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# ON A QUESTION OF HASKELL P. ROSENTHAL CONCERNING A CHARACTERIZATION OF $c_0$ AND $\ell_p$

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Abstract

The following property of a normalized basis in a Banach space is considered: any normalized block sequence of the basis has a subsequence equivalent to the basis. Under uniformity or other natural assumptions, a basis with this property is equivalent to the unit vector basis of $c_0$ or $\ell_p$. An analogous problem concerning spreading models is also addressed.

## 1. *Introduction*

Haskell P. Rosenthal posed the following problem on basic sequences in a Banach space (see [**12**]).

Problem 1. Let $X$ have a normalized basis $\{e_i\}$ with the property that every normalized block basis admits a subsequence equivalent to $\{e_i\}$. Is $\{e_i\}$ equivalent to the unit vector basis of $c_0$ or $\ell_p$ for some $p \in [1, \infty)$?

We recall a well-known theorem of Zippin (see [**8**]), which states that a normalized basis of a Banach space such that all normalized block bases are equivalent (to the original basis) must be equivalent to the unit vector basis of $c_0$ or $\ell_p$ for some $p \in [1, \infty)$.

The problem posed by Rosenthal is of particular interest in that it is of a 'mixed' Ramsey type, in the sense that it links two types of 'subbases' of a given basis: namely, subsequences and block sequences. An instance of a theorem that mixes a property concerning subsequences and a property concerning block bases was given by the second author in [**12**]. She proved that a Banach space saturated with subsymmetric sequences must contain a minimal subspace.

We note that this mixing is necessary to make Rosenthal's problem significant. Indeed, a weakening of the Rosenthal property would be to assume that every subsequence has a further subsequence equivalent to the basis. An application of Ramsey's theorem would show that the basis is subsymmetric – but, obviously, not every subsymmetric basis is equivalent to the unit vector basis of $c_0$ or $\ell_p$ (take the basis of Schlumprecht's space S, for example; see [**13**]).

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On the other hand, we may weaken the Rosenthal property by requiring only that any block sequence have a further block sequence equivalent to $\{e_i\}$. We call a basis with this property *block equivalence minimal*. The correct setting for such a property is W. T. Gowers's block version of Ramsey's theorem (see [**4**]). A standard diagonalization yields the information that for some constant $C > 0$, any block sequence has a further block sequence that is $C$-equivalent to $\{e_i\}$, and by Gowers's dichotomy theorem we obtain the existence of a winning strategy for Player 2 in Gowers's game to produce sequences that are $(C + \varepsilon)$-equivalent to $\{e_i\}$. Again, however, Schlumprecht's space is a non-trivial example (see [**13**] or [**2**]). Actually, by [**12**, proof of Theorem 3.4], any Banach space saturated with subsymmetric sequences contains a block equivalence minimal basic sequence.

Rosenthal's problem is closely related to a problem posed by S. Argyros, concerning spreading models (see [**1**]): if all the spreading models of a Banach space are equivalent, must these spreading models be equivalent to the unit vector basis of $c_0$ or $\ell_p$? Inspired by results of G. Androulakis, E. Odell, T. Schlumprecht and N. Tomczak-Jaegermann regarding Argyros's question (see [**1**]), we prove that the answer to Rosenthal's question is positive if uniformity is assumed (Theorem 1), or when 1 is in the Krivine set of the basis (Corollary 6). In addition, we show that the answer is also positive if $X$ and $X^*$ satisfy Rosenthal's property (Proposition 7), or when the selection of the subsequence in the definition of Rosenthal's property can be chosen to be continuous (Proposition 8). We note that this is in contrast to the case of block equivalence minimality, where uniformity comes as a consequence of the definition, as well as continuity (because of the previous remark using Gowers's dichotomy theorem). In Section 3 we investigate the case of spreading models. After relating Rosenthal's problem to Agyros's question, we show how results from descriptive set theory may be used to obtain a dichotomy concerning the number of non-equivalent spreading models in a Banach space (Proposition 9). We also show that $c_0$ or $\ell_p$ embeds in $X$ if there exists a continuous way of choosing the subsequences that generate spreading models (Proposition 10).

## 2. *Results on Rosenthal bases*

We give a definition of the main property used in this paper: a normalized basis $\{e_i\}$ such that any normalized block basis of $\{e_i\}$ has a subsequence that is equivalent to $\{e_i\}$ is said to have *Rosenthal's property* or, for short, to be a *Rosenthal basis*.

First, some easy remarks are appropriate. We note that a Rosenthal basis must be subsymmetric. Indeed, let $\mathcal{A} \subset [\omega]^{\aleph_0}$ be the set of subsequences of $\omega$ giving a subsequence of $\{e_i\}$ equivalent to $\{e_i\}$. Then $\mathcal{A}$ is clearly Borel, and therefore by the Galvin–Prikry theorem (see [**7**]), there is an infinite subset $H$ of $\omega$ such that either $[H]^{\aleph_0} \subset \mathcal{A}$ or $[H]^{\aleph_0} \subset \mathcal{A}^C$. The latter possibility clearly contradicts Rosenthal's property. Therefore $\{e_i\}_\omega \sim \{e_i\}_H$ and $\{e_i\}_H$ is subsymmetric, and hence so is $\{e_i\}_\omega$.

