The Kučera - Gács Theorem

Lemma There is a recursive function

$l : \mathbb{Q} \cap]1, \infty[\times \mathbb{N} \longrightarrow \mathbb{N}$ such that

for any Martingale d and any $\sigma \in 2^{<\mathbb{N}}$

$$\#\{\tau \in 2^{l(\delta,k)} \mid d(\sigma\tau) \leq \delta \cdot d(\sigma)\} \geq k.$$

Proof Notice that for any d and l

we have $$d(\sigma) = 2^{-l} \sum_{\tau \in 2^l} d(\sigma\tau)$$

Thus, for $\delta > 1$,

$$2^{-l} \cdot \sum_{\{\tau \in 2^l \mid d(\sigma\tau) > \delta \cdot d(\sigma)\}} d(\sigma\tau) \leq d(\sigma)$$

and hence $\#\{\tau \in 2^l \mid d(\sigma\tau) > \delta d(\sigma)\}$

$$\leq \frac{2^k}{\delta} .$$

Since $\delta > 1$, also $1 - \frac{1}{\delta} > 0$ and hence

when $l(\delta, k) \geq \log\left(k\delta(\delta-1)^{-1}\right)$, then

$$\#\{\tau \in 2^{l(\delta,k)} \mid d(\sigma\tau) \leq \delta d(\sigma)\}$$

$$\geq 2^{l(\delta,k)} - \frac{2^{l(\delta,k)}}{\delta} = \frac{(\delta-1)\, 2^{l(\delta,k)}}{\delta}$$

$$\geq \frac{(\delta-1)\cdot k\ \delta\, (\delta-1)^{-1}}{\delta} = k \quad . \qquad \square$$

<u>Theorem</u> (Kučera - Gács) Every real $x \in 2^{\mathbb{N}}$ is Turing reducible to a 1-random real.

<u>Proof</u> Let $r_0 > r_1 > r_2 > \ldots > 1$ be a sequence of strictly positive rationals such that if $b_i = \prod_{j \leq i} r_j$, then (b_i) converges to some b.

Put $l_s = l(r_s, 2)$.

By the lemma, we have for any σ and martingale d, at least 2 strings $\tau \in 2^{l_s}$ such that $d(\sigma\tau) \leq r_s d(\sigma)$.

Let d be the universal effective martingale. Thus, for β to be 1-random it is enough that d does not succeed on β. We can assume that $d(\emptyset) = 1$.

So given $\alpha \in 2^{\mathbb{N}}$ we construct $\beta \geq_T \alpha$ such that d does not succeed on β.

We now construct $\sigma_0 \subseteq \sigma_1 \subseteq \dots \subseteq \beta$ as follows.

- Stage $s = 0$: Notice that there are at least two strings σ such that

  $d(\sigma) \leq r_0 \cdot d(\emptyset) = r_0$

  If $\alpha(0) = 0$ let σ_0 be the lexicographically least such and if $\alpha(0) = 1$ the lexicographically greatest such.

- Stage $s+1$: There are at least two τ such that $d(\sigma_s \tau) \leq r_{s+1} \cdot d(\sigma_s)$ and we let $\sigma_{s+1} = \sigma_s \tau$, where τ is chosen as before.

Then by induction $d(\sigma_s) \leq b_s$, $b_s \to b$.
So $\liminf d(\beta|n) < \infty$ and β is 1-random. $\square$