Rabin automata

Let $T = \{0,1\}^*$ denote the set of all finite binary strings. For $x, y \in T$, we let

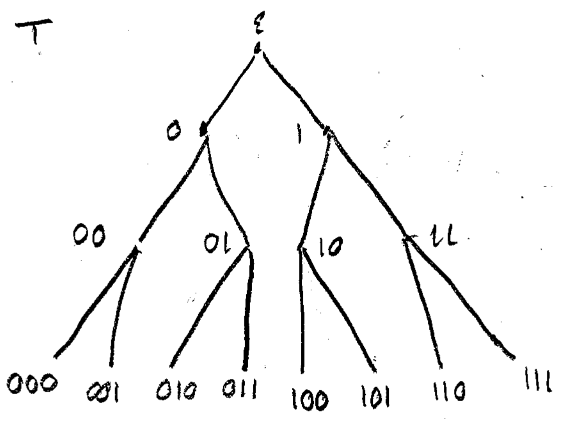
$$x \sqsubseteq y \iff x \text{ is a prefix of } y.$$

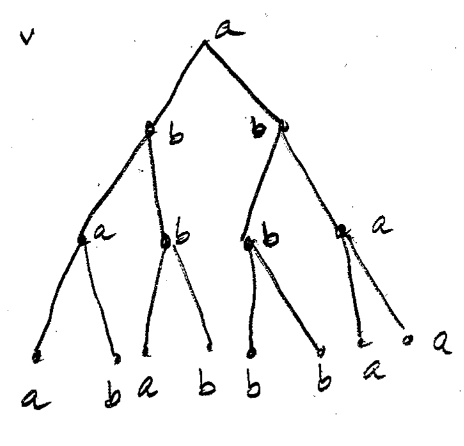
Thus T is a tree with the ordering $\sqsubseteq$ and root ε.

<u>Definition</u> A <u>Σ-valuation</u> on T is a function $v : T \to \Sigma$. Denote by V_Σ the set of all Σ-valuations on T.

A <u>language</u> is any subset $L \subseteq V_\Sigma$.

<u>Note</u> We can represent Σ-valuations simply by labellings of the infinite binary tree:





Definition A *Rabin automaton* over Σ is a tuple $\mathfrak{A} = (S, I, M, \mathcal{F})$ where

- S is a finite set of *states*
- $I \subseteq S$ is the set of *initial states*
- M is a map from $S \times \Sigma$ to $\mathcal{P}(S \times S)$ called the *transition table*
- $\mathcal{F} \subseteq \mathcal{P}(S)$ are the *designated subsets* of S.

We shall now define computations of Rabin automata on Σ-valuations. Informally, the automaton $\mathfrak{A}$ will non-deterministically do computations along each branch of the tree representing v.

Definition A *run* or *computation* of $\mathfrak{A}$ on a Σ-valuation v is a function $\sigma : T \to S$ such that

$$\sigma(\varepsilon) \in I$$

and for all $x \in T$

$$(\sigma(x0), \sigma(x1)) \in M(\sigma(x), v(x)).$$

In other words, $\mathfrak{A}$ produces a labelling (or computation) σ of T by the states S by first labelling ε by some $s_0 \in I$. Also, if $\mathfrak{A}$ has labeled $x \in T$ by $s \in S$, then $\mathfrak{A}$ has to label the pair of nodes $(x0, x1)$ by some pair of states (t_0, t_1) belonging to $M(s, v(x))$.