Take any 1-unconditional basis $\{e_i\}$ that is invariant under spreading (1-equivalent to its subsequences). It has been observed by A. Brunel and L. Sucheston that for a normalized basic sequence $\{t_i\}$, invariant under spreading, the difference sequence $\{t_{2i+1} - t_{2i}\}$ is suppression unconditional (that is, the norm decreases as the support diminishes); see, for example, [**6**]. By Rosenthal's property, $\{e_{2i+1} - e_{2i}\}$ is equivalent to $\{e_i\}$, and is also invariant under spreading. Hence we may always assume that a Rosenthal basis is both invariant under spreading and suppression unconditional. We note at this point that according to Schlumprecht's terminology, spaces with a Rosenthal basis are exactly spaces of Class 1 with a subsymmetric basis. Therefore, for once, our favorite non-trivial example $S$ will not do: it is of Class 2 (see [**14**])!

We fix some notational matters: we say that a block basis $\{x_i\}$ over $\{e_i\}$ is *identically distributed* if there are scalars $r_0, \dots, r_k$ and natural numbers $m_0 < m_0 + k < m_1 < m_1 + k < m_2 < m_2 + k < \dots$ such that $x_i = r_0 e_{m_i} + \dots + r_k e_{m_i+k}$ for all $i$.

We show that with a uniformity condition added in the hypothesis, the answer is positive. In fact we obtain rather more, as shown in the following theorem.

THEOREM 1. *Let $\{e_i\}$ be a normalized basic sequence, and let $K \geqslant 1$ be a constant such that any identically distributed normalized block basis admits a subsequence that is $K$-equivalent to $\{e_i\}$. Then $\{e_i\}$ is equivalent to the unit vector basis of $c_0$ or $\ell_p$.*

*Proof.* By virtue of the remarks made just before the theorem, and the proofs cited there, we may assume, without loss of generality, that $\{e_i\}$ is both invariant under spreading and suppression unconditional. Under these conditions, Krivine's theorem now takes a particularly simple form (see [**9**]), as follows.

LEMMA 2 (J. L. Krivine). *Let $\{t_i\}$ be a suppression unconditional basis that is invariant under spreading. Then there is a $p \in [1, \infty]$ such that for all $k \in \mathbb{N}$, $0 < \varepsilon$, there are identically distributed blocks $x_1 < x_2 < \dots < x_k$ that are $(1+\varepsilon)$-equivalent to the unit vector basis of $\ell_p^k$.*

The set of values of $p$ satisfying this assertion is called the *Krivine set*. Take a $p$ in this set for our basis $\{e_i\}$. Then, for any $k$, there is a norm-1 block $x(k)$ such that taking $k$ successive copies of this vector $x_1(k) < x_2(k) < \dots < x_k(k)$ gives a sequence that is 2-equivalent to the unit vector basis in $\ell_p^k$.

We now take infinitely many copies of $x(k) : x_1(k) < x_2(k) < \dots$, and we observe that the sequence is identically distributed, so (as before) it must be $K$-equivalent to $\{e_i\}$. This means, however, that $\{e_1, \dots, e_k\}$ is $2K$-equivalent to the unit vector basis of $\ell_p^k$; therefore, as $k$ was arbitrary, $\{e_i\}$ must be $2K$-equivalent to the unit vector basis of $\ell_p$ or $c_0$ if $p = +\infty$. □

REMARK. The uniformity condition is necessary in this result. Indeed, if we take any 1-unconditional basis $\{e_i\}$ that is invariant under spreading (such as our usual example of the unit basis of Schlumprecht's space, which is not equivalent to the unit vector basis of $c_0$ or $\ell_p$), then any identically distributed sequence $\{x_i\} = \{r_0 e_{m_i} + \dots + r_k e_{m_i+k}\}$ is equivalent to $\{e_i\}$, as proved by the relation

$$\Big(\max_j |r_j|\Big) \Big\|\sum \lambda_i e_i\Big\| \leqslant \Big\|\sum \lambda_i x_i\Big\| \leqslant \Big(\sum_{j=0}^{k} |r_j|\Big) \Big\|\sum \lambda_i e_i\Big\|,$$

for all finite sequences $(\lambda_i)$.

Notice also that this is in opposition to the property of block equivalence minimality, where uniformity is a direct consequence of the property.

We now study Rosenthal's problem without the uniformity condition. First, we notice that the only relevant case is the reflexive one, as shown by the next lemma. We need some notation about spreading models (see, for example, [**6**]). A sequence $\{x_i\}$ in a Banach space $X$ is called *seminormalized* if there exist real numbers $0<c<C$ such that $c<\|x_i\|<C$, for all $i$. Let $\{x_i\}$ in a Banach space $X$ be a seminormalized basic sequence. Suppose that for any $r_1, \ldots, r_k \in \mathbb{R}$ there is $t \in \mathbb{R}$ such that for any $\varepsilon>0$ there exists $N$ with the following property: for any $N < l_1 < \ldots < l_k$, we have

$$\big|\,\|r_1 x_{l_1} + \ldots + r_k x_{l_k}\| - t\big| < \varepsilon$$

(or, more intuitively, $\lim_{l_1<\ldots<l_k,\, l_1\to\infty} \|r_1 x_{l_1} + \ldots + r_k x_{l_k}\|$ exists). Then we say that $\{x_i\}$ *generates a spreading model* $\{\tilde{x}_i\}$ with the norm defined as follows:

$$\|r_1\tilde{x}_1 + \ldots + r_k\tilde{x}_k\| := \lim_{l_1\ldots<l_k,\, l_1\to\infty} \|r_1 x_{l_1} + \ldots + r_k x_{l_k}\|.$$

