NOTES ON DETERMINACY

SHERWOOD HACHTMAN AND JUSTIN PALUMBO

Fix a set $A \subseteq \mathbb{R}$. Consider a game G_A where two players, call them Player I and Player II, collaborate to produce the decimal expansion of a real number a as follows: Player I picks an integer a_0, the integer part of a. Then Player II picks an integer a_1 between 0 and 9, Player I picks a_2 between 0 and 9, Player II picks a_3 between 0 and 9, and so on. If the players survive through infinitely many rounds, they have produced a real number $a = a_0.a_1a_2a_3 \cdots = \Sigma_{n=0}^{\infty} a_n/10^n$, and the game is over.

The winner of G_A is decided as follows: If $a \in A$, then Player I wins. If $a \notin A$, Player II wins.

$$\begin{array}{c|ccccccccccccc} \mathrm{I} & 3 & & 4 & & 5 & & 2 & & 5 & & 6 & \ldots \\ \hline \mathrm{II} & & 1 & & 1 & & 9 & & 6 & & 3 & & \ldots \end{array}$$

FIGURE 1. A sample play of the game G_A.

We draw attention to a few features of this game. First, there are infinitely many moves. Second, there are two players who move sequentially, and have complete control over the moves they make (that is, there is no element of chance). Third, each player has perfect knowledge of the set A, as well as the history of the game at each turn. We call such a game an **infinite two-person game of perfect information.**

Notice that some of your favorite (finite) games can be represented as games of the form G_A for the correct choice of $A \subseteq \mathbb{R}$. It's easy to see how to code tic-tac-toe as such a game. And you should convince yourself that more complicated finite games, such as chess, can be similarly encoded.

However, the class of infinite games we have defined is substantially more general. As an example, consider the case that A is countably infinite. In the course of play, Player II can enforce a Cantor-style diagonalization, by playing a_{2n+1} so that a disagrees with the nth element of A. Since this always ensures $a \notin A$, this describes a winning strategy for Player II. Similary, if $\mathbb{R} \setminus A$ is countable, then Player I can always win G_A. A few questions are immediate:

- For which sets $A \subseteq \mathbb{R}$ is there a strategy for one of the players to always win G_A?
- How hard is it to show such strategies exist? Or to describe them?
- What does the existence of a winning strategy in G_A tell us about A?
- What use could such games possibly have?

These are among the questions that we consider in these notes. As we will see, the answers are surprisingly deep, with implications for the theory of definability, for the structure theory of sets of real numbers, and for the foundations of mathematics.

§1. Games, Determinacy, and the Gale-Stewart Theorem. We begin by introducing some notions that will allow us to define games in a bit greater generality.

Let X, Y be sets. X^Y denotes the set of functions $f : Y \to X$. For $n \in \omega$, we regard a function $s : n \to X$ as a finite sequence $\langle s_0, \dots, s_{n-1}\rangle$, and write X^n for the set of all such sequences. (Note then $1^0 = \{\varnothing\} = 1$!) Similarly, $f : \omega \to X$ is regarded as an infinite sequence, and X^ω is the collection of such. We denote the collection $\bigcup_{n\in\omega} X^n$ of all finite sequences in X by $X^{<\omega}$.

So for example, if $X = \omega$, then $X^{<\omega}$ consists of objects like $\langle 8, 6, 7, 5, 3, 0, 9\rangle$, or $\langle 0, 0, 0, 0\rangle$. The empty sequence $\varnothing$ is also a sequence, and always belongs to $X^{<\omega}$, regardless of X (even if $X = \varnothing$!).

For $s \in X^{<\omega}$, we write $\ell(s)$ for the length of s, so e.g. $\ell(\langle 8, 6, 7, 5, 3, 0, 9\rangle) = 7$. We write $s \subseteq t$ to mean that s is an initial segment of t, that is, $s(i) = t(i)$ for all $i < \ell(s)$ (so $\langle 8, 6, 7\rangle \subseteq \langle 8, 6, 7, 5, 3, 0, 9\rangle$, but $\langle 7, 5, 3\rangle \nsubseteq \langle 8, 6, 7, 5, 3, 0, 9\rangle$). $s^\frown t$ denotes the concatenation of s and t: So $\langle 8, 6, 7\rangle^\frown\langle 5, 3, 0, 9\rangle = \langle 8, 6, 7, 5, 3, 0, 9\rangle$.

These definitions extend naturally to infinite sequences. For example, $s \subseteq f$ for $f \in X^\omega$ if s is equal to the restriction $f \restriction \ell(s) = \langle f(0), f(1), \dots, f(\ell(s) - 1)\rangle$.

