Büchi automata and languages of infinite strings

Languages recognised by automata are somewhat more sensitive to the exact choice of automaton allowed, so we will be a bit more careful than in the definition of DFA's and NFA's.

Fix a finite alphabet Σ.

Definition A Büchi automaton is a tuple $\mathfrak{A} = (S, I, T, F)$ where

- S is a finite set of states
- $I \subseteq S$ is the set of initial states
- $F \subseteq S$ is the set of final states
- $T \subseteq S \times \Sigma \times S$ is the transition table.

Definition The automaton is complete if for all $s \in S$ and $a \in \Sigma$ there is some $t \in S$ with $(s, a, t) \in T$.

It is deterministic if I is a singleton, and for all $s \in S$ and $a \in \Sigma$ there is exactly one $t \in S$ with (s, a, t).

$(s, a, t) \in T$ are called transitions and correspond to the rule that when in state s and reading a, the automaton can change to state t.

Recall that Σ^ω is the set of infinite strings $\alpha = a_1 a_2 a_3 \dots$, $a_i \in \Sigma$.

<u>Definition</u> Given $\alpha \in \Sigma^\omega$, a <u>run</u> or <u>computation</u> on α is an infinite string

$\sigma = s_1 s_2 s_3 \dots \in S^\omega$ such that

- $s_1 \in I$
- for all $i \in \mathbb{N}$, $(s_i, a_i, s_{i+1}) \in T$.

We say that the automaton $\mathfrak{A} = (S, I, T, F)$ <u>accepts</u> $\alpha \in \Sigma^\omega$ if there is a run $\sigma = s_1 s_2 \dots$ on α such that $s_i \in F$ for infinitely many $i \in \mathbb{N}$. In this case, we say that σ is a <u>successful</u> run on α.

Again we let $L(\mathfrak{A}) \subseteq \Sigma^\omega$ be the set of all strings accepted by $\mathfrak{A}$ and say that $\mathfrak{A}$ <u>recognizes</u> $L(\mathfrak{A})$.

Theorem The class of Büchi recognizable languages is closed under finite union and intersection.

Proof Assume $L_1, L_2 \subseteq \Sigma^\omega$ are recognized by $\mathfrak{A}_1$ and $\mathfrak{A}_2$ respectively. We shall construct an automaton recognizing $L_1 \cap L_2$.

Ideally, we would just run $\mathfrak{A}_1$ and $\mathfrak{A}_2$ on the same input, but then we have the problem that even though both enter final states infinitely often, they might not do so simultaneously. So instead we will ensure that they alternate in accepting states.

So let $\mathfrak{A}_i = (S_i, I_i, T_i, F_i)$ and define

$S = S_1 \times S_2 \times \{1,2,3\}$, $I = S_1 \times S_2 \times \{1\}$,

$F = S_1 \times S_2 \times \{3\}$,

$$\begin{aligned} T = &\{ ((s,t,1),a,(p,q,1)),\ ((s,t,2),a,(p,q,2)),\\ &\quad ((s,t,3),a,(p,q,1)) \mid (s,a,p) \in T_1,\ (t,a,q) \in T_2 \}\\ &\cup \{ ((s,t,1),a,(p,q,2)) \mid (s,a,p) \in T_1,\ (t,a,q) \in T_2\ \&\ p \in F_1 \}\\ &\cup \{ ((s,t,2),a,(p,q,3)) \mid (s,a,p) \in T_1,\ (t,a,q) \in T_2\ \&\ q \in F_2 \} \end{aligned}$$

To see that $\mathfrak{A} = (S, I, T, F)$ recognises $L_1 \cap L_2$, note that if $\mathfrak{A}$ has a successful run π on α, then we can write

$$\pi = (s_1, t_1, m_1)(s_2, t_2, m_2)(s_3, t_3, m_3)\cdots$$

where $s_i \in S_1$, $t_j \in S_2$, $m_i \in \{1,2,3\}$ and $m_i = 3$ for infinitely many i.

Now, by looking at the transition table T, we see that m_i must have a subsequence (m_{i_j}) of the form

$$1\,2\,3\,1\,2\,3\,1\,2\,3\cdots$$

from which we deduct that $s_{i_j} \in F_1$ for $j \equiv 2 \pmod 3$, $t_{i_j} \in F_2$ for $j \equiv 0 \bmod 3$.