The spreading model $\{\tilde{x}_i\}$ is then a basic sequence, invariant under spreading. Furthermore, it is easily seen that the basis constant of $\{\tilde{x}_i\}$ is majorized by that of $\{x_i\}$. Moreover, any subsequence of $\{x_i\}$ generates the same spreading model.

LEMMA 3. *Let $\{e_i\}$ be a Rosenthal basis for a Banach space $X$. Then $\{e_i\}$ is equivalent to the unit vector basis of $c_0$ or $\ell_1$, or $X$ is reflexive. In the last case, all spreading models in $X$ are equivalent to $\{e_i\}$.*

*Proof.* As before, we can assume that $\{e_i\}$ is suppression unconditional and invariant under spreading. Now, by James's theorem, $X$ is reflexive or contains a subspace isomorphic to $c_0$ or $\ell_1$. In the latter case, we may assume that $c_0$ or $\ell_1$ is equivalent to a block subspace of $X$, so that $\{e_i\}$ is itself equivalent to the unit vector basis of $c_0$ or $\ell_1$. If $X$ is reflexive, any spreading model is generated by a weakly null sequence, and thus by a block basic sequence. Hence, once again by Rosenthal's property, any spreading model is equivalent to $\{e_i\}$. □

Let $\{x_i\}$ and $\{y_i\}$ be basic sequences, and let $K>0$. We say that $\{x_i\}$ $K$-*dominates* $\{y_i\}$ (and we denote this by $\{x_i\} \geqslant^K \{y_i\}$) if, for any $(a_i) \in c_{00}$,

$$K\Big\|\sum a_i x_i\Big\| \geqslant \Big\|\sum a_i y_i\Big\|,$$

holds, where $c_{00} \subset c_0$ denotes the family of sequences that are eventually zero.

We need the following recent result of Androulakis, Odell, Schlumprecht and Tomczak-Jaegermann.

PROPOSITION 4 [**1**]. *Let $\{x_i^n\}_i$, $n \in \mathbb{N}$, be a sequence of normalized basic weakly null sequences in a Banach space $X$ which have spreading models $\{\tilde{x}_i^n\}_i$. Then there exists a seminormalized basic weakly null sequence $\{y_i\}$ in $X$ with spreading model $\{\tilde{y}_i\}$, such that for any $n \in \mathbb{N}$ and any $(a_i) \in c_{00}$, we have*

$$2^n\Big\|\sum a_i\tilde{y}_i\Big\| \geqslant \Big\|\sum a_i\tilde{x}_i^n\Big\|.$$

COROLLARY 5. *Suppose that $X$ is a Banach space with a basis $\{e_i\}$ having Rosenthal's property. Then there exists $K > 0$ such that $\{e_i\}$ $K$-dominates any identically distributed normalized block basis of $\{e_i\}$.*

*Proof.* We may assume that $\{e_i\}$ is a normalized suppression unconditional basic sequence, invariant under spreading. Assume that for any $n \in \mathbb{N}$ there is a normalized basic weakly null sequence $\{x_i^n\}$ in $X$ with a spreading model $\{\tilde{x}_i^n\}$ such that $\{e_i\}$ does not $4^n$-dominate $\{\tilde{x}_i^n\}$. Take $\{y_i\}$ as in Proposition 4, and note that its spreading model must be equivalent to $\{e_i\}$, by Lemma 3. Hence there is some $K$ such that $\{e_i\} \geqslant^K \{\tilde{y}_i\} \geqslant^{2^n} \{\tilde{x}_i^n\}$; if we take $n$ large enough, we obtain a contradiction. Therefore there exists some $K$ such that $\{e_i\}$ $K$-dominates any spreading model generated by a normalized basic weakly null sequence. Now any identically distributed normalized block basis is invariant under spreading, and thus it is its own spreading model, which proves the result. □

COROLLARY 6. *Suppose that $\{e_i\}$ is a normalized suppression unconditional basic sequence, invariant under spreading, with Rosenthal's property. If* 1 *is in the Krivine set of $\{e_i\}$, then $\{e_i\}$ is equivalent to the unit vector basis of $\ell_1$.*

*Proof.* For any $k$, we pick some norm-1 block $x(k)$ of $\{e_i\}$, such that any $k$ successive copies $x_1(k) < x_2(k) < \ldots < x_k(k)$ form a sequence that is 2-equivalent to the unit vector basis of $\ell_1^k$. An infinite sequence of successive copies of this vector $x_1(k) < x_2(k) < \ldots$ must be $K$-dominated by $\{e_i\}$, where $K$ is the constant given by Corollary 5. Therefore, for any $k$, $\{e_i\}_{i=1}^k$ $2K$-dominates $\ell_1^k$, but must itself, by the triangle inequality, be 1-dominated by $\ell_1^k$. Hence $\{e_i\}$ is equivalent to the unit vector basis of $\ell_1$. □

The following proposition states that the answer to Rosenthal's question is positive if we also assume that Rosenthal's property holds in the dual.

