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On the algebraic structure of the unitary group

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Abstract

We consider the unitary group $\mathbb{U}$ of complex, separable, infinite-dimensional Hilbert space as a discrete group. It is proved that, whenever $\mathbb{U}$ acts by isometries on a metric space, every orbit is bounded. Equivalently, $\mathbb{U}$ is not the union of a countable chain of proper subgroups, and whenever $\mathbb{E} \subseteq \mathbb{U}$ generates $\mathbb{U}$, it does so by words of a fixed finite length.

A property of uncountable groups that has recently been studied by a number of authors is a strengthening of the property of uncountable cofinality that originated in the work of Jean-Pierre Serre on actions of groups on trees [7]. Here an uncountable group G is said to have ***uncountable cofinality*** if G is not the union of a countable increasing chain of proper subgroups. Serre proved this to be one of the three conditions in his reformulation of when a group does not have fixed point free actions without inversions on trees and it has subsequently been confirmed for a great number of profinite groups (see, e.g., Koppelberg and Tits [4]) and groups of permutations of $\mathbb{N}$. The strengthening of this property, in which we are interested, comes from considering the additional condition on G that whenever E is a symmetric generating set for G

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containing the identity there is some finite n such that $G = E^n$, i.e., any element of G can be written as a word of length n using elements of E. We denote this condition by *Cayley boundedness* and it indeed corresponds to any Cayley graph for G being of bounded diameter in the word metric. George M. Bergman [1] originally proved that the conjuction of uncountable cofinality and Cayley boundedness holds for the infinite symmetric group S_∞ and subsequently this has been verified for a number of other groups. The purpose of this paper is to prove it for the unitary group of complex separable infinite-dimensional Hilbert space ℓ_2. However, before we begin, we should mention some of the equivalent formulations of these properties.

Consider a group G. It is fairly straightforward to see that G has uncountable cofinality and is Cayley bounded if and only if it satisfies the following property:

> Whenever $\mathbb{W}_0 \subseteq \mathbb{W}_1 \subseteq \mathbb{W}_2 \ldots \subseteq G = \bigcup_n \mathbb{W}_n$ is an increasing and exhaustive sequence of subsets of G there is some n and some k such that $\mathbb{W}_n^k = G$.

More interestingly is the fact that it is also equivalent to:

> Whenever G acts by isometries on a metric space (X, d) every orbit is bounded.

Or for another variation:

> Any left-invariant metric on G is bounded.

There are several names for these equivalent properties in the literature, but the two most common seem to be *Bergman property* and *strong uncountable cofinality*. Moreover, if the constant k appearing above can be chosen independently of the sequence $(\mathbb{W}_i)$, then we say that the group is k-Bergman.

The reformulation in terms of isometric actions discloses that we are really dealing with a property of geometric group theory and indeed it does have quite evident geometric implications for other types of actions. For example, one can quite easily prove that any group with this property must also have properties (FH) and (FA), meaning that any action of the group by affine isometries on a real Hilbert space has a fixed point and similarly, any action of the group on a tree without inversions fixes a vertex. We should stress the fact that despite that we will use a bit of topology in our proof, the result is really about the unitary group as a *discrete* group. Thus there are no topological assumptions being made. We shall denote the unitary group of separable infinite-dimensional complex Hilbert space ℓ_2 by $\mathbb{U} = \mathbb{U}(\ell_2)$. We recall that the inverse of a unitary operator is its adjoint, $U^{-1} = U^*$. To avoid confusion, we will use the blackboard bold font type $\mathbb{A}, \mathbb{B}, \ldots$. to denote subsets of $\mathbb{U}$ and standard fonts $A, B, \ldots, a, b, \ldots$ to denote individual operators in $\mathbb{B}(\ell_2)$. Thus $\mathbb{A}^* = \{A^* \mid A \in \mathbb{A}\}$.

Theorem 1

The unitary group $\mathbb{U}$ of separable, infinite-dimensional, complex Hilbert space is Cayley bounded and has uncountable cofinality. More precisely, $\mathbb{U}$ is k-Bergman for some k.

The proof will proceed through a sequence of reductions showing that the sets $\mathbb{W}_n$ of an exhaustive sequence of subsets are big in both an algebraic and an analytic sense. We first need the following result.

Theorem 2 (Brown and Pearcy [2])

Any $T \in \mathbb{U}$ is a multiplicative commutator, i.e., $T = ABA^*B^*$ for appropriate $A, B \in \mathbb{U}$.