It follows that $\sigma = s_1 s_2 \cdots$ and $\tau = t_1 t_2 \cdots$ are both successful runs of $\mathfrak{A}_1$, resp. $\mathfrak{A}_2$, on α, whence $\alpha \in L_1 \cap L_2$.

Conversely, a pair of successful runs σ and τ of $\mathfrak{A}_1$, resp. $\mathfrak{A}_2$, on α can easily be merged into a successful run of $\mathfrak{A}$ on α.

Union is easy. $\square$

The Büchi Theorem

The goal now is to give a more algebraic characterization of Büchi recognizable languages.

Recall if $L \subseteq \Sigma^*$ is a language, we set

$$L^\omega = \{ w_1 w_2 w_3 \cdots \mid w_i \in L \}.$$

<u>Theorem</u> (Büchi) A language $L \subseteq \Sigma^\omega$ is Büchi recognizable if and only if there are regular languages $V_i, W_i \subseteq \Sigma^*$ such that

$$L = \bigcup_{i=1}^{n} V_i W_i^\omega .$$

<u>Example</u> Using this theorem, we immediately see that

$$\{ x \in \{0,1\}^\omega \mid x \text{ contains only finitely many } 1\text{'s} \}$$

$$= \{0,1\}^* 0^\omega$$

is Büchi recognizable.

<u>Exercise</u> Construct a Büchi automaton recognizing $\{0,1\}^* 0^\omega$

<u>Question</u> Is there a deterministic Büchi automaton recognizing $\{0,1\}^* 0^\omega$?

Lemma Suppose $W \subseteq \Sigma^*$ is regular. Then W^ω is Büchi recognizable.

Proof Fix a deterministic Büchi automaton $\mathfrak{A} = (S, \{s_0\}, T, F)$ recognizing W. Wlog we can assume that there is no transition (s, a, s_0) in T.

We let $\mathcal{B} = (S, \{s_0\}, T', \{s_0\})$ where

$$T' = T \cup \{(s, a, s_0) \mid \text{there is } t \in F \text{ with } (s, a, t) \in T\}.$$

Thus, $\mathcal{B}$ works like $\mathfrak{A}$ except that whenever s is a state and $a \in \Sigma$ is s.t. $(s, a, t) \in T$ for some final state $t \in F$, we allow $\mathfrak{A}$ instead to jump to s_0 reading a. Clearly then $L(\mathcal{B}) = L(\mathfrak{A})^\omega = W^\omega$. $\square$

An elaboration of the above argument gives us:

Lemma If $V, W \subseteq \Sigma^*$ are regular languages, then VW^ω is Büchi recognizable.

<u>Exercises</u> Construct Büchi automata recognizing VW^ω for

(a) $V = a^*$, $W = b^*$

(b) $V = \{x \in \{a,b\}^* \mid |x|_a \text{ is even}\}$, $W = \{ab, ba\}$

(c) $V = \{x \in \{a,b\}^* \mid x \text{ begins and ends with } b\}$,

$W = \{x \in \{a,b\}^* \mid x \text{ contains no subword } aaa\}$.

<u>Proof of theorem</u>:

Suppose $L \in \Sigma^\omega$ is recognized by a Büchi automaton $\mathcal{O}\!\mathcal{L} = (S, I, T, F)$.

Then if $\alpha \in L$, there is a successful run $\sigma = s_1 s_2 s_3 \ldots$ of $\mathcal{O}\!\mathcal{L}$ on α. Now some $t_1 \in F$ must appear infinitely often in σ, so we must be able to cut up α into words

$\alpha = v w_1 w_2 w_3 \ldots$ s.t. $\mathcal{O}\!\mathcal{L}$ can go from an initial state t_0 to t_1 reading v and go from t_1 to t_1 reading w_i.

So define for $s, t \in S$

$$W_{s,t} = \{ w \in \Sigma^* \mid \mathcal{O}\!\mathcal{L} \text{ can go from state } s \text{ to state } t \text{ while reading } w \}.$$

Clearly each $W_{s,t}$ is regular, and so

$$L = \bigcup_{\substack{s \in I \\ t \in F}} W_{s,t} W_{t,t}^{\omega}. \qquad \Box$$

Complementation of Büchi recognisable languages.

