Computing power series expansions of modular forms

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q-expansions

A classical modular form f of weight $k \in 2\mathbb{Z}_{\geq 0}$ for $\Gamma_0(N)$ satisfies the translation invariance $f(z+1) = f(z)$ for $z \in \mathcal{H}$, so f admits a Fourier expansion (or *q-expansion*)

$$f(z) = \sum_{n=0}^{\infty} a_n q^n$$

at the cusp ∞, where $q = e^{2\pi i z}$. If further f is a normalized eigenform for the Hecke operators T_n, then the coefficients a_n are the eigenvalues of T_n for n relatively prime to the level of f.

Modular forms, no cusps!

Let $\Gamma \leq \mathrm{PSL}_2(\mathbb{R})$ be a cocompact Fuchsian group. A *modular form* f of weight $k \in 2\mathbb{Z}_{\geq 0}$ for Γ is a holomorphic map $f : \mathcal{H} \to \mathbb{C}$ satisfying

$$f(gz) = j(g, z)^k f(z)$$

for all $g \in \Gamma$, where $j(g, z) = cz + d$ if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

As Γ is cocompact, the quotient $X = \Gamma \backslash \mathcal{H}$ has no cusps, so there are no q-expansions!

However, not all is lost: such a modular form f still admits a power series expansion in the neighborhood of a point $p \in \mathcal{H}$.

Power series expansions in unit disc

A q-expansion is really just a power series expansion at ∞ in the parameter q, convergent for $|q| < 1$. So it is natural to consider a neighborhood of p normalized so the expansion also converges in the unit disc $\mathcal{D}$ for a parameter w. So we map

$$\begin{aligned} w : \mathcal{H} &\to \mathcal{D} \\ z &\mapsto w(z) = \frac{z - p}{z - \overline{p}}. \end{aligned}$$

We then consider series expansions of the form

$$f(z) = (1 - w)^k \sum_{n=0}^{\infty} b_n w^n$$

where $w = w(z)$. The term

$$(1 - w(z))^k = \left(\frac{p - \overline{p}}{z - \overline{p}} \right)^k$$

is the automorphy factor arising by slashing by linear fractional transformation $w(z)$.

Shimura-Maass derivatives

Like Taylor coefficients, the coefficients b_n in the expansion

$$f(z) = (1-w)^k \sum_{n=0}^{\infty} b_n w^n$$

are given by derivatives.

However, the derivative of a modular form is no longer a modular form (unless $k = 0$)! Instead, we consider an operator which preserves modularity but destroys holomorphicity.

A function $f : \mathcal{H} \to \mathbb{C}$ is said to be *nearly holomorphic* if

$$f(z) = \sum_{d=0}^{m} \frac{f_d(z)}{(z-\overline{z})^d}$$

where each $f_d : \mathcal{H} \to \mathbb{C}$ is holomorphic. Let $M_k^*(\Gamma)$ be the space of nearly holomorphic modular forms of weight k for Γ.

Shimura-Maass derivatives

Define the *Shimura-Maass differential operator* by

$$\partial_k = \frac{1}{2\pi i}\left(\frac{d}{dz} + \frac{k}{z-\overline{z}}\right).$$

Then $\partial_k : M_k^*(\Gamma) \to M_{k+2}^*(\Gamma)$ preserves modularity. Abbreviate $\partial_k^n = \partial_{k+2(n-1)} \circ \cdots \circ \partial_{k+2} \circ \partial_k$.

Lemma

If $f : \mathcal{H} \to \mathbb{C}$ is holomorphic at $p \in \mathcal{H}$, then f has the expansion

$$f(z) = (1-w)^k \sum_{n=0}^{\infty} b_n w^n$$

with

$$b_n = \frac{(\partial^n f)(p)}{n!}(-4\pi y)^n$$

and $y = \mathrm{Im}(p)$.

