Borel bireducible, *written $E \sim_B F$.*

For example, if we let $\mathfrak{B}$ be the standard Borel space of separable Banach spaces, then we will see that the relation of isomorphism is analytic as a subset of $\mathfrak{B}^2$, i.e., is the image by a Borel function of a standard Borel space. Since most other natural isomorphism relations are also analytic or even Borel one usually restricts the attention to this subclass. Thus, if we have two classes $\mathcal{A}$ and $\mathcal{B}$ of mathematical objects, that we have identified with standard Borel spaces $\mathcal{A} \sim X$ and $\mathcal{B} \sim Y$, respectively, and we denote by E and F the corresponding isomorphism relations on X and Y, then a reduction $\phi : X \to Y$ of E to F can be viewed as a classification of the objects in $\mathcal{A}$ by the objects in $\mathcal{B}$. To sum up, one can say that the theory of complexity of equivalence relations is the study of which invariants one can use for various mathematical objects.

We should also briefly mention a slightly other way of viewing Borel reducibility. If $\phi : X \to Y$ is a Borel reduction of E to F, then it is easy to see that ϕ induces an injection $\hat{\phi} : X/E \to Y/F$ and thus that the cardinality of X/E is smaller than that of Y/F. In fact, one can use this to define a notion of *effective cardinality* for quotient spaces, a notion that refines Cantor's concept of cardinality. Thus, for example, the effective cardinals are not wellordered or even linearly ordered.

A large part of the general theory on analytic equivalence relations has concerned the structure of $\leqslant_B$, i.e., the hierarchy of analytic equivalence relations under this ordering, and the place of naturally occurring isomorphism relations in the hierarchy. It is now known that $\leqslant_B$ is extremely complex as an ordering, while, on the other hand, most naturally occurring classification problems tend to be bireducible with a fairly small class of equivalence relations for which there is also a nice structure theory. We now proceed to describe some of these.

Apart from equivalence relations with only countable many classes the simplest Borel equivalence relation is the relation $(\mathbb{R}, =)$ of equality of real numbers. In fact, a result of Silver [46] implies that any Borel equivalence relation either has countably many classes, and thus is just a countable partition of the underlying space into Borel sets, or $(\mathbb{R}, =)$ is Borel reducible to it, and thus has continuum many classes. Equivalence relations reducible to $(\mathbb{R}, =)$, called *smooth*, are simply those "isomorphism" relations that admit real numbers as complete invariants. One outstanding example is Ornstein's theorem that entropy is a complete invariant for Bernoulli shifts.

A deep result due to Harrington, Kechris, and Louveau [28] shows that among non-smooth Borel equivalence relations there is a minimum (with respect to $\leqslant_B$) one, which is called E_0. It is the relation of eventual agreement between infinite binary sequences, i.e., for $x, y \in 2^{\mathbb{N}}$,

$$xE_0y \leftrightarrow \exists n\, \forall m \geqslant n\; x_m = y_m.$$

Besides being minimum above $(\mathbb{R}, =)$, E_0 is also characterised as being maximum for *hyperfinite* Borel equivalence relations, i.e., those that can be written as an increasing union of countably many Borel equivalence relations with finite classes. All Borel actions of $\mathbb{Z}$ give rise to such orbit equivalence relations.

Of special interest among analytic equivalence relations are those that appear as the orbit equivalence relation of a continuous action of a Polish group, i.e., a topological group whose topology is Polish. Since any countable or locally compact second countable group is Polish, this class encompasses most of the orbit equivalence relations usually studied in analysis. It turns out that for each Polish group H there is a maximum, with respect to $\leqslant_B$, orbit equivalence relation induced by H, and even among all orbit equivalence relations induced by actions of countable (or what turns out to be the same, locally compact second countable) groups there is a maximum one denoted by E_∞. It is characterised as the maximum Borel equivalence relation all of whose classes are countable. Similarly, there is a maximum orbit equivalence relation among all those induced by Polish groups, which we denote by E_G.

We should also mention two other equivalence relations of complexity between E_∞ and E_G: the relation $=^+$ on infinite sequences of complex numbers enumerating the same sets, i.e., for $(x_n), (y_n) \in \mathbb{C}^{\mathbb{N}}$,

$$(x_n)_n =^+ (y_n)_n \leftrightarrow \{x_n\}_n = \{y_n\}_n,$$

and the $\leqslant_B$-maximum orbit equivalence E_{S_∞} induced by the infinite symmetric group S_∞. One easily sees that $=^+$ is induced by an action of S_∞, which is Polish, and thus also E_{S_∞} reduces to E_G. And on the other hand, if E is a Borel equivalence relation with countable classes, then for each z in the domain of E, we can enumerate $[z]_E$

in some way depending on z, which will reduce E to $=^+$. So also $E_\infty \leqslant_B =^+$. The discrete spectrum theorem of Halmos and von Neumann says that two ergodic measure preserving automorphisms T and S of Lebesgue spaces with discrete spectrum are isomorphic if and only if they have the same countable set of eigenvalues, $\Lambda(T)$ and $\Lambda(S)$. Thus, if we fix a Borel function ϕ that to each T picks out an enumeration (λ_n) of the set of eigenvalues $\Lambda(T)$, then we see that T and S are isomorphic if and only if $\phi(T) =^+ \phi(S)$ and hence isomorphism reduces to $=^+$. On the other hand, there is no way of constructing ϕ so that it makes the same choice of enumeration of $\Lambda(T)$ and $\Lambda(S)$ provided the two *sets* are the same. This has to do with the fact that the quotient space $\mathbb{C}^N/=^+$ is not countably separated, or in our terminology that $=^+$ is non-smooth. And in fact, isomorphism of measure preserving automorphisms with discrete spectrum is Borel bireducible with $=^+$ (see Foreman [21]).

An interesting discovery due to Kechris and Louveau [32] is that there are analytic equivalence relations that are not reducible to orbit equivalence relations, or equivalently, to E_G. In fact, there is one, minimal among Borel equivalence relations having this property, namely E_1, which is the relation of eventual agreement between infinite sequences of real numbers. I.e., for $x, y \in \mathbb{R}^\mathbb{N}$,

$$xE_1y \leftrightarrow \exists n\, \forall m \geqslant n\; x_m = y_m.$$

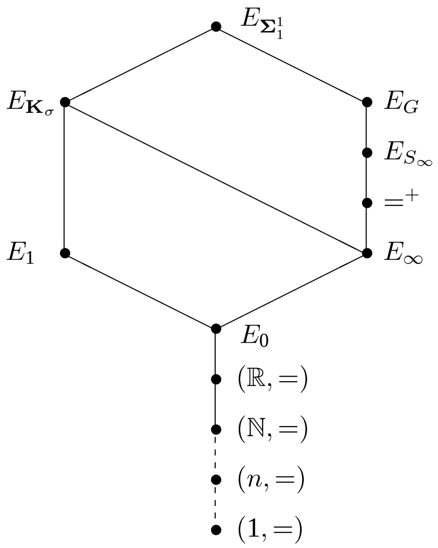
To this moment, E_1 is the only known obstruction for Borel equivalence relations to being reducible to an orbit equivalence relation. As with E_0, E_1 is not only characterised by its minimality property, but also by the fact that it is maximum among Borel equivalence relations that can be written as a union of a countable increasing chain of smooth equivalence relations, called *hypersmooth*.

Beyond E_1 there is the relation $E_{\boldsymbol{K}_\sigma}$ maximum among all $\boldsymbol{K}_\sigma$ equivalence relations, i.e., those that can be written as an increasing union of compact sets. It is defined on the space $\prod_{n=1}^\infty\{1,\ldots,n\}$ by the following formula

$$xE_{\boldsymbol{K}_\sigma}y \leftrightarrow \exists N\, \forall n\; |x_n - y_n| \leqslant N.$$

Another realisation of this important degree is, for example, a growth relation on functions $f:\mathbb{N}\to\mathbb{N}$ [43]. It is not too hard to see that $E_\infty \leqslant_B E_{\mathbf{K}_\sigma}$, while, on the other hand, $=^+ \not\leqslant_B E_{\mathbf{K}_\sigma}$.

As a last example we shall mention the most complex of all analytic equivalence relations, namely the *complete analytic equivalence relation* $E_{\mathbf{\Sigma}^1_1}$. It is simply characterised by being maximum among all analytic relations. Combinatorial realisations of this relation may be found in [36] and [17].



Simplified diagram of complexity of analytic equivalence relations.

2 Standard Borel spaces

For us to consider the class of separable Banach spaces as a standard Borel space, we notice that up to isometry they are all represented as closed subspaces of some metrically universal separable space such as $C(2^{\mathbb{N}})$. We now use a by now standard way to make the closed subspaces of $C(2^{\mathbb{N}})$ into a standard Borel space. First we denote by $F(X)$ the set of all closed subsets of $X = C(2^{\mathbb{N}})$ and equip $F(X)$ with its so called Effros–Borel structure, which is the σ-algebra generated by the sets on the form

$$\{F \in F(X) \bigm| F \cap U \neq \emptyset\},$$

where U varies over open subsets of X. Equipped with the Effros–Borel structure, $F(X)$ becomes a standard Borel space, i.e., isomorphic as a measure space with $\mathbb{R}$ given its standard Borel algebra. One then easily checks that various relations on $F(X)^2$ and $X \times F(X)$ are Borel, e.g., for $x \in X$ and $F \in F(X)$,

$$x \in F \leftrightarrow \forall n\ (x \in U_n \to F \cap U_n \neq \emptyset),$$

where $\{U_n\}$ is a fixed basis for the topology on X. So the relation '$\in$' is Borel in $X \times F(X)$. We can now also verify that the set $\mathfrak{B}$ of closed *linear* subspaces of X form a Borel set in $F(X)$. To do this, notice that for $Y \in F(X)$, Y is a linear subspace if and only if

$$0 \in Y \ \&\ \forall n, m\ \forall p, q \in \mathbb{Q}$$
$$\big(Y \cap U_n \neq \emptyset \ \&\ Y \cap U_m \neq \emptyset \to Y \cap (p \cdot U_n + q \cdot U_m) \neq \emptyset\big).$$

As all the quantifiers are over countable sets, $\mathfrak{B}$ is a Borel set in $F(X)$ and thus a standard Borel space in its own right and we designate this as *the* standard Borel space of separable Banach spaces. One could of course have constructed this space in other ways, e.g., as the set of norms on a countable set of vectors, but experience shows that any other way of proceeding leads to equivalent results.