As we shall see, the Büchi recognisable language $\{\alpha \in \{a,b\}^{\omega} \mid \alpha$ contains only finitely many b's$\}$ is not recognised by any deterministic Büchi automaton. So we cannot prove that complements of Büchi recognisable languages are Büchi recognisable by the same easy means as for regular languages. Instead we must proceed to a deeper combinatorial investigation.

We fix in the following a complete Büchi automaton $\mathcal{A} = (S, I, T, F)$ on a finite alphabet Σ.

Define for $s,t \in S$ the set $W_{s,t}$ as before and

$W_{s,t}^{F} = \{w \in \Sigma^* \mid$ there is a finite run $\sigma = s_0 s_1 s_2 \ldots s_{n-1} s_n$ of $\mathcal{A}$ on $w = a_1 a_2 \ldots a_n$ such that $s_0 = s$, $s_n = t$ and $s_i \in F$ for some $0 \le i \le n$. $\}$.

We also define an equivalence relation $\sim$ on Σ^* by

$$w \sim v \iff \forall s,t \in S \ (w \in W_{s,t} \leftrightarrow v \in W_{s,t})$$
$$\& \ \forall s,t \in S \ (w \in W^F_{s,t} \leftrightarrow v \in W^F_{s,t}).$$

Since S is finite, $\sim$ has finitely many classes. Also, $\sim$ is a <u>congruence</u> <u>relation</u> on Σ^*, i.e., for all $w, v, x \in \Sigma^*$,

$$w \sim v \implies wx \sim vx \qquad (*)$$

To see this, note that if $w \sim v$ and, e.g., $wx \in W_{s,t}$, then there is $q \in S$ such that $w \in W_{s,q}$ & $x \in W_{q,t}$. But then also $v \in W_{s,q}$ & $x \in W_{q,t}$, whence $vx \in W_{s,t}$. Similarly for $W^F_{s,t}$.

Again, each $W_{s,t}$ and $W^F_{s,t}$ are easily seen to be regular, so each $\sim$-equivalence class, being a Boolean combination of these sets, is regular.

<u>Lemma</u> Each $\sim$-class is a regular language.

We can now indicate how we will prove that Büchi recognisable languages are closed under complementation:

<u>Lemma 1</u>: If V and W are $\sim$-classes and

$$L(\mathfrak{A}) \cap VW^{\omega} \neq \emptyset, \text{ then } VW^{\omega} \subseteq L(\mathfrak{A}).$$

<u>Lemma 2</u> For any word $\alpha \in \Sigma^{\omega}$ there are $\sim$-classes V, W such that $\alpha \in VW^{\omega}$.

From Lemma 1 and 2, we see that

$$L(\mathfrak{A}) = \bigcup \{ VW^{\omega} \mid V, W \text{ are } \sim\text{-classes and } VW^{\omega} \cap L(\mathfrak{A}) \neq \emptyset \}$$

and

$$\Sigma^{\omega} \setminus L(\mathfrak{A}) = \bigcup \{ VW^{\omega} \mid V, W \text{ are } \sim\text{-classes and } VW^{\omega} \cap L(\mathfrak{A}) = \emptyset \}.$$

Since $\sim$ has only finitely many classes, it will follow from the Büchi characterization that $\Sigma^{\omega} \setminus L(\mathfrak{A})$ is Büchi recognizable.

Proof of Lemma 1

Suppose $\alpha \in L(\mathcal{Ol}) \cap VW^\omega$ and let σ be a successful run of $\mathcal{Ol}$ on α.

Since $\alpha \in VW^\omega$, it can be decomposed as

$\alpha = v w_1 w_2 w_2 \cdots$ where $v \in V$ and $w_i \in W$.

So, as W and V are $\sim$ equivalence classes, it follows that the w_i belong to the same sets $W_{s,t}$ and $W^F_{s,t}$.