Example

Let $f \in S_2(\Gamma_0(11))$ be defined by

$$f(z) = q \prod_{n=1}^{\infty} (1 - q^n)^2 (1 - q^{11n})^2 = q - 2q^2 - q^3 + 2q^4 + \ldots.$$

The point $p = (-9 + \sqrt{-7})/22 \in \mathcal{H}$ is a CM point on $X_0(11)$ for $K = \mathbb{Q}(\sqrt{-7})$. From the q-expansion, we obtain

$$\begin{aligned} f(z) &= (1-w)^2 \sum_{n=0}^{\infty} b_n w^n = f(p)(1-w)^2 \sum_{n=0}^{\infty} \frac{c_n}{n!} (\Theta w)^n \\ &= -\sqrt{3 + 4\sqrt{-7}}\,\Omega^2 (1-w)^2 \cdot \\ &\quad \cdot \left(1 + \Theta\omega + \frac{5}{2!}(\Theta w)^2 - \frac{123}{3!}(\Theta w)^3 - \frac{59}{4!}(\Theta w)^4 - \ldots\right) \end{aligned}$$

where

$$\Theta = -4\pi y \frac{(\partial_2 f)(p)}{f(p)} = \frac{-4 + 2\sqrt{-7}}{11} \pi \Omega^2$$

and $\Omega = 0.500491\ldots$ is the Chowla-Selberg period for K.

Arithmetic groups

Let F be a totally real number field with ring of integers $\mathbb{Z}_F$. Let B be a quaternion algebra over F with a unique split real place $\iota_\infty : B \hookrightarrow \mathrm{M}_2(\mathbb{R})$. Let $\mathcal{O} \subset B$ be a maximal order and let $\mathcal{O}_1^*$ denote the group of units of reduced norm 1 in $\mathcal{O}$. Then the group

$$\Gamma^B(1) = \iota_\infty(\mathcal{O}_1^*/\{\pm 1\}) \subset \mathrm{PSL}_2(\mathbb{R})$$

is a Fuchsian group with $X = \Gamma\backslash\mathcal{H}$ of finite area. A Fuchsian group Γ is *arithmetic* if it is commensurable with $\Gamma^B(1)$ for some choice of B. Let $\mathfrak{N}$ be an ideal of $\mathbb{Z}_F$. Define

$$\mathcal{O}(\mathfrak{N})_1^\times = \{\gamma \in \mathcal{O}_1^* : \gamma \equiv 1 \pmod{\mathfrak{N}\mathcal{O}}\}$$

and let $\Gamma^B(\mathfrak{N}) = \iota_\infty(\mathcal{O}(\mathfrak{N})_1^*)/[\pm 1]$. A Fuchsian group Γ is *congruence* if it contains $\Gamma^B(\mathfrak{N})$ for some $\mathfrak{N}$.

If F has narrow class number 1, the space $M_k(\Gamma)$ has an action of Hecke operators $T_\mathfrak{p}$ indexed by the prime ideals $\mathfrak{p} \nmid \mathfrak{D}\mathfrak{N}$.

Algebraicity

Let K be a totally imaginary quadratic extension of F that embeds in B, and let $\nu \in B$ be such that $F(\nu) \cong K$. Let $p \in \mathcal{H}$ be a fixed point of $\iota_\infty(\nu)$. Then we say p is a *CM point* for K.

Theorem (Shimura)

There exists $\Omega \in \mathbb{C}^\times$ such that for every CM point p for K, every congruence subgroup Γ commensurable with $\Gamma^B(1)$, and every $f \in M_k(\Gamma)$ with $f(p) \in \overline{\mathbb{Q}}$, we have for all $n \in \mathbb{Z}_{\geq 0}$ that

$$\frac{(\partial^n f)(p)}{\Omega^{2n}} \in \overline{\mathbb{Q}}.$$

Rodriguez-Villegas and Zagier link the coefficients b_n to square roots of central values of the Rankin-Selberg L-function $L(s, f \times \theta^n)$, where θ is associated to a Hecke character for K. Many authors have pursued this further, including O'Sullivan-Risager, Bertolini-Darmon-Prasanna, Mori, ...

