Infinite games

We will consider games with perfect information by two players on the integers

$$\begin{array}{c|ccccccc} \mathrm{I} & a_0 & & a_2 & & a_4 & & \cdots \\ \hline \mathrm{II} & & a_1 & & a_3 & & a_5 & \cdots \end{array}$$

So player I and II alternately play $a_i \in \mathbb{N}$ and hence a run of the game corresponds to a function $f : \mathbb{N} \longrightarrow \mathbb{N}$ given by $f(n) = a_n$.

Given a set of total functions, i.e., $\mathcal{A} \subseteq \mathbb{N}^{\mathbb{N}}$, and a run of the game $f \in \mathbb{N}^{\mathbb{N}}$, we say that I <u>wins</u> the game if $f \in \mathcal{A}$.

Thus any set $\mathcal{A}$ determines a game with two players which we denote by $G(\mathcal{A})$. In this case, $\mathcal{A}$ is said to be the winning condition.

A <u>strategy</u> s for player I in this game is a function

$$s : \mathbb{N}^{<\mathbb{N}} \longrightarrow \mathbb{N}$$

i.e., from finite strings of numbers.

We say that I plays <u>according</u> to this strategy if for $a_0, a_1, a_2, \ldots$, the outcome of the game we have

$$a_{2i} = s(a_0, a_1, \ldots, a_{2i-1}).$$

In other words, the strategy tells I what to play depending on what II plays.

The notion of a strategy for II is defined similarly.

Given $A \subseteq \mathbb{N}^{\mathbb{N}}$, a strategy s for I (resp. for II) is said to be <u>winning</u> for the game $G(A)$ if I wins whenever he plays according to the strategy.

We say that a game $G(\mathcal{A})$ is determined if one of the two players have a winning strategy.

<u>Definition</u> The AXIOM of DETERMINACY (AD) is the statement that $G(\mathcal{A})$ is determined for every $\mathcal{A} \subseteq \mathbb{N}^{\mathbb{N}}$.

<u>Theorem</u> Let $\mathcal{A} \subseteq \underset{\sim}{D}$ and let $\mathcal{A}^* = \{ A \mid [A]_T \in \mathcal{A} \}$. If $G(\mathcal{A}^*)$ is determined, then $\mathcal{A}$ either contains or is disjoint from a cone.

<u>Proof</u> Consider first the case where I has a winning strategy $s : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$ in the game $G(\mathcal{A}^*)$.

Since s is a sequence (σ_n, k_n) of pairs of finite strings and natural numbers we can recursively code s as a subset of $\mathbb{N}$. Thus s has a Turing degree $\underset{\sim}{a}$.

We claim that $\underset{\sim}{D}(\geq \underset{\sim}{a}) \subseteq \mathcal{A}$.

To see this, pick any $\underset{\sim}{b} \geq \underset{\sim}{a}$ and function $g = (a_1, a_3, a_5, a_7, \ldots) \in \underset{\sim}{b}$.

In the game $G(\mathcal{A}^{\#})$ we let II play $a_1, a_3, a_5, \ldots$ and let I play $a_0, a_2, \ldots$ according to its strategy s.

Clearly, $a_0, a_2, \ldots$ can be computed from s and $a_1, a_3, a_5, \ldots$.

Thus if $\underset{\sim}{c}$ is the degree of $a_0, a_2, \ldots$ then $\underset{\sim}{c} \leq \underset{\sim}{b} \vee \underset{\sim}{a} = \underset{\sim}{b}$ (since $\underset{\sim}{a} \leq \underset{\sim}{b}$).

Also since I wins $G(\mathcal{A}^{\#})$, the sequence $a_0, a_1, a_2, a_3, \ldots$ belongs to $\mathcal{A}$ and so its degree $\underset{\sim}{b} \vee \underset{\sim}{c} = \underset{\sim}{b}$ belongs to $\mathcal{A}$.

Therefore, $\underset{\sim}{b} \in \mathcal{A}$ and $\underset{\sim}{D}(\geq \underset{\sim}{a}) \subseteq \mathcal{A}$.