Now from the decomposition of α we find a substring $t_0 q_1 q_2 q_3 \cdots$ of $\sigma = s_1 s_2 s_3 \cdots$ such that

$$v \in W_{t_0 q_1}, \quad w_i \in W_{q_i, q_{i+1}}.$$

Also, since σ is a successful run, we actually have $w_i \in W^F_{q_i, q_{i+1}}$ for infinitely many and hence all i.

Now, if $\beta \in VW^\omega$ is any other word, write

$\beta = v' w'_1 w'_2 \cdots$ s.t. $v' \in V$ and $w'_i \in W$.

Then $v' \in W_{t_0 q_1}$ and $w'_i \in W^F_{q_i q_{i+1}}$. It follows that $\mathcal{Ol}$ will have a successful run τ on β. So $\beta \in L(\mathcal{Ol})$. $\square$

Proof of Lemma 2:

This is essentially a reworking of the proof of the infinite version of Ramsey's Theorem.

We fix $\alpha \in \Sigma^\omega$, $\alpha = a_1 a_2 a_3 \ldots$, $a_i \in \Sigma$.

Also for $n \leq m$, set $\alpha[n,m[= a_n a_{n+1} \cdots a_{m-1} \in \Sigma^*$.

We now inductively define infinite $A_i \subseteq \mathbb{N}$ and $n_i \in \mathbb{N}$ such that

$$A_1 \supsetneq A_2 \supsetneq A_3 \supsetneq \ldots \quad , \quad n_i = \min A_i \,,$$

$$n_1 < n_2 < n_3 < \ldots$$

and for all i and $k, l \in A_{i+1}$

$$\alpha[n_i, k[\sim \alpha[n_i, l[\,.$$

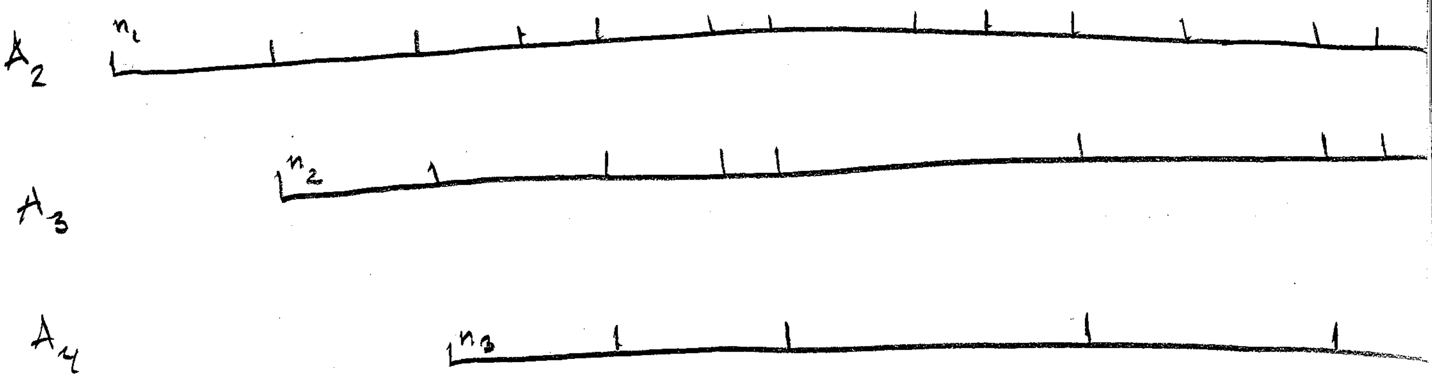
Set $n_1 = 1$ and $A_1 = \mathbb{N}$.

We note that $\{\alpha[n_1, k[\mid k \in A_1\}$ is covered by finitely many $\sim$-classes, so for some infinite $A_2 \subseteq A_1$, $n_1 < \min A_2$, we have

$$\alpha[n_1, k[\sim \alpha[n_1, l[\quad \text{whenever } k, l \in A_2.$$

Set $n_2 = \min A_2$. Again, we find $A_3 \subseteq A_2$ infinite, $n_2 < \min A_3$, such that for $k, l \in A_3$

$$\alpha[n_2, k[\sim \alpha[n_2, l[\quad \text{etc.}$$



So $n_j \in A_i$ for $i \le j$ and thus for all $i < j < k$, $\alpha[n_i, n_j[\; \sim \; \alpha[n_i, n_k[$.

