Büchi - Gurevich - Harrington

We fix a game automaton $\mathcal{M} = (S, I, R, F)$.

Suppose $v \in V_\Sigma$ and p is a position in $\mathcal{G}_v^{\mathcal{M}}$.

The *node* of p is the string $Node(p)$ of *even* positions in p, i.e., for $p = s_0 d_1 s_1 \dots d_n$ or $p = s_0 d_1 s_1 \dots d_n s_n$,

$$Node(p) = d_1 \cdots d_n \in \{0,1\}^*$$

Given $x \in \{0,1\}^*$, we define the *x-residue* of v by

$$v_x : \{0,1\}^* \longrightarrow \Sigma \quad , \quad v_x(y) = v(xy).$$

Definition Suppose $p = s_0 d_1 s_1 \cdots d_n s_n$ or $p = s_0 d_1 s_1 \cdots d_n s_n d_n$ is a position in $\mathcal{G}_v^{\mathcal{M}}$ and let $Q = \{s_0, s_1, \dots, s_n\}$ be the set of states appearing in p (note that $s_0, s_1, \dots, s_n$ may have repetitions).

We define the *last appearence record* $LAR(p)$ by

$$LAR(p) = s_{i_1} s_{i_2} \cdots s_{i_k}, \quad i_1 < i_2 < \dots < i_k,$$

where i_l is the last appearence of some $s \in Q$ in the string $s_0 s_1 \cdots s_n$.

For example, if $S = \{r, s, t\}$, then

$$LAR(s0s1\underline{t}1\underline{s}) = ts$$

$$LAR(s1s1\underline{s}1) = s$$

$$LAR(s1r0\underline{s}1\underline{r}1\underline{t}0) = srt .$$

<u>Theorem</u> (Forgetful determinacy)

Fix a game automaton $\mathcal{M} = (S, I, R, \mathcal{F})$ and a Σ-valuation $v : \{0,1\}^* \to \Sigma$.

Then one of the following two conditions hold:

(i) I has a winning strategy ϕ in $G_v^{\mathcal{M}}$ such that if p, q are positions of even length in $G_v^{\mathcal{M}}$ (where I is to play),

$v_{Node(p)} = v_{Node(q)}$ and $LAR(p) = LAR(q)$,

then $\phi(p) = \phi(q)$

(ii) II has a winning strat. ψ in $G_v^{\mathcal{M}}$ such that for positions p, q of odd length, if

$v_{Node(p)} = v_{Node(q)}$ and $LAR(p) = LAR(q)$, then

$\psi(p) = \psi(q)$.

Corollary Suppose Σ is a one-letter alphabet. Then one of the two players I and II in G_v^{dl} has a strategy ϕ s.t. if p, q are positions where the winner is to make moves and $LAR(p) = LAR(q)$, then $\phi(p) = \phi(q)$.

Pf Just note that if $\Sigma = \{a\}$, then $v(x) = a$ for any $x \in \{0,1\}^*$ and so $v_x = v_y$ for any $x, y \in \{0,1\}^*$. $\square$

Corollary There is an algorithm that given any Rabin automaton $\mathfrak{A}$ decides whether $L(\mathfrak{A}) \neq \emptyset$.

Proof First replace $\mathfrak{A}$ by a game automaton $\mathcal{B} = (S_1, I_1, R_1, \mathcal{F}_1)$ such that $L(\mathfrak{A}) = L(\mathcal{B})$. Let Σ be the alphabet of $\mathfrak{A}$ and $\mathcal{B}$. We now construct a new game automaton $\mathcal{C} = (S_2, I_2, R_2, \mathcal{F}_2)$ over the single letter alphabet $\Sigma_2 = \{a\}$ such that

$$L(\mathcal{B}) \neq \emptyset \iff L(\mathcal{C}) \neq \emptyset.$$

To do this, let

$$S_2 = S_1$$

$$(s,a) \in I_2 \iff \exists b \in \Sigma \quad (s,b) \in I_1$$

$$(s,d,a,s') \in R_2 \iff \exists b \in \Sigma \quad (s,d,b,s') \in I_2$$

$$\mathcal{F}_2 = \mathcal{F}_1 .$$

Then $\mathcal{C}$ fits the bill. We therefore only need to see whether $v \in V_{\Sigma_2}$, $v \equiv a$, belongs to $L(\mathcal{C})$. But if $v \in L(\mathcal{C})$, then I has a winning strategy ϕ in $G_v^{\mathcal{C}}$ such that for any positions p,q in $G_v^{\mathcal{C}}$ of even length, if $LAR(p) = LAR(q)$, then $\phi(p) = \phi(q)$.

In other words, $\phi(p)$ only depends on $LAR(p)$. Since there are only finitely many strings of the form $LAR(p)$, there are only finitely many such winning strategies possible. List these as $\phi_1, \dots, \phi_n$.