Similarly, $\underset{\sim}{D}(\geq \underset{\sim}{a}) \cap \mathcal{A} = \emptyset$ if II has a w.s.

$\square$

<u>Corollary</u> (AD) Every set of degrees either contains or is disjoint from a cone.

<u>Theorem</u> (AD) Every map $\Theta : \underset{\sim}{D} \to \mathbb{R}$ is constant on a cone.

<u>Proof</u> For every $q \in \mathbb{Q}$ let

$$\mathcal{A}_q = \{ \underset{\sim}{a} \in \underset{\sim}{D} \mid \Theta(\underset{\sim}{a}) \geq q \}.$$

List $\mathbb{Q}$ as $q_0, q_1, q_2, \ldots$.
We define inductively $\underset{\sim}{a}_0 \leq \underset{\sim}{a}_1 \leq \underset{\sim}{a}_2 \leq \ldots$ such that $\underset{\sim}{D}(\geq \underset{\sim}{a}_n)$ is either contained in or disjoint from $\mathcal{A}_{q_n}$.

$n=0$: $\mathcal{A}_{q_0}$ is a set of degrees, so either contains or is disjoint from some cone $\underset{\sim}{D}(\geq \underset{\sim}{b})$. So let $\underset{\sim}{a}_0 = \underset{\sim}{b}$.

$n=k+1$: Suppose $\underset{\sim}{a}_k$ is defined and find a cone $\underset{\sim}{D}(\geq \underset{\sim}{b})$ which is either contained in or disjoint from $\mathcal{A}_{q_n}$.

Then as $\underset{\sim}{D}(\geq \underset{\sim}{b} \vee \underset{\sim}{a}_k) \subseteq \underset{\sim}{D}(\geq \underset{\sim}{b})$, $\underset{\sim}{a}_n = \underset{\sim}{b} \vee \underset{\sim}{a}_k$ works.

Pick $A_n \in \underset{\sim}{a}_n$ and let $A = \bigoplus_n A_n$, $\underset{\sim}{a} = [A]_T$. Then $\underset{\sim}{a}_0 \leq \underset{\sim}{a}_1 \leq \dots \leq \underset{\sim}{a}$ and thus for every n,

$$\underset{\sim}{D}(\geq \underset{\sim}{a}) \cap \mathcal{A}_{q_n} = \emptyset \quad \text{or}$$

$$\underset{\sim}{D}(\geq \underset{\sim}{a}) \subseteq \mathcal{A}_{q_n}.$$

In particular, for every $\underset{\sim}{b} \geq \underset{\sim}{a}$ and $q \in \mathbb{Q}$

$$\Theta(\underset{\sim}{b}) \geq q \iff \Theta(\underset{\sim}{a}) \geq q.$$

Whence $\Theta(\underset{\sim}{b}) = \Theta(\underset{\sim}{a})$.

So Θ is constant on $\underset{\sim}{D}(\geq \underset{\sim}{a})$. $\Box$

<u>Corollary</u> (AD) There is no injective function from $\underset{\sim}{D}$ to $\mathbb{R}$.

In particular, the axiom of choice fails.

<u>Theorem</u> (AD) There is a countably additive non-atomic measure on $\underset{\sim}{D}$. Moreover, the measure is two-valued.

Recall first that a ctbl additive measure on a set X is a function

$$\mu : \mathcal{P}(X) \longrightarrow [0,1] \quad \text{satisfying}$$

(i) $\mu(\emptyset) = 0$, $\mu(X) = 1$

(ii) $Y \subseteq Z \subseteq X \Rightarrow \mu(Y) \leq \mu(Z)$

(iii) If $Y_n \subseteq X$ are disjoint, then

$$\mu\left(\bigcup_{n \in \mathbb{N}} Y_n\right) = \sum_{n \in \mathbb{N}} \mu(Y_n).$$

(iv) μ is <u>non-atomic</u> if $\mu(\{x\}) = 0$ for every $x \in X$.

Proof We simply define

$$\mu(\mathcal{A}) = \begin{cases} 0 & \text{if } \mathcal{A} \text{ is disjoint from a cone} \\ 1 & \text{if } \mathcal{A} \text{ contains a cone} . \end{cases}$$

Clearly, (i), (ii) and (iv) hold.