Triangle groups

For $a, b, c \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$ with $1/a + 1/b + 1/c < 1$, define the (a, b, c)-triangle group to be the subgroup of orientation-preserving isometries in the group generated by reflections in the sides of a hyperbolic triangle with angles $\pi/a, \pi/b, \pi/c$. Power series expansions for a uniformizing function at the vertices of the fundamental triangle are obtained as the inverse of the ratio of ${}_2F_1$-hypergeometric functions.

This case was also of great classical interest, and has been taken up again more recently by Bayer, Bayer-Travesa, V, and Baba-Granath; this includes the well-studied case where the Fuchsian group arises from the quaternion algebra of discriminant 6 over $\mathbb{Q}$, corresponding to the $(2, 4, 6)$-triangle group.

Our result

We exhibit a general method for numerically computing power series expansions of modular forms for cocompact Fuchsian groups. Our method has generalizations to a wide variety of settings (noncongruence groups, real analytic modular forms, higher dimensional groups) and applies equally well for arithmetic Fuchsian groups over any totally real field F.

(There is another recent method, due to Nelson, which directly computes the Shimizu lift of a modular form on a Shimura curve over $\mathbb{Q}$ to a classical modular curve!)

Our method is inspired by the method of Stark and Hejhal, who used the same basic principle to compute Fourier expansions for Maass forms on $\mathrm{SL}_2(\mathbb{Z})$ and the Hecke triangle groups.

Basic idea

Let Γ be a cocompact Fuchsian group. Let $D \subset \mathcal{D}$ be a fundamental domain for Γ contained in a circle of radius $\rho > 0$. Let $f \in S_k(\Gamma)$. We consider an approximation

$$f(z) \approx f_N(z) = (1-w)^k \sum_{n=0}^{N} b_n w^n$$

valid for all $|w| \leq \rho$ to some precision $\epsilon > 0$.

For a point $w = w(z) \notin D$, there exists $g \in \Gamma$ such that $z' = gz \in D$; by the modularity of f we have

$$f_N(z') \approx f(z') = j(g,z)^k f(z)$$

$$(1-w')^k \sum_{n=0}^{N} b_n (w')^n \approx j(g,z)^k (1-w)^k \sum_{n=0}^{N} b_n w^n,$$

imposing a (nontrivial) linear relation on the unknowns b_n.

Better idea

Use the Cauchy integral formula:

$$b_n = \frac{1}{2\pi i} \oint \frac{f(z)}{w^{n+1}(1-w)^k}\, dw.$$

We take the contour to be a circle of radius ρ, apply automorphy, and again obtain linear relations among the coefficients b_n.

Let $w_m = \rho e^{2\pi m i/Q}$ and $z'_m = g_m z_m$ with $z'_m \in D$, we obtain

$$b_n \approx \frac{1}{Q} \sum_{m=1}^{Q} \frac{j(g_m, z_m)^{-k} f_N(z'_m)}{w_m^n (1-w_m)^k}$$

and expanding $f_N(z)$ we obtain a relation of the form

$$b_n \approx \sum_{r=0}^{N} K^{\mathrm{c}}_{nr} b_r.$$

The matrix K^{c} with entries K^{c}_{nr} can be obtained by a matrix multiplication of size $(N+1) \times Q$ by $Q \times (N+1)$. The column vector b satisfies $(K^{\mathrm{c}} - 1)b \approx 0$.

Computing a fundamental domain

For a point $p \in \mathcal{H}$, we denote by $\Gamma_p = \{g \in \Gamma : g(p) = p\}$ the stabilizer of p in Γ.

Theorem (V)

There exists an algorithm that, given as input a cocompact Fuchsian group Γ and a point $p \in \mathcal{H}$ with $\Gamma_p = \{1\}$, computes as output a fundamental domain $D(p) \subset \mathcal{H}$ for Γ and an algorithm that, given $z \in \mathcal{H}$ returns a point $z' \in D(p)$ and $g \in \Gamma$ such that $z' = gz$.