PROPOSITION 7. *Let $X$ be a Banach space with a Rosenthal basis, and such that $X^*$ has a Rosenthal basis. Then $X$ is isomorphic to $c_0$ or $\ell_p$ for some $p > 1$ (and any Rosenthal basis of $X$ is equivalent to the unit vector basis of $c_0$ or $\ell_p$ for some $p > 1$).*

*Proof.* By Lemma 3, we may assume that $X$ is reflexive. Let $\{e_i\}$ be a Rosenthal basis of $X$. By renorming, we may assume that the basis is suppression unconditional and invariant under spreading. The biorthogonal basis $\{e_i^*\}$ satisfies these properties as well; in particular, it is its own spreading model. Hence, by Lemma 3, $\{e_i^*\}$ is equivalent to any Rosenthal basis of $X^*$; so it has Rosenthal's property. By Corollary 5, there exists $K > 0$ such that any normalized identically distributed block basis in $X$ is $K$-dominated by $\{e_i\}$, and also any normalized identically distributed block basis in $X^*$ is $K$-dominated by $\{e_i^*\}$. Fix a normalized identically distributed block basis $\{x_n\}$ in $X$. By the 1-unconditionality and 1-subsymmetry of $\{e_i\}$, there is a normalized identically distributed block basis $\{x_n^*\}$ in $X^*$ such that each $x_n^*$ satisfies $x_n^*(x_n) = 1$ and has support no larger than the support of $x_n$. By the assumption, $\{x_n^*\}$ is $K$-dominated by $\{e_i^*\}$. It follows that

$\{x_n\}$ thus $1/K$-dominates $\{e_i\}$. Indeed, for $(a_i) \in c_{00}$,

$$K\Big\|\sum a_i x_i\Big\| \geqslant K \sup_{(b_i)\in c_{00}} \frac{(\sum b_i x_i^*)(\sum a_i x_i)}{\|\sum b_i x_i^*\|} \geqslant \sup_{(b_i)\in c_{00}} \frac{\sum b_i a_i}{\|\sum b_i e_i^*\|}$$
$$= \Big\|\sum a_i e_i\Big\|.$$

Hence any identically distributed normalized block $\{x_i\}$ of $\{e_i\}$ is $K^2$-equivalent to $\{e_i\}$. By Theorem 1, the sequence $\{e_i\}$ must be equivalent to the unit basis of $\ell_p$ for some $p>1$. □

We now prove that if the selection of the subsequence in Rosenthal's property is continuous, then the answer to the problem is also positive. In fact, we obtain more: it is enough to find a continuous selection of subsequences dominating the basis. We let $bb(\{e_i\})$ (or $bb(X)$) be the set of normalized block bases of $\{e_i\}$. Denote by $bb_D(X)$ the same set equipped with the product of the discrete topology on $X$, and by $bb_E(X)$, the same set equipped with the 'Ellentuck–Gowers topology' (see [**4**]). Basic open sets in this latter topology are of the form $[a, A]$, with $a = (a_1, \dots, a_n)$ a finite normalized block sequence, and $A$ an infinite normalized block sequence, where

$$[a, A] = \{a^\frown x,\ x \in bb(A),\ a < x\}.$$

Here, $a^\frown x \in bb(X)$ denotes the concatenation of $a$ and $x$, and $bb(A)$ denotes the set of normalized block bases of $A$. Proposition 8 uses the weakest notion of continuity combining the two topologies. The proof that we present here is easier to understand than our original one, and was suggested by the referee.

PROPOSITION 8. *Assume that $X$ is a Banach space with a Rosenthal basis $\{e_i\}$. Let $\phi : bb(X) \longrightarrow bb(X)$ be a $bb_E(X) - bb_D(X)$ continuous map such that for any normalized basic sequence $x = \{x_i\}$ in $bb(X)$, the sequence $\phi(x)$ is a subsequence of $x$ that dominates $\{e_i\}$. Then $\{e_i\}$ is equivalent to the unit vector basis of $c_0$ or $\ell_p$.*

*Proof.* We may assume that $\{e_i\}$ is invariant under spreading, and also suppression unconditional. From [**9**, Theorem 3], we may find a $p$ in the Krivine set of $\{e_i\}$ that is also in the Krivine set of any normalized identically distributed block of $\{e_i\}$. We give the proof in the case where $p < +\infty$. We define, by induction, sequences $\{x_i^n\}_{i\in\mathbb{N}}$, $n \in \mathbb{N}$. Put $\{x_i^0\}_{i\in\mathbb{N}} = \{e_i\}_{i\in\mathbb{N}}$. For any $n \in \mathbb{N}$, let $\{x_i^{n+1}\}_{i\in\mathbb{N}}$ be an identically distributed normalized block basis of $\{x_i^n\}_{i\in\mathbb{N}}$, such that any sequence of $n+1$ elements of $\{x_i^{n+1}\}_{i\in\mathbb{N}}$ is 2-equivalent to the unit vector basis of $\ell_p^{n+1}$.

