Plain complexity and randomness

Definition Define $G : \mathbb{N} \longrightarrow \mathbb{N}$ by

$$G(n) = \begin{cases} K_{s+1}(t^*) & \text{if } n = 2^{\langle s,t\rangle} \text{ and } K_{s+1}(t^*) \neq K_s(t^*) \\ n & \text{otherwise.} \end{cases}$$

Then $$\sum_{n\in\mathbb{N}} 2^{-G(n)} \leq \sum_{n\in\mathbb{N}} 2^{-n} + \sum_{t\in\mathbb{N}} \sum_{m\geq K(t^*)} 2^{-m}$$

$$\leq 2 + 2\cdot\sum_{t\in\mathbb{N}} 2^{-K(t^*)} \leq 2+2 < \infty.$$

Remark: Here $K_s(t^*)$ is the smallest value among $|\rho|$ for $\rho \in 2^{\leq s}$ such that $\exists i \leq s$ $\hat{\Phi}_i^s(\rho) = t^*$. So $s,t \longmapsto K_s(t^*)$ is partial recursive and G is thus total recursive.

Theorem For $\alpha \in 2^{\mathbb{N}}$ the following are equivalent:

(i) α is 1-random

(ii) $\forall n$ $C(\alpha|n) \geq n - K(n^*) - O(1)$

(iii) $\forall n$ $C(\alpha|n) \geq n - g(n) - O(1)$ for all partial recursive $g : \mathbb{N} \to \mathbb{N}$ such that $\sum_{n\in\mathbb{N}} 2^{-g(n)} < \infty$

(iv) $\forall n$ $C(\alpha|n) \geq n - G(n) - O(1)$.

Proof (ii) $\Rightarrow$ (iii): Notice that if $g : \mathbb{N} \to \mathbb{N}$ is partial recursive with

$$\sum_{n \in \mathbb{N}} 2^{-g(n)} = \delta < \infty ,$$ then

$$\sum_{n \in \mathbb{N}} 2^{-g(n) - \log \delta} = \frac{1}{\delta} \cdot \sum_{n \in \mathbb{N}} 2^{-g(n)} = 1$$

so $\hat{P}(n^*) = g(n) + \lceil \log \delta \rceil$ is an information content measure, and thus

$$K(n^*) \leq \hat{P}(n^*) + O(1) \leq g(n) + O(1).$$

Thus, (ii) $\Rightarrow$ (iii).

That (iii) $\Rightarrow$ (iv) is trivial.

(i) $\Rightarrow$ (ii) Let $I_k = \{\alpha \in 2^{\mathbb{N}} \mid \exists n \;\; C(\alpha|_n) < n - K(n^*) - k\}$

Then $\alpha \in I_k \iff \exists n \; \exists s \;\; C_s(\alpha|_n) + K_s(n^*) < n - k$

Now, $\# \{\sigma \in 2^n \mid C(\sigma) < n - K(n^*) - k\}$

$\leq 2^{n-K(n^*)-k}$, so

$$\mu(I_k) \leq \sum_{n \in \mathbb{N}} \mu(\{\alpha \mid C(\alpha|_n) < n - K(n^*) - k\})$$

$$\leq \sum_{n \in \mathbb{N}} \left(2^{n-K(n^*)-k} \cdot 2^{-n}\right) = \sum_{n \in \mathbb{N}} 2^{-K(n^*)-k}$$

$$= 2^{-k} \cdot \sum_{n \in \mathbb{N}} 2^{-K(n^*)} = 2^{-k} \sum_{\sigma \in 2^{<\mathbb{N}}} 2^{-K(\sigma)} \leq 2^{-k}.$$

Thus, if

$$(\sigma, k) \in T \iff \exists s \;\; C_s(\sigma) + K_s(|\sigma|^*) < |\sigma| - k$$

then $\bigcup_{\sigma \in T(k)} N_\sigma = I_k$ and thus T is a Martin-Löf test. So if α is 1-random, then $\alpha \notin \bigcap I_k$ and so for some k and all n, $C(\alpha|_n) \geq n - K(n^*) - k$.

(iv) $\Rightarrow$ (i) : We construct a partial recursive function $h : 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$ by enumerating its graph.

First, find by the counting theorem a constant c such that $\forall k, t \in \mathbb{N}$

(2.) $\# \{ \tau \in 2^t \mid K(\tau) \leq t - k \} \leq 2^{t - K(t^*) - k + c}$

Fix $s, t \in \mathbb{N}$ and let $n = 2^{\langle s,t \rangle}$.
Strings of length between $\frac{n}{2} + c + 1$ to $n + c$ in the domain of h will be devoted to the pair s, t. So different pairs do not have the same strings associated.

By some systematic procedure, we go through all $s, t, k, i \in \mathbb{N}$ and $\sigma \in 2^n$:

Assume that at some stage we consider s, t, k, i, σ.

Let $m = n - K_{s+1}(t^*) - k + c \leq n + c$.

Assume that

(i) σ has not been put in range(h) at some previous stage s, t, k, j $(j \neq i)$

(ii) $\frac{n}{2} + c + 1 \leq m$

(iii) $K_i(\sigma|_t) \leq t - k$

Then, if possible, we take some unused ρ of length m and put $h(\rho) = \sigma$.

Otherwise, we go to the next stage.

Notice that if $K_{s+1}(t^*) = K(t^*)$, then $m = n - K(t^*) - k + c$ and so the number of $\sigma \in 2^n$ such that $K(\sigma|_t) \leq t - k$ is less than $2^{|\sigma| - t} \cdot 2^{t - K(t^*) - k + c} = 2^{n - K(t^*) - k + c}$ $= 2^m$. So in this case, we will indeed find $\rho \in 2^m$ such that $h(\rho) = \sigma$.

Now, assume that α is not 1-random and hence that

$$\forall N \, \exists t \quad K(\alpha|_t) < t - N.$$

Then

$$\forall k \, \exists^\infty t \quad K(\alpha|_t) < t - k.$$

Let $\psi(0^t 1) = t^*$ and $\psi(\sigma) \nearrow$ otherwise.

Then ψ is prefix free and hence

$$K(t^*) \leq C_\psi(t^*) + O(1) = t + O(1).$$

So for every k, we can find t such that $K(t^*) \leq 2^{t-1} - k - 1$ and

$$K(\alpha|_t) < t - k.$$

Now, let s be minimal with $K_s(t^*) = K(t^*)$
(+1)?

and let $n = 2^{\langle s,t \rangle}$.

Then $m = n - K(t^*) - k + c \geq$

$$n - 2^{t-1} + c + 1 \geq \frac{n}{2} + c + 1$$

since $n = 2^{\langle s,t\rangle} \geq 2^t$.

Thus, there is some $q \in 2^m$ with

$h(q) = \alpha|_n$

Notice that $G(n) = K_{s+i}(t^*) = K(t^*)$,

so $C(\alpha/n) \leq C_n(\alpha/n) + O(1)$

$\leq m + O(1) = n - K(t^*) - k + c + O(1)$

$= n - G(n) - k + (c + O(1))$

where the constant $c + O(1)$ is independent of α, n and k.

Thus, as k was arbitrary, we see that

$$\liminf_n C(\alpha/n) - n + G(n) = -\infty$$

Therefore, by contraposition, if

$$(\forall n) \quad C(\alpha/n) \geq n - G(n) - O(1),$$

then α must be 1-random. $\square$