Proposition 3

Suppose $\mathbb{W}_0 \subseteq \mathbb{W}_1 \subseteq \ldots \subseteq \mathbb{U} = \bigcup_n \mathbb{W}_n$. Then there is a decomposition $\ell_2 = X \oplus X^\perp$ into closed infinite-dimensional subspaces, such that for some $l \in \mathbb{N}$,

$$\mathbb{D} := \{T \in \mathbb{U} \mid T[X] = X\} = \mathbb{U}(X) \oplus \mathbb{U}(X^\perp) \subseteq \mathbb{W}_l^{20}$$

Proof. Suppose $\mathbb{W}_0 \subseteq \mathbb{W}_1 \subseteq \ldots \subseteq \mathbb{U} = \bigcup_n \mathbb{W}_n$. By instead considering the sequence

$$\mathbb{W}_0 \cap \mathbb{W}_0^* \subseteq \mathbb{W}_1 \cap \mathbb{W}_1^* \subseteq \ldots \subseteq \mathbb{U} = \bigcup_n \mathbb{W}_n \cap \mathbb{W}_n^*$$

we can suppose that each $\mathbb{W}_n$ is symmetric, $\mathbb{W}_n = \mathbb{W}_n^*$. Now, write ℓ_2 as the direct sum of infinitely many infinite-dimensional closed subspaces $\ell_2 = (\sum_n \oplus X_n)_{\ell_2}$. Then for some n and all $T \in \mathbb{U}$, if $T[X_n] = X_n$ then there is some $S \in \mathbb{W}_n$ with $T \restriction X_n = S \restriction X_n$. If not, we would be able to find for each n some unitary operator T_n of X_n such that for all $S \in \mathbb{W}_n$, $T_n \neq S \restriction X_n$. But then the infinite direct sum $T = \oplus_m T_m$ is such that for all n and $S \in \mathbb{W}_n$, $T \restriction X_n = T_n \neq S \restriction X_n$. In particular, $T \notin \bigcup_n \mathbb{W}_n$, contradicting our supposition.

So suppose this holds for n. Find infinite-dimensional closed subspaces $X, Y, Z \subseteq \ell_2$ such that $X = X_n$ and $X \oplus Y \oplus Z = \ell_2$. Let also $N, M \in \mathbb{U}$ be such that $N[X] = X \oplus Y$, $N[Y \oplus Z] = Z$, $M[X] = X \oplus Z$ and $M[Y \oplus Z] = Y$.

Clearly, if $A \in \mathbb{U}$ is such that $A[X \oplus Y] = X \oplus Y$, then for some $S \in \mathbb{W}_n$, $S[X] = X$, we have

$$A \restriction X \oplus Y = NSN^* \restriction X \oplus Y$$

and similarly, if $B \in \mathbb{U}$, $B[X \oplus Z] = X \oplus Z$, then there is $R \in \mathbb{W}_n$, $R[X] = X$, with

$$B \restriction X \oplus Z = MRM^* \restriction X \oplus Z.$$

Now suppose that $T \in \mathbb{U}$, $T[X] = X$ and $T \restriction X^\perp = T \restriction Y \oplus Z = \mathrm{id} \restriction X^\perp$. By the theorem of Brown and Pearcy, there are unitary operators A and B on X such that $T \restriction X = ABA^*B^*$. Extend now A and B to all of ℓ_2 by letting $A \restriction X^\perp = B \restriction X^\perp = \mathrm{id} \restriction X^\perp$.

Moreover, find S and R in $\mathbb{W}_n$ as above, whence for $\hat{A} = NSN^*$ and $\hat{B} = MRM^*$ we have

$$\hat{A} \restriction X \oplus Y = NSN^* \restriction X \oplus Y = A \restriction X \oplus Y$$
$$\hat{B} \restriction X \oplus Z = MRM^* \restriction X \oplus Z = B \restriction X \oplus Z$$

whence

$$\hat{A} \restriction Y = \mathrm{id} \restriction Y,\ \hat{B} \restriction Z = \mathrm{id} \restriction Z,\ \hat{A}[Z] = Z \ \text{ and } \ \hat{B}[Y] = Y.$$

Thus

$$\hat{A}\hat{B}\hat{A}^*\hat{B}^* \restriction X = ABA^*B^* \restriction X = T \restriction X,$$
$$\hat{A}\hat{B}\hat{A}^*\hat{B}^* \restriction Y = \hat{B}\hat{B}^* \restriction Y = \mathrm{id} \restriction Y = T \restriction Y,$$

and

$$\hat{A}\hat{B}\hat{A}^*\hat{B}^* \restriction Z = \hat{A}\hat{A}^* \restriction Z = \mathrm{id} \restriction Z = T \restriction Z.$$

Therefore,

$$\begin{aligned} T = \hat{A}\hat{B}\hat{A}^*\hat{B}^* &= (NSN^*)(MRM^*)(NS^*N^*)(MR^*M^*) \\ &= NS(N^*M)R(M^*N)S^*(N^*M)R^*M^* \in \mathbb{W}_m^9 \end{aligned}$$

provided that $m \geqslant n$ is large enough such that $N, M, N^*M \in \mathbb{W}_m$. Notice however, that N and M do not depend on T, so $\mathbb{V} = \{T \in \mathbb{U} \mid T \restriction X^\perp = \mathrm{id} \restriction X^\perp\} \subseteq \mathbb{W}_m^9$. Find now a $K \in \mathbb{U}$ such that $K[X] = X^\perp$. Then clearly, if $T \in \mathbb{U}$ satisfies $T[X] = X$, we have $T \in \mathbb{V}K\mathbb{V}K^*$. Let now $l \geqslant m$ be sufficiently big that $K \in \mathbb{W}_l$. Then we have $\mathbb{D} \subseteq \mathbb{W}_l^9\mathbb{W}_l\mathbb{W}_l^9\mathbb{W}_l = \mathbb{W}_l^{20}$. □