Definition 1.1. Let X be a set. A **tree on** X is a non-empty set $T \subseteq X^{<\omega}$ so that if $s \subseteq t$ and $t \in T$, then $s \in T$.

So a tree is just a set $T \subseteq X^{<\omega}$ closed under taking initial segments. For example, $X^{<\omega}$ itself is a tree. As another example, consider the tree T_0 on ω consisting of all sequences without two consecutive 1's, or the tree T_1 consisting of decreasing sequences in ω. By definition, $\varnothing \in T$ for all trees T; we define trees to be non-empty to avoid annoying trivialities.

We call the elements $s \in T$ **nodes** (or sometimes **positions**) in T. A node s of T is **terminal** if s has no immediate successor in T, that is, $s \in T$ but $s^\frown\langle x\rangle \notin T$ for all $x \in X$. For example, $s \in T_1$ is terminal if and only if $s_{\ell(s)-1} = 0$. For $s \in T$, we let T_s denote the **subtree of** T **with stem** s,

$$T_s = \{t \in T \mid s \subseteq t \text{ or } t \subseteq s\}.$$

Note that if $s \subseteq t$ then $T_t \subseteq T_s$; and for all $s, t \in T$, $T_s \cup T_t = T_{s\cap t}$.

Definition 1.2. Let T be a tree on X. $f \in X^\omega$ is a(n infinite) **branch** through T if $f \restriction n \in T$ for all $n \in \omega$. The **body of** T, denoted $[T]$, is the set of infinite branches through T.

So $f = \langle 0, 1, 0, 1, 0, 1, \dots\rangle$ is a branch through T_0, but T_1 has no infinite branch.

Definition 1.3. Let T be a tree on a set X, and let $A \subseteq [T]$. The **game on** T **with payoff** A, denoted $G(A; T)$, is played as follows: two players, Player I and Player II, alternate choosing elements of X,

I	x_0		x_2	$\dots$	x_{2n}		$\dots$
II		x_1		$\dots$		x_{2n+1}	$\dots$

so that for all n, $\langle x_0, \dots, x_{n-1}\rangle$ is an element of T. The game ends if either a terminal node of T is reached, or if an infinite branch $\langle x_0, x_1, \dots\rangle \in [T]$ is produced. A sequence s is a **play in** T if s is terminal in T or $s \in [T]$ is an infinite branch. Player I **wins the play** s if either

- $s \in T$ is a terminal node, and $\ell(s)$ is odd;

- $s \in [T]$ is an infinite branch, and $s \in A$.

Otherwise, Player II wins the play s.

When $T = \omega^{<\omega}$, we write $G(A)$ for the game $G(A;T) = G(A;\omega^{<\omega})$.

In the game $G(A;T)$, Player I is trying to produce a branch f through T with $f \in A$; Player II is trying to ensure $f \notin A$. If a terminal node is reached, then the last player who made a move is the winner.

Intuitively, a strategy for Player I in the game $G(A;T)$ should be a function that takes positions s of even length as input, and tells Player I what move to make next. There are a number of equivalent ways to formalize this. We elect to regard strategies as *trees.*

Definition 1.4. Let T be a tree. A **strategy for Player I in** T is a set $\sigma \subseteq T$ so that

1. σ is a tree.
2. If $s \in \sigma$ is a position of odd length and $x \in X$ is such that $s^\frown\langle x\rangle \in T$, then $s^\frown\langle x\rangle \in \sigma$.
3. If $s \in \sigma$ is a position of even length, then there is a unique $x \in X$ so that $s^\frown\langle x\rangle \in \sigma$.

We say an infinite play f is **compatible with** σ if $f \in [\sigma]$; a strategy σ is **winning for Player I in** $G(A;T)$ if $[\sigma] \subseteq A$ (that is, every play compatible with σ is winning for Player I).

Strategies τ for Player II are defined similarly (exchanging "even" with "odd"); τ is winning for Player II in $G(A;T)$ if $[\tau] \cap A = \varnothing$.

So a strategy for Player I is a subtree σ of T that picks out moves for Player I, but puts no restrictions on moves for Player II. We will often abuse notation and regard σ as a function, writing $\sigma(s) = x$ for the unique element guaranteed by (3).

Note that (3) implies that no finite play in a strategy is won by the opponent. It is then not obvious at this stage that given a tree T, a strategy in T (winning or not) exists for *either* player!

Definition 1.5. Let T be a tree on X with $A \subseteq [T]$. If one of the players has a winning strategy in $G(A;T)$, then we say the game is **determined**.

When $T = \omega^{<\omega}$, we often say simply that $A \subseteq \omega^\omega$ **is determined**.