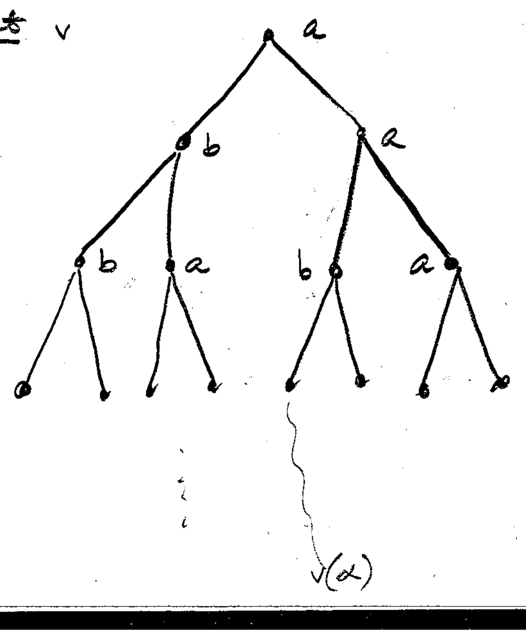
__Example__

Consider $\mathfrak{A} = (\{s,t\}, \{s\}, M, \mathfrak{F})$ on $\Sigma = \{a,b\}$ where M is given by

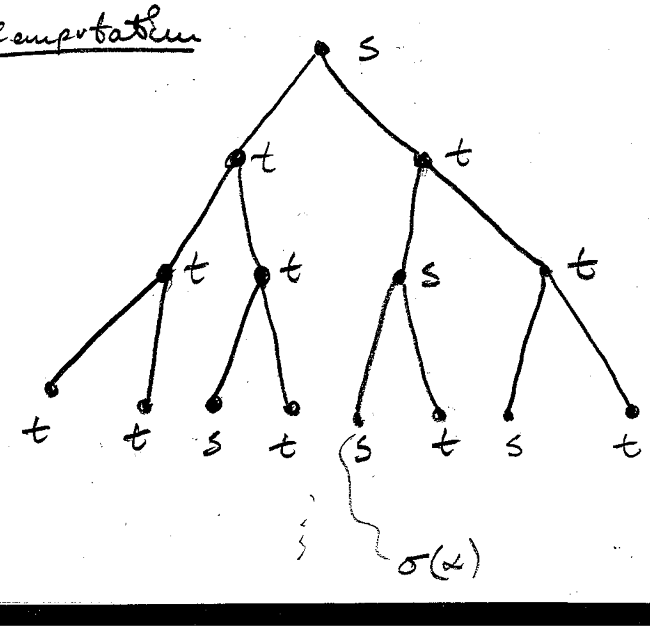
$$M(s,a) = M(t,b) = \{(t,t)\}$$
$$M(t,a) = M(s,b) = \{(s,t)\}$$

Thus $\mathfrak{A}$ is deterministic.

__Input__ v



__Computation__



Definition A run or computation $\sigma : T \to S$ of $\mathfrak{A}$ on $v \in V_\Sigma$ is said to be **successful** if for any $\alpha \in \{0,1\}^\omega$ we have

$$\{ s \in S \mid \exists^\infty n \ \ \sigma(\alpha|_n) = s \} \in \mathcal{F}.$$

So the run is successful if along each branch of T it satisfies the success criteria of Müller automata.

The language $L \subseteq V_\Sigma$ **recognised** by a Rabin automaton $\mathfrak{A}$ is the set of all $v \in V_\Sigma$ on which $\mathfrak{A}$ has a successful run.

Definition The Rabin automaton $\mathfrak{A} = (S, I, M, \mathcal{F})$ is **complete** if $M(s,a) \neq \emptyset$ for all $s \in S$ and $a \in \Sigma$. It is **deterministic** if I is a singleton and $M(s,a)$ is a singleton for all $s \in S$ and $a \in \Sigma$.

Proposition For any Rabin automaton $\mathfrak{A}$ there is a complete Rabin automaton $\mathfrak{B}$ with a single initial state such that

$$L(\mathfrak{A}) = L(\mathfrak{B}).$$

Proof To see how to reduce I to a singleton, suppose $\mathcal{A} = (S, I, M, \mathcal{F})$ is given and let $s_0 \notin S$. We set $\mathcal{B} = (S \cup \{s_0\}, \{s_0\}, M', \mathcal{F})$ where

$$M'(s_0, a) = \bigcup_{s \in I} M(s, a)$$

and $M'(s, a) = M(s, a)$ for $s \in S$.