Similarly, if $v \notin L(\mathcal{C})$, II has a winning strategy ψ in $G_v^{\mathcal{C}}$ such that $\psi(p)$ only depends on $LAR(p)$. List these potential functions as $\psi_1, \dots, \psi_m$.

It follows that

(i) $v \in L(\mathcal{C}) \neq \emptyset$ if and only if there is an i s.t. ϕ_i wins every play against the strategies $\psi_1, \ldots, \psi_m$

(ii) $v \notin L(\mathcal{C}) = \emptyset$ if and only if there is an i s.t. ψ_i wins every play against the strategies $\phi_1, \ldots, \phi_n$.

So to decide whether $L(\mathcal{C}) \neq \emptyset$, we just need to decide whether a ϕ_i wins against a ψ_j.

So fix i, j and let

$$s_0 d_1 s_1 d_2 s_2 \ldots$$

be the play consistent with ϕ_i and ψ_j.

Then there are minimal $l < k$ s.t.

$$LAR(s_0 d_1 s_1 \ldots d_l) = LAR(s_0 d_1 s_1 \ldots d_k)$$

whence

$$s_l = \phi_i(s_0 d_1 s_1 \ldots d_l) = \phi_i(s_0 d_1 s_1 \ldots d_k) = s_k .$$

Thus also $LAR(s_0 d_1 s_1 \ldots d_l s_l) = LAR(s_0 d_1 s_1 \ldots d_k s_k)$ and hence

$$d_{l+1} = \psi_j(s_0 d_1 s_1 \ldots d_l s_l) = \psi_j(s_0 d_1 s_1 \ldots d_k s_k) = d_{k+1}$$

etc. So the play is eventually periodic and

$$\{s \in S \mid \exists^\infty e \;\; s_e = s\} = \{s_l, s_{l+1}, \ldots, s_{k-1}\} .$$

so to decide whether II wins, it suffices to compute $s_0 d_1 s_1 d_2 s_2 \ldots$ up to $s_0 d_1 s_1 \ldots d_k$.

$\square$

Theorem (Complementation)

The class of languages, i.e., subsets of V_Σ, accepted by Rabin automata is closed under complementation.

We shall deduce this from

Corollary If $\mathcal{M}$ is a game automaton on Σ and $v \in V_\Sigma$, then the winning player of $G_v^{\mathcal{M}}$ has a strategy ϕ such that $\phi(p)$ only depends on Node(p) and LAR(p).

Pf of theorem:

Fix $\mathcal{M} = (S, I, R, \mathcal{F})$, a game automaton, and let $D = \{x \in S^* \mid$ each state $s \in S$ occurs at most once in $x\}$.

Let Σ_0 = set of functions $\sigma : D \to \{0, 1\}$.

Also, for any $s \in S$ and $x \in D$ set

$$\text{Exposure}_s(x) = \begin{cases} xs & \text{if } s \text{ does not occur in } x \\ yzs & \text{if } x = ysz \end{cases}$$

Now if $w \in V_{\Sigma_0}$ is a Σ_0-valuation, we define the following strategy ψ_w for II in $G_v^{\mathcal{M}}$ (for any $v \in V_\Sigma$):

If p is a position of odd length, we let II play
$$w(\mathrm{Node}(p))(\mathrm{LAR}(p)) \in \{0,1\}.$$

Note that since $\mathrm{Node}(p) \in \{0,1\}^*$, $w(\mathrm{Node}(p)) : D \to \{0,1\}$.

From the corollary above we get the following:

<u>Lemma</u>

For any $v \in V_\Sigma$, the following are equivalent:

(i) $v \notin L(\mathcal{M})$

(ii) there is $w \in V_{\Sigma_0}$ such that ψ_w is a winning strategy for II in $G_v^{\mathcal{M}}$.

Note now that if $v \in V_\Sigma$, $w \in V_{\Sigma_0}$, then ψ_w is a winning strategy for II in $G_v^{\mathcal{M}}$ if and only if for any $\alpha \in \{0,1\}^\omega$, setting $\sigma_n = w(\alpha|n)$, the following holds:

(*) if $s_0, s_1, \ldots \in S$, $r_0, r_1, \ldots \in D$ satisfy

$r_0 = s_0$, $(v(\varepsilon), s_0) \in I$, $\alpha(n+1) = \sigma_n(r_n)$,

$(s_n, d_{n+1}, v(\alpha|n+1), s_{n+1}) \in R$, $r_{n+1} = \mathrm{Exposure}_{s_{n+1}}(r_n)$,

then $\{s \in S \mid \exists^\infty n \; s_n = s\} \notin \mathcal{F}$.