Let us now see that, e.g., isometry of separable Banach spaces is an analytic equivalence relation.

$$\begin{aligned} Y \cong_{\mathrm{i}} Z \leftrightarrow & \exists (y_n) \in X^{\mathbb{N}}\ \exists (z_n) \in X^{\mathbb{N}} \\ & \forall m\ \big(Y \cap U_m \neq \emptyset \to \exists n\ y_n \in U_m\big) \\ & \&\ \forall m\ \big(Z \cap U_m \neq \emptyset \to \exists n\ z_n \in U_m\big) \\ & \&\ \forall n\ \big(y_n \in Y\big)\ \&\ \forall n\ \big(z_n \in Z\big) \\ & \&\ \forall n, m\ \|y_n - y_m\| = \|z_n - z_m\|, \end{aligned}$$

which simply expresses that two separable spaces are isometric if and only if they have countable dense subsets which are isometric.

In many cases, we are not interested in all separable Banach spaces, but only in a Borel set of spaces. The most common situation is when we consider only the subspaces of a particular space. But it is not hard to see that if $X \in \mathfrak{B}$, then $\mathfrak{B}(X) = \{Y \in \mathfrak{B} \mid Y \subseteq X\}$ is Borel and we can thus talk about the complexity of the isomorphism relation restricted to this set.

If we are interested in a relation between subsequences of a given basis $(e_n)_n$ of a space X, e.g. isomorphism of the closed linear spans or equivalence of the subsequences, we identify the space $[\mathbb{N}]^{\mathbb{N}}$ of infinite subsets of $\mathbb{N}$ with the set of subsequences of $(e_n)_n$. The associated embedding of $[\mathbb{N}]^{\mathbb{N}}$ into $\mathfrak{B}(X)$ is Borel and therefore we can also compute the complexity of isomorphism etc. between subspaces spanned by subsequences of the basis. The case of block-subspaces is more complicated and will be developed later.

3 Relations between separable Banach spaces

Certainly among the relations of similarity between (infinite-dimensional) Banach spaces a few stick out as being of outmost importance, namely, linear isomorphism, (linear) isometry, Lipschitz isomorphism, and uniform homeomorphism. It is well-known that the project of classifying separable Banach spaces up to isomorphism is essentially an impossible task and the tendency nowadays is to settle for something less, namely to find a "basis" for the class of separable Banach spaces, i.e., a list of recognisable spaces such that every space contains a copy of some space in the list.

Another natural question however is also what the complexity of the various classification problems is in the hierarchy of analytic equivalence relations. For example, if one can show that the relation of isomorphism is of high complexity then this lends mathematical sense to the feeling that this relation remains intractable.

Concerning these various relations, Gao and Kechris [22] have shown that isometry between separable complete metric spaces is bireducible with the most complex orbit equivalence relation, E_G, and thus the relation of isometry between separable Banach spaces is at most of this complexity. However, recently, Melleray [38] has been able to show that isometry on $\mathfrak{B}$ is itself Borel bireducible with E_G. This should be contrasted with the result in [36] saying that the relation of (linear) isometric biembeddability between separable Banach spaces is a complete analytic equivalence relation.

Concerning the complexity of isomorphism, lower bounds were successively obtained by Bossard (E_0, [7]), Rosendal (E_1, [41]), and Ferenczi-Galego (the product $E_{\boldsymbol{K}_\sigma} \otimes =^+$, [15]). And the complexity was finally determined by a very recent and yet unpublished result of Ferenczi, Louveau, and Rosendal [17].

Theorem 2 (Ferenczi–Louveau–Rosendal [17]). *The relations of isomorphism, Lipschitz isomorphism, (complemented) biembeddability, and Lipschitz biembeddability between separable Banach spaces are analytic complete, i.e., are maximum among analytic equivalence relations in the Borel reducibility ordering $\leqslant_B$. The same holds for the relation of permutative equivalence of unconditional basic sequences.*

This result thus has the surprising consequence that it is possible to assign in a Borel manner to each separable Banach space X an unconditional basic sequence (e_i^X) such that two spaces X and Y are isomorphic if and only if (e_i^X) and (e_i^Y) are permutatively equivalent. This seems to contradict the feeling that it is somehow easier to check permutative equivalence rather than isomorphism. However, the proof of this result gives no hint as to how this assignment could be computed and obviously the basis (e_i^X) does not itself has to be related to the space X. It would certainly be very interesting to find an "explicit" such assignment which could potentially be of use in applications.

We shall not go into the proof of this result, but only mention that it relies on an elaborate construction due to S. Argyros and P. Dodos [4] that allows one to construct spaces containing any specified analytic set of ℓ_p's as its minimal subspaces.

In the fundamental paper [24] W. T. Gowers proved his now famous dichotomy theorem stating that any infinite-dimensional Banach space contains either an unconditional basic sequence or an HI subspace. This result, in combination with another result of A. Komorowski–N. Tomczak-Jaegermann [33], also solved the homogeneous space problem: ℓ_2 is the only (infinite-dimensional) Banach space which is isomorphic to all its infinite-dimensional closed subspaces.

Given a separable Banach space X which is not isomorphic to ℓ_2, the question remains as to what the possible complexity of isomorphism (and also biembeddability etc.) is on the set of subspaces of X. This is in line with the general question of Gowers concerning the structure of the set of subspaces of a separable Banach space under the quasiorder of embeddability. Even for classical spaces these questions remain unsolved, only lower bounds are obtained.

We shall say that a separable Banach space is *analytic complete* if isomorphism between its subspaces is analytic complete. It is said to be *ergodic* [20] if isomorphism between its subspaces reduces the relation E_0. Analytic complete spaces are those spaces on which isomorphism between subspaces reflects the complexity of isomorphism between all separable Banach spaces (equivalently, of the most complex analytic equivalence relation). Ergodic spaces are such that isomorphism between their subspaces is not smooth though it is not clear whether this is also sufficient for a space to be ergodic.

Question 3. *Let X be a separable Banach space which is not isomorphic to ℓ_2. Is X ergodic? Is X analytic complete?*

By the proof of [17], the universal unconditional basis of Pełczyński spans an analytic complete space. The spaces c_0 and ℓ_p, $1 \leqslant p < 2$ are ergodic [15], and, in fact, isomorphism between their subspaces has complexity at least $E_{\boldsymbol{K}_\sigma}$; isomorphism between subspaces of L_p, $1 \leqslant p < 2$, has complexity at least $E_{\boldsymbol{K}_\sigma} \otimes =^+$. Concerning $\ell_p, p > 2$, it is only known that there are uncountably many non-isomorphic subspaces [34]. Isomorphism between subspaces of Tsirelson's space T, as well as of its dual T^*, has complexity at least E_1 [41].

When the space X is equipped with a Schauder basis, it is natural to restrict the question of complexity of isomorphism to the class of block-subspaces of X. A natural topological setting for this is the space $bb(X)$ of normalised block-sequences of X, seen as a subspace of $X^{\mathbb{N}}$ where X is equipped with the norm topology. The relation of isomorphism induces a relation denoted $\simeq$ on $bb(X)$, and the canonical map from $bb(X)$ into $\mathfrak{B}(X)$ is Borel.

In this setting, the spaces c_0 and ℓ_p, $1 \leqslant p < +\infty$, with their canonical bases, are the natural homogeneous examples; their bases are *block-homogeneous*, meaning that all their block-subspaces are isomorphic.

Concerning T with its canonical basis, it follows from [41] and [43] that

$$E_1 \leqslant_B (bb(T), \simeq) \leqslant_B E_{K_\sigma},$$

but the exact complexity of $(bb(T), \simeq)$ remains unknown. Note that the basis of T is strongly asymptotic ℓ_1, where a basis (e_i) is *strongly asymptotic* ℓ_p if for some $C < \infty$ and some increasing function $f : \mathbb{N} \to \mathbb{N}$, every normalised sequence $(x_i)_{i=1}^n$ ($n \in \mathbb{N}$) of disjointly supported vectors from $[(e_i)_{i=f(n)}^\infty]$ is C-equivalent to the unit vector basis of ℓ_p^n. For example, the convexification T_p of Tsirelson's space, $1 < p < +\infty$, has a strongly asymptotic ℓ_p basis, and T^* has a strongly asymptotic ℓ_∞ basis. It is more generally proved in [9] that for any space X with a strongly asymptotic ℓ_p basis ($1 \leqslant p \leqslant +\infty$), that is not equivalent to the canonical basis of c_0 or ℓ_p, E_0 is Borel reducible to isomorphism between block-subspaces of X (even to isomorphism between spaces spanned by subsequences of the basis).

It is unknown whether c_0 and ℓ_p, $1 \leqslant p < +\infty$, are the only spaces with a block-homogeneous basis. This would be a natural generalisation of Zippin's theorem about perfectly homogeneous bases [47]. Note that a positive answer would imply the homogeneous Banach space theorem of Gowers and Komorowski–Tomczak-Jaegermann, via the fact that every Banach space contains a basic sequence and the non-trivial fact that the spaces c_0 and ℓ_p, $p \neq 2$, are not homogeneous. The following question could be easier to solve than the first part of Question 3.

Question 4. *Let X be a separable Banach space with a Schauder basis which is not isomorphic to c_0 or ℓ_p, $1 \leqslant p < +\infty$. Does it follow that $E_0 \leqslant_B (bb(X), \simeq)$?*

4 Isomorphism and spaces with an unconditional basis

The general idea of this section is that spaces with an unconditional basis which are not ergodic must satisfy some algebraic properties (isomorphism with their square, their hyperplanes etc.). They therefore resemble Hilbert space more than a generic separable Banach space. Quite similar ideas in the context of the Schröder-Bernstein property for Banach spaces were first developed by N. Kalton [30].

4.1 Subspaces spanned by subsequences of the basis

Given a space X with a Schauder basis (e_i), we first look at the space $ss(e_i)$ of subsequences of the basis, which we identify with the space $[\mathbb{N}]^{\mathbb{N}}$ of infinite subsets of $\mathbb{N}$ considered as a subset of $2^{\mathbb{N}}$ with its induced topology. The relation induced by isomorphism of the corresponding linear spans will be denoted $\simeq$, and for $K \geqslant 1$, $\simeq^K$ denotes the relation induced by isomorphism with constant at most K.