The fundamental domain $D(p)$ is the *Dirichlet domain*

$$D(p) = \{z \in \mathcal{H} : d(z, p) \leq d(gz, p) \text{ for all } g \in \Gamma\}$$

where d is the hyperbolic distance. The set $D(p)$ is a closed, connected, and hyperbolically convex domain whose boundary consists of finitely many geodesic segments.

Computing the numerical kernel

Having assembled our linear relations into an $M \times (N+1)$ matrix A with $Ab \approx 0$, we now seek to compute the numerical kernel of A. We compute the singular value decomposition (SVD) of the matrix A, writing

$$A = USV^*$$

where U and V are $M \times M$ and $(N+1) \times (N+1)$ unitary matrices and S is diagonal. The diagonal entries of the matrix S are the *singular values* of A, and singular values that are approximately zero correspond to column vectors of V that are in the numerical kernel of A.

Confirming the output

Although we cannot prove that our results are correct, there are several tests that allow one to be quite convinced that they are correct. (See also Booker-Strömbergsson-Venkatesh.)

First we simply decrease the error ϵ and see if the coefficients b_n converge. The second is to look at the singular values to see that the approximately nonzero eigenvalues are sufficiently large.

More seriously, we can also verify that f is modular at point $w \notin D$ with $|w| \leq \rho$. This shows that the computed expansion transforms like a modular form of weight k for Γ.

Finally, when f is an eigenform for a congruence group Γ, we can check that f is indeed numerically an eigenform (with the right eigenvalues) and that the normalized coefficients appear to be algebraic using the LLL-algorithm.

Example

Let $F = \mathbb{Q}(a) = \mathbb{Q}(\sqrt{5})$ where $a^2 + a - 1 = 0$, and let $\mathbb{Z}_F$ be its ring of integers. Let $\mathfrak{p} = (5a+2)$, so $N\mathfrak{p} = 31$. Let B be the quaternion algebra ramified at $\mathfrak{p}$ and the real place sending $\sqrt{5}$ to its positive real root: we take $B = \left(\dfrac{a, 5a+2}{F}\right)$. We consider $F \hookrightarrow \mathbb{R}$ embedded by the second real place, so $a = (1-\sqrt{5})/2 = -0.618033\ldots$.

A maximal order $\mathcal{O} \subset B$ is given by

$$\mathcal{O} = \mathbb{Z}_F \oplus \mathbb{Z}_F\alpha \oplus \mathbb{Z}_F\frac{a + a\alpha - \beta}{2} \oplus \mathbb{Z}_F\frac{(a-1) + a\alpha - \alpha\beta}{2}.$$

Let ι_∞ be the splitting at the second real place given by

$$\begin{aligned} \iota_\infty &: B \hookrightarrow \mathrm{M}_2(\mathbb{R}) \\ \alpha, \beta &\mapsto \begin{pmatrix} 0 & \sqrt{a} \\ \sqrt{a} & 0 \end{pmatrix}, \begin{pmatrix} \sqrt{5a+2} & 0 \\ 0 & -\sqrt{5a+2} \end{pmatrix} \end{aligned}$$

Example

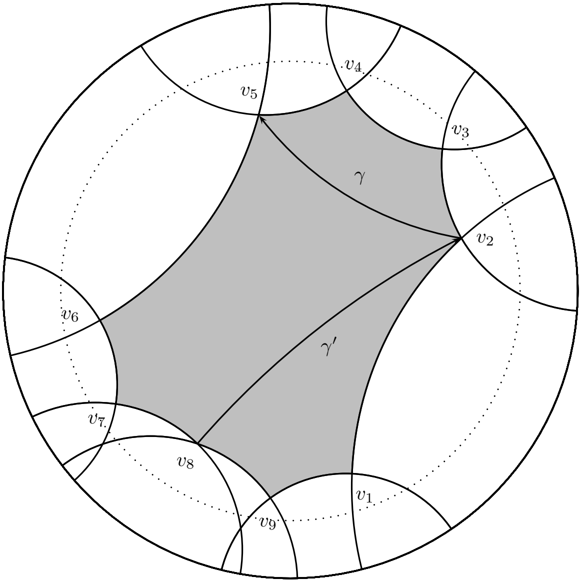
Let $\Gamma = \iota_\infty(\mathcal{O}_1^\times)/\{\pm 1\} \subseteq \mathrm{PSL}_2(\mathbb{R})$. Then Γ has signature $(1; 2^2)$, so $X = \Gamma \backslash \mathcal{H}$ can be given the structure of a compact Riemann surface of genus 1. Consequently, the space $S_2(\Gamma)$ of modular forms on Γ of weight 2 is 1-dimensional, and it is this space that we will compute.