Let AD_X denote the statement that for every set $A \subseteq X^{<\omega}$, the game $G(A;X^{<\omega})$ is determined. The **Axiom of Determinacy**, denoted AD, is AD_ω: *Every set $A \subseteq \omega^\omega$ is determined.*

Note that every strategy in $\omega^{<\omega}$ is a subset of $\omega^{<\omega}$, so that the collection of strategies in $\omega^{<\omega}$ has size at most $\mathfrak{c}$, where $\mathfrak{c} = |\mathbb{R}|$ is the cardinality of the continuum; furthermore, for each strategy σ in $\omega^{<\omega}$, the set $[\sigma]$ of plays compatible with σ has size $\mathfrak{c}$.

Our first observation is that not all sets are determined.

Theorem 1.6. *Assume $\mathcal{P}(\omega)$ can be well-ordered. Then there is a set $B \subseteq \omega^\omega$ so that $G(B)$ is not determined; in particular, if the Axiom of Choice holds, then AD fails.*

Proof. By hypothesis, there exist enumerations $\langle\sigma_\alpha\rangle_{\alpha<\mathfrak{c}}$, $\langle\tau_\alpha\rangle_{\alpha<\mathfrak{c}}$ of all strategies for Player I and Player II, respectively, in $\omega^{<\omega}$.

We define disjoint sequences $\langle a_\alpha\rangle_{\alpha<\mathfrak{c}}, \langle b_\alpha\rangle_{\alpha<\mathfrak{c}}$ by transfinite recursion. Suppose $\alpha < \mathfrak{c}$ is such that for all $\xi < \alpha$, a_ξ, b_ξ are defined. Since there are $\mathfrak{c}$-many plays compatible with σ_α, we have that $[\sigma_\alpha] \setminus \{b_\xi \mid \xi < \alpha\}$ is non-empty. Therefore let $a_\alpha \in \omega^\omega$ be a play compatible with σ_α so that $a_\alpha \neq b_\xi$ for all $\xi < \alpha$. Similarly, let b_α be compatible with τ_α so that $b_\alpha \neq a_\xi$ for all $\xi \leq \alpha$.

Let $A = \{a_\alpha \mid \alpha < \mathfrak{c}\}$ and $B = \{b_\alpha \mid \alpha < \mathfrak{c}\}$. Note that $A \cap B = \varnothing$. We claim $G(B)$ is not determined. For suppose towards a contradiction that σ is a winning strategy for Player I. Then $\sigma = \sigma_\alpha$ for some $\alpha < \mathfrak{c}$, and we have $a_\alpha \in A \cap [\sigma_\alpha]$, by definition. So the play a_α is compatible with σ, but not in B, contradicting the assumption that σ was winning for Player I.

Similarly, if τ is a strategy for Player II, then $\tau = \tau_\alpha$ for some $\alpha < \mathfrak{c}$. We have by definition that $b_\alpha \in B$ is compatible with τ, but is not won by Player II. Then τ is not winning for Player II in $G(B)$.

We have that neither player has a winning strategy in $G(B)$. $\dashv$

Theorem 1.7 (Gale-Stewart). *(Using the Axiom of Choice.) Let T be a tree. Then $G([T];T)$ is determined.*

For reasons that will become clear later, this theorem is often called **closed determinacy**. To help us prove this theorem, we introduce a more general notion of strategy.

Definition 1.8. Let T be a tree. A **quasistrategy for Player I in** T is a tree $S \subseteq T$ satisfying (1) and (2) in Definition 1.4, but instead of (3), satisfying

3′. If $s \in S$ has odd length, then there is some $x \in X$ so that $s^\frown\langle x\rangle \in S$.

A quasistrategy can be thought of as a "multi-valued strategy". Quasistrategies are typically obtained from the following lemma:

Lemma 1.9. *(Using the Axiom of Choice.) Let T be a tree on X, $A \subseteq [T]$, and suppose Player II does not have a winning strategy in $G(A;T)$. Define*

$$S = \{s \in T \mid (\forall i \leq \ell(s)) \text{ Player II has no winning strategy in } G(A; T_{s\restriction i})\}.$$

Then S is a quasistrategy for Player I in T, the ***non-losing quasistrategy for I in*** *$G(A;T)$.*

Proof. Clause (1) of the definition of quasistrategy is immediate; closure under initial segment follows from the definition of S, and the assumption that Player II has no winning strategy ensures S is non-empty, a requirement for S to be a tree.

For clause (2), suppose $s \in S$ has odd length. If $s^\frown\langle x\rangle \in T \setminus S$ for some $x \in X$, then by definition of S, there is some strategy $\tau \subseteq T_{s^\frown\langle x\rangle}$ that is winning for Player II in the game $G(A; T_{s^\frown\langle x\rangle})$. But clearly $\tau \subseteq T_s$ is also winning for Player II in $G(A;T_s)$, contradicting our assumption that $s \in S$.

