Finitely generated groups

Example Let $\Sigma = \{a, a^{-1}, b, b^{-1}\}$ and let

L = all reduced words in Σ.

Then it is easy to see that L has finitely many cone types, so L is regular.

The word problem:

Let G be a group and $\Sigma = \Sigma^{-1} \subseteq G$ a symmetric generating set. We define

$\pi : \Sigma^* \longrightarrow G$ be the canonical semigroup homomorphism, where the operation on Σ^* is concatenation.

Assume Σ is finite.

We let $WP(G, \Sigma) = \{ w \in \Sigma^* \mid \pi(w) = 1 \}$

Theorem If G is a finitely generated group, Σ a finite, symmetric, generating set. Then $L = WP(G, \Sigma)$ is regular if and only if G is finite.

Proof Note that if $w, \sigma \in \Sigma^*$, then

$$\mathrm{cone}_L(w) = \mathrm{cone}_L(\sigma) \iff \pi(w) = \pi(\sigma)$$ $\square$

Normal forms

Suppose Σ is a finite, symmetric, generating set for a group G. Let $\pi : \Sigma^* \to G$ be the canonical semigroup homomorphism.

Definition A <u>normal form</u> for (G, Σ) is a language $L \subseteq \Sigma^*$ such that $\pi : L \to G$ is bijective.

Examples • The freely reduced words are a regular normal form for F_2 (since there are only finitely many cone types).

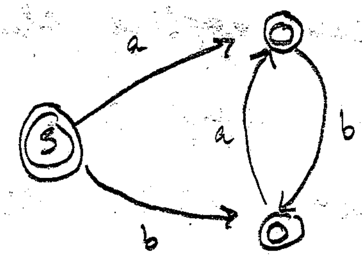
• Let $\mathbb{Z}^2$ be the free abelian group on generators a and b. Then $L = \{a^i b^j \mid i, j \in \mathbb{Z}\}$ is a regular normal form.

Proposition Suppose Σ and Λ are finite, symmetric, generating sets for a group G. Then G has a regular normal form wrt Σ if and only if it has a regular normal form wrt Λ.

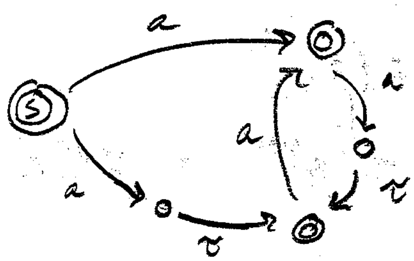
Pf. Suppose $L \subseteq \Sigma^*$ is a regular normal form. For every $s \in \Sigma$ let $\omega_s \in \Lambda^*$ be a rewriting of s in base Λ. So if $\mathcal{M}$ is an FSA accepting L, we can build a generalized FSA by replacing labels s on $\mathcal{M}$ by new labels ω_s. The language $L' \subseteq \Lambda^*$ accepted is a regular normal form for G wrt Λ. $\square$

Example $D_\infty = \mathbb{Z}_2 * \mathbb{Z}_2$ is generated by two reflections a and b, but is also generated by a reflection a and a translation $\tau = ab$. Since $a = a^{-1}$, $b = b^{-1}$, $\Sigma = \{a, b\}$, $\Lambda = \{a, \tau, \tau^{-1}\}$ are symmetric generating sets.

Also, $L = \{(ab)^n, (ab)^n a, b(ab)^n\}_{n \in \mathbb{N}}$ is a regular normal form:



Now $b = a\tau$, so $L' = \{(aa\tau)^n, (aa\tau)^n a, a\tau(aa\tau)^n\}_{n \in \mathbb{N}}$ is a regular normal form:



Proposition If G, H have regular normal forms, then so do $G \oplus H$ and $G * H$. In particular, if G, H are finite, then $G * H$ has a regular normal form.

Proof If L, K are regular normal forms for G and H respectively, then LK and

$$(LK)^* \cup (LK)^* L \cup K(LK)^*$$

are regular

normal forms for $G \oplus H$ and $G * H$ respectively. $\square$

<u>Thm</u> (R. Gilman)

Suppose G is an infinite group with a regular normal form. Then G has an element of infinite order.

<u>Pf</u> Let Σ be the finite symmetric generating set and $L \subseteq \Sigma^*$ the regular normal form. Then by the pumping lemma, which applies since L is infinite, there is some $\alpha w \beta \in L$ s.t. $w \neq \varepsilon$ and $\alpha w^n \beta \in L$ for all $n \geq 0$. Since L is a normal form, $\pi(\alpha w^n \beta) \neq \pi(\alpha w^m \beta)$ for $n \neq m$, whence

$$\pi(w)^n = \pi(\alpha)^{-1} \pi(\alpha w^n \beta) \pi(\beta)^{-1}$$
$$\neq \pi(\alpha)^{-1} \pi(\alpha w^m \beta) \pi(\beta)^{-1} = \pi(w)^m$$

for all $m \neq n$. $\square$

<u>Finitely generated subgroups of free groups</u>

Suppose G is a group with a finite, symmetric, generating set $\Sigma \subseteq G$. A subgroup $H \leq G$ is said to be an <u>image of a regular language over Σ</u> if there is a regular language $L \subseteq \Sigma^*$ such that $\pi(L) = H$.