Theorem 5. *Let X be a Banach space with an unconditional basis (e_i). Then $E_0 \leqslant_B (ss(e_i), \simeq)$, or X is uniformly isomorphic to $X \oplus Y$, for all Y generated by a finite or infinite subsequence of the basis - and therefore isomorphic to its square and to its hyperplanes - and, moreover, isomorphic to an infinite direct sum of uniformly isomorphic copies of itself.*

Proof. The proof uses the following lemma. We need to define the relation E_0' between infinite subsets of $\mathbb{N}$: two sets $A, B \subseteq \mathbb{N}$ are E_0' equivalent if $|(A \cup B) \setminus A| = |(A \cup B) \setminus B| < \infty$, i.e., if A and B have the same finite co-cardinality in $A \cup B$. If (e_i) is a basic sequence and we identify a subset $A \subseteq \mathbb{N}$ with the subsequence it generates, then one easily sees that the relation of equivalence between subsequences of (e_i) is E_0'-invariant.

Lemma 6 (cf. [42]). *Let E be an analytic equivalence relation on $[\mathbb{N}]^{\mathbb{N}}$ that is E_0'-invariant. Then either $E_0 \leqslant_B E$ or E has a comeagre class. In the latter case there are $A, B \in [\mathbb{N}]^{\mathbb{N}}$ such that $AE(\sim A)$ and $BE(B \setminus \min B)$.*

Proof. If E is meagre as a subset of $[\mathbb{N}]^{\mathbb{N}} \times [\mathbb{N}]^{\mathbb{N}}$, then it is not difficult to build finite successive subsets a_n^0 and a_n^1 of $\mathbb{N}$ such that $|a_n^0| = |a_n^1|$ and such that the map $T : 2^{\mathbb{N}} \to [\mathbb{N}]^{\mathbb{N}}$ defined by $T(\alpha) = \bigcup_{k \in \mathbb{N}} a_k^{\alpha(k)}$ Borel reduces E_0 to $\simeq$. If on the other hand E is non-meagre, then by a classical result of Kuratowski–Ulam, [31] Theorem 8.41, some E-class $\mathbb{A}$ is non-meagre. Therefore $\mathbb{A} \cap U$ is comeagre in U for some basic open set U. By E_0'-invariance it follows that $\mathbb{A}$ is comeagre.

Now the map $c : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ defined by $c(A) = \sim A$ is a homeomorphism of $2^{\mathbb{N}}$ and as $[\mathbb{N}]^{\mathbb{N}}$ is comeagre in $2^{\mathbb{N}}$, also $\mathbb{A}$ is comeagre in $2^{\mathbb{N}}$. But then also $c(\mathbb{A})$ is comeagre in $2^{\mathbb{N}}$ and hence $\mathbb{A} \cap c(\mathbb{A})$ is comeagre too. So pick some $A \in \mathbb{A} \cap c(\mathbb{A})$ which is co-infinite. Then $AE(\sim A)$.

Similarly, for each $n \in \mathbb{N}$, the map $C \in [\mathbb{N}]^{\mathbb{N}} \mapsto C\triangle\{n\} \in [\mathbb{N}]^{\mathbb{N}}$ is an involution homeomorphism of $[\mathbb{N}]^{\mathbb{N}}$ and thus there must be some $B \in \mathbb{A}$ such that for all n, $B\triangle\{n\} \in \mathbb{A}$. In particular, $BE(B \setminus \min B)$. □

Going back to the proof of Theorem 5, assuming E_0 is not Borel reducible to $\simeq$ on $[\mathbb{N}]^{\mathbb{N}}$, we apply Lemma 6 and obtain a comeagre $\simeq$-class $\mathbb{A}$. Classical Banach space arguments about perturbation by finite-dimensional subspaces then ensure that the set $\mathbb{A}_K = \{A \in [\mathbb{N}]^{\mathbb{N}} : A \simeq^K A_0\}$ is comeagre for some fixed $A_0 \in \mathbb{A}$ and some $K \geqslant 1$.

If a and b are non empty subsets of $\mathbb{N}$ and a is finite, recall that a and b are successive, $a < b$, when $\max(a) < \min(b)$. If b is a finite subset of $\mathbb{N}$, say that $A \in [\mathbb{N}]^{\mathbb{N}}$ *passes through* b if $A = a \cup b \cup C$ for some finite subset a of $\mathbb{N}$ and some $C \in [\mathbb{N}]^{\mathbb{N}}$ such that $a < b < C$. We now use the following classical characterization of comeagre subsets of $[\mathbb{N}]^{\mathbb{N}}$, see, e.g., [19].

Lemma 7. *Let $\mathbb{A}$ be a subset of $[\mathbb{N}]^{\mathbb{N}}$. Then $\mathbb{A}$ is comeagre if and only if there exists a countable family $\{a_k, k \in \mathbb{N}\}$ of finite successive subsets of $\mathbb{N}$ such that whenever $A \in [\mathbb{N}]^{\mathbb{N}}$ passes through a_k for infinitely many k's, then A belongs to $\mathbb{A}$.*

Choose such a family $\{a_k, k \in \mathbb{N}\}$ for our set $\mathbb{A}$. To conclude the proof of Theorem 5, we fix $A_0 \in \mathbb{A}_K$ such that $(\sim A_0) \in \mathbb{A}_K$, and we obtain by unconditionality of the basis $(e_n)_{n\in\mathbb{N}}$ of X,

$$X \simeq [e_n, n \in A_0] \oplus [e_n, n \notin A_0] \simeq [e_n, n \in A_0]^2,$$

and

$$[e_n, n \in A_0] \simeq [e_n, n \in a_k, k \in \mathbb{N}] = [e_n, n \in a_{2k}, k \in \mathbb{N}] \oplus [e_n, n \in a_{2k+1}, k \in \mathbb{N}],$$

therefore $[e_n, n \in A_0] \simeq [e_n, n \in A_0]^2 \simeq X$.

Fix $k_0 \in \mathbb{N}$, then $\{1, \ldots, k_0\} \cup \bigcup_{k>k_0} a_k$ and $\bigcup_{k>k_0} a_k$ belong to $\mathbb{A}_K$. Therefore

$$[e_n, n \in \bigcup_{k>k_0} a_k] \simeq [e_1, \ldots, e_{k_0}] \oplus [e_n, n \in \bigcup_{k>k_0} a_k],$$

and by taking a direct sum with the appropriate subspace, we obtain

$$X \simeq [e_n, n > k_0].$$

Whenever I is a subset of $\mathbb{N}$ and $Y = [e_n, n \in I]$, we may find a partition $\{I_1, I_2\}$ of I and two infinite disjoint subsets N_1 and N_2 of $\mathbb{N}$ such that $I_1 \cap (\bigcup_{k\in N_2}[\min a_k, \max a_k]) = \emptyset$ and $I_2 \cap (\bigcup_{k\in N_1}[\min a_k, \max a_k]) = \emptyset$. It follows that $I_1 \cup (\bigcup_{k\in N_2} a_k)$ and $I_2 \cup \bigcup_{k\in N_2} a_k$ belong to $\mathbb{A}_K$ and therefore

$$X \simeq X \oplus [e_n, n \in I_1]$$

and

$$X \simeq X \oplus [e_n, n \in I_2].$$

Thus

$$X \simeq X \oplus [e_n, n \in I_1] \oplus [e_n, n \in I_2] \simeq X \oplus Y.$$

Finally, let $\{B_i, i \in \mathbb{N}\}$ be a partition of $\mathbb{N}$ in infinite subsets and let for $i \in \mathbb{N}$, $C_i = \bigcup_{k \in B_i} a_k$. Then $C_i \in \mathbb{A}_K$ for all i, while also $\bigcup_{i \in \mathbb{N}} C_i \in \mathbb{A}_K$. Therefore

$$X \simeq [e_n, n \in \bigcup_{i \in \mathbb{N}} C_i] \simeq \oplus_{i \in \mathbb{N}} [e_n, n \in C_i],$$

with $[e_n, n \in C_i] \simeq X$, for all $i \in \mathbb{N}$.

All the isomorphisms are obtained with uniform constants depending only on K and the unconditional constant of the basis. □

Note that these results may be generalized to spaces with unconditional decompositions in the same spirit as in [30], see [16].

4.2 Subspaces spanned by block-sequences of the basis

We consider here a space X with a Schauder basis (e_i) and the class of subspaces generated by block-bases, i.e., sequences of vectors with successive supports (called successive vectors). Rather than $bb(X)$, the correct setting for the main theorem here (Theorem 8) seems to be the space denoted $bb_{\mathbf{Q}}(X)$ defined as follows. Let first $\mathbf{Q}$ be a countable subfield of $\mathbb{R}$ such that any finite $\mathbf{Q}$-linear combination of the basis vectors has norm in $\mathbf{Q}$ and let $\mathbb{D}$ be the set of non-zero blocks with coefficients in $\mathbf{Q}$, $\mathbb{D}_1$ be the set of norm 1 vectors in $\mathbb{D}$. The assumption on $\mathbf{Q}$ allows us to normalise while staying in $\mathbf{Q}$. The set $bb_{\mathbf{Q}}(X)$ is then the set of block-bases of vectors in $\mathbb{D}_1$, equipped with the product topology of the discrete topology on $\mathbb{D}_1$. The relation of isomorphism induces a relation $\simeq$ on $bb_{\mathbf{Q}}(X)$. As $\mathbb{D}_1$ is countable, this topology is Polish and epsilon matters may be forgotten until the applications. When we deal with isomorphism classes, they are not relevant since a small enough perturbation preserves the class. Note that the canonical map of $bb_{\mathbf{Q}}(X)$ into $\mathfrak{B}(X)$ is Borel, and this allows us to forget about the Effros-Borel structure when computing the complexity on block-subspaces. Note also that the reduction of E_0 to $\simeq$ on $bb_{\mathbf{Q}}(X)$ in Theorem 8 will provide a reduction to $\simeq$ on $bb(X)$ as well.

The set $fbb_{\mathbf{Q}}(X)$ denotes the set of finite successive sequences of blocks in $\mathbb{D}_1$. The support $supp(a)$ of such a sequence a is the union of the supports of the blocks composing the sequence.