The key part of the proof is clause (3′). So suppose $s \in S$ has even length. We claim $s^\frown\langle x\rangle \in S$ for some $x \in X$. Suppose instead for a contradiction, that for each $x \in X$, there is some strategy τ_x in $T_{s^\frown\langle x\rangle}$ that is winning for Player II in $G(A; T_{s^\frown\langle x\rangle})$. Define a strategy $\tau \subseteq T_s$ for Player II by setting

$$t \in \tau \iff t \subseteq s \text{ or } (\exists x)(s^\frown\langle x\rangle \subseteq t \text{ and } t \in \tau_x).$$

Note τ does not restrict Player I's move at s, so τ is a strategy for Player II. If f is a play compatible with τ, then we have $s^\frown\langle x\rangle \subseteq f$ for some $x \in X$, so that f is compatible with τ_x. It follows that f is a win for II in $G(A;T_s)$, and τ is a winning strategy for Player II in $G(A;T_s)$, contradicting our assumption that $s \in S$. $\dashv$

Proof of Theorem 1.7. Let T be a tree, and suppose Player II does not have a winning strategy in $G([T];T)$. By Lemma 1.9, we obtain a quasistrategy S for Player II. This can be refined to a strategy $\sigma \subseteq S$ for I, by choosing a single successor node at each $s \in S$ of even length. It is clear that σ is winning for Player I. $\dashv$

The uses of the Axiom of Choice in this theorem can be weakened somewhat; in particular, it is sufficient to assume X can be well-ordered, so that the determinacy of $G([T];T)$ follows without choice if e.g. $T \subseteq \omega^{<\omega}$.

§2. The Axiom of Choice and Cardinal Numbers. The results of the last section indicate there is some tension between choice and determinacy. We would like to study the consequences of AD for analysis, but many standard results about the reals rely on the Axiom of Choice. It is impossible to prove without some choice, for example, that the countable union of countable sets is countable!

We therefore isolate some weakenings of choice which are compatible with AD, but still strong enough to obtain a reasonable theory of the real numbers. First, recall the statement of the Axiom of Choice.

Definition 2.1. The Axiom of Choice states that whenever $\{A_i\}_{i\in I}$ is a collection of non-empty sets, there is a function $f : I \to \bigcup_{i\in I} A_i$ such that $f(i) \in A_i$ for each $i \in I$. Such a function is called a **choice function** for the family $\{A_i\}_{i\in I}$.

We can restrict the Axiom of Choice by requiring the sets A_i to be a subset of some fixed set A, or by fixing the index set I. The following definition allows for both of these restrictions.

Definition 2.2. Let $\mathsf{AC}_X(A)$ be the axiom which states: whenever $\{A_i\}_{i\in X}$ is a collection of non-empty subsets of A, there is a function $f : X \to A$ such that $f(i) \in A_i$ for all $i \in X$.

The **Axiom of Countable Choice**, abbreviated AC_ω, states: for all sets A, $\mathsf{AC}_\omega(A)$ holds.

Let's see AC_ω in action. For the following theorem, recall that a set B is infinite if there is no surjection $f : n \to B$ with $n \in \omega$.

Theorem 2.3. *Assume* AC_ω. *If B is infinite, there is an injection $f : \omega \to B$.*

Proof. For each $n \in \omega$, let $\mathcal{B}_n$ be the collection of subsets of B containing exactly 2^n elements; note that $\mathcal{B}_n$ is non-empty for each n since B is infinite. By AC_ω, there is a choice function $g : \omega \to \bigcup_{n\in\omega} \mathcal{B}_n$ so that for all n, $g(n) \in \mathcal{B}_n$.

Now define A_n by

$$A_n = g(n) \setminus \bigcup_{i<n} g(i).$$

Note that $|\bigcup_{i<n} g(i)| \leq \Sigma_{i<n}|g(i)| = \Sigma_{i<n}2^i = 2^n - 1$. Since $|g(n)| = 2^n$, it follows that A_n is non-empty, and the A_n are pairwise disjoint. Then again by AC_ω, we obtain a choice function $f : \omega \to \bigcup_{n\in\omega} A_n$, which is the desired injection into B. $\dashv$

Some choice is necessary to prove this! Ditto the next theorem:

Theorem 2.4. *Assume* AC_ω. *Let $\{A_n\}_{n\in\omega}$ be a collection of countable sets. Then $\bigcup_{n\in\omega} A_n$ is countable.*

Proof. We may assume some A_n is non-empty; then we need to find a surjection $f : \omega \to \bigcup_{n\in\omega} A_n$. We know for each n there is some surjection $g : \omega \to A_n$; the problem is picking a collection of such g_n for all n simultaneously.

