Relative computability

Definition A set of partial functions is *recursively closed* if it contains S, P_d^i and is closed under composition, definition by recursion and the μ-operator.

So if f is any partial function, there is a least recursively closed set containing f.

Similarly, let χ be a variable for a function with inputs from $\mathbb{N}$ and outputs in $\{0,1\}$. In the same fashion one can define the least set of programs containing the basic programs S, P_d^i, χ and closed under composition, definition by recursion and the μ-operator.

Again, we can proceed to an arithmetization of these programs by assigning some new initial index $\langle 0, 1 \rangle$ to χ.

However, the outcome of a program depends on how we interpret χ.

- On the other hand if some fixed program converges on input x for a particular interpretation of χ, then the calculation is finite and so only some finite $s \subseteq \chi$ has been used. So we can construct computation trees for all terminating computations of some program with index e, input x and at most using some $\sigma \subseteq \chi$ finite.

Let therefore $T(e, \sigma, x, y) \iff$ y is the computation tree of a terminating computation with input x of the program with index e and using <u>at most</u> $\sigma \subseteq \chi$.

Similarly, we let $\mathcal{U}(y)$ be the output of the computation if y is a computation tree and 0 otherwise.

<u>Proposition</u> $T \subseteq \mathbb{N} \times 2^{<\mathbb{N}} \times \mathbb{N} \times \mathbb{N}$ is primitive recursive and $\mathcal{U} : \mathbb{N} \to \mathbb{N}$ is primitive recursive.

<u>Proposition</u> If $e, x, y \in \mathbb{N}$, $s \subseteq t$ and $T(e,s,x,y)$, then also $T(e,t,x,y)$.

<u>Definition</u> For $s \in 2^{<\mathbb{N}}$, $e, x, z \in \mathbb{N}$ put

$$\{e\}^s(x) \simeq z \iff \exists y \quad T(e,s,x,y) \ \& \ \mathcal{U}(y) = z$$

Moreover, for $A \subseteq \mathbb{N}$ let

$$\{e\}^A(x) \simeq z \iff \exists s \subseteq A \quad \{e\}^s(x) \simeq z$$

<u>Basic theorem</u> For any $A \subseteq \mathbb{N}$ the following three sets coincide:

(i) $\{\phi_e^A\}_{e \in \mathbb{N}}$

(ii) $\{\{e\}^A\}_{e \in \mathbb{N}}$

(iii) The recursive closure of $\{\chi_A\}$.

<u>Definition</u> • $\{e\}^s_n(x) \simeq z \iff \exists y < n\,(T(e,s,x,y)$ & $U(y) = z)$

• $\{e\}^A_n(x) \simeq z \iff \exists s \subseteq A \quad \{e\}^s_n(x) \simeq z$.

The use function u:

• $u(A, e, x, n) = \mu y < n\,(\exists s \subseteq A \quad T(e,s,x,y))$

Notice that u is "primitive recursive", since if some $s \subseteq A$ can be found such that $T(e,s,x,y)$, then one can take $|s| < y$. And 2^y is finite.

In other words, if $\{e\}^A_n(x)\downarrow$ then

$\{e\}^{A|u(A,e,x,n)}_n(x)\downarrow$.

<u>Master enumeration theorem</u>: The set

$\{(e,s,x,n) \mid \{e\}^s_n(x)\downarrow\}$ is recursive.

<u>Theorem</u> (Use principle)

(i) $\{e\}^A(x) = y \Rightarrow \exists n\, \exists s \subseteq A \quad \{e\}^s_n(x) = y$

(ii) $\{e\}^s_n(x) = y \Rightarrow \forall t \supseteq s\ \forall m \geq n \quad \{e\}^t_m(x) = y$

(iii) $\{e\}^s(x) = y \Rightarrow \forall A \supseteq s \quad \{e\}^A(x) = y$

In the same manner one can define T-predicates for all finite dimensions and therefore enumerate all partial A-recursive functions of n-variables $(A \subseteq \mathbb{N}, n \in \mathbb{N})$.

We will also use the notations

$\{e\}^A$ and ϕ_e^A interchangeably,

knowing that there is a recursive bijection σ such that $\{\sigma(e)\}^A \simeq \phi_e^A$.