Thus, we can colour the n_i according to the equivalence class of $\alpha[n_i, n_j]$ for $j > i$. And so we can extract a subsequence $(m_i)_{i=1}^{\infty}$ of $(n_i)_{i=1}^{\infty}$ such that for all $i < j$ and $k < l$, we have

$$\alpha[m_i, m_j[\; \sim \; \alpha[m_k, m_l[\; .$$

Let now $v = \alpha[1, m_1[$ and $w_i = \alpha[m_i, m_{i+1}[$. Then $w_i \sim w_j$ for all i, j, so

$$\alpha = v w_1 w_2 w_3 \cdots \in V W^{\omega}$$

for some $\sim$ classes V and W. $\square$

Determinism

We shall now characterise the languages recognised by deterministic Büchi automata.

Definition Suppose $W \subseteq \Sigma^*$. Then we set

$$\vec{W} = \{ \alpha \in \Sigma^\omega \mid \alpha \text{ has infinitely many prefixes in } W \}.$$

Once we have this definition, the following theorem at once becomes evident.

Thm A language $L \subseteq \Sigma^\omega$ is recognised by a deterministic Büchi automaton if and only if there is a regular language $W \subseteq \Sigma^*$ with

$$L = \vec{W}.$$

Thm The language $\{a,b\}^* a^\omega$ is not recognised by a deterministic Büchi automaton.

Proof Just note that for any $W \subseteq \Sigma^*$, $\vec{W} \subseteq \Sigma^\omega$ is a G_δ subset of Σ^ω, so if dense, it must also be comeagre. However, $\{a,b\}^* a^\omega$ is dense, so not comeagre in $\{a,b\}^\omega$. $\square$

Note however that the complement of $\{a,b\}^* a^\omega$ is recognised by a deterministic Büchi automaton.

Müller automata

We shall now replace Büchi automata by another computational model which will allow us to recognise all Büchi recognisable languages by a deterministic procedure.

Definition A Müller automaton over Σ is a tuple $\mathfrak{A} = (S, s_0, T, \mathcal{F})$, where

- S is a finite set of states
- $s_0 \in S$ is the initial state
- $T \subseteq S \times \Sigma \times S$ is the set of transitions such that $\forall s \in S \ \forall a \in \Sigma \ \exists! t \quad (s, a, t) \in T$
- $\mathcal{F} \subseteq \mathcal{P}(S)$ is the set of final subsets.

Runs (or computations) are defined as for Büchi automata. Note that the run σ is determined by α.

Definition A run $\sigma = s_1 s_2 s_3 \ldots$ of $\mathfrak{A}$ on $\alpha = a_1 a_2 \ldots \in \Sigma^\omega$ is successful if $\{t \in S \mid \exists^\infty i \quad s_i = t\} \in \mathcal{F}$.

As opposed to the case of Büchi automata, the following result is now almost immediate.

<u>Theorem</u> The class of Müller recognisable languages $L \subseteq \Sigma^\omega$ is a Boolean algebra.

<u>Theorem</u> A language $L \subseteq \Sigma^\omega$ is Müller recognisable if and only if L belong to the Boolean algebra generated by languages $\vec{W}$, where $W \subseteq \Sigma^*$ is regular.

<u>Corollary</u> A Müller recognisable language $L \subseteq \Sigma^\omega$ is $\underset{\sim}{\Delta}^0_3$ and is Büchi recognisable.

<u>Proof of theorem</u>

Note first that if $L = \vec{W}$ for W regular, then L is recognised by a deterministic Büchi automaton and so also by a Müller automaton. So Müller automata also recognise languages that are Boolean combinations of languages $\vec{W}$, W regular.

Conversely, suppose $\mathfrak{A} = (S, s_0, T, \mathcal{F})$ is a Müller automaton and $L = L(\mathfrak{A})$. For every $q \in S$, define $\mathfrak{A}_q = (S, \{s_0\}, T, \{q\})$, which then is a deterministic Büchi automaton.