We leave it as an exercise how to make the automaton complete. $\square$

Proposition If $L_1, L_2 \subseteq V_\Sigma$ are Rabin recognisable, then so is $L_1 \cup L_2$.

Proof Suppose $\mathcal{A}_i = (S_i, I_i, M_i, \mathcal{F}_i)$ and $L_i = L(\mathcal{A}_i)$. Then let $\mathcal{B} = (S, I, M, \mathcal{F})$ where

$$S = S_1 \sqcup S_2$$

$$I = I_1 \sqcup I_2$$

$$M(s, a) = \begin{cases} M_1(s, a) & \text{if } s \in S_1 \\ M_2(s, a) & \text{if } s \in S_2 \end{cases}.$$

$$\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \subseteq \mathcal{P}(S_1 \sqcup S_2).$$

It follows that $L(\mathcal{B}) = L(\mathcal{A}_1) \cup L(\mathcal{A}_2) = L_1 \cup L_2$.

$\square$

Proposition If $L_1, L_2 \subseteq V_\Sigma$ are recognized by Rabin automata $\mathfrak{A}_1$ and $\mathfrak{A}_2$ as before, also $L_1 \cap L_2$ is recognized a Rabin automaton.

Proof Let $\mathcal{C} = (S_1 \times S_2, I_1 \times I_2, M, \mathcal{F})$, where

$$((t_1, t_2), (q_1, q_2)) \in M((s_1, s_2), a) \iff$$

$$(t_1, q_1) \in M_1(s_1, a) \;\&\; (t_2, q_2) \in M_2(s_2, a)$$

and for $F \subseteq S_1 \times S_2$ we have

$$F \in \mathcal{F} \iff \mathrm{proj}_1(F) \in \mathcal{F}_1 \;\&\; \mathrm{proj}_2(F) \in \mathcal{F}_2.$$

Again, $L(\mathcal{C}) = L_1 \cap L_2$. ∎

Game automata

In order to show that Rabin recognizable languages are closed under complementation we will make appeal to another game theoretic model of computation.

The idea is that to see if a Rabin automaton has a successful run $\sigma : T \to S$ on an input $v : T \to \Sigma$ one should just be able to inductively show how σ is constructed along branches of T.

Definition

A *game automaton* is a tuple $\mathfrak{A} = (S, I, R, \mathcal{F})$ where

- S is a finite set of *states*
- $I \subseteq \Sigma \times S$ is the set of *initial rules*
- $R \subseteq S \times \{0,1\} \times \Sigma \times S$ is the set of rules
- $\mathcal{F} \subseteq \mathcal{P}(S)$ is the set of winning conditions.

Given a game automaton $\mathfrak{A} = (S, I, R, \mathcal{F})$ and a Σ-valuation $v : T \to \Sigma$ we can define the game $G_v^{\mathfrak{A}}$ between players I and II by

I	s_0	s_1	s_2	...
II		d_1	d_2	d_3 ...

where $s_i \in S$ and $d_i \in \{0,1\}$ are subject to the conditions:

$$(*) \begin{cases} (v(\varepsilon), s_0) \in I \\ (s_n, d_{n+1}, v(d_1 d_2 \ldots d_{n+1}), s_{n+1}) \in R \quad \text{for all } n \geq 0. \end{cases}$$

We say that I *wins* a run of the

game $G_v^{\mathfrak{A}}$ if $\{ s \in S \mid \exists^\infty n \ s_n = s \} \in \mathcal{F}$.
Otherwise, II wins the game.

<u>Definition</u> A (legal) <u>position</u> of $G_v^{\mathfrak{A}}$ is a finite sequence $p = s_0 d_1 s_1 d_2 s_2 \ldots d_n s_n$ or $p = s_0 d_1 s_1 \ldots d_n$ such that (*) hold.

A <u>strategy</u> for I in $G_v^{\mathfrak{A}}$ is a function ϕ associating to every legal position $p = s_0 d_1 s_1 \ldots d_n$ of <u>even</u> length some $\phi(p) \in S$ such that $s_0 d_1 s_1 \ldots d_n \phi(p)$ is a legal position.