Theorem 8 (Ferenczi–Rosendal [20]). *Let X be a Banach space with an unconditional basis. Then E_0 is Borel reducible to $(bb_{\mathbf{Q}}(X), \simeq)$ or there exists a block-subspace X_0 of X which is uniformly isomorphic to $X_0 \oplus Y$ for all block-subspaces Y of X.*

Proof. The argument being similar to the case of subsequences, we shall just sketch the proof. If E_0 is not Borel reducible to $(bb_{\mathbf{Q}}(X), \simeq)$, then there exists a comeagre $\simeq$-class $\mathbb{A} \subset bb_{\mathbf{Q}}(X)$. Now comeagre subsets of $bb_{\mathbf{Q}}(X)$ may be characterized up to small perturbations by the next lemma from [20]. For a a finite block-sequence and b a finite or infinite block-sequence such that $supp(a) < supp(b)$, $a^\frown b$ denotes the finite or infinite block-sequence which is the concatenation of a and b. If $b \in fbb_{\mathbf{Q}}(X)$, and $A \in bb_{\mathbf{Q}}(X)$, then A *passes through* b if $A = a^\frown b^\frown C$ for some $a \in fbb_{\mathbf{Q}}(X)$ and some $C \in bb_{\mathbf{Q}}(X)$. If $\mathbb{A}$ is a subset of $bb_{\mathbf{Q}}(X)$ and $\Delta = (\delta_n)_{n\in\mathbb{N}}$ is a sequence of strictly positive real numbers, written $\Delta > 0$, we denote by $\mathbb{A}_\Delta$ the Δ-expansion of $\mathbb{A}$ in $bb_{\mathbf{Q}}(X)$, that is $x = (x_n) \in \mathbb{A}_\Delta$ if and only if there exists $y = (y_n) \in \mathbb{A}$ such that $\|y_n - x_n\| < \delta_n, \forall n \in \mathbb{N}$.

Lemma 9. *Let $\mathbb{A}$ be comeagre in $bb_{\mathbf{Q}}(X)$. Then for all $\Delta > 0$, there exist successive finite block-sequences $a_n, n \in \mathbb{N}$ in $fbb_{\mathbf{Q}}(X)$ such that any element of $bb_{\mathbf{Q}}(X)$ passing trough infinitely many of the a_n's is in $\mathbb{A}_\Delta$.*

By classical perturbation arguments, $\mathbb{A} = \mathbb{A}_\Delta$ for some Δ small enough. Let $a_n = (a_n^1, \ldots, a_n^{m_n})$, $n \in \mathbb{N}$ be given by Lemma 9 and let $X_0 = [A_0]$, for some $A_0 \in \mathbb{A}$.

Fix $Y = (y_n)_{n\in\mathbb{N}}$ in $bb_{\mathbf{Q}}(X)$. We may find a partition $(I_k)_{k\in\mathbb{N}}$ of $\mathbb{N}$ in successive intervals and an increasing sequence (n_k) of integers such that for all k in $\mathbb{N}$,

$$supp(y_n, n \in I_k) < supp(a_{n_{k+1}}) < supp(y_n, n \in I_{k+2}).$$

Therefore the block sequence $A = (z_n)_{n\in\mathbb{N}}$ defined by

$$\{z_n, n \in \mathbb{N}\} = \bigcup_{k\in\mathbb{N}} \{y_n, n \in I_{2k+1}\} \bigcup \bigcup_{k\in\mathbb{N}} \{a_{n_{2k}}^1, \ldots, a_{n_{2k}}^{m_{2k}}\},$$

belongs to $\mathbb{A}$. It follows that

$$X_0 \simeq X_0 \oplus [y_n, n \in \bigcup_{k\in\mathbb{N}} I_{2k+1}],$$

and likewise

$$X_0 \simeq X_0 \oplus [y_n, n \in \bigcup_{k\in\mathbb{N}} I_{2k}],$$

whence finally

$$X_0 \simeq X_0 \oplus Y.$$

Additional care in the proof guarantees uniformity. □

5 Embeddability, biembeddability, and isomorphism

In the paper [24] W.T. Gowers proved his dichotomy theorem stating that any infinite-dimensional Banach space contains either an unconditional basic sequence or an HI subspace. Actually Gowers proved more refined structure results that set the stage for a detailed list of inevitable classes of subspaces. To simplify notation in the following we shall use $X \sqsubseteq Y$ to denote that a space X embeds isomorphically into Y. Moreover, we shall assume all spaces considered are infinite-dimensional.

A space is said to be *minimal* if it is $\sqsubseteq$-minimal among its subspaces, *quasi-minimal* if any two subspaces have a common $\sqsubseteq$-minorant, and *strictly quasi-minimal* if it is quasi-minimal but does not contain a minimal subspace. On the other hand, we shall say that a space with a Schauder basis has the *strong Casazza property* (in reference to a property defined by Casazza, see [23]) if no two disjointly supported block-subspaces are isomorphic. Two spaces are said to be *incomparable* in case neither of them embeds into the other, and *totally incomparable* if no space embeds into both of them.

Theorem 10 (Gowers [24]). *Let X be an infinite dimensional Banach space. Then X contains a subspace Y with one of the following properties, which are all possible and mutually exclusive.*

1. Y is hereditarily indecomposable,

2. Y has an unconditional basis with the strong Casazza property,

3. Y has an unconditional basis and is strictly quasi-minimal,

4. Y has an unconditional basis and is minimal.

Type (1) spaces were discovered by Gowers and Maurey [25] in 1991, and a type (2) space was constructed by Gowers in [23] and further analysed in [26]. Tsirelson's space T, the precursor of Banach spaces with "exotic" properties such as Gowers and Maurey's examples is a typical example of a type (3) space. The spaces c_0, ℓ_p for $1 \leqslant p < +\infty$, the dual T^* of Tsirelson's space, and Schlumprecht's space S [2] are the main known examples of spaces of type (4).

In each case one can ask what the structure of the relations of embeddability, $\sqsubseteq$, biembeddability, $\equiv$, and isomorphism, $\cong$, is on the subspace in question. In his paper [24] Gowers had asked the question of what quasiorders could be realised as the set of subspaces of a separable Banach space ordered by $\sqsubseteq$.

Theorem 11. *Let X be a separable infinite-dimensional Banach space belonging to one of the four types given by Gowers' Theorem 10. Then for each of the relations $\sqsubseteq$, $\equiv$, and $\cong$, we have lower bounds on the complexity as given in the following diagram.*

Type	$\sqsubseteq$	$\equiv$	$\cong$
(1)	$\mathbb{R}$, ω_1, *and* ω_1^**-chains, uncountable Borel antichain*	E_0	E_0
(2)	$\mathbb{R}$, ω_1, *and* ω_1^**-chains,* *uncountable Borel set of totally incomparable spaces*	E_0	E_0
(3)	ω_1^**-chain, uncountable Borel antichain*	E_0	E_0
(4)	*trivial*	*trivial*	*none*

We should mention that by classical results, all uncountable Borel sets are of the size of the continuum, so we get, for example, continuum size antichains in the case of strictly quasiminimal spaces.

Proof. In order to prove this result, we will show how to get it from, in each case, stronger and more refined results that also have somewhat larger scopes.

We may first deduce the results for $\equiv$ and $\cong$ for type (1) and (2) from Lemma 6. For if we are given a Schauder basis (e_i), we can for $\equiv$ and $\cong$ consider the corresponding analytic equivalence relation on $[\mathbb{N}]^{\mathbb{N}}$ obtained by identifying $A \in [\mathbb{N}]^{\mathbb{N}}$ with the subspace $[e_i]_{i\in A}$. As an HI space is non-isomorphic with all of its proper subspaces, we thus see that for (e_i) of type (1) there is no $B \in [\mathbb{N}]^{\mathbb{N}}$ such that $B \equiv B \setminus \min B$ or $B \cong B \setminus \min B$, whence E_0 must reduce to both $\equiv$ and $\cong$. Similarly, in case of type (2), there is no A such that $A \equiv \sim A$ or $A \cong \sim A$, and thus again E_0 reduces to both $\equiv$ and $\cong$.

In the case of type (2), there is also an explicit reduction by the map $\alpha \in 2^{\mathbb{N}} \mapsto [e_{2n+\alpha(n)}, n \in \mathbb{N}]$.

Consider now the chains in the case of type (1) and type (2). Fix first a basic sequence either spanning an HI space or such that any two disjointly supported spaces are totally incomparable.

Assume now that for infinite sets $A, B \subseteq \mathbb{N}$ we have $|A \setminus B| < \infty$, but $|B \setminus A| = \infty$, which we denote by $A \subsetneq^* B$. Then we can find some $B' E_0' B$ such that $A \subseteq B'$ and $|B' \setminus A| = \infty$, whence $[e_i]_{i\in A} \sqsubseteq [e_i]_{i\in B'} \cong [e_i]_{i\in B}$. On the other hand, in the case of HI spaces, $[e_i]_{i\in B} \cong [e_i]_{i\in B'} \not\sqsubseteq [e_i]_{i\in A}$, and in case of type (2), $[e_i]_{i\in A}$ and $[e_i]_{i\in B'\setminus A}$ are totally incomparable and hence $[e_i]_{i\in B} \cong [e_i]_{i\in B'} \not\sqsubseteq [e_i]_{i\in A}$ again. In any case,

$$A \subsetneq^* B \Rightarrow [e_i]_{i\in A} \sqsubseteq [e_i]_{i\in B} \;\&\; [e_i]_{i\in B} \not\sqsubseteq [e_i]_{i\in A}.$$

By simple diagonalisation it is now easy to construct a sequence $(A_\xi)_{\xi<\omega_1}$ such that if $\xi < \zeta < \omega_1$, then $A_\xi \subsetneq^* A_\zeta$, whence $([e_i]_{i\in A_\xi})_{\xi<\omega_1}$ gives a ω_1-chain in the ordering $\sqsubseteq$. Similarly for an ω_1^*-chain. Now to construct the $\mathbb{R}$-chain, we identify $\mathbb{Q}$ with $\mathbb{N}$ and $\mathbb{R}$ with the left parts of the corresponding Dedekind cuts. Thus, if $r < s$ belong to $\mathbb{R}$, then they correspond to subsets $A_r \subsetneq^* A_s$ of $\mathbb{N}$. Thus, $([e_i]_{i\in A_r})_{r\in\mathbb{R}}$ forms an $\mathbb{R}$-chain in $\sqsubseteq$.

To get an uncountable Borel set of totally incomparable subspaces, i.e. such that no subspace of one embeds into the other, in the case of type (2), we simply pick an uncountable Borel set of almost disjoint subsets of $\mathbb{N}$ and notice that if A and B are almost disjoint, i.e., they are both infinite with finite intersection, then $[e_i]_{i\in A}$ and $[e_i]_{i\in B}$ are totally incomparable.