The ***strong operator topology*** on $\mathbb{U}$ is the topology of pointwise convergence on ℓ_2, i.e., $U_i \to U$ if for all $x \in \ell_2$, $U_i x \to Ux$. In this topology $\mathbb{U}$ becomes a ***Polish*** space, i.e., a separable space whose topology is induced by a complete metric. Actually $\mathbb{U}$ is a G_δ in $\mathbb{B}(\ell_2)$ under this topology. A subset A of a Polish space X is said to be ***analytic*** if it is the image of another Polish space by a continuous function. Analytic sets have the Baire property, meaning that they differ from an open set by a meagre set (see, e.g., Kechris [3] for the basics of descriptive set theory). The following result was proved in [6] as a byproduct of other computations, but for the readers convenience we include a simple proof here.

Proposition 4

Assume that $\mathbb{F}$ *is a symmetric subset of* $\mathbb{U}$, *closed in the strong operator topology, and that* $U_0, U_1, U_2, \ldots$ *is a sequence of unitary operators such that* $\mathbb{U}$ *is generated as a group by* $\{U_n\}$ *and* $\mathbb{F}$. *Then there is some finite* n *such that*

$$(\mathbb{F} \cup \{1, U_0, U_0^*, \ldots, U_n, U_n^*\})^n = \mathbb{U}.$$

Proof. Define $\mathbb{W}_n = (\mathbb{F} \cup \{1, U_0, U_0^*, \ldots, U_n, U_n^*\})^n$ and notice that $(\mathbb{W}_n)$ is an increasing, exhaustive sequence of symmetric, analytic subsets of $\mathbb{U}$. Since the sets have the Baire property there is some k such that $\mathbb{W}_k$ is comeagre in an open set and hence by Pettis' Theorem (see Kechris [3]), $\mathbb{W}_k\mathbb{W}_k^* = \mathbb{W}_k^2$ contains an open neighborhood of the identity in the strong operator topology. This implies that we can find some finite dimensional space $X \subseteq \ell_2$ such that if $U \restriction X = 1$, then $U \in \mathbb{W}_k^2$. Now find a unitary operator V such that $Y := V[X] \subseteq X^\perp$. Then $V\mathbb{W}_k^2V^*$ contains all unitaries U such that $U \restriction Y = 1$. Suppose now that U is an arbitrary unitary operator and find some finite dimensional space $Z \subseteq (X \oplus Y)^\perp$ such that $U[X] \subseteq X \oplus Y \oplus Z$. Find now some $W_0 \in \mathbb{W}_k^2$ such that $W_0[Y] \subseteq (X \oplus Y \oplus Z)^\perp$, while $W_0 \restriction X \oplus Z = I$. Then

$$W_0U[X] \subseteq W_0[X \oplus Y \oplus Z] \subseteq X \oplus W_0[Y] \oplus Z \subseteq Y^\perp.$$

There is therefore some $W_1 \in V\mathbb{W}_k^2 V^*$ such that $W_1 W_0 U \restriction X = I$. Thus we get $W_1 W_0 U \in \mathbb{W}_k^2$ and therefore

$$U \in W_0^* W_1^* \mathbb{W}_k^2 \subseteq \mathbb{W}_k^2 V \mathbb{W}_k^2 V^* \mathbb{W}_k^2$$

and

$$\mathbb{U} = \mathbb{W}_k^2 V \mathbb{W}_k^2 V^* \mathbb{W}_k^2.$$

Find now some sufficiently big $m > k$ such that $V, V^* \in \mathbb{W}_m$, then

$$\mathbb{U} = \mathbb{W}_k^2 V \mathbb{W}_k^2 V^* \mathbb{W}_k^2 = \mathbb{W}_k^2 \mathbb{W}_m \mathbb{W}_k^2 \mathbb{W}_m \mathbb{W}_k^2 = \mathbb{W}_m^8 = \mathbb{W}_{8m}.$$

So $n = 8m$ works. $\square$

We now fix some increasing, exhaustive sequence $(\mathbb{W}_n)$ of subsets of $\mathbb{U}$, and note that by considering instead $(\mathbb{W}_n \cap \mathbb{W}_n^*)$ we can assume they are symmetric. By Proposition 3, we also fix a decomposition $\ell_2 = X \oplus X^\perp$ into closed infinite-dimensional subspaces such that $\mathbb{D} = \mathbb{U}(X) \oplus \mathbb{U}(X^\perp) \subseteq \mathbb{W}_l^{20}$ for some l.

Proposition 5

$\mathbb{U}$ is finitely generated over $\mathbb{D}$.

This means that $\mathbb{U}$ is generated by $\mathbb{D}$ and a finite number of elements (actually 8), say U_i's.