By the continuity of $\phi$ at $\{x_i^1\}_{i\in\mathbb{N}}$, we find integers $m_1 \leqslant n_1$, and an infinite normalized block sequence $A_1$ such that $\{x_i^1\}_{i\in\mathbb{N}} \in [x_1^1, \dots, x_{n_1}^1, A_1]$ and

$$\phi\big([x_1^1, \dots, x_{n_1}^1, A_1]\big) \subset \big[x_{m_1}^1, X\big].$$

Without loss of generality, we may assume that $A_1 = [(x_i^1)_{i>n_1}]$.

By the continuity of $\phi$ at $(x_1^1, \dots, x_{n_1}^1, x_{n_1+1}^2, x_{n_1+2}^2, \dots)$, we find an integer $n_2 > n_1$, a subsequence $(x_{m_1}^1, \dots, x_{m_{k_1}}^1)$ of $(x_1^1, \dots, x_{n_1}^1)$, extending $\{x_{m_1}^1\}$, and integers $m_{k_1+1}, m_{k_1+2}$ such that, if

$$a_2 = \big(x_1^1, \dots, x_{n_1}^1, x_{n_1+1}^2, \dots, x_{n_2}^2\big) \quad \text{and} \quad A_2 = \big[\{x_i^2\}_{i>n_2}\big],$$

then

$$\phi([a_2, A_2]) \subset \left[x^1_{m_1}, \ldots, x^1_{m_{k_1}}, x^2_{m_{k_1+1}}, x^2_{m_{k_1+2}}, X\right].$$

The terms of the indices $m_{k_1+1}$ and $m_{k_1+2}$ will ensure that $k_2 - k_1 \geqslant 2$. Repeat by induction to find increasing sequences of integers $(n_j)_{j\in\mathbb{N}}$, $(k_j)_{j\in\mathbb{N}}$ and $(m_i)_{i\in\mathbb{N}}$ such that, if

$$A_j = \{x^j_i\}_{i>n_j} \quad \text{and} \quad a_j = \left(x^1_1, \ldots, x^1_{n_1}, x^2_{n_1+1}, \ldots, x^2_{n_2}, \ldots, x^j_{n_{j-1}+1}, \ldots, x^j_{n_j}\right),$$

then

$$\phi([a_j, A_j]) \subset \left[x^1_{m_1}, \ldots, x^1_{m_{k_1}}, \ldots, x^j_{m_{k_{j-1}+1}}, \ldots, x^j_{m_{k_j}}, x^{j+1}_{m_{k_j}+1}, \ldots, x^{j+1}_{m_{k_j}+j+1}, X\right].$$

By construction, $a_{j+1}$ extends $a_j$ for each $j$. Also, for all $j$, $k_j - k_{j-1} \geqslant j$. Let $a$ be the infinite block sequence $a = \bigcup_{j\in\mathbb{N}} a_j$. That is,

$$a = \left(x^1_1, \ldots, x^1_{n_1}, x^2_{n_1+1}, \ldots, x^2_{n_2}, \ldots, x^j_{n_{j-1}+1}, \ldots, x^j_{n_j}, \ldots\right).$$

Then

$$\phi(a) = \left(x^1_{m_1}, \ldots, x^1_{m_{k_1}}, x^2_{m_{k_1+1}}, \ldots, x^2_{m_{k_2}}, \ldots, x^j_{m_{k_{j-1}+1}}, \ldots, x^j_{m_{k_j}}, \ldots\right).$$

By the definition of $\phi$, $\{e_i\}$ is $C$-dominated by $\phi(a)$ for some $C > 0$. Therefore, as $k_j - k_{j-1} \geqslant j$, it follows that

$$\begin{aligned} l^j_p &\geqslant^2 \left(x^j_{m_{k_{j-1}+1}}, \ldots, x^j_{m_{k_{j-1}+j}}\right) \\ &\geqslant^C \left(e_{m_{k_{j-1}+1}}, \ldots, e_{m_{k_{j-1}+j}}\right), \end{aligned}$$

for all $j$. By the 1-subsymmetry of $\{e_i\}$, it follows that $(e_1, \ldots, e_j)$ is $2C$-dominated by $l^j_p$ for all $j$. So $\{e_i\}$ is dominated by the unit vector basis of $l_p$.

On the other hand, by Corollary 5, there exists $K > 0$ such that each $\{x^n_i\}_{i\in\mathbb{N}}$ is $K$-dominated by $\{e_i\}$. It follows that $l^n_p$ is $2K$-dominated by $(e_1, \ldots, e_n)$, for all $n$, and so the unit vector basis of $l_p$ is dominated by $\{e_i\}$. □

We remark that, once again, there is an opposition between Rosenthal's property and block equivalence minimality. Indeed, for any block equivalence minimal basis, Gowers's theorem implies the existence of a winning strategy to produce block sequences that are ($C$-)equivalent to $\{e_i\}$. Consider Gowers's game, defined by J. Bagaria and J. Lopez-Abad, which is actually equivalent to the original game defined by W. T. Gowers (see [**3**]). In this game, Player 1 plays block vectors, and Player 2 sometimes chooses a vector in the finite-dimensional space defined by the blocks played by Player 1. The winning strategy defines a continuous map from block sequences to further block sequences that are ($C$-)equivalent to $\{e_i\}$.

