# Finitely presented groups

Suppose $a_1, a_2, \ldots, a_n$ are distinct symbols and let $a_1^{-1}, a_2^{-1}, \ldots, a_n^{-1}$ be their formal inverses. Given any set $R$ of words in the alphabet $\Sigma = \{a_1, \ldots, a_n, a_1^{-1}, \ldots, a_n^{-1}\}$, we define the group

$$G = \langle \Sigma \mid R \rangle = \langle a_1, \ldots, a_n \mid R \rangle$$

to be the freest group generated by $a_1, \ldots, a_n, a_1^{-1}, \ldots, a_n^{-1}$ subject to the relations

$$a_i a_i^{-1} = 1 \quad \text{and} \quad r = 1 \text{ for all } r \in R.$$

__Definition__ $\langle \Sigma \mid R \rangle$ is a __presentation__ of the group $G$.

__Example__ The following is a presentation of the trivial group:

$$G = \langle a, b \mid \{a^2 b^{-3}, abab^{-1}a^{-1}b^{-1}\} \rangle.$$

To see this, note that $a^2 b^{-3} = 1$ and $abab^{-1}a^{-1}b^{-1} = 1$ can be rewritten as $a^2 = b^3$ and $aba = bab$.

We see that

$$b^7 = b^3 b b^3 = a^2 b a^2 = a(aba)a = ab\underline{aba}$$

$$= ab^2ab, \quad \text{so} \quad a^4 = b^6 = ab^2a \quad \text{and}$$

hence $b^3 = a^2 = b^2$, whence $b = 1$.

It follows that $a = bab = aba = a^2$ and so also $a = 1$. Therefore, $G$ is generated by $1$, i.e., $G = \{1\}$.

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<u>Notation</u> We shall sometimes simplify notation and write $\langle \Sigma \mid x_1 = y_1, \ldots, x_n = y_n \rangle$ to denote $\langle \Sigma \mid \{x_1 y_1^{-1}, \ldots, x_n y_n^{-1}\} \rangle$.

To construct a group $G = \langle \Sigma \mid R \rangle$ we can see $G$ as a quotient of the free group. That is, if $\Sigma = \{a_1, \ldots, a_n, a_1^{-1}, \ldots, a_n^{-1}\}$, we let $\mathbb{F}(\Sigma) = \mathbb{F}(a_1, \ldots, a_n)$ be the free group with basis $a_1, \ldots, a_n$. Denote by $\overline{R}$ the normal subgroup of $\mathbb{F}(\Sigma)$ generated by $R$.

Then $G \cong \mathbb{F}(\Sigma) / \overline{R}$.

Let $\phi : \Sigma^* \to \mathbb{F}(\Sigma)$ and $\pi : \Sigma^* \to G$ denote the canonical monoid homomorphisms. Then we see that for any $w \in \Sigma^*$,

$$\pi(w) = 1 \iff \phi(w) \in \overline{R} \iff$$

$$\exists r_1, \ldots, r_k \in R \ \exists x_1, \ldots, x_k \in \Sigma^* \quad \phi(w) = \phi\left( \prod_{i \le k} x_i r_i x_i^{-1} \right).$$

Definition Let $a_1, \dots, a_n$ be distinct letters with formal inverses $a_1^{-1}, \dots, a_n^{-1}$ and set

$$\Sigma = \{a_1, \dots, a_n, a_1^{-1}, \dots, a_n^{-1}\}.$$

Suppose $R \subseteq \Sigma^*$ and denote by

$$\phi : \Sigma^* \longrightarrow \mathbb{F}(\Sigma), \quad \pi : \Sigma^* \longrightarrow \langle \Sigma / R \rangle$$

the canonical homomorphisms.

Then for any $w \in \ker \pi \subseteq \Sigma^*$, we define the combinatorial area of $w$ to be

$$A(w) = \min k \quad \text{s.t. } \exists r_1, \dots, r_k \in R, \exists x_1, \dots, x_k \in \Sigma^*$$

$$\phi(w) = \phi\left( \prod_{i=1}^{k} x_i r_i x_i^{-1} \right).$$

The isoperimetric function $f$ corresponding to the presentation $\langle \Sigma | R \rangle$ is the function

$$f(m) = \max \left( A(w) \;\middle|\; w \in \ker \pi \ \& \ |w| \leq m \right).$$

**Lemma 1** Suppose $(\Sigma, L)$ is an automatic structure on a group $G$. Then there is a constant $M$ s.t. for any $w \in \Sigma^*$ there is $x \in L$ with $|x| \leq |w| \cdot M + M$ and $\pi(x) = \pi(w)$.

**Proof** Let $x_0 \in L$ be s.t. $\pi(x_0) = 1$ and let $M > |x_0|$ be a constant such for any $a \in \Sigma \cup \{\varepsilon\}$ and $y, z \in \Sigma^*$ with $|y| < |z|$, if $\text{rightcone}_{L_a}(y, z) \neq \emptyset$ then there is $v \in \Sigma^*$ with $|v| < M$ s.t.

$(y, zv) \in L_a$.

Now, given $w = a_1 a_2 \cdots a_m \in \Sigma^*$, $a_i \in \Sigma$, find $x_1, \ldots, x_m \in L$ of minimal length such that $\pi(x_i) = \pi(a_1 \ldots a_i)$.