We will now reformulate (*) to see that it can be recognized by a Büchi automaton.

So let

$$\Sigma_1 = S \times D \times \{\varepsilon, 0, 1\} \times \Sigma \times \Sigma_0 ,$$

and let $L_1 \subseteq \Sigma_1^\omega$ be set of all quintuples

$$\big((s_0, s_1, \dots), (r_0, r_1, \dots), (\varepsilon, d_1, d_2, \dots), (a_0, a_1, \dots), (\sigma_0, \sigma_1, \dots)\big)$$

such that

$$(**)\ \left\{ \begin{array}{l} r_0 = s_0 ,\ (a_0, s_0) \in I ,\ \text{and for all } n \in \mathbb{N},\ d_{n+1} = \sigma_n(r_n) \\ (s_n, d_{n+1}, a_{n+1}, s_{n+1}) \in R ,\ r_{n+1} = \mathrm{Explore}_{s_{n+1}}(r_n) . \end{array} \right.$$

Clearly we can devise a Büchi automaton $\mathcal{B}_1$ on the alphabet Σ_1 recognizing L_1.

<u>Exercise</u>. If Λ_1, Λ_2 are alphabets and $K \subseteq \Lambda_1^\omega \times \Lambda_2^\omega$ is a Büchi recognizable language, then also

$$\mathrm{proj}_1(K) = \{ \alpha \in \Lambda_1^\omega \mid \exists \beta \in \Lambda_2^\omega \ (\alpha, \beta) \in K \}$$

is Büchi recognizable.

• Similarly,

$$\forall\text{-}\mathrm{proj}_1(K) = \{ \alpha \in \Lambda_1^\omega \mid \forall \beta \in \Lambda_2^\omega \ (\alpha, \beta) \in K \}$$

is Büchi recognizable.

Also, let $L_2 \subseteq \Sigma_1^\omega$ be the set of all quintuples

$((s_0, s_1, \ldots), (r_0, r_1, \ldots), (\varepsilon, d_1, d_2, \ldots), (a_0, a_1, \ldots), (\sigma_0, \sigma_1, \ldots))$

such that

$$\{ s \in S \mid \exists^\infty n \;\; s_n = s \} \in \mathcal{F}$$

Then clearly L_2 is Müller and thus also Büchi recognizable.

It follows that if $\Sigma_3 = \{\varepsilon, 0, 1\} \times \Sigma \times \Sigma_0$

$L_3 = \forall\text{-proj}_{\Sigma_3^\omega} (\sim L_1 \cup \sim L_2)$

$=$ set of all triples $((\varepsilon, d_1, d_2, \ldots), (a_0, a_1, \ldots), (\sigma_0, \sigma_1, \ldots))$ s.t. for all $s_0, s_1, \ldots \in S$, $r_0, r_1, \ldots \in D$, if $(**)$ holds, then $\{ s \in S \mid \exists^\infty n \;\; s_n = s \} \notin \mathcal{F}$.

is Büchi recognizable.

So let $\mathcal{C} = (S_1, s_1, T_1, \mathcal{F}_1)$ be a Müller automaton recognizing L_3 and let $\mathcal{D} = (S_1, I_1, R_1, \mathcal{F}_1)$ be the game automaton over the alphabet $\Sigma \times \Sigma_0$ defined by

$I_1 = \{ (s, (a, \sigma)) \in S_1 \times (\Sigma \times \Sigma_0) \mid (s_1, (\varepsilon, a, \sigma), s) \in T_1 \}$

$R_1 = \{ (s, d, (a, \sigma), t) \mid (s, (d, a, \sigma), t) \in T_1 \}$.

<u>Note</u> If $v \in V_\Sigma$, $w \in V_{\Sigma_0}$, then, since the Müller automaton $\mathcal{C}$ is deterministic, whenever I is to play in $\mathcal{G}^{\mathcal{D}}_{(v,w)}$ there is only one play possible.

We now see by the properties of $\mathcal{D}$ that

ψ_w is a winning strategy in $\mathcal{G}^{\mathcal{M}}_v$

$\Updownarrow$ the unique strategy for I in $\mathcal{G}^{\mathcal{D}}_{(v,w)}$ is winning

$\Updownarrow$ $(v,w) \in L(\mathcal{D})$.

So finally we have

$v \notin L(\mathcal{M})$

$\Updownarrow$ $\exists w \in V_{\Sigma_0} \quad (v,w) \in L(\mathcal{D})$.

The following exercise thus finishes the proof.

<u>Exercise</u> If Λ_1, Λ_2 are alphabets and $K \subseteq V_{\Lambda_1} \times V_{\Lambda_2}$ is Rabin recognizable, then also $\mathrm{proj}_{V_{\Lambda_1}}(K)$ is Rabin recognizable.