Similarly, a <u>strategy</u> for II is a function ψ associating to every legal position $p = s_0 d_1 s_1 \ldots d_n s_n$ of <u>odd</u> length some $\phi(p) \in \{0,1\}$.

A run of the game $G_v^{\mathfrak{A}}$: $\begin{array}{ccccccc} \text{I} & s_0 & & s_1 & & s_2 & \ldots \\ \text{II} & & d_1 & & d_2 & & \ldots \end{array}$ is <u>consistent</u> with a strategy ϕ for I in case $s_n = \phi(s_0 d_1 s_1 \ldots d_n)$ for all n.

A strategy ϕ for I is <u>winning</u> if every run consistent with ϕ is winning for I.
Similarly for II.

<u>Definition</u> A game automaton $\mathfrak{A}$ <u>accepts</u> a Σ-valuation $v \in V_\Sigma$ if I has a winning strategy ϕ in $G_v^{\mathfrak{A}}$.

Let $L(\mathfrak{A}) = \{ v \in V_\Sigma \mid v$ is accepted by $\mathfrak{A} \}$.

<u>Theorem</u> For any Rabin automaton $\mathfrak{A}$ there is a game automaton $\mathcal{B}$ with $L(\mathfrak{A}) = L(\mathcal{B})$.

<u>Proof</u> Wlog, we can assume that $\mathfrak{A} = (S, \{s_0\}, M, \mathcal{F})$ is a complete Rabin automaton with a single initial state. We construct the game automaton $\mathcal{B} = (S_1, I_1, R_1, \mathcal{F}_1)$ as follows:

$$S_1 = S \times S \times S$$

$$I_1 = \{ (a, (t_0, t_1, s_0)) \mid (t_0, t_1) \in M(s_0, a) \}$$

$$((t_0, t_1, t_2), d, a, (q_0, q_1, q_2)) \in R$$

$$\iff (q_0, q_1) \in M(t_d, a) \ \& \ q_2 = t_d$$

and for $F \subseteq S_1 = S \times S \times S$,

$$F \in \mathcal{F}_1 \iff \mathrm{proj}_3(F) \in \mathcal{F}$$

Now, to see that $L(\mathfrak{A}) = L(\mathcal{B})$, suppose first that $v \in L(\mathfrak{A})$ and let $\sigma : T \to S$ be a successful run of $\mathfrak{A}$ on v.

We shall use σ to construct a winning strategy ϕ for I in $G_v^{\mathcal{B}}$.

First, as σ is successful, we have for any $\alpha \in \{0,1\}^\omega$,

$$\{s \in S \mid \exists^\infty n \ \sigma(\alpha|_n) = s\} \in \mathcal{F}.$$

Now if $p = q_0 d_1 q_1 d_2 \cdots q_{n-1} d_n$, $q_i \in S_1$, $d_i \in \{0,1\}$, is any legal position in $G_v^{\mathcal{B}}$, we let

$$\phi(p) = (\sigma(d_1 d_2 \cdots d_n 0), \sigma(d_1 d_2 \cdots d_n 1), \sigma(d_1 d_2 \cdots d_n)).$$

To see that this is a legal move, note first that

$$\phi(\varepsilon) = (\sigma(0), \sigma(1), \sigma(\varepsilon)) = (\sigma(0), \sigma(1), s_0)$$

and $(v(\varepsilon), (\sigma(0), \sigma(1), s_0)) \in I$ since

$$(\sigma(0), \sigma(1)) \in M(v(\varepsilon), s_0) = M(v(\varepsilon), \sigma(\varepsilon)).$$

Also, for any $n \geq 0$, we have

$$(q_{n-1}, d_n, v(d_1 d_2 \cdots d_n), \phi(p))$$

$$= ((\sigma(d_1 \cdots d_{n-1} 0), \sigma(d_1 \cdots d_{n-1} 1), \sigma(d_1 \cdots d_{n-1})), d_n, v(d_1 d_2 \cdots d_n),$$
$$(\sigma(d_1 \cdots d_n 0), \sigma(d_1 \cdots d_n 1), \sigma(d_1 \cdots d_n))) \in R$$

since by assumption on σ

$$\left(\sigma(d_1 d_2 \ldots d_n 0), \sigma(d_1 \ldots d_n 1)\right) \in M\left(v(d_1 \ldots d_n), \sigma(d_1 \ldots d_n)\right).$$