The last fairly simple part of the picture is the existence of the ω_1^*-chain in the case of type (3). Again this is a direct set theoretical diagonalisation. We use here the well-known fact that the embeddability relation on quasiminimal spaces is downwards σ-directed, i.e., that any countable family of subspaces have a common subspace up to isomorphism. This is easy to see, for if X is quasiminimal and (Y_m) is a countable family of infinite dimensional subspaces, where we suppose Y_0 has a basis (e_i), then by quasiminimality, we can inductively pick block sequences $(x_n^{(m)})$ such that $(x_n^{(m+1)})$

is a block of $(x_n^{(m)})$, $[x_n^{(m)}] \sqsubseteq Y_m$ and then take the diagonal sequence $(x_{2n}^{(n)})_n$. Then $[x_{2n}^{(n)}]_n \sqsubseteq [x_n^{(m)}]_n \sqsubseteq Y_m$ for all m. By diagonalisation and using the fact that no subspace is minimal one now constructs an ω_1^* sequence in the ordering $\sqsubseteq$.

The final four results, namely the existence of uncountable Borel antichains in type (1) and (3) and the reduction of E_0 to $\equiv$ and $\cong$ in type (3), however, seems to necessitate more advanced techniques, in particular, metamathematics and methods of effective descriptive set theory. Moreover, also the determinacy result of Gowers [24] is here used in its full force, i.e., for analytic sets. To our knowledge, this is one of the only known applications of his result other than for closed/open sets.

To begin, let us first state a general Ramsey principle for equivalence relations that is specifically adapted for the geometry of Banach spaces.

Theorem 12 (Rosendal [42]). *Let E be an analytic equivalence relation on $[\mathbb{N}]^{\mathbb{N}}$ that is E_0'-invariant, i.e., $E_0' \subseteq E$. Then either $E_0 \leqslant E$ or for some infinite subset $A \subseteq \mathbb{N}$, the set $[A]^{\mathbb{N}}$ is contained in a single E-class.*

For example, if (e_i) is a basis and E the induced relation on $[\mathbb{N}]^{\mathbb{N}}$ of isomorphism between spaces spanned by subsequences of the basis, then E is easily E_0'-invariant. We should mention that the only known proof of Theorem 12 uses metamathematical methods and it would be interesting to find a more topological proof of this. Probably, one would have to find some uniformity that could allow for standard methods. As an application of the statement, we see that if E_0 does not reduce to isomorphism on the set of subspaces of $[e_i]$, then (e_i) has an isomorphically homogeneous subsequence, i.e., a subsequence all of whose further subsequences span isomorphic spaces. Nevertheless, this principle does not in itself appear to be enough to get E_0 in type (3). For that we will need a better result due to Ferenczi, relying on the one hand on Theorem 12 and on the other hand on the methods of Pelczar [40], who herself was inspired by the closed case of Gowers' determinacy result.

Theorem 13 (Ferenczi [14]). *Let X be a separable Banach space saturated with isomorphically homogeneous basic sequences. Then X contains a minimal space. In fact, it is enough to suppose that X is saturated with basic sequences whose closed span embed into the closed span of any of their subsequences.*

Pelczar's original result was quite similar, but had a considerably stronger hypothesis, namely, that X was saturated with subsymmetric basic sequences.

Consider now the case of spaces of type (3). If X is a space of type (3), and E_0 does not reduce to isomorphism between its subspaces, then any basic sequence in X has an isomorphically homogeneous subsequence and hence X is saturated by such sequences. By Theorem 13, X contains a minimal space, which is impossible. Similarly, if E_0 does not reduce to biembeddability between the subspaces of X, then X is saturated by sequences embeddable into all of the spaces spanned by their subsequences and again X must contain a minimal space, which is a contradiction. Therefore, E_0 reduces in each case.

For the record, we spell out the argument in the following corollary.

Corollary 14. *Let X be a separable infinite-dimensional Banach space. Then either E_0 Borel reduces to isomorphism between its subspaces or X contains a minimal space.*

The result we need for the existence of antichains is quite similar, but differs rather by its proof. Again it does not rely heavily on geometric properties of Banach spaces, but mostly on methods of combinatorics and descriptive set theory, and, in particular, not only Gowers' determinacy theorem, but also on the solution to the distortion problem by Odell and Schlumprecht [39].

Theorem 15 (Rosendal [42]). *Let X be an infinite-dimensional separable Banach space. Either X contains a minimal space or there is an uncountable Borel set of incomparable subspaces.*

This immediately implies the existence of uncountable Borel antichains in type (1) and type (3) spaces and thus finishes the proof. □

The remaining case is that of minimal spaces. The structures of $\sqsubseteq$ and $\equiv$ are trivial on such spaces. Concerning $\cong$, there is of course no general result saying that there are many non-isomorphic subspaces in this case, as the space could be Hilbertian, but if the space is not homogeneous, it seems plausible that there must be many isomorphism classes. Indeed, c_0 and ℓ_p, $1 \leqslant p < 2$, [15], as well as Tsirelson's dual space T^* (by the method of [41]) are ergodic. Note also that any non-reflexive minimal space must contain c_0 or ℓ_1 and therefore be ergodic, and that Theorem 5 and Theorem 8 may provide other classes of minimal ergodic spaces.

Question 16. *Does there exist an analytic complete minimal Banach space?*

We end this section by sketching the proofs of Theorem 15 and of Theorem 13 in the $\cong$-homogeneous case. In each case we may assume that the space X has a Schauder basis (e_i). All vectors will belong to the set $\mathbb{D}$ of non-zero finite $\mathbf{Q}$-linear combinations of the basis vectors (e_i), where $\mathbf{Q}$ is a countable subfield of $\mathbb{R}$ closed under computing norms. We recall that $\mathbb{D}_1$ denotes the set of normalized blocks in $\mathbb{D}$, that the space of infinite $\mathbb{D}_1$-block bases is denoted by $bb_{\mathbf{Q}}(X)$ and the set of finite $\mathbb{D}_1$-block bases is denoted $fbb_{\mathbf{Q}}(X)$.

Sketch of the proof of Theorem 15: We notice first that $\sqsubseteq$ restricted to the standard Borel space of subspaces of X is an analytic quasiorder. So Theorem 15 amounts to saying that either $\sqsubseteq$ has a minimal element or an uncountable Borel antichain. The idea of the proof is to replace $\sqsubseteq$ by a *Borel* quasiorder R containing $\sqsubseteq$ and sufficiently reflecting the properties of the latter. Then one can employ the analysis by Harrington, Marker, and Shelah [29, 27] of Borel quasiorders to deduce the result. The exact result we need can be deduced from [27].

Theorem 17 (Harrington–Marker–Shelah [27]). *If R is a downwards σ-directed Borel quasiorder on a standard Borel space, then either R has an uncountable Borel antichain or a minimal element.*

Here R is downwards σ-directed if any countable family in P has a common minorant. We have the following standard observation.

Lemma 18. *Suppose X is quasi-minimal. Then $\sqsubseteq$ is downwards σ-directed on $bb_{\mathbf{Q}}(X)$.*

For $Y = (y_i)$, $Z = (z_i) \in bb_{\mathbf{Q}}(X)$, let $Y \leqslant Z$ if Y is a blocking of Z and put $Y \leqslant^* Z$ if for some k, $(y_i)_{i \geqslant k} \leqslant Z$. Also, if $\Delta = (\delta_i)$ is an infinite sequence of strictly positive reals, write $d(Y, Z) < \Delta$ if $\forall i \; \|y_i - z_i\| < \delta_i$. Set $Y =^* Z$ if $\exists k \; \forall i \geqslant k \; y_i = z_i$. Evidently, $Y =^* Z$ implies $Y \sim Z$.

For a subset $\mathbb{A} \subseteq bb_{\mathbf{Q}}(X)$ let $\mathbb{A}^* = \{Y \in bb_{\mathbf{Q}}(X) \mid \exists Z \in \mathbb{A} \; Z =^* Y\}$ and $\mathbb{A}_\Delta = \{Y \in bb_{\mathbf{Q}}(X) \mid \exists Z \in \mathbb{A} \; d(Z, Y) < \Delta\}$. Notice that if $\mathbb{A}$ is analytic so are both $\mathbb{A}^*$ and $\mathbb{A}_\Delta$. We also set $[Y] = \{Z \in bb_{\mathbf{Q}}(X) \mid Z \leqslant Y\}$ and notice that $[Y]$ is a Borel subset of $bb_{\mathbf{Q}}(X)$. $\mathbb{A}$ is said to be *large* in $[Y]$ if for any $Z \in [Y]$ we have $[Z] \cap \mathbb{A} \neq \emptyset$.

For $\mathbb{A} \subseteq bb_{\mathbf{Q}}(X)$ and $Y \in bb_{\mathbf{Q}}(X)$, the Bagaria–Gowers–López-Abad game $\Game_Y^{\mathbb{A}}$ is defined as follows: Player I plays in the k'th move of the game a vector $z_k \in \mathbb{D}_1$ such that $z_{k-1} < z_k$. Player II responds by either doing nothing or playing a vector $v \in \mathbb{D}_1$ such that $v \in [z_{l+1}, \ldots, z_k]$ where l was the last move where II played a vector. We say that player II wins the game if in the end she has produced a block-basis $V = (v_i) \in \mathbb{A}$. A slight variant of this game is shown in [6] to be equivalent to the game studied by Gowers in [24]. It follows from Gowers' determinacy result in [24] that if $\mathbb{A} \subseteq bb_{\mathbf{Q}}(X)$ is analytic, large in $[Y]$ and Δ is given, then for some $Z \in [Y]$, II has a winning strategy in the game $\Game_Z^{\mathbb{A}_\Delta}$. However, due to the complexity of the sets we are dealing with, we need to have a stronger determinacy result, which holds under stronger set-theoretical assumptions.

Lemma 19. (MA $+ \neg$CH) *Suppose $\mathbb{W} \subseteq bb_{\mathbf{Q}}(X)$ is a $\mathbf{\Sigma}^1_2$ set, large in some $[Y]$ and $\Delta > 0$. Then II has a winning strategy in $\Game_Z^{\mathbb{W}^*_\Delta}$ for some $Z \in [Y]$.*

Lemma 20. (MA $+ \neg$CH) *Suppose that X does not contain a minimal subspace. Then for any $W \in bb_{\mathbf{Q}}(X)$ there is $Y \in [W]$ and a Borel function $\phi \colon [Y] \to [Y]$ such that for all $Z \in [Y]$,*

$$\phi(Z) \leqslant Z$$

and

$$Z \not\sqsubseteq \phi(Z).$$

Proof. We can assume that $W = X$. Also, as c_0 is minimal, X does not contain c_0 and therefore, by the solution to the distortion problem by Odell and Schlumprecht [39], we can by replacing X by a block-subspace suppose that we have two positively separated sets F_0, F_1 of the unit sphere, such that for any $Y \in bb_{\mathbf{Q}}(X)$ there are $x, y \in \mathbb{D}_1$ such that $x \in F_0$, $y \in F_1$, and $x, y \in Y$. We call such sets *inevitable*.