Also, if one tried to build a non-trivial Rosenthal basis, one would probably want to adapt the techniques of Gowers and B. Maurey (see [**5**]). There are two major difficulties in this. The first difficulty is to find a way of passing from selecting further block sequences to selecting subsequences. For the second, note that the Gowers–Maurey constructions yield winning strategies in the previous sense: technically, the $\ell^n_1$-averages used to build interesting vectors in their space may at each step of the construction be chosen in an arbitrary block subspace. This means that one would also have to add new methods to suppress the continuity of the selection map.

## 3. *Results on spreading models*

We investigate the relation between Rosenthal's question and the problem posed by Argyros, which can be stated as follows.

PROBLEM 2 [**1**]. Let $X$ be a Banach space such that all spreading models in $X$ are equivalent. Must these spreading models be equivalent to the unit vector basis of $c_0$ or $\ell_p$ for some $p \geqslant 1$?

For example, the spaces $\ell_p$ have unique spreading models up to equivalence. Indeed, in the reflexive case, all spreading models are generated by weakly null sequences. In $\ell_1$, any spreading model is generated by a $\ell_1$-sequence, or (by Rosenthal's $\ell_1$-theorem) by a weakly Cauchy sequence. In the second case, the difference sequence is weakly null, and so it generates $\ell_1$, and it follows that the spreading model is equivalent to the unit vector basis of $\ell_1$. However, this does not generalize to the case of $c_0$, since the unit basis of $c_0$ and the summing basis generate non-equivalent spreading models; nevertheless, all the spreading models generated by weakly null sequences are clearly equivalent to the unit vector basis of $c_0$.

Lemma 3 shows that a positive answer to the problem set by Argyros implies a positive answer to the problem identified by Rosenthal. In actual fact, Androulakis, *et al.*, have proved that the answer to Agyros's problem is positive under the additional assumption of uniformity, or that 1 is in the Krivine set of some basic sequence. Our methods are inspired by their results. A natural generalization of Agyros's question is mentioned in their article: if a Banach space contains only countably many spreading models up to equivalence, must one of them be equivalent to the unit vector basis of $c_0$ or $\ell_p$? In the other direction, the following remark about Banach spaces with more than countably many spreading models is a straightforward consequence of a well-known result due to J. Silver [**7**].

PROPOSITION 9. *Let $X$ be a separable Banach space. Then either $X$ contains continuum many non-equivalent spreading models, or $X$ contains at most countably many non-equivalent spreading models. If $X^*$ is separable, the same dichotomy holds for spreading models generated by weakly null basic sequences. If $X$ has a Schauder basis, the same dichotomy holds for spreading models generated by block basic sequences.*

*Proof.* In what follows, $\sim^C$ denotes the usual $C$-equivalence between basic sequences. We consider the set $\mathcal{S}$ of semi-normalized basic sequences generating spreading models. The set $\mathcal{S}$ can be described as the set of semi-normalized basic sequences $\{x_i\}$ such that for any $k \in \mathbb{N}$ and any $\varepsilon > 0$, there exists an $N$ with the following property: for any $N < l_1 < \ldots < l_k$ and $N < l'_1 < \ldots < l'_k$, we have $\{x_{l_i}\}_{i=1}^k \sim^{1+\varepsilon} \{x_{l'_i}\}_{i=1}^k$. This set is clearly a Borel subset of the Polish space $X^\omega$. Now consider the equivalence relation $\simeq$ on $\mathcal{S}$, which means that the two sequences generate spreading models that are equivalent in the usual $\sim$ sense. That is, $\{y_n\} \simeq \{z_n\}$ if and only if there is $C > 0$ such that for any $k \in \mathbb{N}$ there exists an $N$ with the following property: for any $N < l_1 < \ldots < l_k$, we have $\{y_{l_i}\}_{i=1}^k \sim^C \{z_{l_i}\}_{i=1}^k$. This equivalence relation is Borel as well. By a theorem of

Silver's [**7**, Theorem 35.20], a Borel (or even a coanalytic) equivalence relation on a Borel subset of a Polish space has only countably many classes, or else there exists a Cantor set of mutually non-equivalent elements. As two spreading models are $\sim$-equivalent if and only if any two semi-normalized basic sequences that generate them are $\simeq$-equivalent, the result follows. If $X^*$ is separable, the same proof holds for the set of weakly null semi-normalized basic sequences generating spreading models, which is also Borel in $X^\omega$. If $X$ has a basis, an analogous proof holds for the set of block basic sequences generating spreading models. □

Remark. It is also a consequence of Silver's theorem that a Schauder basis of a Banach space $X$ has continuum many non-equivalent subsymmetric block basic sequences, or else only countably many equivalence classes. Indeed, the set of normalized block bases equipped with the product topology on $X$ is Polish, and the set of subsymmetric normalized block basic sequences is an $F_\sigma$ subset of it.

A result analogous to Proposition 8 also holds for spreading models. We consider a map that selects from basic sequences, subsequences that generate spreading models in a strong sense, as described below. We show that if the map is continuous, then the space has a unique asymptotic structure.