We notice that this will indeed be enough to prove Theorem 1. For combining Propositions 4 and 5, we get that $\mathbb{U}$ has bounded length with respect to $\mathbb{D}$ and the family $U_1, U_1^*, ..., U_8, U_8^*$ that we will prove generates $\mathbb{U}$ over $\mathbb{D}$, in other words, there is an integer k such that any unitary U in $\mathbb{U}$ can be written as a product $U = V_1 ... V_k$ where V_i either belongs to $\mathbb{D}$ or is one of U_i or U_i^*.

We also note also that, the k in the definition of the Bergman property does not depend on the sequence $(\mathbb{W}_i)$, as $\mathbb{D}$ is unique up to inner automorphism. It is known that this uniformity does not hold in all groups with the Bergman property, as, e.g., it fails in the full group of a countable measure preserving equivalence relation (see B.D. Miller [5]).

From now on, we will use identifications of both X and $X^\perp$ with $\ell_2 = \ell_2(\mathbb{N})$ with its canonical basis and see endomorphisms of $X \oplus X^\perp$ as two by two matrices with entries in $\mathbb{B}(\ell_2)$.

A partial isometry is a map u on ℓ_2 so that $u^* u = p$ and $uu^* = q$ are orthogonal projections. We say that u is a partial isometry with initial space Im p and final space Im q (or from Im p to Im q), where Im T denotes the closure of the range of an operator T, and notice that, in this case, u is actually an isometric bijection between Im p and Im q. We recall also the polar decomposition of an operator: Every operator T can be decomposed as $T = u|T|$, where $|T| = (T^*T)^{1/2}$ is positive, self-adjoint, Ker $|T| =$ Ker T, and u is a partial isometry from Im $|T| =$ (Ker $T)^\perp$ to Im T.

Let S be the unilateral shift operator on ℓ_2 ($S(e_i) = e_{i+1}$), and L be an isometry from ℓ_2 to an infinite codimensional subspace ($L(e_i) = e_{2i}$ for instance). For any isometry u, p_u will be the projection $1 - uu^*$. Let $\mathbb{W}$ be the subgroup of $\mathbb{U}$ generated by $\mathbb{D}$ and the matrices

$$\begin{bmatrix} u & p_u \\ 0 & u^* \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & u^* \\ u & -uu^* + \sqrt{2} p_u \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & u^{*2} \\ u^2 & -u^2 u^{*2} + \sqrt{2} p_{u^2} \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

where u is either 1, S or L.

$\mathbb{W}$ is generated by $\mathbb{D}$ and eight unitaries, that we denote by $U_1, ..., U_8$. We will show that $\mathbb{W} = \mathbb{U}$.

Lemma 6

Let $\begin{bmatrix} A \\ B \end{bmatrix}$ *be an isometry from* ℓ_2 *to* $\ell_2 \oplus \ell_2$, *then there is a partial isometry* v *from* Im $1 - A^*A$ *to* Im B *so that* $B = v(1 - A^*A)^{1/2}$.

Proof. Since $\begin{bmatrix} A \\ B \end{bmatrix}$ is an isometry, we get that $A^*A + B^*B = 1$. So we conclude that $|B| = (1 - A^*A)^{1/2}$, the last part follows from the polar decomposition and the fact that for any positive operator T, Im T = Im $T^{1/2}$. □

Lemma 7

$\mathbb{W}$ *contains all matrices of the form*

$$\begin{bmatrix} u & p_u \\ 0 & u^* \end{bmatrix}, \qquad \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & u^* \\ u & -uu^* + \sqrt{2} p_u \end{bmatrix},$$

where u *is an isometry.*

Proof. First, remark that up to multiplications by unitaries any isometry u is determined only by the dimension of Im p_u. So multiplying by elements in $\mathbb{D}$, we only need to prove the lemma for $u = S^k$, $u = L$ and $u = 1$. The only non-trivial cases are for $u = S^k$. For the first case, we notice that

$$\begin{bmatrix} S & p_S \\ 0 & S^* \end{bmatrix}^k = \begin{bmatrix} S^k & A \\ 0 & S^{*k} \end{bmatrix}.$$

Thus $\begin{bmatrix} A \\ S^{*k} \end{bmatrix}$ and $\begin{bmatrix} S^{*k} \\ A^* \end{bmatrix}$ are isometries from ℓ_2 to $\ell_2 \oplus \ell_2$. Hence, as S^{*k} is an isometry on $(\text{Im } p_{S^k})^\perp = [e_{k+1}, e_{k+2}, \ldots]$, we have $A \restriction (\text{Im } p_{S^k})^\perp = A^* \restriction (\text{Im } p_{S^k})^\perp = 0$, while, as $S^{*k} \restriction \text{Im } p_{S^k} = 0$, A and A^* must be isometries on Im p_{S^k}. Therefore, A is a partial isometry with initial and final space Im p_{S^k}. Letting $U = (1 - p_{S^k}) + A$, we see that U is unitary and that

$$\begin{bmatrix} S & p_S \\ 0 & S^* \end{bmatrix}^k \begin{bmatrix} 1 & 0 \\ 0 & U^* \end{bmatrix} = \begin{bmatrix} S^k & p_{S^k} \\ 0 & S^{*k} \end{bmatrix}.$$