Let now

$$\mathbb{A} = \{Y = (y_i) \in bb_{\mathbf{Q}}(X) \mid \forall i \; y_i \in F_0 \cup F_1\}$$

and for $Y \in \mathbb{A}$ let $\alpha(Y) \in 2^{\mathbb{N}}$ be defined by

$$\alpha(Y)(i) = 0 \Leftrightarrow y_i \in F_0.$$

Then $\alpha\colon \mathbb{A} \to 2^{\mathbb{N}}$ is continuous. Fix an uncountable closed set $P \subseteq 2^{\mathbb{N}}$ of almost disjoint subsets of $\mathbb{N}$ and let

$$\mathbb{B} = \{Y \in \mathbb{A} \mid \alpha(Y) \in P\}.$$

As P is closed, so is $\mathbb{B}$ and by the inevitability of F_0 and F_1, $\mathbb{B}$ is large in every $[Y]$.

By fixing a Borel bijection between P and the set of infinite sequences of finite non-zero **Q**-tuples, we can see each $p \in P$ as coding a way to construct an infinite **Q**-block sequence of any $Y \in bb_{\mathbf{Q}}(X)$. Denote this infinite block sequence by Y^p and notice that Y^p is a blocking of Y though not necessarily normalised.

Consider now the set

$$\mathbb{W} = \{Y = (y_i) \in bb_{\mathbf{Q}}(X) \mid (y_{2i}) \in \mathbb{B} \ \wedge\ [y_{2i+1}] \not\sqsubseteq \big[(y_{2i+1})^{\alpha(y_{2i})}\big]\}.$$

So $\mathbb{W}$ consists of the blocks $(y_i) \in bb_{\mathbf{Q}}(X)$ such that (y_{2i}) codes a subspace of $[y_{2i+1}]$ into which $[y_{2i+1}]$ does not embed. First of all, $\mathbb{W}$ is clearly coanalytic, and again, using the inevitability of F_0 and F_1 and the fact that X contains no minimal subspace, one can verify that $\mathbb{W}$ is large in $bb_{\mathbf{Q}}(X)$.

Take now some $\Delta = (\delta_i)$ depending on the basis constant such that $d(Y, Y') < \Delta$ implies $Y \sim Y'$, and, moreover, such that $\delta_i < \frac{1}{2} d(F_0, F_1)$. By Lemma 19 we can find a $Y \in bb_{\mathbf{Q}}(X)$ such that II has a winning strategy σ in the game $\Game_Y^{\mathbb{W}^*_\Delta}$.

We shall now show how the function $\phi\colon [Y] \to [Y]$ is defined. For this, let $Z \in [Y]$ be given and suppose I plays the sequence $(z_i) = Z$ in the game $\Game_Y^{\mathbb{W}^*_\Delta}$. Then using the strategy σ, II will respond to Z by playing some $V = (v_i) \leqslant Z$, $V \in \mathbb{W}^*_\Delta$. There is therefore some $W = (w_i) \in \mathbb{W}$ such that $V \in \{W\}^*_\Delta$. This W might not, however, be Borel in V. Nevertheless, we can in a Borel manner compute $\alpha(w_{2i}) \in P$, because for almost all i

$$d(v_{2i}, w_{2i}) < \delta_{2i} < \frac{1}{2} d(F_0, F_1)$$

and $w_{2i} \in F_0 \cup F_1$. So by letting $p(i) = 0$ if $d(v_{2i}, F_0) \leqslant d(v_{2i}, F_1)$ and $p(i) = 1$ otherwise, we see that p and $\alpha(w_{2i})$ differ in finitely many coordinates. Moreover, as different elements of P differ in infinitely many coordinates, $\alpha(w_{2i})$ is the unique element of P that differs from p in finitely many coordinates and hence $\alpha(w_{2i})$ is Borel in V. Also by the assumption on $\Delta > 0$,

$$(v_{2i+1}) \sim (w_{2i+1}),$$

and thus

$$(v_{2i+1})^{\alpha(w_{2i})} \sim (w_{2i+1})^{\alpha(w_{2i})}.$$

Now $W \in \mathbb{W}$, so

$$[w_{2i+1}] \not\sqsubseteq \big[(w_{2i+1})^{\alpha(w_{2i})}\big],$$

whence also

$$[v_{2i+1}] \not\sqsubseteq \big[(v_{2i+1})^{\alpha(w_{2i})}\big].$$

Renormalising the blocking $(v_{2i+1})^{\alpha(w_{2i})}$, we finally find a $U = (u_i) \leqslant V$ such that $[v_i] \not\sqsubseteq [u_i]$. Thus, $\phi(V) = U$ works. ∎

The idea of coding with inevitable sets was originally used by J. López-Abad to give a new proof of Gowers' determinacy Theorem [35]. The import of its use here is to impose a relationship between Z and $\phi(Z)$. One is tempted to just apply Gowers' determinacy result directly to get $\phi(Z)$ from Z, but Gowers' Theorem only allows use to force $\phi(Z)$ to belong to a certain set, not such that $\phi(Z)$ stands in a certain relation to Z.

Now to finish the proof of the theorem, we can suppose that X is quasiminimal, but does not contain a minimal subspace. In that case, $(bb_{\mathbf{Q}}(X), \sqsubseteq)$ is a downwards σ-directed analytic quasiorder without a minimal element. Moreover, by replacing X with a subspace, we can suppose that this latter fact is testified by a Borel function ϕ that to each $Y \in bb_{\mathbf{Q}}(X)$ picks out a subspace of Y into which Y does not embed.

The fact that non-minimality is witnessed by a Borel function allows us now to reflect this property to a Borel quasiorder R on $bb_{\mathbf{Q}}(X)$ such that $Y \sqsubseteq Z \Rightarrow YRZ$. But, as R has no minimal element, it must have an uncountable Borel antichain, which thus also is an uncountable Borel antichain for $\sqsubseteq$.

This proves the result under (MA+$\neg$CH), but additional work, again using Gowers' determinacy result and coding with inevitable sets, allows us to show that the property of having a minimal subspace is actually $\mathbf{\Sigma}^1_2$ and not just its face value $\mathbf{\Sigma}^1_3$. Similarly, having a continuum of incomparable subspaces is easily $\mathbf{\Sigma}^1_2$, and thus the statement of the theorem is itself $\mathbf{\Sigma}^1_2$. By Shoenfield's absoluteness theorem, the additional set-theoretical assumptions can thus be eliminated from the proof. $\square$

Before we prove Theorem 13, we introduce some other notation. We denote by $\mathcal{G}_{\mathbf{Q}}(X)$ the set of subspaces of X spanned by elements of $bb_{\mathbf{Q}}(X)$ and by $Fin_{\mathbf{Q}}(X)$ the set of subspaces spanned by elements of $fbb_{\mathbf{Q}}(X)$. Standard notation will be used concerning successive vectors (respectively finite dimensional subspaces) on (e_i). For $L, M \in \mathcal{G}_{\mathbf{Q}}(X)$, $L \subset^* M$ means that $L = [l_i, i \in \mathbb{N}]$, where $l_i \in M$ for all but finitely many i's.

Sketch of the proof of Theorem 13: Let X be a space which is saturated with isomorphically homogeneous sequences.

By a standard use of Ramsey's Theorem and a diagonalisation, we may assume that there exists $K \geqslant 1$ such that every block-sequence in $bb_{\mathbf{Q}}(X)$ has a further block-sequence in $bb_{\mathbf{Q}}(X)$ in which is K-isomorphically homogeneous. We fix some $C > K$.

For L, M two block-subspaces in $\mathcal{G}_{\mathbf{Q}}(X)$, define the infinite game $G_{L,M}$ between two players as follows; for each $k \in \mathbb{N}$, m_k, n_k are integers, x_k is a vector in $\mathbb{D}_1$, y_k a vector in $\mathbb{D}$, and F_k belongs to $Fin_{\mathbf{Q}}(X)$.

$1:$		$n_1 < x_1 \in L,$ m_1		$n_2 < x_2 \in L,$ m_2	$\ldots$
$2:$	n_1		$m_1 < F_1 \subseteq M,$ $y_1 \in F_1, n_2$		$m_2 < F_2 \subseteq M,$ $y_2 \in F_1 + F_2, n_3$ $\ldots$

Player 2 wins the game $G_{L,M}$ if $(y_n)_{n\in\mathbb{N}}$ is C-equivalent to $(x_n)_{n\in\mathbb{N}}$.

We shall now provide a stabilising subspace on which Player 2 has "sufficiently many" winning strategies in games $G_{L,M}$. The reader may look for more about this type of proof in the survey by S. Todorcevic [5] where it is called "combinatorial forcing". A crucial point in the definition of $G_{L,M}$ is that the moves of the players are asymptotic, in the sense that one player can force the other to play "far enough" along the basis, and this will allow the stabilisation; on the other hand, the use of finite-dimensional spaces F_k leaves enough room for Player 2 to pick vectors y_k which are not necessarily successive on the basis. There are indeed some spaces with a basis, such as T^*, where minimality cannot be proved by finding copies of the basis as successive vectors, and therefore the apparent technicality of the definition is necessary (at least in the case of the weaker hypothesis in Theorem 13).

A *state* is a couple (a, b) with $a \in fbb_{\mathbf{Q}}(X)$ and $b \in (Fin_{\mathbf{Q}}(X) \times \mathbb{D})^{<\omega}$ such that $|a| = |b|$ or $|a| = |b| + 1$. The set S of states is countable, and corresponds to the possible states of a game $G_{L,M}$ after a finite number of moves were made, restricted to elements which do affect the outcome of the game from that state (i.e m_k's and n_k's are forgotten). Thus for $s \in S$, we may define $G_{L,M}(s)$ as the game $G_{L,M}$ starting from the state s. For example, if $s = (a, b)$ with $|a| = 2, |b| = 1$, the game $G_{L,M}(s)$ will start with 1 playing some integer m_2, then 2 playing (F_2, y_2, n_3), etc.

We require a classical "stabilisation lemma" used by Maurey in [37].