Let $\mathcal{F}_n = \{h \in {A_n}^\omega \mid h \text{ is surjective}\}$. By assumption, each $\mathcal{F}_n$ is non-empty, so by AC_ω, we obtain a choice function $g : \omega \to \bigcup_{n\in\omega} \mathcal{F}_n$. Fix some $a \in \bigcup_{n\in\omega} A_n$. Let f be defined by setting

$$f(n) = \begin{cases} g(i)(j) & \text{if } n = 2^i 3^j \text{ for some } i, j \in \omega; \\ a & \text{otherwise.} \end{cases}$$

Then it is easy to see that $f : \omega \to \bigcup_{n\in\omega} A_n$ is onto. ⊣

It turns out AC_ω is compatible with AD, and most instances we need actually follow from AD. From now on, we take the Axiom of Countable Choice for granted, and won't draw attention to its use.

There is a stronger choice axiom which is also compatible with AD.

Definition 2.5. The **Principle of Dependent Choices**, abbreviated DC, states: Suppose R is a binary relation on a non-empty set X so that for all $x \in X$ there is an element $y \in X$ with $x\,R\,y$. Then there is a sequence $\langle x_n\rangle_{n\in\omega}$ of elements of X so that $x_n\,R\,x_{n+1}$ for all $n \in \omega$.

First observe this follows from AC: if R is such a relation, then for each $x \in X$, let $A_x = \{y \in X \mid x\,R\,y\}$. Each A_x is non-empty, and so we obtain a choice function $f : X \to X$ with $x\,R\,f(x)$ for all $x \in X$. The desired sequence is obtained by letting $x_0 \in X$ be arbitrary, and inductively letting $x_{n+1} = f(x_n)$.

Theorem 2.6. *Assume* DC. *Then* AC_ω *holds.*

Proof. Let $\{A_n\}_{n\in\omega}$ be a collection of non-empty sets. Let X be the set of functions f so that each $f \in X$ has $\mathrm{dom}(f) \in \omega$, and $f(n) \in A_n$ for each n.

Now define R on X by letting $f\,R\,g$ if $\mathrm{dom}(g) = \mathrm{dom}(f) + 1$ and $f \subseteq g$. It's easy to see that given $f \in X$, we may fix $a \in A_{\mathrm{dom}(f)}$ and obtain $f\,R\,f^\frown\langle a\rangle$. By DC, we obtain a sequence $\langle f_n\rangle_{n\in\omega}$ with $f_n\,R\,f_{n+1}$ for all n. Then $f = \bigcup_{n\in\omega}$ is the desired choice function. ⊣

Heuristically, DC is like a "dynamic" version of AC_ω: it says that we can make countably many choices without knowing in advance where the choices have to come from. The next lemma is one of the most important consequences of DC. (It does not follow from AC_ω alone!)

Definition 2.7. A tree T on a set X is **finitely branching** if for all $s \in T$, the set $\{x \in X | s^\frown\langle x\rangle \in T\}$ of immediate successors of s in T is finite.

Lemma 2.8 (König). *Let T be an infinite finitely branching tree. Then T has an infinite branch.*

Proof. Let $S \subseteq T$ be the set of elements s of T so that T_s is infinite. Define R on S by setting $s\,R\,t$ if $s \subseteq t$.

Since $T_s = \bigcup\{T_{s^\frown\langle x\rangle} \mid s^\frown\langle x\rangle \in T\}$ and T is finitely branching, we have for each $s \in S$ some x so that $T_{s^\frown\langle x\rangle}$ is infinite. Then $s^\frown\langle x\rangle \in S$, and $s\,R\,s^\frown\langle x\rangle$.

By DC, there is an infinite sequence $\langle s_n\rangle_{n\in\omega}$ so that $s_n \in S$, and $s_n \subseteq s_{n+1}$ for all n. Then $f = \bigcup_{n\in\omega} s_n$ is an infinite branch through T. ⊣

Recall that the Axiom of Choice is equivalent to the statement that every set can be well-ordered; equivalently, for all sets X, there is an ordinal α and a bijection $f : \alpha \to X$. If κ is the least ordinal for which such a bijection exists, we say κ is the **cardinality** of X, and write $\kappa = |X|$. A **cardinal** is an ordinal κ so that $\kappa = |\kappa|$.

