Some number theory associated to automorphic forms on SL_2

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Other number fields: assume something more, for instance that $\pi = \otimes \pi_v$ is such that some π_v is either special or supercuspidal at some finite place, or a discrete series at some real place

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(Original motivation for the question: find a quadratic twist that is nonzero modulo p for some given p.)

This non-vanishing result was proven by Bump and Murty-Murty for the classical case, and by Waldspurger for general automorphic forms over number fields. Note that this result is false for general π and F – no such χ need exist, unless we make some additional hypothesis.

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One uses the approximate functional equation to obtain a rapidly convergent expression for (a variant of) this sum, and then one computes the limit as N grows; it turns out to be given by the value of the symmetric square of π at $s = 1$, which is nonzero.

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In the classical case: he showed that such a χ exists if and only if there exists a half integer weight form which corresponds to the given newform f. He then uses results about the trace formula to show that there exists a matching automorphic representation on the metaplectic side to deduce that such a χ exists.

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Note that there's no easy way to actually produce such a χ, even if we know such a thing exists. All the standard proofs are not constructive.

Another way to look at this result.

Consider the collection of the representations $\pi \otimes \chi$ for fixed π and varying χ. Observe that these all restrict to the same representation of SL_2, since χ factors through det. Furthermore, as automorphic representations, for automorphic forms which realize each $\pi \otimes \chi$ restrict to precisely the same space of functions on $SL_2(\mathbf{A})$. They are all the same thing on SL_2.

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Main point: maybe the non vanishing of some twist at the centre is equivalent to the unconditional non-vanishing of some single invariant of automorphic forms on SL_2.

Period integrals: Let $\tilde{\pi}$ be an automorphic cuspidal representation of SL_2 which occurs as a constituent of the restriction of π. Let T denote the maximal torus of PGL_2 given by diagonal elements, and let $\tilde{T}$ denote the diagonal torus in SL_2.

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Theorem: The functional ℓ is nonzero on π if and only there exists some quadratic character χ such that $L(\pi \otimes \chi, 1/2)$ is nonzero.

Can we actually construct a particular χ so that $L(\pi \otimes \chi, 1/2) \neq 0$? Equivalently, can we find specific ϕ such that $\ell(\phi) \neq 0$?

Something funny happens:

Take your favourite vector $\phi = \otimes \phi_v$. Then ϕ_v is spherical for almost all v. But maybe you're unlucky and $\ell(\phi) = 0$.

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This is quite weird: somehow more or less straightforward local modifications of the test vector are controlling vanishing (or not) of a global invariant.

Digression: Mazur-Rubin gave a different kind of proof for the existence of non vanishing twists, in the context of Selmer groups. They show that for an elliptic curve E over $\mathbf{Q}$, there exists a quadratic character χ such that the 2-Selmer group of $E \otimes_\chi$ is trivial.

Idea: $E[2]$ is the same for E and any twist, and the Selmer groups differ only by local conditions at finitely many places. Thus local changes compute the change in the global invariant (can make it go up or down by changing local conditions).

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But we still can't prove anything: unlike the Selmer group, the functional ℓ is not factorizable: $\ell \neq \otimes \ell_v$, with $\tilde{T}_v$ invariant functionals ℓ_v.

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Question: what is the analog of the 2-Selmer group on the analytic side? Can we replace the period integral with a factorizable function with the same order of vanishing?

This leads us to the general study of factorizable functionals of the form $\prod \ell_v$ where each v is a $\tilde{T}_v$ invariant functional on $\tilde{\pi}_v$.

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Such functionals can be constructed directly, they are related to the symmetric square L function which we saw earlier.

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The period integral is the linear functional $\ell : \tilde{\pi} \to \mathbf{C}$ defined by

$$\phi \mapsto \int_{\tilde{T}(F)\backslash \tilde{T}(\mathbf{A})} \phi(t) dt.$$

More generally, can consider

$$\ell(\phi, s) = \int_{\tilde{T}(F)\backslash \tilde{T}(\mathbf{A})} \phi(t)|t|^s dt.$$

This is convergent for all complex s.

The functional $\ell(\cdot, s)$ is a linear functional on π which transforms under the action of $\tilde{T}$ according to the character $t \mapsto |t|^s$. At $s = 0$, it is a $\tilde{T}$-invariant functional.

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However it does not factor as $\ell(\cdot, s) = \prod \ell_v(\cdot, s)$.

The point here is that $\ell(\cdot, s)$ depends crucially on the embedding of $\tilde{\pi}$ in to the space of automorphic forms. There's no apparent way to characterize the period integral in the space of all $\tilde{T}$-invariant linear functionals.

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The converse theorem for $\pi = \otimes \pi_v$ on GL_2 says that if we know the functional

$$\phi \mapsto \int_{T(F)\backslash T(\mathbf{A})} \phi(t)|t|^s dt.$$

for every s and ϕ, and we know the appropriate functional equation, then we can actually recover the representation π, and furthermore, that we can actually produce an embedding of π in to the space of automorphic forms.