Theorem Suppose G is a group with a finite symmetric generating set $\Sigma \subseteq G$.
Then, a subgroup $H \leq G$ is the image of a regular language L over Σ if and only if H is finitely generated.

Proof If H is finitely generated, let S be a generating set for H, $S = S^{-1}$. Also, for every $s \in S$, let $w_s \in \Sigma^*$ be a word with $\pi(w_s) = s$. Then $L = \{w_s \mid s \in S\}^* \subseteq \Sigma^*$ is regular and $\pi(L) = H$.

Conversely, suppose L is a regular language in Σ with $\pi(L) = H$. Pick a finite selector $T_0 \subseteq \Sigma^*$ of all non-empty cone types of L, i.e., for every $w \in \Sigma^*$ with $\mathrm{cone}(w) \neq \emptyset$, there is $\sigma \in T_0$ with $\mathrm{cone}(\sigma) = \mathrm{cone}(w)$. Let $n = \max(|w| : w \in T_0)$ and set

$$S = \pi(L \cap \Sigma^{\leq n}) \cup \{\pi(\omega)\pi(\sigma)^{-1} \mid \sigma \in \Sigma^{n+1},\ \omega \in T_0 \text{ and } \mathrm{cone}(\omega) = \mathrm{cone}(\sigma)\}.$$

First, to see that $S \subseteq H$, note that if $\sigma \in \Sigma^*$, $\omega \in T_0$ and $\mathrm{cone}(\omega) = \mathrm{cone}(\sigma)$, then, in particular, $\mathrm{cone}(\omega) \neq \emptyset$. So for some $\beta \in \Sigma^*$,

$\omega\beta$, $\sigma\beta \in L$, i.e., $\pi(\omega)\pi(\beta)$, $\pi(\sigma)\pi(\beta) \in H$, whence $\pi(\omega)\pi(\sigma)^{-1} = \pi(\omega)\pi(\beta)\cdot\pi(\beta)^{-1}\pi(\sigma)^{-1} \in H$.

Now, we claim that if $\alpha \in L$, $|\alpha| > n$, then there is $\beta \in L$, $|\beta| < |\alpha|$, such that

$$\pi(\alpha) \in S^{-1}\cdot\pi(\beta).$$

To see this, let $\alpha = \sigma\delta$, $|\sigma| = n+1$, and find $\omega \in T_0$ s.t. cone(ω) = cone(σ). Then $\pi(\omega)\pi(\sigma)^{-1} \in S$ and $\beta = \omega\delta \in L$, whence

$$\pi(\omega)\pi(\sigma)^{-1}\pi(\alpha) = \pi(\omega)\pi(\sigma)^{-1}\pi(\sigma)\pi(\delta) = \pi(\omega\delta) = \pi(\beta).$$

It thus follows that $H = \langle S \rangle$. $\square$

<u>Proposition</u> Suppose Σ is a finite symmetric subset of a group G and L is a regular language over Σ. Then if R is the set of all words obtained by freely reducing a word in L, R is regular.

<u>Proof</u> Suppose L is the language accepted by some finite automaton M. Let N be the automaton obtained from M as follows: If there is a path from state s_1 to state s_2 labeled ww^{-1} for $w, w^{-1} \in \Sigma^*$, add an edge from s_1 to s_2 labeled ε. Then if S is the language accepted by N, a freely reduced word in Σ belongs to S if and only if it belongs to R. So R is the intersection of S with the regular language of freely reduced words. $\square$

Howson's Theorem The intersection $H \cap K$ of two finitely generated subgroups H, K of $\mathbb{F}_n$ is finitely generated.

Proof Let $\Sigma = \{a_1, a_1^{-1}, \ldots, a_n, a_n^{-1}\}$ be the basis for $\mathbb{F}_n$. Since H and K are finitely generated, there are regular languages $L_H, L_K \subseteq \Sigma^*$ s.t. $H = \pi(L_H)$ and $K = \pi(L_K)$. Let R_H, R_K be the regular languages obtained by freely reducing words in L_H and L_K respectively. Then since $H, K \leq \mathbb{F}_n$ and Σ is the symmetric basis for $\mathbb{F}_n$, $\pi : R_H \longrightarrow H$ and $\pi : R_K \longrightarrow K$ are bijections, whence $\pi : R_H \cap R_K \longrightarrow H \cap K$ is a bijection too. Since $R_H \cap R_K$ is regular, $H \cap K$ is finitely generated. $\square$

Automata on pairs of strings

Suppose Σ and Λ are finite alphabets and $\$$ is a symbol not occurring in any of Σ, Λ.

For $(\omega, \sigma) \in \Sigma^* \times \Lambda^*$, let

$$(\omega,\sigma)^{\$} = \begin{cases} (\omega,\sigma) & \text{if } |\omega| = |\sigma| \\ (\omega, \sigma \$^n) & \text{if } |\omega| = |\sigma| + n \\ (\omega \$^n, \sigma) & \text{if } |\omega| + n = |\sigma|. \end{cases}$$