For the other case, we notice that by definition

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & S^{*k} \\ S^k & -S^kS^{*k}+\sqrt{2}p_{S^k} \end{bmatrix}$$

is in $\mathbb{W}$ for $k=1$ and $k=2$. Now

$$\begin{bmatrix} S & p_S \\ 0 & S^* \end{bmatrix}^* \begin{bmatrix} 1 & S^{*k} \\ S^k & -S^kS^{*k}+\sqrt{2}p_{S^k} \end{bmatrix} \begin{bmatrix} S & p_S \\ 0 & S^* \end{bmatrix} = \begin{bmatrix} 1 & S^{*(k+2)} \\ S^{k+2} & X \end{bmatrix},$$

where $X = p_S + p_S S^{*(k+1)} + S^{k+1}p_s - S^{k+1}S^{*(k+1)} + \sqrt{2}Sp_SS^*$.

Let $V = -S^{k+2}S^{*(k+2)} + \sqrt{2}p_{S^{k+2}}$ and let u be the unitary operator given by $u(e_1) = \frac{e_1+e_{k+2}}{\sqrt{2}}$, $u(e_{k+2}) = \frac{e_1-e_{k+2}}{\sqrt{2}}$, while $u(e_i)=e_i$ for all $i \neq 1, k+2$. One can check, by, e.g., considering the action on vectors, that $X = uV$ and, as u is the identity on $[e_1, e_{k+2}]^\perp$, $u^*S^{k+2} = S^{k+2}$. Thus

$$\begin{bmatrix} 1 & S^{*(k+2)} \\ S^{k+2} & V \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & u^* \end{bmatrix} \begin{bmatrix} 1 & S^{*(k+2)} \\ S^{k+2} & X \end{bmatrix},$$

and the lemma follows by induction. □

The matrices appearing the preceding lemma are actually special cases of the more general form of a unitary that will appear in the following.

Proof of Proposition 5: We have to prove that any unitary belongs to $\mathbb{W}$. The strategy is to start with an arbitrary unitary viewed as a 2 by 2 matrix

$$U = \begin{bmatrix} T & M \\ N & K \end{bmatrix}$$

and to multiply it by elements in $\mathbb{W}$ to get simpler forms.

First, we can assume that T is positive. To see this, notice that by replacing U by U^* we can assume that dim Ker $T \leqslant$ dim Ker T^*. We have $T = u|T|$ for some partial isometry u from Im $|T| = (\text{Ker } T)^\perp$ to Im $T = (\text{Ker } T^*)^\perp$, and our hypothesis dim Ker $T \leqslant$ dim Ker T^* implies that u can be extended to an isometry $\tilde{u}$ so that $T = \tilde{u}|T|$. Then

$$\begin{bmatrix} \tilde{u} & p_{\tilde{u}} \\ 0 & \tilde{u}^* \end{bmatrix}^* U = \begin{bmatrix} |T| & * \\ * & * \end{bmatrix},$$

which proves our claim.

Applying Lemma 6 twice, we only need to prove that a unitary of the form

$$U = \begin{bmatrix} T & (1-T^2)^{1/2}v^* \\ u(1-T^2)^{1/2} & X \end{bmatrix}$$

is in $\mathbb{W}$, where u and v are partial isometries with initial space $\text{Im}\,(1-T^2)^{1/2} = (\text{Ker}\,(1-T^2))^\perp = (\text{Ker}\,(1-T))^\perp$ (as T is positive). So $u^*u = v^*v$ is the projection onto $\text{Im}\,(1-T^2)^{1/2}$. Let $uu^* = 1-p$ and $vv^* = 1-q$ and note that $u^*p = pu = v^*q = qv = 0$.

The fact that U is a unitary means that $U^*U = UU^* = 1$, and writing down the computation on matrices, this implies that

(i) $u(1-T^2)u^* + XX^* = 1$,
(ii) $v(1-T^2)v^* + X^*X = 1$,
(iii) $u(1-T^2)^{1/2}T + Xv(1-T^2)^{1/2} = 0$,
(iv) $v(1-T^2)^{1/2}T + X^*u(1-T^2)^{1/2} = 0$.

Now $u^*u = v^*v$ is the projection onto $\mathrm{Im}\,(1-T^2)^{1/2}$, so as T and $(1-T^2)^{1/2}$ commute, we have by (iii)

$$uT(1-T^2)^{1/2} + Xv(1-T^2)^{1/2} = 0,$$

whence $-uT$ and Xv agree on $\mathrm{Im}\,(1-T^2)^{1/2} = \mathrm{Im}\ v^*$. So

$$-uTv^* = Xvv^* = X(1-q).$$

Similarly, using (iv) we have

$$-vTu^* = X^*uu^* = X^*(1-p).$$

Thus $-uTv^* = X(1-q) = (1-p)X$.