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Implicit in all of this is the multiplicity one theorem for GL_2 – that there's only one way to embed π in to the space of automorphic

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This seems to be a deep fact (Ramakrishnan). The main point is that the period integral involves the Fourier coefficients a_p for all p, but Hecke theory for SL_2 only tells us a_{p^2}. There's an ambiguity of sign.

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We can define a linear functional on π_v as
$\ell_v(\psi_v, s) : \int_{\tilde{T}_v} W_\phi(t)|t|^s dt$.

This converges for large s and can be continued as a meromorphic function for all s.

Here W_ϕ is the vector in the (unique) ψ_v-Whittaker model for π_v which corresponds to ϕ. One can evaluate such functionals explicitly on spherical vectors, using the formula for such vectors. The result is closely tied to the value of the symmetric square L-function for π (the rep on GL_2).

There is a similar construction for GL_2, where we take the integral over the torus T_v instead; this gives the local factor of the standard GL_2 L-function.

In terms of the representation $\tilde{\pi}$, the value of this integral is the local factor at v for the degree 3 L-function L_3 associated to the representation $PGL_2 \to GL_3$ given by the conjugation action on trace zero matrices (when everything is unramified)

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Now take a global character $\psi = \prod \psi_v$. The factorizable functional $\prod_v \ell_v(\psi_v, s)$ is the L-function $L_3(\tilde{\pi}, s)$, up to some finite number of factors.

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Again, we can do the same thing for GL_2.

Now, in the GL_2 case, the analytic properties of the product of the local integrals imply automorphy of the representation π.

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Let $\tilde{\pi} = \otimes \pi_v$ be a representation of $SL_2(\mathbf{A})$ such that almost all π_v are unramified and unitary. Let X be the set of global characters of $F(\mathbf{A})^\times / F^\times$. If $\chi \in X$, then let let $L_3(\tilde{\pi}, \chi, s)$ denote the χ-twisted L-function of $\tilde{\pi}$.

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Here twisted L-function just means the naive thing: replace ℓ_v

by

$$\ell_v(\psi_v, \chi_v, s) : \int_{\tilde{T}_v} W_\phi(t) \chi_v(t) |t|^s dt.$$

in the unramified case.

A more intrinsic definition: at the unramified places, we form a local factor by taking the 3 dimensional representation of $PGL_2(\mathbf{C})$ in to $GL_3(\mathbf{C})$. Recall here that PGL_2 is the dual group of SL_2. One can check by calculation that this recovers the integral of the Whittaker function in the unramified case.

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At the remaining primes, the local representation $\tilde{\pi}_v$ determines a representation π_v of GL_2, where $tilde\pi$ is the (finite) sum of conjugated representations $\tilde{\pi}^g$, where g runs through $PGL_2(F_v)/SL_2(F_v) \cdot F_v^\times$.

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Then we can define the Euler factor at v by following Gelbart-Jacquet: take $\frac{L_v(\pi \times \check{\pi}, s)}{\zeta_v(s)}$, and similarly for twists.

Theorem (converse theorem for SL_2): Suppose that the L-functions $L(\pi, \chi, s)$ have analytic continuation to all s, for every $\chi \in X$. Suppose further that we have the functional equations $L_3(\pi, \chi, s) = \epsilon(\pi, \chi, s) L_3(\tilde{\pi}, \chi^{-1}, 1 - s)$ where $\epsilon(\pi, \chi, s)$ is a certain ϵ-factor. Then π is automorphic.

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One important difference with the GL_2 case is that we can't actually produce an embedding in to the space of automorphic forms from the given L-functions. The proof here is indirect, and entirely un-constructive. to get a an embedding one would somehow have to re-prove multiplicity one, so it's probably difficult.

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Again the definition of the epsilon factors used here is somewhat ad-hoc. It would be interesting to define these factors canonically.

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Step 5: Conclude that $\tilde{\pi}$ itself is also automorphic

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To apply the converse theorem we need highly ramified characters χ. Can we get by with just quadratic characters? I don't know the answer.

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To do this, we have to characterize the image of the representations of GL_2. This can be done by using a theorem of Ginzburg and others; the point is that the symmetric square of Π has a pole, and such representations can be descended back to GL_2

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This is pretty easy. The point is that by construction the symmetric square L-function of π is the degree 3 L-function of $\tilde{\pi}$, so each pi_v determines the same local L-packet as $\tilde{\pi}_v$.

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It remains only to show that $\tilde{\pi}$ itself is automorphic.

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L-L show that if one representation in a global packet is automorphic, then so are all the others, unless the packets consists of representations of CM type.

This case is excluded by our assumption that all twists of the degree 3 L-function be holomorphic, since it's well known that for dihedral representations, the symmetric square is reducible and so some twist will have a pole.