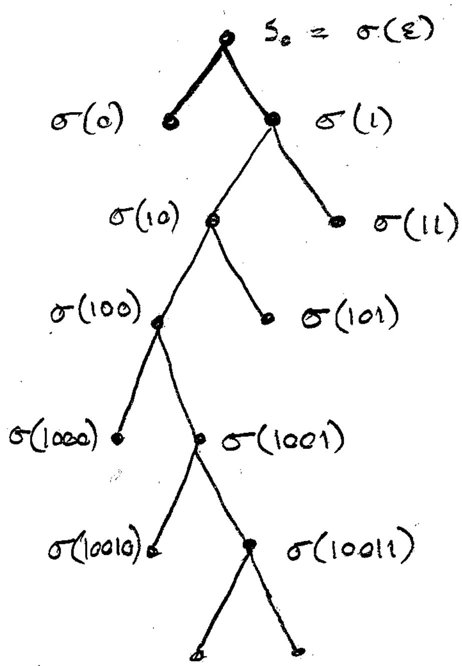
So ϕ is indeed a strategy for I in $G_v^{\mathcal{B}}$.

We note the function of ϕ:

For any given position $p = q_0 d_1 \ldots d_n$,

the last coordinate of $\phi(p)$, i.e., $\sigma(d_1 d_2 \ldots d_n)$ gives the current state of $\mathcal{O}$ when running the computation σ along $d_1 d_2 \ldots d_n \in T$.

The first two coordinates of $\phi(p)$, namely $\sigma(d_1 \ldots d_n 0)$ and $\sigma(d_1 \ldots d_n 1)$, give the two next possible states.



Play of I using ϕ against

$d_1 d_2 d_3 d_4 d_5 \ldots = 10011\ldots$

played by II:

$(\sigma(0), \sigma(1), \sigma(\varepsilon)), (\sigma(10), \sigma(11), \sigma(1)),$

......

To see that ϕ is winning for I, suppose $\alpha = d_1 d_2 d_3 \ldots$ is being played by II. Then using ϕ, I responds with

$$(\sigma(0), \sigma(1), \sigma(\varepsilon)), \ (\sigma(d_1 0), \sigma(d_1 1), \sigma(d_1)), \ \ldots$$

and hence the sequence of third coordinates of I's play is just

$$\sigma(\varepsilon), \sigma(\alpha|_1), \sigma(\alpha|_2), \ldots .$$

Since σ is successful,

$$\{ s \in S \mid \exists^\infty n \ \ \sigma(\alpha|_n) = s \} \in \mathcal{F}$$

showing that I wins the game.

Thus, ϕ is a winning strategy for I in $G_v^{\mathcal{B}}$ and so $v \in L(\mathcal{B})$. Hence $L(\mathcal{A}) \subseteq L(\mathcal{B})$.

The converse implication is similar. For suppose $v \in L(\mathcal{B})$ and let ϕ be a winning strategy for I in $G_v^{\mathcal{B}}$. Then if $d_1 d_2 \ldots d_n \in T$ let $q_0 d_1 q_1 d_2 q_2 \ldots d_n q_n$ be the corresponding run consistent with ϕ. We then set

$$\sigma(d_1 d_2 \ldots d_n) = \text{third coordinate of } q_n .$$

Then σ is successful and so $L(\mathcal{B}) \subseteq L(\mathcal{A})$. $\square$

Theorem For any game automaton $\mathfrak{A} = (S, I, R, \mathcal{F})$ there is a Rabin automaton $\mathcal{B}$ with $L(\mathfrak{A}) = L(\mathcal{B})$.