Using (ii) we have $|X| = (X^*X)^{1/2} = (1 - v(1-T^2)v^*)^{1/2} = (q + vTv^*)^{1/2}$. Now, q is the projection onto $(\mathrm{Im}\ v)^\perp$, so $qvT^2v^* = 0 = vT^2v^*q$, and, using that $m^{1/2}$ is the operator limit of polynomials in m, we therefore have (as v^*v commutes with T)

$$(q + vT^2v^*)^{1/2} = q^{1/2} + (vT^2v^*)^{1/2} = q + vTv^*.$$

Hence, by the polar decomposition, there is a partial isometry α with initial space $\mathrm{Im}\,(q+vTv^*)^{1/2}$ so that $X = \alpha(q + vT^2v^*)^{1/2}$. The initial space of α is $\mathrm{Im}\ q + vTv^* = \mathrm{Im}\ q + \mathrm{Im}\ vTv^*$ and therefore $\alpha^*\alpha \geqslant q$. Now,

$$X = Xq + X(1-q) = \alpha(q + vTv^*)q - uTv^* = \alpha q - uTv^*.$$

Similarly, using (i), we see that $X = (p + uTu^*)\beta^*$, where β is a partial isometry with initial space $\mathrm{Im}\ p + uTu^* = \mathrm{Im}\ p + \mathrm{Im}\ uTu^*$, whence $\beta^*\beta \geqslant p$, and

$$X = pX + (1-p)X = p\beta^* - uTv^*.$$

So $p\beta^* = \alpha q = pX = Xq = pXq$.

Using $\alpha^*\alpha \geqslant q$, we see that $\alpha q = p\beta^*$ is an isometry with initial space $\mathrm{Im}\ q$, and, as $\beta^*\beta \geqslant p$, $\alpha q = p\beta^*$ has final space $\mathrm{Im}\ p$.

To summarise,

$$U = \begin{bmatrix} T & (1-T^2)^{1/2}v^* \\ u(1-T^2)^{1/2} & -uTv^* + \delta \end{bmatrix},$$

where u is a partial isometry from $(\mathrm{Ker}\ 1-T)^\perp$ to $\mathrm{Im}\ 1-p$, v is a partial isometry from $(\mathrm{Ker}\ 1-T)^\perp$ to $\mathrm{Im}\ 1-q$ and $\delta = \alpha q = p\beta^*$ is a partial isometry from $\mathrm{Im}\ p$ to $\mathrm{Im}\ q$.

Consequently, one can find a unitary s so that $v = su$ and that coincides with δ on $\mathrm{Im}\ p$. So

$$U' = U\begin{bmatrix} 1 & 0 \\ 0 & s \end{bmatrix} = \begin{bmatrix} T & (1-T^2)^{1/2}u^* \\ u(1-T^2)^{1/2} & -uTu^* + p \end{bmatrix}.$$

In particular, U' is self-adjoint, and note that $\mathrm{Ker}\ 1 - uTu^* = \{0\}$.

The next step is to show that we can suppose that u is an isometry. To do this, it suffices to write $T' = T - q'$ where q' is the projection onto Ker $1 - T$. So then Ker $1 - T' = \{0\}$. Then $T'q' = 0$, $q'T' = q' - q'T = q' - (Tq')^* = q' - q' = 0$, and $q'u^* = 0$, so $(1-T^2)^{1/2}u^* = (1-T'^2)^{1/2}u^*$ and

$$U' = \begin{bmatrix} T' + q' & (1-T'^2)^{1/2}u^* \\ u(1-T'^2)^{1/2} & -uT'u^* + p \end{bmatrix}$$

where u is a partial isometry from Im $1 - q'$ onto Im $1 - p$ and Ker $1 - T' = \{0\}$.

The previous formula is now symmetric in terms of T and uTu^* as Ker $1 - T' =$ Ker $1 - uT'u^* = \{0\}$. We mean that $T = T' + q'$ and $-uTu^* + p = -uT'u^* + p$ play exactly the same role in the above formula. So, using a conjugation with $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ to reverse the roles of p and q', we therefore can assume that dim Im $p \geqslant$ dim Im q'. Using other conjugations with elements in $\mathbb{D}$, we can assume that $q' \leqslant p$ and u is a partial isometry from $1 - q'$ to $1 - p$.

With these choices, it is then easy to find an isometry w extending u, that is $(w - u)(1 - q') = 0$, so that replacing $T' + q'$ by T', we can suppose that U' is of the form

$$\begin{bmatrix} T' & (1-T'^2)^{1/2}u^* \\ u(1-T'^2)^{1/2} & -uT'u^* + p \end{bmatrix} \qquad (*)$$

were u is an isometry from ℓ_2 to Im $1 - p$. Note that the matrices of Lemma 7 are of this form.

