The jump operator

Definition (i) $A \equiv_T B$ if $A \leq_T B \leq_T A$.

(ii) The Turing degree of A (or the degree of unsolvability) is the set

$$\deg(A) = \{ B \mid B \equiv_T A \}.$$

(iii) $\deg(A) \cup \deg(B) = \deg(A \oplus B)$

(iv) Degrees, i.e., $\equiv_T$-equivalence classes are denoted by smaller case letters $\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}$ in boldface and $\underset{\sim}{D}$ denotes the class of degrees.

(v) $(\underset{\sim}{D}, \leq)$ is defined by $\underset{\sim}{a} \leq \underset{\sim}{b}$ iff

$$\exists A \in \underset{\sim}{a} \; \exists B \in \underset{\sim}{b} \; A \leq_T B.$$

(vi) $\underset{\sim}{a} \in \underset{\sim}{D}$ is *recursively enumerable* if there is an r.e. set $A \in \underset{\sim}{a}$

(vii) $\underset{\sim}{a}$ is r.e. in $\underset{\sim}{b}$ iff $\exists A \in \underset{\sim}{a} \; \exists B \in \underset{\sim}{b}$ A is r.e. in B.

Definition $K^A = \{x \mid \phi_x^A(x)\downarrow\}$.

This set is called the jump of A and is also denoted by A'.

By the Myhill isomorphism theorem one can see that $K^A \cong K_0^A = \{\langle x,y\rangle \mid \phi_x^A(y)\downarrow\}$.
It is enough to notice that $K_0^A \equiv_1 K^A$, which follows by relativizing $K_0 \equiv_1 K$.

Theorem (i) A' is r.e. in A

(ii) $A' \not\leq_T A$,

(iii) If A is r.e. in B and $B \leq_T C$, then A is r.e. in C.

(iv) B is r.e. in A iff $B \leq_1 A'$

(v) $B \leq_T A \iff B' \leq_1 A'$

(vi) $B \equiv_T A \implies B' \equiv_1 A' \implies B' \cong A' \implies B' \equiv_T A'$

(vii) A is r.e. in $B \iff A$ is r.e. in $\complement B$.

(viii) $A \leq_1 A'$ and $\complement A \leq_1 A'$.

<u>Proof</u> (i) is clear, as $A' = K^A =$

$$\{x \mid \exists y, s\, (\underbrace{T(x, s, x, y)\ \&\ s \subseteq A})\}$$

and the matrix is recursive in A.

(ii) this follows from relativising the proof of $K \not\leq_T \emptyset$: To see what this latter says, notice that $K \leq_T \emptyset$ means that χ_K belongs to the recursive closure of $\{\chi_\emptyset\}$, which is just the set of partial recursive functions. But χ_K is not total recursive, so $K \not\leq_T \emptyset$. Working in the class of partial A-recursive functions, one sees that $K^A \not\leq_T A$.

(iv) this is also the relativised version of "B is r.e. iff $B \leq_1 K$".

(iii) If $A \neq \emptyset$ is r.e. in B, then $A = rg(f)$ for some total B-recursive function f. Moreover, if $B \leq_T C$ then f belongs to the recursive closure of $\{\chi_B\}$ and χ_B belongs to the recursive closure of $\{\chi_C\}$, whence f belongs to the recursive closure of $\{\chi_C\}$, witnessing that A is r.e. in C.

(v) If $B \leq_T A$ then B' is r.e. in B (by (i)) and hence B' is r.e. in A (by (iii)) and $B' \leq_1 A'$ (by (iv)).

Conversely, if $B' \leq_1 A'$, then by (viii) $B, CB \leq_1 B' \leq_1 A'$ and so by (iv) B, CB are r.e. in A. Therefore, $B \leq_T A$.

(vi) $B \equiv_T A \Rightarrow B' \equiv_1 A' \Rightarrow B' \cong A'$ are known. So if h is a recursive permutation of $\mathbb{N}$ such that $h(B') = A'$, notice that $\chi_B = \chi_A \circ h$, $\chi_A = \chi_B \circ h^{-1}$.

I.e., $B' \cong A' \Rightarrow A \equiv_T B$.

(vii) Notice that $B \equiv_T CB$ (by $\chi_B = 1 \dot{-} \chi_{CB}$) so the result follows from (iii).

(viii) Relativise the proof that K is 1-complete, i.e., that any r.e. set C, $C \leq_1 K$. Now, both A and CA are recursive in A, so r.e. in A and hence

$A, CA \leq_1 K^A = A'$.

$\square$

<u>Definition</u> Let $\underset{\sim}{a}' = \deg(A')$ for any/some $A \in \underset{\sim}{a}$. This is well-defined by (vi) above.

Moreover, put $\underset{\sim}{0} = \deg(\emptyset)$, so $\underset{\sim}{0}' = \deg(K)$.

To see this one can use the following

<u>Proposition</u> If $A \leq_T B$, then there is a recursive function $h : \mathbb{N} \to \mathbb{N}$ such that

$$\phi_e^A = \phi_{h(e)}^B \quad \text{for all } e \in \mathbb{N}.$$

<u>Proof</u> Suppose $\chi_A = \phi_a^B$. Then h will be the function that to the program with index e calculates an index of the program such that:

calculate ϕ_e^A but such that whenever a value $\chi_A(n)$ is needed, calculate instead ϕ_a^B.

Notice that h does <u>not</u> depend on A, B. $\square$

By padding one can moreover suppose that h is injective. So as for acceptable systems of indices, one can show that if $A \equiv_T B$ then there is a recursive bijection σ such that $\phi_{\sigma(e)}^A = \phi_e^B$, $\forall e$.