We note that $(x_i, x_{i+1}) \in L_{a_{i+1}}$ for all $i < m$ and so either $|x_{i+1}| \leq |x_i|$ or $|x_i| < |x_{i+1}|$ and $\text{rightcone}_{L_{a_{i+1}}}(x_i, x_{i+1}|_{|x_i|+1}) \neq \emptyset$. In the second case we find $v \in \Sigma^*$ with $|v| < M$ such that $(x_i, x_{i+1}|_{|x_i|+1} {}^\frown v) \in L_a$, whence $x_{i+1}|_{|x_i|+1} {}^\frown v \in L$ and $\pi(x_{i+1}|_{|x_i|+1} {}^\frown v) = \pi(x_i a_{i+1})$ $= \pi(x_{i+1}) = \pi(a_1 \cdots a_{i+1})$.

By the minimality of $|x_{i+1}|$, we have $|x_{i+1}| < |x_i| + 1 + M$, i.e., $|x_{i+1}| \leq |x_i| + M$.

It follows that $|x_m| \leq m \cdot M + M = |w| \cdot M + M$. $\square$

<u>**Lemma 2**</u> Suppose $(\Sigma, L)$ is an automatic structure on a group $G$ and $L$ has the fellow traveler's condition with constant $K$. Then for any $a \in \Sigma \cup \{\varepsilon\}$, if $(x,y) \in L_a$ with $\ell = \max\{|x|, |y|\}$, there are $v_0, \dots, v_\ell, z_0, \dots, z_\ell \in \Sigma^*$ with $\pi(v_i) = 1$, $|v_i| \leq 2K+2$ such ~~that~~

$$\phi\left(\prod_{i=0}^{\ell} z_i v_i z_i^{-1}\right) = \phi(x a y^{-1}).$$

<u>**Proof**</u> Let $x_i = \begin{cases} \text{the } i\text{'th letter of } x & \text{for } i \leq |x| \\ \varepsilon & \text{otherwise} \end{cases}$

and similarly for $y$.

Now, since $L$ has the fellow traveler's condition with constant $K$, we see that for any $i$ there is a word $u_i \in \Sigma^*$ with $|u_i| \leq K$ such that $\pi(x/_i\, u_i) = \pi(y/_i)$.

Note now that, modulo $\phi$, we have

$$x a y^{-1} = x_1 x_2 \cdots x_\ell a y_\ell^{-1} \cdots y_1^{-1}$$

$$= (x_1 u_1 y_1^{-1}) \cdot y_1 (u_1^{-1} x_2 u_2 y_2^{-1}) y_1^{-1} \cdot y_1 y_2 (u_2^{-1} x_3 u_3 y_3^{-1}) y_2^{-1} y_1^{-1} \cdot$$

$$\cdots \cdot y_1 y_2 \cdots y_{\ell-1} (u_{\ell-1}^{-1} x_\ell u_\ell y_\ell^{-1}) y_{\ell-1}^{-1} \cdots y_2^{-1} y_1^{-1}$$

$$\cdot\ y_1 y_2 \cdots y_\ell (u_\ell^{-1} a) y_\ell^{-1} \cdots y_2^{-1} y_1^{-1} .$$

Note also that since $\pi(x_1 u_1) = \pi(x|_1 u_1) = \pi(y|_1) = \pi(y_1)$ we have $\pi(x_1 u_1 y_1^{-1}) = 1$ and similarly, $\pi(u_i^{-1} x_{i+1} u_{i+1} y_{i+1}^{-1}) = 1$ and $\pi(u_\ell^{-1} a) = 1$.

Also, since $a, x_i, y_i \in \Sigma \cup \{\varepsilon\}$ and $|u_i| \le K$, we have $|x_1 u_1 y_1^{-1}|, |u_i^{-1} x_{i+1} u_{i+1} y_{i+1}^{-1}|, |u_\ell^{-1} a| \le 2K+2$.

$\square$

<u>Theorem</u> Suppose $\Sigma = \{a_1, \ldots, a_n, a_1^{-1}, \ldots, a_n^{-1}\}$ is a finite set of letters and their formal inverses and that $(\Sigma, L)$ is an automatic structure on a group $G$. Then $G$ has a finite presentation $\langle \Sigma \mid R \rangle$ whose isoperimetric function is bounded above by a quadratic function.

<u>Proof</u> Let $L$ has the traveler's condition with constant $K$ and let $M$ be the constant of Lemma 1. We set

$$R = \{ u \in \Sigma^* \mid \pi(u) = 1 \ \& \ |u| \leq 2K+2 \}.$$

Suppose $w = b_1 b_2 \cdots b_m \in \Sigma^*$ and let $x_i \in L$ be words of minimal length such that $\pi(x_i) = \pi(b_1 \cdots b_i)$. By Lemma 1, $|x_i| \leq (m+1)M$. Also, since $\pi(x_i b_{i+1}) = \pi(x_{i+1})$, we have $(x_i, x_{i+1}) \in L_{b_{i+1}}$ and $\pi(x_i b_{i+1} x_{i+1}^{-1}) = 1$.

Thus, if $\pi(w) = 1$, then we have $\pi(x_m) = \pi(x_0) = 1$, so we can assume that $x_m = x_0$, whence, modulo $\Phi$,

$$x_0 w x_0^{-1} = x_0 b_1 x_1^{-1} \cdot x_1 b_2 x_2^{-1} \cdot \ldots \cdot x_{m-1} b_m x_m^{-1}.$$

Now, by Lemma 2, modulo $\Phi$, each $x_i b_{i+1} x_{i+1}^{-1}$ can be written as a product of at most $(m+1)M + 1$ conjugates of elements of $R$.

It follows that $\langle \Sigma | R \rangle$ is really a presentation of $G$ and that

$$A(w) \leq m \cdot \big( (m+1)M + 1 \big). \qquad \square$$