T' is a self-adjoint positive contraction on ℓ_2, so using the functional calculus, define $A = \exp(i \arccos T')$ and $B = \exp(-i \arccos T')$. These are two unitaries with

$$A + B = 2\cos(\arccos T') = 2T' \quad \text{and} \quad A - B = 2i \sin(\arccos T') = 2i(1-T'^2)^{1/2}.$$

Notice also that $uBu^* - p$ is unitary, actually $p = p_u$ with the notations of Lemma 7. Finally,

$$\begin{aligned} &\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & u^* \\ u & -uu^* + \sqrt{2}p \end{bmatrix} \cdot \begin{bmatrix} A & 0 \\ 0 & uBu^* - p \end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & u^* \\ u & -uu^* + \sqrt{2}p \end{bmatrix} \\ &= \begin{bmatrix} T' & i(1-T'^2)^{1/2}u^* \\ iu(1-T'^2)^{1/2} & uT'u^* - p \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix} \cdot \begin{bmatrix} T' & (1-T'^2)^{1/2}u^* \\ u(1-T'^2)^{1/2} & -uT'u^* + p \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix} \end{aligned}$$

So $U \in \mathbb{W}$. □

The drawback of using a Baire category argument is that it does not provide an explicit bound for the number k for which $\mathbb{U}$ is k-Bergman. Actually it is not hard to modify the proof of Lemma 7 to get it. It relies on the following observation: assume that there is a decomposition $\ell_2 = X \oplus X^\perp$ into two infinite dimensional subspaces that are reducing for all block elements of a matrix $U = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, then conjugating U

with an element in $\mathbb{D}$, we can assume that X is fixed and another conjugation with the unitary V of $\ell_2 \oplus \ell_2$ that identifies $(X \oplus X^\perp) \oplus (X \oplus X^\perp)$ with $(X \oplus X) \oplus (X^\perp \oplus X^\perp)$, we end up with an element in $\mathbb{D}$. We can add V to the list of unitaries $U_1, ..., U_8$. This can be applied to the matrices

$$\begin{bmatrix} u & p_u \\ 0 & u^* \end{bmatrix}, \qquad \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & u^* \\ u & -uu^* + \sqrt{2}p_u \end{bmatrix},$$

where u is an even power of the shift (with $X = \text{span}\{e_{2i},\ i \geqslant 0\}$), this is enough to conclude to the bounded length of $\mathbb{U}$ over $\mathbb{D}$. The resulting estimate $k \leqslant 200$ is probably far from being optimal.

Along the same lines, one can ask if the Cayley graph of $\mathbb{U}$ is uniformly bounded with respect to any generating set. The answer is of course negative as one can take $E_n = \{e^{2i\pi H}; H = H^*,\ \|H\| \leq 1/n\}$. E_n is clearly a generating set and $E_n^n = \mathbb{U}$. From the triangular inequality,

$$\text{dist}(Id, E_n^k) \leq k.(e^{2\pi/n} - 1)$$

so if $-Id \in E_n^k$ then $k \geqslant n/e\pi$.

We have focused on the unitary group, but one can easily adapt the arguments for the orthogonal group $\mathbb{O}$. Indeed, the result of Brown and Pearcy still holds for $\mathbb{O}$. The only place in the proof where we used the orthogonal group is at the very end in the diagonalization procedure. We briefly explain how to adapt it to the case of $\mathbb{O}$. We start with any orthogonal matrix O on the form $(*)$ with respect to a decomposition $\ell_2 = Y \oplus Y$. Since T' is self-adjoint, there is a decomposition $Y = X \oplus X^\perp$ into infinite dimensional reducing subspaces. Conjugating with elements in $\mathbb{D}$ with can assume that X, u and p depend only on the dimension of Im p. Moreover, as u is an isometry, it means that there is a decomposition of $\ell_2 = Y \oplus Y \oplus \text{Im } p$, so that the matrix of O with respect to it is

$$\begin{bmatrix} T' & (1 - T'^2)^{1/2} & 0 \\ (1 - T'^2)^{1/2} & -T' & 0 \\ 0 & 0 & \text{Id} \end{bmatrix}.$$

As explained above, the 3×3 matrix of O is then block diagonal for the decomposition $(X \oplus X) \oplus (X^\perp \oplus X^\perp) \oplus \text{Im } p$. Since X and $X^\perp$ are infinite dimensional, there is an orthogonal transformation identifying $(X \oplus X) \oplus \big((X^\perp \oplus X^\perp) \oplus \text{Im } p\big)$ with $Y \oplus Y$, which means that conjugating O with a matrix that depends only on Im p, we end up in $\mathbb{D}$. Hence, we have that $\mathbb{O}$ is generated by $\mathbb{D}$, a finite number of orthogonal maps and another sequence of orthogonal transformations corresponding to the possible values of the dimension of Im p. So Proposition 4 allows us to conclude the k-Bergman property for $\mathbb{O}$. On the other hand, as $GL(\ell_2)$ acts unboundedly on ℓ_2 by Lipschitz maps, it is fairly easy to check that the Bergman property fails for $GL(\ell_2)$ (see [6]).

As a possible extension of the main result, one can wonder what happens for the group of invertible isometries $\mathbb{U}(X)$ on a separable Banach space X. In some particular cases, one can give a precise answer.

If $X = \ell_p$ with $1 \leqslant p \neq 2 < \infty$ or c_0, it is well known that $\mathbb{U}(X)$ is the semi-direct product of $\mathbb{T}^{\mathbb{N}}$ by S_∞ with the natural action $\sigma.(z_i) = (z_{\sigma(i)})$ for $\sigma \in S_\infty$ and

$(z_i) \in \mathbb{T}^{\mathbb{N}}$. Using Bergman's theorem saying that S_∞ is 17-Bergman, Proposition 3 can be adapted to show that $\mathbb{U}(X)$ has the 72-Bergman property. We give a brief sketch of it. We keep the notations $\mathbb{T}^{\mathbb{N}}$ and S_∞ for their copy in $\mathbb{T}^{\mathbb{N}} \rtimes S_\infty = \mathbb{U}(X) = \mathbb{U}$.