We introduce some terminology. Given a sequence $\varepsilon = \{\varepsilon_i\}$, $\varepsilon_i \searrow 0$, and a basic sequence $\{x_i\}$ in a Banach space, we say that $\{x_i\}$ *$\varepsilon$-generates* a spreading model $\{\tilde{x}_i\}$ if, for any $k < n_1 < n_2 < \cdots < n_k$, we have $(x_{n_1}, \dots, x_{n_k}) \sim^{1+\varepsilon_k} (\tilde{x}_1, \dots, \tilde{x}_k)$. Obviously, every basic sequence for any sequence $\varepsilon$ of non-zero scalars has a subsequence $\varepsilon$-generating a spreading model. We use the notation introduced after Proposition 7.

We recall the notion of asymptotic spaces (see [**10**]). Let $X$ be a Banach space with a basis $\{e_i\}$. A *tail subspace* means here a block subspace of $X$, of a finite codimension.

We consider an asymptotic game between two players, in which player I picks tail subspaces, and player II picks block vectors from the subspaces chosen by player I. We say that a normalized basic sequence $\{a_i\}_{i=1}^n$ is *asymptotic in* $X$ if and only if for any $\delta > 0$, player II has a winning strategy in choosing a normalized block sequence of vectors that is $(1+\delta)$-equivalent to $\{a_i\}_{i=1}^n$. In other words, a normalized basic sequence $\{a_i\}_{i=1}^n$ is asymptotic in $X$ if, for any $\delta > 0$, we have

$$\forall k_1 \ \exists x_1 > e_{k_1} \ \forall k_2 \ \exists x_2 > e_{k_2} \ \dots \ \forall k_n \ \exists x_n > e_{k_n},$$

so that $\{x_i\}_{i=1}^n$ is a normalized block sequence that is $(1+\delta)$-equivalent to $\{a_i\}_{i=1}^n$. We denote by $\{X\}_n$ the set of basic sequences of length $n$ that are asymptotic in $X$.

Since a block sequence has a subsequence generating an unconditional spreading model, by Krivine's theorem there is some $1 \leqslant p \leqslant \infty$ such that $\ell_p^n$ is asymptotic in $X$ for any $n \in \mathbb{N}$ (that is, the unit basic vectors in $\ell_p^n$ form asymptotic sequences for any $n \in \mathbb{N}$).

Proposition 10. *Let $X$ be a Banach space with a basis $\{e_i\}$. Fix a sequence $\varepsilon = \{\varepsilon_i\}$, $\varepsilon_i \searrow 0$. Assume that there is a continuous map $\phi : bb_D(X) \longrightarrow bb_D(X)$ such that for any normalized basic sequence $x = \{x_i\}$ in $bb(X)$, the sequence $\phi(x)$ is a subsequence of $x$ that $\varepsilon$-generates a spreading model. Then there is $1 \leqslant p \leqslant \infty$ such that $\{X\}_n = \ell_p^n$ for all $n \in \mathbb{N}$.*

*Proof.* Let $X$ satisfy the assumptions of the proposition. Choose $1 \leqslant p \leqslant \infty$ such that $\ell_p^n$ is asymptotic in $X$ for all $n$. We present the argument for $n = 2$; for $n > 2$, an analogous reasoning process, applying asymptotic games with $n$ moves, yields the result.

We now show that any asymptotic pair $(a_1, a_2)$ of $X$ is 1-equivalent to the unit basic vectors of $\ell_p^2$.

Fix $\delta > 0$ and choose any asymptotic pair $(a_1, a_2)$. Select $i \in \mathbb{N}$, $i > 1$, such that $(1 + \varepsilon_{i-1})^3 < 1 + \delta$. Consider the asymptotic game for $(1 + \varepsilon_{i-1})$ and $(a_1, a_2)$. Let $x = (x_1, x_2, \dots)$ be a block sequence consisting of vectors chosen by player II in the first move in some game. (Player I can produce such a sequence by choosing tail subspaces of arbitrarily large codimension in the first move.) Let $\phi(x) = (x_{j_1}, x_{j_2}, \dots)$. By the continuity of $\phi$, there is some $J > j_i$ such that $\phi([(x_1, \dots, x_J), X]) \subset [(x_{j_1}, \dots, x_{j_i}), X]$.

Now consider the sequence $y = (x_1, \dots, x_J, y_1, y_2, \dots)$, where $(y_1, y_2, \dots)$, with $x_J < y_1$, is a block sequence of vectors chosen by player II in the second move in some game for $(1+\varepsilon_{i-1})$ and $(a_1, a_2)$, in which player II chose the vector $x_{j_i}$ in the first move. (Again, player I can produce such a sequence by choosing tail subspaces of arbitrary large codimension in the second move.) Let

$$\phi(y) = \big(x_{j_1}, \dots, x_{j_i}, \dots, x_{j'}, y_{k_1}, y_{k_2}, \dots\big).$$

Again by the continuity of $\phi$, there is some $K > k_1$ such that

$$\phi([(x_1, \dots, x_J, y_1, \dots, y_K), X]) \subset \big[\big(x_{j_1}, \dots, x_{j_i}, \dots, x_{j'}, y_{k_1}\big), X\big].$$