The first few cardinal numbers are $0, 1, 2, 3$. The least infinite cardinal is ω. Using choice, we have that $\mathcal{P}(\omega)$ can be well-ordered, and so there is an uncountable cardinal. The least such is denoted ω_1. Can we show ω_1 exists without using choice? The answer is yes, with "Hartogs' trick".

Proposition 2.9. *(Without using the Axiom of Choice.) Let κ be an infinite cardinal. Then there is a cardinal $\lambda > \kappa$.*

Proof. Let κ be a cardinal. Consider the set

$$H = \{R \in \mathcal{P}(\kappa \times \kappa) \mid R \text{ well-orders some subset of } \kappa\}.$$

For each $R \in H$, there is a unique ordinal η so that the well-order R has order-type η; that is, there is a bijection $f : \mathrm{dom}(R) \to \eta$ so that $\alpha \, R \, \beta$ if and only if $f(\alpha) \in f(\beta)$, for all $\alpha, \beta \in \mathrm{dom}\, R$ (check this!).

By the Axiom of Replacement, let λ be the image of H under the map sending R to its order-type. Note that λ is an ordinal with $\lambda > \kappa$. We claim λ is a cardinal. Suppose instead that $\alpha = |\lambda| < \lambda$; we have a bijection $f : \alpha \to \lambda$. Since $\alpha < \lambda$, we have a well-order R of some subset of κ with order-type κ, hence a surjection $g : \kappa \to \alpha$. Composing with f, we have a surjection $f \circ g : \kappa \to \lambda$. This gives rise to a well-order of a subset of κ with order-type λ. Then $\lambda \in \lambda$, a contradiction! $\dashv$

Evidently the λ we defined in the proof is the least cardinal greater than κ, its so-called **cardinal successor**. We write $\lambda = \kappa^+$. Using this proposition, we can generate a list of all the infinite cardinals.

Definition 2.10. $\aleph_0$ is the least infinite cardinal, $\aleph_0 = |\omega|$.
Given $\aleph_\alpha$, $\aleph_{\alpha+1}$ is the least cardinal greater than $\aleph_\alpha$: $\aleph_{\alpha+1} = \aleph_\alpha^+$.
For limit ordinals λ, $\aleph_\lambda = \sup_{\alpha<\lambda} \aleph_\alpha$.

We also write ω_α for $\aleph_\alpha$. Typically we use the former either when we are emphasizing the nature of ω_α as an ordinal, rather than a cardinal, or if we don't feel like drawing a $\aleph$.

An infinite cardinal κ is a **successor cardinal** if $\kappa = \aleph_{\alpha+1}$ for some α; it is a **limit cardinal** if $\kappa = \aleph_\lambda$ for a limit ordinal λ.

The least limit cardinal is the ωth cardinal, $\aleph_\omega$. Notice that it is a union of ω many cardinals each smaller than $\aleph_\omega$. This turns out to be a very special property.

Definition 2.11. A cardinal κ is **singular** if it is the sup of fewer than κ many cardinal smaller than κ, that is, if $\kappa = \lim_{\xi<\alpha} \eta_\xi$, for some $\alpha < \kappa$ and $\eta_\xi < \kappa$ for $\xi < \alpha$.

A cardinal that is not singular is called **regular**.

Clearly ω is a regular cardinal. The Axiom of Choice implies every successor cardinal is regular. And we have just seen that $\aleph_\omega$ is singular. So is $\aleph_{\omega_1}$, and $\aleph_{\omega_2}$, and so on; indeed, $\aleph_\lambda$ is singular whenever $\lambda < \aleph_\lambda$. Can there be a regular limit cardinal?

Definition 2.12. A cardinal κ is called **weakly inaccessible** if κ is a regular limit cardinal.

What must such a cardinal look like? Clearly we have $\kappa = \aleph_\kappa$, but this is not a sufficient criterion for inaccessibility: Consider the sequence $\omega, \omega_\omega, \omega_{\omega_\omega}, \dots$. If κ is the limit of this sequence, then $\kappa = \aleph_\kappa$, but is singular.

Our intuition is that weakly inaccessible cardinals are difficult to reach from below. We have one more stronger notion of inaccessibility:

DEFINITION 2.13. (Using the Axiom of Choice.) A cardinal κ is **strong limit** if $|2^\alpha| < \kappa$ for all $\alpha < \kappa$.

κ is (strongly) **inaccessible** if κ is a regular strong limit cardinal.

It's easy to come up with strong limit cardinals. For example, set $\kappa_0 = \omega$, and for $n < \omega$, set $\kappa_{n+1} = |2^{\kappa_n}|$. Then $\lambda = \sup \kappa_n$ is strong limit, but also singular.

It's hard to come up with examples of (weakly) inaccessible cardinals, and there is a reason for this: In ZFC, one cannot prove they exist! We'll explore this when the time is right.