The field $K = F(\sqrt{-7})$ embeds in $\mathcal{O}$ with

$$\mu = -\frac{1}{2} - \frac{5a+10}{2}\alpha - \frac{a+2}{2}\beta + \frac{3a-5}{2}\alpha\beta \in \mathcal{O}$$

satisfying $\mu^2 + \mu + 2 = 0$ and $\mathbb{Z}_F[\mu] = \mathbb{Z}_K$ the maximal order with class number 1. We take $p = -3.1653\ldots + 1.41783\ldots \in \mathcal{H}$ to be the fixed point of μ, a CM point of discriminant -7.

Fundamental domain



Finding the numerical kernel

We have $\rho = 0.71807\ldots$ so for $\epsilon = 10^{-20}$ we take $N = 150$.

We use the relations coming from the Cauchy integral formula.

The $(N+1) \times (N+1)$-matrix has largest singular value $4.01413\ldots$ and one singular value which is $< \epsilon$. The next largest singular value is $0.499377\ldots$, showing that the numerical kernel is gratifyingly one-dimensional.

An expansion for the form

$$
\begin{aligned}
f(z) = (1-w)^2 \Bigg(1 + (\Theta w) - \frac{70a+114}{2!}(\Theta w)^2 \\
- \frac{8064a+13038}{3!}(\Theta w)^3 + \frac{174888a+282972}{4!}(\Theta w)^4 \\
- \frac{13266960a+21466440}{5!}(\Theta w)^5 \\
- \frac{1826784288a+2955799224}{6!}(\Theta w)^6 \\
- \frac{2388004416a+3863871648}{7!}(\Theta w)^7 + \dots \Bigg)
\end{aligned}
$$

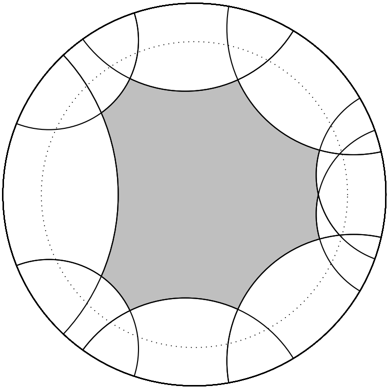
where

$$\Theta = 0.04621857952920849918\ldots - 0.07598731753183256835 1\ldots i$$

is a period related to the CM abelian variety given by the point p.

The conjugate curve

We further compute the other embedding of this form by repeating the above with an algebra ramified at $\mathfrak{p}$ and the other real place.



The coefficients agree with the conjugates under the nontrivial element of $\mathrm{Gal}(\mathbb{Q}(\sqrt{5})/\mathbb{Q})$.

Finding an equation

We can identify the equation of the Jacobian J of the curve X by computing the associated periods. We first identify the group Γ using the sidepairing relations coming from the computation of $D(p)$:

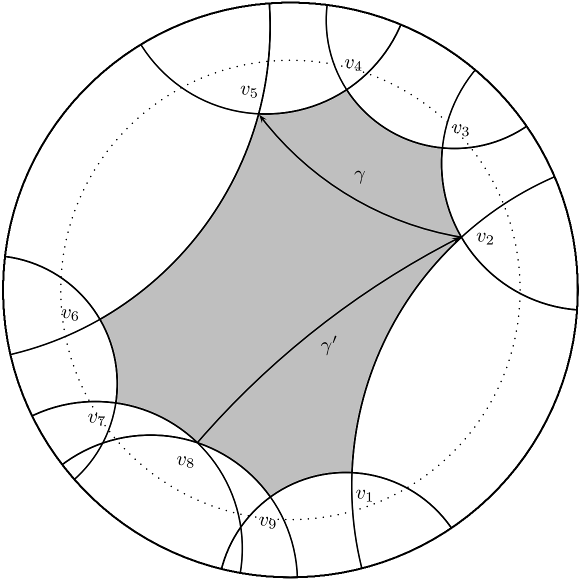
$$\Gamma \cong \langle \gamma, \gamma', \delta_1, \delta_2 \mid \delta_1^2 = \delta_2^2 = \gamma^{-1}\gamma'^{-1}\delta_1\gamma\gamma'\delta_2 = 1 \rangle$$

where

$$\begin{aligned} \gamma &= \frac{a+2}{2} - \frac{2a+3}{2}\alpha + \frac{a+1}{2}\alpha\beta \\ \gamma' &= \frac{2a+3}{2} + \frac{7a+10}{2}\alpha + \frac{a+2}{2}\beta - (3a+5)\alpha\beta \end{aligned}$$

generate the free part of the maximal abelian quotient of Γ.

Fundamental domain again



Finding an equation

Therefore, we compute two independent periods ω_1, ω_2

$$\omega_1 = \int_{v_2}^{v_5} f(z) \frac{dw}{(1-w)^2} \approx \left(\sum_{n=0}^{N} \frac{b_n}{n+1} w^{n+1} \right) \Bigg|_{v_2}^{v_5}$$

$$= -0.654017\ldots + 0.397799\ldots i$$

$$\omega_2 = \int_{v_8}^{v_2} f(z) \frac{dw}{(1-w)^2} = 0.952307\ldots + 0.829145\ldots i.$$

We then compute the j-invariant

$$j(\omega_1/\omega_2) = -18733.423\ldots$$

$$= -\frac{11889611722383394a + 8629385062119691}{31^8}.$$

We identify the elliptic curve J as

$$y^2 + xy - ay = x^3 - (a-1)x^2 - (31a+75)x - (141a+303).$$

Heegner point

Finally, we compute the image on J of a degree zero divisor on X.

The fixed points w_1, w_2 of the two elliptic generators δ_1 and δ_2 are CM points of discriminant -4. Let $K = F(i)$ and consider the image of $[w_1] - [w_2]$ on J given by the Abel-Jacobi map as

$$\int_{w_1}^{w_2} f(z) \frac{dw}{(1-w)^2} \equiv -0.177051\ldots - 0.291088\ldots i \pmod{\Lambda}$$

where $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ is the period lattice of J. Evaluating the elliptic exponential, we find the point

$$(-10.503797\ldots, 5.560915\ldots - 44.133005\ldots i) \in J(\mathbb{C})$$

which matches to the precision computed $\epsilon = 10^{-20}$ the point

$$Y = \left(\frac{-81a - 118}{16}, \frac{(358a + 1191)i + (194a + 236)}{64} \right) \in J(K).$$

We have $J(K) \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}$ and Y generates the free quotient.

Additional considerations

We compute the whole space at once in the kernel, but we find better numerical results when we cut down to a one-dimensional space using the action of Hecke operators. (These relations are again linear.) Can also turn this around!

The coefficients b_n for $n > N$ are approximately determined by the coefficients b_n for $n \leq N$. In this way, they can be computed using integration and without any further linear algebra step.

This expression of the coefficients b_n in terms of derivatives implies that they can also be given as (essentially) the constant terms of a sequence of polynomials satisfying a recurrence relation (Rodriguez-Villegas, Zagier).

Conclusion

The coefficients of a power series expansion of a modular form f encode interesting information about f that is of independent interest.

We have exhibited a general method for numerically computing power series expansions of modular forms for cocompact Fuchsian groups with good results in practice.

The potential for this algorithm to transport familiar algorithms for modular curves to the more general setting of Shimura curves (even quaternionic Shimura varieties) is promising.

Better methods in numerical linear algebra are needed.