Now consider the asymptotic game for $(1+\varepsilon_{i-1})$ and asymptotic $\ell_p^2$. By repeating the previous procedure for such $\ell_p$ or $c_0$, we can extend the finite sequence $(x_1, \dots, x_J, y_1, \dots, y_K)$ by suitable block sequences that $(1 + \varepsilon_{i-1})$-realize $\ell_p^2$ in $X$. In this way, we obtain the finite block sequences

$$b = (x_1, \dots, x_J, y_1, \dots, y_K, v_1, \dots, v_L, z_1, \dots, z_M)$$

and

$$c = \big(x_{j_1}, \dots, x_{j_i}, \dots, x_{j'}, y_{k_1}, \dots, y_{k'}, v_{l_1}, \dots, v_{l'}, z_{m_1}\big),$$

with $(x_{j_i}, y_{k_1}) \sim^{1+\varepsilon_{i-1}} (a_1, a_2)$, $(v_{l_1}, z_{m_1}) \sim^{1+\varepsilon_{i-1}} \ell_p^2$, and $\phi([b, X]) \subset [c, X]$.

By the definition of $\phi$, we know that $(x_{j_i}, y_{k_1}) \sim^{1+\varepsilon_{i-1}} (v_{l_1}, z_{m_1})$. Hence, by the choice of $i$, we have $(a_1, a_2) \sim^{1+\delta} \ell_p^2$. Since $\delta$ was chosen arbitrarily small, $(a_1, a_2)$ is 1-equivalent to the unit vector basis of $\ell_p^2$, which completes the proof. □

Recall that a Banach space $X$ contains almost isometric copies of $c_0$, if for any $\delta > 0$, $X$ has a subspace that is $(1+\delta)$-isomorphic to $c_0$. Analogously, a Banach space $X$ contains almost isometric copies of $\ell_p$ if, for any $\delta > 0$, $X$ has a subspace that is $(1 + \delta)$-isomorphic to $\ell_p$. Notice that Proposition 10, by the argument presented in [**11**, 6.4], implies the following corollary.

COROLLARY 11. *If $X$ satisfies the assumptions of Proposition* 10*, then $X$ contains almost isometric copies of $c_0$ or $\ell_p$ for some $1 \leqslant p < \infty$.*

REMARK. Notice that by this corollary one cannot choose sequences producing spreading models (in the sense defined above) in a continuous way in $\ell_p$, $1 < p < \infty$, endowed with a distorting norm; however, any block sequence of the unit basic vectors in $\ell_p$ is subsymmetric.

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## *References*


1. G. Androulakis, E. Odell, T. Schlumprecht and N. Tomczak-Jaegermann, 'On the structure of the spreading models of a Banach space', preprint, arXiv:math.FA/0305082.
2. G. Androulakis and T. Schlumprecht, 'The Banach space $S$ is complementably minimal and subsequentially prime', *Studia Math.* 156 (2003) 227–242.
3. J. Bagaria and J. Lopez-Abad, 'Weakly Ramsey sets in Banach spaces', *Adv. Math.* 160 (2001) 133–174.
4. W. T. Gowers, 'An infinite Ramsey theorem and some Banach-space dichotomies', *Ann. of Math.* 156 (2002) 797–833.
5. W. T. Gowers and B. Maurey, 'The unconditional basic sequence problem', *J. Amer. Math. Soc.* 6 (1993) 851–874.
6. S. Guerre-Delabrière, *Classical sequences in Banach spaces* (Marcel Dekker, New York, 1992).
7. A. Kechris, *Classical descriptive set theory*, Grad. Texts in Math. 156 (Springer, New York, 1995).
8. J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces I* (Springer, New York, 1977).
9. B. Maurey, 'Type, cotype and $K$-convexity', *Handbook on the geometry of Banach spaces*, vol. 2 (Elsevier, Amsterdam, 2003).
10. B. Maurey, V. Milman and N. Tomczak-Jaegermann, 'Asymptotic infinite-dimensional theory of Banach spaces', *Geometric Aspects of Functional Analysis* (*Israel, 1992–1994*), Oper. Theory Adv. Appl. 77 (Birkhäuser, Basel, 1995) 149–175.
11. V.Milman and N.Tomczak-Jaegermann, 'Asymptotic $\ell_p$ spaces and bounded distortions', *Banach spaces*, Contemp. Math. 144 (Amer. Math. Soc., Providence, RI, 1993) 173–196.
12. A.Pelczar, 'Subsymmetric sequences and minimal spaces', *Proc. Amer. Math. Soc.* 131 (2003) 765–771.
13. T.Schlumprecht, 'A complementably minimal Banach space not containing $c_0$ or $\ell_p$', *Seminar notes in functional analysis and partial differential equations, Baton Rouge, LA, 1992* (Dept. of Math., Louisiana State Univ., Baton Rouge, LA, 1992).
14. T.Schlumprecht, 'How many operators exist on a Banach space?', *Trends in Banach spaces and operator theory,* (*Memphis, TN, 2001*), Contemp. Math. 321 (Amer. Math. Soc., Providence, RI, 2003).


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