We close this section by defining a special class of important subsets of cardinals.

DEFINITION 2.14. Let $A \subseteq \kappa$ be a set of ordinals. A is **unbounded in** λ for limit $\lambda \leq \kappa$ if whenever $\alpha < \lambda$, there is some $\beta \in A$ with $\alpha < \beta < \lambda$. A is **closed** (in κ) if $\lambda \in A$ whenever A is unbounded in $\lambda < \kappa$.

A set $C \subseteq \kappa$ is **club in** κ if it is closed and unbounded in κ.

Notice that a set is closed precisely when it is a closed set of ordinals with respect to the order topology on κ (basic open sets are of the form $(\alpha, \beta) = \{\xi \in \mathrm{ON} \mid \alpha < \xi < \beta\}$, or $\{0\}$). Our intuition is that clubs are "large". We give two examples.

EXAMPLE 2.15. Let κ be regular, and let $A \subseteq \kappa$ be unbounded. Then the set of limit points of A,

$$A' = \{\alpha < \kappa \mid (\forall \beta < \alpha)(\exists \xi)\beta < \xi < \alpha \text{ and } \xi \in A\}$$

is club in κ.

EXAMPLE 2.16. Let $F : \kappa \to \kappa$ be a function with κ regular. Then the set

$$C_F = \{\alpha \in \kappa \mid (\forall \xi < \alpha) F(\xi) < \alpha\}$$

is club in κ. C_F is called the set of **closure points of** F.

§3. Baire Space and Cantor Space. In this section, we depart from the more general setting and focus on set theory of the reals. We will gain more insight into the reals by working with "set theorists' reals:" sequences $f : \omega \to \omega$.

We have already encountered the set ω^ω of functions $f : \omega \to \omega$. We regard ω^ω as a topological space by taking as a basis all sets of the form

$$N_s = \{x \in \omega^\omega \mid s \subseteq x\} = [(\omega^{<\omega})_s],$$

where $s \in \omega^{<\omega}$. So endowed, ω^ω is called **Baire space**.

Every open set U in Baire space is then of the form $U = \bigcup_{s \in B} N_s$ with $B \subseteq \omega^{<\omega}$. We regard open subsets of ω^ω as the simplest subsets of ω^ω, because they come with a finite certificate for membership: $x \in U$ if and only if $x \upharpoonright n \in B$ for some $n \in \omega$.

As an example, the set $U_0 = \{x \in \omega^\omega \mid (\exists n) x(n) = 3\}$ is open. Its complement, $U_1 = \{x \in \omega^\omega \mid (\forall n) x(n) \neq 3\}$, is not open, since every $s \in \omega^{<\omega}$ extends to some $x \in U_0$. Similarly, the set $U_2 = \{x \in \omega^\omega \mid \Sigma_{n<\omega} x(n) > 9{,}000\}$ is open, but its complement is not.

We may similarly put a topology on 2^ω by using as a basis the sets N_s, for $s \in 2^{<\omega}$. Note that 2^ω is the set of branches through the infinite binary tree. We call 2^ω with this topology **Cantor space**.

A few remarks are in order regarding our basic open sets N_s. If $s \subseteq t$, then clearly $N_s \supseteq N_t$. If it is not the case that $s \subseteq t$ or $t \subseteq s$, then there must be some $i < \min\{\ell(s), \ell(t)\}$ so that $s(i) \neq t(i)$. In this situation, $N_s \cap N_t = \varnothing$. We write $s \perp t$, and say s, t are **incompatible**.

We now isolate a number of important topological properties of the spaces $\omega^\omega, 2^\omega$.

Definition 3.1. A subset C of a topological space is **clopen** if it is both closed and open. A topological space is **totally disconnected** if it has a basis of clopen sets.

Proposition 3.2. *ω^ω and 2^ω are totally disconnected.*

Proof. Note that $\omega^\omega \setminus N_s = \bigcup_{t \perp s} N_t$. So N_s is clopen for all $s \in \omega^{<\omega}$. So ω^ω is totally disconnected; similarly for 2^ω. ⊣

We have the following simple characterization of convergence.

Proposition 3.3. *Let $\langle x_n \rangle_{n\in\omega}$ be a sequence in Baire space (or Cantor space). Then $\lim_{n\to\infty} x_n = x$ if and only if for all $m \in \omega$, there is some N so that $x_n \restriction m = x \restriction m$ for all $n \geq N$.*

Let X be a topological space. A set $D \subseteq X$ is **dense in** X if $U \cap D \neq \varnothing$ whenever $U \subseteq X$ is open. X is called **separable** if it has a countable dense subset. A set $D \subseteq \omega^\omega$ is dense if and only if $N_s \cap D \neq \varnothing$ for all $s \in \omega^{<\omega}$; that is, if for all s there is some $x \in D$ with $s \subseteq x$. It follows that if D_0 is the set of eventually zero sequences in ω^ω, then D_0 is dense, and clearly countable. So ω^ω (and 2^ω) is separable.