Lemma 21. *Let N be a countable set and let $\mu : \mathcal{G}_{\mathbf{Q}}(X) \to 2^N$ be a $(\subset^*, \subset)$-monotone map. Then there exists a stabilising subspace $M_0 \in \mathcal{G}_{\mathbf{Q}}(X)$, i.e., such that $\mu(M) = \mu(M_0)$ for any $M \subset^* M_0$.*

Let now $\tau : \mathcal{G}_{\mathbf{Q}}(X) \to 2^S$ be defined by $s \in \tau(M)$ if there exists $L \subset M$ such that Player 2 has a winning strategy for the game $G_{L,M}(s)$. By the asymptotic nature of the game, τ is $(\subset^*, \subset)$-increasing, and therefore there exists M_0 which is stabilizing for τ. We then define a map $\rho : \mathcal{G}_{\mathbf{Q}}(X) \to 2^S$ by setting $s \in \rho(L)$ if Player 2 has a winning strategy for the game $G_{L,M_0}(s)$. Then ρ is $(\subset^*, \subset)$-decreasing, and therefore there exists a block-subspace $L_0 \in \mathcal{G}_{\mathbf{Q}}(X)$ of M_0 which is stabilising for ρ. Finally, we check that $\rho(L_0) = \tau(L_0) = \tau(M_0)$. We may assume that $L_0 = [f_n, n \in \mathbb{N}]$, with (f_n) K-isomorphically homogeneous.

We prove that L_0 is minimal. Fix M a block subspace of L_0. We use induction to construct a subsequence $(f_{n_k})_k$ of (f_n), and a sequence $(F_k, y_k)_k$ such that for all k, $F_k \subset M, y_k \in F_1 + \cdots + F_k$,

$$s_k = ((f_{n_1}, \ldots, f_{n_k}), (y_1, \ldots, y_k, F_1, \ldots, F_k)) \in \rho(L_0).$$

Then we are done, since $(f_{n_1}, \ldots, f_{n_k}) \sim^C (y_1, \ldots, y_k)$ for all k, so $(f_{n_k})_k$ is C-equivalent to $(y_k)_k$, and M contains a CK-isomorphic copy of L_0.

Given $s_k = ((f_{n_1}, \ldots, f_{n_k}), (y_1, \ldots, y_k, F_1, \ldots, F_k)) \in \rho(L_0)$, Player 2 has a winning strategy for $G_{L_0,M_0}(s_k)$, therefore if we pick some $n_{k+1} > n_k$ large enough, the state

$$s'_k = ((f_{n_1}, \ldots, f_{n_{k+1}}), (y_1, \ldots, y_k, F_1, \ldots, F_k))$$

belongs to $\rho(L_0) = \tau(M)$, and 2 has a winning strategy for $G_{L,M}(s'_k)$ for some $L \subset M$. Therefore there exist $F_{k+1} \subset M$ and $y_{k+1} \in F_1 + \cdots + F_{k+1} \subset M$, such that

$$s_{k+1} = ((f_{n_1}, \ldots, f_{n_{k+1}}), (y_1, \ldots, y_{k+1}, F_1, \ldots, F_{k+1}))$$

belongs to $\tau(M) = \rho(L_0)$.

It therefore only remains to initiate the induction, i.e. prove that the empty state $(\emptyset, \emptyset)$ belongs to $\rho(L_0)$. To obtain this result, one refines the notion of $\cong$-homogeneity in order to imitate the notion of subsymmetry of basic sequences.

Definition 22. *A block-sequence $(x_n)_{n\in\mathbb{N}}$ in X is C-*continuously isomorphically homogeneous *if there exists a continuous map $\phi : [\mathbb{N}]^{\mathbb{N}} \to \mathbb{D}^{\mathbb{N}}$ such for all $A \in [\mathbb{N}]^{\mathbb{N}}$, $\phi(A)$ is a sequence of vectors spanning $[x_n]_{n\in A}$ and is C-equivalent to $(x_n)_{n\in\mathbb{N}}$.*

In this definition, the set $\mathbb{D}^{\mathbb{N}}$ is equipped with the product of the discrete topology on $\mathbb{D}$, which turns it into a Polish space. The following result is a consequence of Ellentuck's and Louveau's proofs of the infinite-dimensional Ramsey theorem.

Lemma 23. *Let $(x_n)_{n\in\mathbb{N}} \in bb_{\mathbf{Q}}(X)$ be a block-sequence which is K-isomorphically homogeneous, and let ϵ be positive. Then some subsequence of $(x_n)_{n\in\mathbb{N}}$ is $K + \epsilon$-continuously isomorphically homogeneous.*

The proof of Theorem 13 ends with the final observation:

Lemma 24. *Assume $(l_n)_{n\in\mathbb{N}}$ is a block-sequence in $bb_{\mathbf{Q}}(X)$, which is C-continuously isomorphically homogeneous, and let $L = [l_n, n \in \mathbb{N}]$. Then Player 2 has a winning strategy in the game $G_{L,L}$, therefore $(\emptyset, \emptyset) \in \tau(L)$.*

Therefore some subsequence of (f_n) is $K+\epsilon$-continuously isomorphically homogeneous, and if $K+\epsilon < C$, spans a block-subspace L_{00} such that $(\emptyset, \emptyset) \in \tau(L_{00}) = \rho(L_0)$. □

6 On the Komorowski–Tomczak-Jaegermann side

R. Anisca [3] developed the techniques of Komorowski–Tomczak-Jaegermann [33] to define finite dimensional decomposition versions of the notion of local unconditional structure and extracted the following consequence.

Theorem 25 (Anisca [3]). *Let X be a separable Banach space with finite cotype and non-isomorphic to ℓ_2. Then for each $k \in \mathbb{N}$, there exists a subspace of $\ell_2(X)$ which has a $k + 1$-uniform FDD but not a k-uniform FDD.*

Corollary 26. *Let X be a separable Banach space non-isomorphic to ℓ_2. Then $\ell_2(X)$ has infinitely many non-isomorphic subspaces.*

Anisca actually obtains this corollary assuming X has finite cotype. If X doesn't have finite cotype, then $\ell_2(X)$ contains, e.g., copies of the spaces $(\oplus_{n\in\mathbb{N}} l_p^n)_2$, $1 \leqslant p \leqslant +\infty$, which are easily seen to be mutually non-isomorphic for $p > 2$ (see [9], Corollary 18), and therefore the result holds as well.

Using the techniques of [19, 42] in the case of spaces with unconditional finite dimensional decomposition, Ferenczi and Galego obtain:

Theorem 27 (Ferenczi–Galego [15]). *Let $1 \leqslant p < +\infty$. Let $X = (\oplus_{n\in\mathbb{N}} F_n)_p$, where the F_n's are finite dimensional. Then X is ergodic or $X \simeq \ell_p(X)$. The similar result holds for c_0-sums.*

The following consequence was observed in [9]. A strongly asymptotic ℓ_p FDD is the obvious generalization of a strongly asymptotic ℓ_p basis; examples are ℓ_p-sums or c_0-sums, as well as Tsirelson sums, of finite dimensional spaces.

Corollary 28 (Dilworth–Ferenczi–Kutzarova–Odell [9]). *Let $1 \leqslant p \leqslant +\infty$. Let X be a Banach space with a strongly asymptotic ℓ_p FDD. Then X is isomorphic to ℓ_2 or X contains infinitely many non-isomorphic subspaces.*

Question 29. *What is the exact complexity of isomorphism between subspaces of c_0 or ℓ_p? Is ℓ_p, $p > 2$, ergodic?*

Question 30. *What is the exact complexity of isomorphism between subspaces of Tsirelson's space T? Between block-subspaces of T?*

Recall that $E_1 \leqslant_B (bb(T), \simeq) \leqslant_B E_{K_\sigma}$; computing the exact complexity of $\simeq$ on $bb(T)$ may not be out of reach.

Question 31. *What is the exact complexity of isomorphism between subspaces of Schlumprecht's space S? Between block-subspaces of S? Is S ergodic?*

Schlumprecht's space is a relevant example by its minimality and the fact that E_0 is Borel reducible to permutative equivalence between its normalised block-sequences [13].

Question 32. *Does there exist a space such that the complexity of isomorphism between its subspaces is exactly E_0? Is there a space with a Schauder basis such that the complexity of isomorphism between its block-subspaces is exactly E_0?*

Note that the complexity of isomorphism is exactly E_0 between subspaces spanned by subsequences of an unconditional basis with the strong Casazza property (i.e., a space of type (2) in Gowers' theorem).

7 Homogeneity questions

From the solution of Gowers and Komorowski–Tomczak-Jaegermann to the homogeneous Banach space problem it is easy to deduce the slightly stronger statement that a space with a Schauder basis which is isomorphic to all its subspaces spanned by Schauder bases must be isomorphic to ℓ_2. Several questions remain open in that direction:

Question 33. *If a Banach space has an unconditional basis and is isomorphic to all its subspaces with an unconditional basis, must it be isomorphic to ℓ_2?*

The next question was already mentioned in the introduction and concerns a stronger statement, modulo the fact that c_0 and ℓ_p, $p \neq 2$, contain a subspace with an unconditional basis which is not isomorphic to the whole space [34]:

Question 34. *If a Banach space has an unconditional block-homogeneous basis, must it be isomorphic to c_0 or ℓ_p?*

Recall that a theorem of Zippin states that a basis which is perfectly homogeneous, i.e., equivalent to all its normalised block-sequences, must be equivalent to the canonical basis of c_0 or ℓ_p, $1 \leqslant p < +\infty$, [47]. Bourgain, Casazza, Lindenstrauss, and Tzafriri extended this result to permutative equivalence [8]. Ferenczi and Rosendal proved that if a normalised Schauder basis is not equivalent to the canonical basis of c_0 or ℓ_p, $1 \leqslant p < +\infty$, then E_0 reduces to equivalence between its normalised block-sequences [19]. Ferenczi [13] obtained that if X has an unconditional basis, then E_0 is Borel reducible to permutative equivalence on $bb(X)$ or every normalised block-sequence has a subsequence equivalent to the unit vector basis of some fixed ℓ_p or c_0.

Some apparently weaker properties turn out to be equivalent to block homogeneity.