Let $\mathbb{W}_i$ be an increasing sequence of symmetric sets such that $\bigcup \mathbb{W}_i = \mathbb{U}$. Since S_∞ has the 17-Bergman property, there is some n so that $\mathbb{W}_n^{17}$ contains S_∞. As in Proposition 3, decompose $\mathbb{N} = \bigcup X_i$ as an infinite partition of infinite sets. Then there must be an i so that $p_i(\mathbb{W}_i \cap \mathbb{T}^{\mathbb{N}}) = \mathbb{T}^{X_i}$ where p_i is the natural projection of $\mathbb{T}^{\mathbb{N}}$ onto $\mathbb{T}^{X_i}$. Let $X_i = X$. Now we have partition $\mathbb{N} = X \cup Y$ with Y infinite and write $((x_l), (y_l))$ for elements in $\mathbb{T}^{\mathbb{N}}$ according to this decomposition. Identify X with $\mathbb{Z}$, and let $\sigma \in S_\infty$ be the permutation corresponding to the bilateral shift on X and the identity on Y. The following equality holds in $\mathbb{U}$:

$$((x_l^{-1}x_{l+1}), (1)) = ((x_l), (y_l))^{-1}.\sigma.((x_l), (y_l)).\sigma^{-1}.$$

As elements on the form $(x_l^{-1}x_{l+1})$ describe all of $\mathbb{T}^X$, we get that $\mathbb{W}_i\sigma\mathbb{W}_i\sigma^{-1}$ contains all elements of the form $(x, (1))$ $x \in \mathbb{T}^X$. Using a permutation τ that exchanges X and Y, we get that

$$\mathbb{T}^{\mathbb{N}} \subset \mathbb{W}_i\sigma\mathbb{W}_i\sigma^{-1}\tau\mathbb{W}_i\sigma\mathbb{W}_i\sigma^{-1}\tau^{-1}.$$

Since any element in $\mathbb{U}$ is the product of one element in $\mathbb{T}^{\mathbb{N}}$ and one element in S_∞, we see that $W_{\max\{i,n\}}^{72} = \mathbb{U}$.

It is also possible to provide Banach spaces whose group of invertible isometries does not have the Bergman property. Consider the space $\mathcal{C}([0,1])$ of continuous functions on $[0,1]$. By the Banach-Stone theorem, a surjective isometry g of $\mathcal{C}([0,1])$ comes from the composition with a continuous homeomorphism ϕ of $[0,1]$ and the multiplication by a unimodular continuous function u, that is $g(f)(x) = u(x)f(\phi(x))$ for $x \in [0,1]$ and $f \in \mathcal{C}([0,1])$. As a group $\mathbb{U}(\mathcal{C}([0,1])$ is the semi-direct product of unimodular continuous functions by continuous homeomorphisms. Of course homeomorphisms of $[0,1]$ have to let $\{0,1\}$ invariant. So, by restriction, there is a surjection of $\mathbb{U}(\mathcal{C}([0,1]))$ onto $\mathbb{T}^2 \rtimes \mathbb{Z}/2\mathbb{Z} = G$, where $\mathbb{Z}/2\mathbb{Z}$ acts by permutation of the coordinates of $\mathbb{T}^2$. But, it is easy to see that G and so $\mathbb{U}(\mathcal{C}([0,1]))$ do not have the Bergman property as $\mathbb{T}$ has countable cofinality.

For more elaborated counterexamples, one can think of X as hereditarily indecomposable. Then, any endomorphism of X is of the form $\lambda Id + S$ with S strictly singular and $\lambda \in \mathbb{C}$. Since strictly singular operators form an ideal, it follows that there is a surjective group homomorphism $\mathbb{U}(X) \to \mathbb{C}$. Since $\mathbb{C}$ does not have the Bergman property, $\mathbb{U}(X)$ also fails it.

References

1. G.M. Bergman, Generating infinite symmetric groups, *Bull. London Math. Soc.* **38** (2006), 429–440.
2. A. Brown and C. Pearcy, Multiplicative commutators of operators, *Canad. J. Math.* **18** (1966), 737–749.
3. A.S. Kechris, *Classical Descriptive Set Theory*, Graduate Texts in Mathematics, **156**, Springer-Verlag, New York, 1995.

4. S. Koppelberg and J. Tits, Une propriété des produits directs infinis de groupes finis isomorphes, *C. R. Acad. Sci. Paris Sér. A* **279** (1974), 583–585.
5. B.D. Miller, *Full Groups, Classification, and Equivalence Relations*, Ph.D. Thesis, UC Berkeley, 2004.
6. C. Rosendal, A topological version of the Bergman property, (preprint).
7. J-P. Serre, *Trees*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003.