Then

$$L(\mathfrak{A}) = \bigcup_{F \in \mathcal{F}} \left(\bigcap_{q \in F} L(\mathfrak{A}_q) \setminus \bigcup_{q \notin F} L(\mathfrak{A}_q) \right),$$

showing that $L(\mathfrak{A})$ is a Boolean combination of languages recognised by deterministic Büchi automata. $\square$

Sequential Rabin automata

A *sequential Rabin automaton* is a tuple $\mathfrak{A} = (S, s_0, T, \Omega)$, where S, s_0, T are as for Müller automata, but

$$\Omega \subseteq \mathcal{P}(S) \times \mathcal{P}(S)$$

Writing $\Omega = \{(N_1, P_1), \ldots, (N_d, P_d)\}$, we say that (N_i, P_i) is an *accepting pair* of $\mathfrak{A}$.

Again Rabin automata are deterministic, so with every $\alpha \in \Sigma^\omega$ there is a unique run σ of $\mathfrak{A}$ on α.

We say that the run $\sigma = s_1 s_2 s_3 \cdots$ is <u>successful</u> if there is some i such that

$$\forall^\infty n \quad s_n \notin N_i \quad \& \quad \exists^\infty n \quad s_n \in P_i .$$

<u>Theorem</u> A language $L \subseteq \Sigma^\omega$ is recognised by a Rabin automaton if and only if it is recognised by a Müller automaton.

<u>Proof</u> suppose $\mathfrak{A} = (S, s_0, T, \Omega)$ is a Rabin automaton and $\Omega = \{(N_1, P_1), \ldots, (N_d, P_d)\}$. Then, letting $\mathcal{F} = \{F \subseteq S \mid \exists i \le d \, \exists s \in P_i \;\; s \in F \;\&\; F \cap N_i = \emptyset\}$, we see that $\mathfrak{A}' = (S, s_0, T, \mathcal{F})$ is a Müller automaton with $L(\mathfrak{A}') = L(\mathfrak{A})$.

Conversely, suppose $\mathfrak{A} = (S, s_0, T, \mathcal{F})$ is a Müller automaton with $\mathcal{F} = \{Q_1, Q_2, \ldots, Q_d\}$, $Q_i \subseteq S$. Consider what happens in a run $\sigma = s_1 s_2 s_3 \cdots$ of $\mathfrak{A}$ on α. For every $A \subseteq S$, we can build a counter $\hat{A}$ that tries to accumulate a copy of A by selecting $s_{i_1}, s_{i_2}, \ldots, s_{i_n}$ st. $A = \{s_{i_1}, \ldots, s_{i_n}\}$ and then, when full, resets and begins over.

Then r is a successful run of $\mathfrak{A}$ iff and only if for some $i \leq d$:

(i) the counter $\hat{Q}_i$ is reset infinitely often,

(ii) there is no $A \supsetneq Q_i$ such that $\hat{A}$ is reset infinitely often.

We now build a Rabin automaton $\mathcal{B}$ incorporating these counters.

First list $\mathcal{P}(S) \setminus \{\emptyset\}$ as $A_1, \ldots, A_k$.

Then $\mathcal{B} = (S', s_0', T', \Omega)$, where

$$S' = \mathcal{P}(A_1) \times \ldots \times \mathcal{P}(A_k) \times S$$

$$s_0' = (\emptyset, \emptyset, \ldots, \emptyset, s_0)$$

$$((B_1, \ldots, B_k, s), a, (C_1, \ldots, C_k, t)) \in T'$$

$$\iff (s, a, t) \in T \quad \&$$

$$C_i = \begin{cases} B_i \cup \{t\} & \text{if } t \in A_i \text{ and } B_i \not\subseteq A_i \\ B_i & \text{if } t \notin A_i \\ \emptyset & \text{if } B_i = A_i . \end{cases}$$

Then $\Omega = \{(N_1, P_1), \ldots, (N_d, P_d)\}$, where

$$N_i = \{(B_1, \ldots, B_{\ell-1}, A_\ell, B_{\ell+1}, \ldots, B_k, t) \mid B_m \subseteq A_m \ \& \ Q_i \subsetneq A_\ell\}$$

$$P_i = \{(B_1, \ldots, B_{i-1}, Q_i, B_{i+1}, \ldots, B_k, t) \mid B_m \subseteq A_m\}.$$

Then $L(\mathfrak{B}) = L(\mathfrak{A})$. $\square$