Theorem 35 (Ferenczi [12]). *Let Y be a Banach space and let X be a space with an unconditional basis such that every sequence of successive finite block-sequences has a subsequence whose concatenation spans a space isomorphic to Y. Then X is block-homogeneous.*

The techniques used for this result are similar to the one used for Theorem 8. They are based on Lemma 9 and the fact that isomorphism classes in $bb_{\mathbf{Q}}(X)$ verify a topological 0-1 law, i.e., they are either meagre or comeagre in $bb_{\mathbf{Q}}(X)$.

Theorem 36 (Rosendal [44], Assuming Projective Determinacy). *Let X be a Banach space which is not ℓ_1-saturated and such that every weakly null tree has a branch which spans a subspace isomorphic to X. Then X has a block-homogeneous basis.*

Here a *weakly-null tree* in X is a sequence of normalised vectors $(x_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$ indexed by $\mathbb{N}^{<\mathbb{N}}$ such that for all $s \in \mathbb{N}^{<\mathbb{N}}$, the sequence $(x_{s^\frown n})_{n\in\mathbb{N}}$ is weakly-null.

Uniformity results might be necessary to answer Questions 33 or 34. This is in line with the question by Gowers whether there exists a direct proof that a homogeneous Banach space must be uniformly homogeneous.

Question 37. *Let X be a space with a block-homogeneous basis. Must it be uniformly block-homogeneous?*

An extremely partial answer was obtained in [12].

Theorem 38 (Ferenczi [12]). *If X has a block-homogeneous unconditional basis (e_i) and X is isomorphic to c_0 or $\ell_p, 1 \leqslant p < +\infty$, then (e_i) is uniformly block-homogeneous.*

8 Spreading models

A fundamental notion in the geometry of Banach spaces is that of a spreading model. We recall that a normalised basic sequence (x_i) is said to generate a spreading model if for all $r_1, \ldots, r_k \in \mathbb{R}$ there is $t \in \mathbb{R}$ such that for any $\epsilon > 0$ there exists N with the following property: for any $N < l_1 < \ldots < l_k$, we have

$$\Big| \, \|r_1 x_{l_1} + \ldots + r_k x_{l_k}\| - t \, \Big| < \epsilon.$$

A more intuitive way of expressing this is by saying that

$$\lim_{l_1 < \ldots < l_k,\, l_1 \to \infty} \|r_1 x_{l_1} + \ldots + r_k x_{l_k}\|$$

exists for all $r_1, \ldots, r_k$. In this case, we can define a 1-subsymmetric basic sequence $(\tilde{x}_i)$ by the formula

$$\|r_1 \tilde{x}_1 + \ldots + r_k \tilde{x}_k\| = \lim_{l_1 < \ldots < l_k,\, l_1 \to \infty} \|r_1 x_{l_1} + \ldots + r_k x_{l_k}\|,$$

and say that (x_i) generates the *spreading model* $(\tilde{x}_i)$. Though the basic sequence $(\tilde{x}_i)$ is very closely related to the space $[x_i]$ it does not necessarily have to be present there, and indeed this is one of the reasons for its interest.

Presumably, the right notion of isomorphism for spreading models is equivalence and the most natural ordering is majoration. Here a basic sequence (e_i) *majorises* a basic sequence (f_i) if there is a constant K such that for all $r_1, \ldots, r_n$

$$\|r_1 e_1 + \ldots + r_n e_n\| \leqslant K \|r_1 f_1 + \ldots + r_n f_n\|.$$

Some of the major problems about spreading models concern the possible sets of spreading models generated by basic sequences of a given space and the structure of this set of spreading models under the quasiorder of majoration. One particular question that has motivated some research, in particular [1], is the following question of S. Argyros.

Question 39 (Argyros [1]). *Let X be a Banach space such that all spreading models in X are equivalent. Must these spreading models be equivalent to the unit vector basis of c_0 or ℓ_p for some $p \geqslant 1$?*

This question is of course motivated by the fact that the spaces ℓ_p have a unique spreading model up to equivalence. For if X is a reflexive space, then any spreading model is generated by a weakly null sequence, and hence in the case of ℓ_p, $1 < p < \infty$, by a normalised block that is itself equivalent with ℓ_p. And, in ℓ_1, any spreading model is, by Rosenthal's ℓ_1-theorem, generated by either an ℓ_1 sequence or a weakly Cauchy sequence. In the case of a weakly Cauchy sequence (x_i), the difference sequence $(x_{2i+1} - x_{2i})$ is weakly null and thus generates ℓ_1 again. This argument, however, does not generalise to c_0 unless one restricts the attention to spreading models generated by weakly null sequences.

Another version of this question had previously been formulated by H. P. Rosenthal in another disguise, namely, as a question concerning characterisations of the standard basis of ℓ_p.

Question 40 (Rosenthal). *Suppose (e_i) is a basic sequence such that any normalised block-sequence has a subsequence equivalent to (e_i). Is (e_i) then equivalent to the unit vector basis of some ℓ_p or c_0?*

It is a fact, shown in [18], that a positive question to Argyros' question leads also to a positive answer to Rosenthal's question. The question of Rosenthal may also open a direction to answer Question 34. For example:

Question 41. *If a normalised unconditional basis is block-homogeneous, does it have a block-sequence, or even a subsequence, with the property defined by Rosenthal?*

Going back to the structure theory of the set of spreading models under the relation of majoration, we shall here show that under the supposition that there is no uncountable Borel antichain, one can prove quite strong structural results at least when X^* is separable. In the following, it will be assumed that all spaces in question are infinite-dimensional.

Definition 42. *Let X be separable infinite-dimensional Banach space and let $\mathcal{S}_w$ be the set of weakly-null, normalised basic sequences (x_i) generating a spreading model $(\tilde{x}_i)$. For (x_i) and (y_i) in $\mathcal{S}_w$, we set $(x_i) \preccurlyeq (y_i)$ if $(\tilde{x}_i)$ is majorised by $(\tilde{y}_i)$. Similarly, we let $(x_i) \approx (y_i)$ if both $(x_i) \preccurlyeq (y_i)$ and $(y_i) \preccurlyeq (x_i)$, i.e, if $(\tilde{x}_i)$ and $(\tilde{y}_i)$ are equivalent.*

We notice that $\preccurlyeq$ is a quasiorder on $\mathcal{S}_w$. It seems perhaps more natural to work directly with the set of spreading models, or even the set of spreading models up to equivalence, instead of the set of sequences generating the spreading models. However, the latter is a standard Borel space when X^* is separable, which is not necessarily the case for the former. Thus, $\mathcal{S}_w$ lends itself to the methods of descriptive set theory.

In the fundamental paper [1] by Androulakis, Odell, Schlumprecht, and Tomzcak-Jaegermann it was proved that $(\mathcal{S}_w, \preccurlyeq)$ is an upper semi-lattice, i.e., any two elements have a common least upper bound. Moreover, it was proved that any countable family has an upper bound, though not necessarily a least upper bound. This line of research was continued by B. Sari, who proved the following result.

Theorem 43 (Sari [45]). *Suppose X is a separable Banach space such that $\mathcal{S}_w$ has an infinite increasing sequence with respect to $\preccurlyeq$. Then it has an increasing ω_1-chain.*

This result puts us in a position where we are able to pursue the analysis lying behind Theorem 15, except that, as the relation of majorisation between bases is Borel and not only analytic, the analysis is completely straightforward.

Theorem 44. *Let X be a Banach space with separable dual. Then*

- *either (i) $\mathcal{S}_w/\approx$ is countable or (ii) there is a continuum size antichain in $(\mathcal{S}_w, \preccurlyeq)$,*
- *either (iii) there is a continuum size antichain and also an increasing ω_1-chain in $(\mathcal{S}_w, \preccurlyeq)$ or (iv) $(\mathcal{S}_w, \preccurlyeq)$ is inversely well-founded with a maximal element and for some ordinal $\alpha < \omega_1$ there are no decreasing α-chains.*

Proof. There are two cases. By Sari's result, either $(\mathcal{S}_w, \preccurlyeq)$ is inversely well-founded or has an increasing ω_1-chain.

In the first case, we have by the Kunen-Martin theorem that there is some countable bound on the length of decreasing sequences in $(\mathcal{S}_w, \preccurlyeq)$ and hence for some $\alpha < \omega_1$ there are no decreasing α-chains in $(\mathcal{S}_w, \preccurlyeq)$. Moreover, by the results of [1], $(\mathcal{S}_w, \preccurlyeq)$ is an upper semi-lattice and hence if inversely well-founded it must have a maximal element or otherwise one could construct an infinite increasing sequence.

In the second case, we can apply the results of Harrington, Marker, and Shelah [29, 27] on Borel quasiorderings as follows. They prove that if a Borel quasiorder on a standard Borel space has an ω_1-chain, then it also has a perfect antichain. This thus shows the dichotomy between (iii) and (iv).

Now to see the dichotomy between (i) and (ii), suppose that $(\mathcal{S}_w, \preccurlyeq)$ does not admit an uncountable Borel set of pairwise incomparable elements. Then $(\mathcal{S}_w, \preccurlyeq)$ is inversely well-founded and for some $\alpha < \omega_1$ there are no decreasing α-chains. Moreover, we again have by the results of Harrington, Marker, and Shelah that there is a partition of the space into countably many Borel sets X_n each of which is linearly ordered by $\preccurlyeq$. Therefore, each $(X_n, \preccurlyeq)$ is an inverse prewellordering of countable length and hence each $X_n/\approx$ is countable, whence also $\mathcal{S}_w/\approx$ is countable. □

We notice that P. Dodos [11] has independently arrived at the same result by essentially the same argument, though his setup is slightly different. He also notices that one can discard of the hypothesis that X^* is separable by applying a result of H. P. Rosenthal [5]. The following is a reformulation of Dodos' argument in our language:

Let (x_n) be a sequence in a Banach space X. We say that (x_n) is a *Brunel-Sucheston* sequence if

- (x_n) is Cesaro summable,
- (x_n) is a normalised basic sequence,
- for all k and $k \leqslant n_1 < \ldots < n_k$, $k \leqslant m_1 < \ldots < m_k$, we have

$$(x_{n_1}, \ldots, x_{n_k}) \sim_{1+1/k} (x_{m_1}, \ldots, x_{m_k}).$$

Dodos then notices that, by the result of Rosenthal, if X is a Banach space, then, apart from possibly ℓ_1, the spreading models generated by normalised weakly-null sequences of X are exactly those generated by Brunel-Sucheston sequences in X. Moreover, as the set of Brunel-Sucheston sequences is clearly Borel, one can just work with this set instead of $\mathcal{S}_w$.

We should mention that Theorem 44 is in response to a question of Dilworth, Odell, and Sari from [10].

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