Proof

We let $\mathcal{B} = (S_1, I_1, M_1, \mathcal{F}_1)$ where

$$S_1 = S \times \{0,1\} \cup \{t_0\}, \text{ where } t_0 \notin S$$

$$I_1 = \{t_0\}$$

$$M_1(t_0, a) = \{ ((s,0),(s,1)) \mid (a,s) \in I \}$$

$$M_1((s,d),a) = \{ ((t,0),(t,1)) \mid (s,d,a,t) \in R \}$$

$$\mathcal{F}_1 = \{ F \subseteq S \times \{0,1\} \mid \mathrm{proj}_1(F) \in \mathcal{F} \}.$$

To see that $L(\mathcal{B}) = L(\mathfrak{A})$, suppose first that $v \in L(\mathfrak{A})$ and let ϕ be a winning strategy for I in $G_v^{\mathfrak{A}}$.

We define a successful run $\sigma : T \to S_1$ of $\mathcal{B}$ on v as follows:

First, $\sigma(\varepsilon) = t_0$, $\sigma(0) = (\phi(\varepsilon), 0)$, $\sigma(1) = (\phi(\varepsilon), 1)$ and let

$$s_0 d_1 s_1 d_2 s_2 \dots d_n s_n$$

has been played according to ϕ, let

$$\sigma(d_1 d_2 \ldots d_n 0) = (s_n, 0)$$
$$\sigma(d_1 d_2 \ldots d_n 1) = (s_n, 1).$$

To see that σ is successful, note that if $\alpha \in \{0,1\}^\omega$, $\alpha = d_1 d_2 d_3 \ldots$, and

$$s_0 d_1 s_1 d_2 s_2 \ldots$$

is played according to ϕ, then

$$\{s \in S \mid \exists^\infty n \ \ proj_1(\sigma(\alpha|_n)) = s\}$$

$$= \{s \in S \mid \exists^\infty n \ \ s_n = s\} \in F,$$

whence σ is successful.

Conversely, suppose $\sigma : T \to S_2$ is a successful run of $\mathcal{B}$ on $v \in V_\Sigma$. Then we can define a winning strategy ϕ for I in $G_v^{\mathcal{A}}$ as follows:

First note that $\sigma(\varepsilon) = t_0$ and

$$(\sigma(0), \sigma(1)) \in M_1(\sigma(\varepsilon), v(\varepsilon)) = M(t_0, v(\varepsilon))$$
$$= \{((s,0),(s,1)) \mid (v(\varepsilon), s) \in I\}$$

so $proj_1(\sigma(0)) = proj_1(\sigma(1))$.

Similarly, $proj_1(\sigma(x0)) = proj_1(\sigma(x1))$ for all $x \in \{0,1\}^+$.

For $d_1 d_2 \dots d_n \in T \setminus \{\varepsilon\}$:

$$\begin{tikzcd}[column sep=small]
& v(d_1 d_2 \dots d_n) = a \quad \sigma(d_1 d_2 \dots d_n) = (s, d_n) \arrow[dl, no head] \arrow[dr, no head] & \\
\sigma(d_1 d_2 \dots d_n 0) = (t, 0) & & \sigma(d_1 d_2 \dots d_n 1) = (t, 1)
\end{tikzcd}$$

where $(s, d_n, a, t) \in R$.

For ε :

$$\begin{tikzcd}[column sep=small]
& v(\varepsilon) = a \quad \sigma(\varepsilon) = t_0 \arrow[dl, no head] \arrow[dr, no head] & \\
\sigma(0) = (s, 0) & & \sigma(1) = (s, 1)
\end{tikzcd}$$

where $(a, s) \in \mathrm{I}$.

Assume $p = s_0 d_1 s_1 d_2 s_2 \dots d_n$ has been played, then we let

$$\phi(p) = \mathrm{proj}_1(\sigma(d_1 d_2 \dots d_n 0)) = \mathrm{proj}_1(\sigma(d_1 d_2 \dots d_n 1)).$$

Again, ϕ can be seen to be winning for I.

$\square$