Relativised s^m_n - Theorem

For every $m, n \geq 1$, there exists a total recursive (in fact primitive recursive) function s^m_n of $m+1$ variables such that for all $A \subseteq \mathbb{N}$ and $e, x_1, \dots, x_m, y_1, \dots, y_n$

$$\phi^A_e(x_1, \dots, x_m, y_1, \dots, y_n) = \phi^A_{s^m_n(e, x_1, \dots, x_m)}(y_1, \dots, y_n)$$

Notice that in the above theorem, s^m_n is indeed <u>recursive</u> and not only A-recursive. This holds as the arithmetization procedure was done uniformly for all $A \subseteq \mathbb{N}$ at the same time.

Exactly as before we also get the

Relativised Recursion Theorem

(i) For all $A \subseteq \mathbb{N}$ and total A-recursive functions $f : \mathbb{N} \to \mathbb{N}$ there is an e such that

$$\phi^A_e \simeq \phi^A_{f(e)} .$$

(ii) In fact, there is a total recursive $h : \mathbb{N} \to \mathbb{N}$ such that for all $A \subseteq \mathbb{N}$, $f = \phi^A_p$ total recursive,

$$\phi^A_{h(p)} \simeq \phi^A_{f(h(p))} .$$

~~Proof~~
Let us give a full proof of this:

Find by the s^m_n-Theorem some total recursive $d : \mathbb{N} \to \mathbb{N}$ defined such that for all $A \subseteq \mathbb{N}$

$$\phi^A_{d(x)} \simeq \begin{cases} \phi^A_{\phi^A_x(x)}(z) & \text{if } \phi^A_x(x)\downarrow \\ \uparrow & \text{otherwise} \end{cases}$$

Given $f = \phi^A_p$ choose an index a such that $\phi^A_a \simeq f \circ d$. We claim that $e = d(a)$ works:

For $$\phi^A_e(z) = \phi^A_{d(a)}(z) = \begin{cases} \phi^A_{\phi^A_a(a)}(z) & \text{if } \phi^A_a(a)\downarrow \\ \uparrow & \text{otherwise.} \end{cases}$$

But as f is total A-recursive and d recursive,

$\phi^A_a(a) = f \circ d(a) \downarrow = f(e)$, whence

$$\phi^A_e(z) = \phi^A_{\phi^A_a(a)}(z) = \phi^A_{f(d(a))}(z) = \phi^A_{f(e)}.$$

Clearly, by the s^m_n-Theorem an index for $f \circ d$ can be found uniformly in an index for f independently of A. $\square$

Most of the results about r.e. sets and partial recursive function relativise, that is, they have exact analogues for A-recursive, A-r.e. and partial A-recursive sets and functions:

Recall that

(i) B is <u>recursive in A</u> (or B is <u>Turing reducible</u> to A, $B \leq_T A$) if $\chi_B = \phi_e^A$ for some e.

(ii) B is <u>recursively enumerable</u> in A if $B = \emptyset$ or $B = rg(\phi_e^A)$ for some total ϕ_e^A.

<u>Proposition</u> Turing reducibility is a partial pre ordering:

<u>Proof</u> Suppose $A \leq_T B \leq_T C$.

Then χ_A belongs to the recursive closure of $\{\chi_B\}$ and χ_B belongs to the recursive closure of $\{\chi_C\}$. So χ_A belongs to the recursive closure of $\{\chi_C\}$ and $A \leq_T C$. $\square$

We halfway showed that

<u>Proposition</u> $B \leq_T A$ iff B and $\complement B$ are r.e. in A.

Let us give an intuitive argument:

Suppose $B \leq_T A$ and construct a total A-recursive f enumerating e.g. $\complement B$: Suppose for simplicity that $\complement B \neq \emptyset$ and fix $b \in \complement B$:

$$f(n) = \begin{cases} n & \text{iff } n \in \complement B \text{ iff } \chi_B(n) = 0 \\ b & \text{otherwise.} \end{cases}$$

Conversely, if $B = rg(f)$, $\complement B = rg(g)$ for total A-recursive f, g, then for any $n \in \mathbb{N}$, compute

$f(0), f(1), f(2), \ldots$
$g(0), g(1), g(2), \ldots$

n will appear in exactly one of the two lists determining whether $n \in B$ or $n \in \complement B$. The procedure is surely recursive in A. $\square$