Now if $\mathcal{A}_n \subseteq \underset{\sim}{D}$ are disjoint then there are no two of them that can contain a cone. For then we would have two disjoint cones, which is impossible.

So $\sum_n \mu(\mathcal{A}_n) \leq 1$.

And if each $\mathcal{A}_n$ is disjoint from a cone $\underset{\sim}{D}(\geq \underset{\sim}{a}_n)$, then we can find $\underset{\sim}{b} \geq \underset{\sim}{a}_n$ for all n, and thus $\mathcal{A}_n \cap \underset{\sim}{D}(\geq \underset{\sim}{b}) = \emptyset$, $\forall n$ whence $\bigcup \mathcal{A}_n$ is disjoint from a cone $\underset{\sim}{D}(\geq \underset{\sim}{b})$.

So $\mu(\bigcup \mathcal{A}_n) = 0$ if $\sum_N \mu(\mathcal{A}_n) = 0$.

$\square$

Definition A linear order $(X, \leq)$ is a well order if there is no infinite descending chain.

Thm Let $(X, \leq)$ and $(Y, \leq)$ be two wellorders. Then either $(X, \leq)$ is isomorphic to an initial segment of $(Y, \leq)$ or the other way around.

Prop No wellorder is embeddable into a proper initial segment of itself.

So wellorders are linearly ordered by the relation of embeddability which coincides with the relation of being isomorphic to an initial segment.

In fact, this wellorders the wellorders.

Evidently there is a bijection between $\mathbb{N}$ and $\mathbb{Q}$, so from a recursion theoretic viewpoint we can identify $2^{\mathbb{N}}$ with $\mathcal{P}(\mathbb{Q})$. Therefore, we can consider $\underset{\sim}{D}$ as $\mathcal{P}(\mathbb{Q})/\equiv_T$.

Let $WO = \{A \subseteq \mathbb{Q} \mid A$ is wellordered under $\leq_{\mathbb{Q}}\}$. Notice that if $A \subseteq \mathbb{Q}$ is wellordered, then its order-type $otp(A)$ is an ordinal $< \omega_1$. So $otp : WO \longrightarrow \omega_1$ is a surjective function since $\mathbb{Q}$ contains a copy of any ctbl linear order.

We can therefore define a function

$$\Theta : \underset{\sim}{D} \longrightarrow \omega_1$$

by

$$\Theta(\underset{\sim}{a}) = \sup\{otp(A) \mid A \in WO \;\&\; [A]_T \leq \underset{\sim}{a}\}$$

⌈Notice that if $\underset{\sim}{a} \leq \underset{\sim}{b}$ then $\Theta(\underset{\sim}{a}) \leq \Theta(\underset{\sim}{b})$. Hence Θ is a monotone function from $(\underset{\sim}{D}, \leq)$ to $(\omega_1, \leq)$.

Now letting $\nu = \Theta_* \mu$, i.e., $\nu(X) = \mu(\Theta^{-1}[X])$, we get a countably additive measure on ω_1. We claim that ν is non-atomic. For otherwise, there would be some $\xi < \omega_1$ such that $\Theta^{-1}(\{\xi\}) = \{\underset{\sim}{a} \in \underset{\sim}{D} \mid \Theta(\underset{\sim}{a}) = \xi\}$ has μ-measure 1, i.e., contains a cone $\underset{\sim}{D}(\geq \underset{\sim}{b})$. But if e.g. $A \subseteq \mathbb{Q}$, $otp(A) = \xi + 1$ and A has degree $\underset{\sim}{c}$, then $\Theta(\underset{\sim}{b} \vee \underset{\sim}{c}) \geq \xi + 1$, which is a contradiction. Here one uses monotonicity.

Theorem (Solovay) If AD holds then ω_1 is a measurable cardinal, i.e., carries a $\{0,1\}$-valued, ω_1-additive measure.

In other words, there is an ultrafilter $\mathcal{U}$ on ω_1 not containing singletons and such that $(\forall n\ A_n \in \mathcal{U}) \Rightarrow \bigcap A_n \in \mathcal{U}$.