Let us also mention that ω^ω and 2^ω can be regarded as metric spaces. Define, for $x, y \in \omega^\omega$,

$$d(x,y) = \begin{cases} 0 & \text{if } x = y; \\ 2^{-n}, \text{where } n \text{ is least so that } x(n) \neq x(y) & \text{if } x \neq y. \end{cases}$$

The reader should verify that this is a metric on ω^ω (2^ω), and that it generates the topology of Baire space (Cantor space).

The following characterization of closed sets in Baire space is fundamental. We say that a tree T is **pruned** if it has no terminal nodes.

Theorem 3.4. *A set $C \neq \varnothing$ is closed in Baire space (or Cantor space) if and only if $C = [T]$ for some pruned tree $T \subseteq \omega^{<\omega}$ ($2^{<\omega}$).*

Proof. If C is closed, set $T = \{\varnothing\} \cup \{x \restriction n \mid x \in C, n \in \omega\}$. It is immediate that T is a pruned tree, and $C \subseteq [T]$. Conversely, suppose $x \in [T]$. For each n, there is some $x_n \in C$ so that $x_n \restriction n = x \restriction n$, by definition of T. Then $x = \lim_{n\to\infty} x_n \in C$, since C is closed.

For the reverse, suppose T is a tree on ω; we need to show $[T]$ is closed as a subset of Baire space. Suppose $\lim_{n\to\infty} x_n = x$, where each $x_n \in [T]$. For each $m \in \omega$, we have some n so that $x_n \restriction m = x \restriction m$; in particular, $x \restriction m \in T$ for all m. This implies $x \in [T]$, as needed. ⊣

Recall that a set K in a topological space is **compact** if every open cover of K admits a finite subcover: that is, if $K \subseteq \bigcup_{i\in I} U_i$ for some collection $\{U_i\}_{i\in I}$ of open sets, then there is some finite $F \subseteq I$ with $K \subseteq \bigcup_{i\in F} U_i$. We mention two facts about compactness:

First, if $C_0 \subseteq K \subseteq X$ with K compact and C_0 closed, then C_0 is compact. Second, if X is a metric space, then whenever K is compact, K is automatically closed (in particular, this holds for ω^ω and 2^ω).

The Heine-Borel Theorem states that a set $K \subseteq \mathbb{R}$ is compact if and only if it is closed and bounded. The following is the analogue for compact subsets of ω^ω, and has a similar proof.

THEOREM 3.5. *A non-empty set $K \subseteq \omega^\omega$ is compact if and only $K = [T]$ for some finitely branching pruned tree T.*

PROOF. Suppose first that K is compact. Then K is closed, and by Theorem 3.4, there is a pruned tree T on ω with $[T] = K$. We claim T is finitely branching. If not, there is some $s \in T$ so that $s^\frown\langle a\rangle \in T$ for infinitely many $a \in \omega$; let $\langle a_n\rangle_{n\in\omega}$ enumerate these a in increasing order. Note that $[T] \cap N_{s^\frown\langle a_n\rangle} \neq \varnothing$ for all n, by the assumption that T is pruned, and the sets $N_{s^\frown\langle a_n\rangle}$ are pairwise disjoint. It follows that $\{N_{s^\frown\langle a_n\rangle}\}_{n\in\omega}$ is an infinite cover of K with no finite subcover. This contradicts compactness of K, so T must be finitely branching.

Conversely, suppose T is a finitely branching pruned tree on ω. We claim $[T]$ is compact. Suppose otherwise. Then there is some collection $\{U_i\}_{i\in I}$ of open sets covering $[T]$, but so that no finite subcover covers T.

We now inductively construct $x \in [T]$ so that for all n, $[T_{x\restriction n}]$ cannot be covered by finitely many of the U_i. This gives the desired contradiction, since $x \in [T]$ implies $x \in U_i$ for some i; then by openness of U_i, there must be some n so that $[T_{x\restriction n}] \subseteq N_{x\restriction n} \subseteq U_i$.

For $n = 0$, we have by assumption that $[T_\varnothing] = [T]$ cannot be covered by finitely many of the U_i. Suppose inductively that we have defined $x \restriction n = \langle x(0), x(1), \dots x(n-1)\rangle$ so that $[T_{x\restriction n}]$ cannot be covered by finitely many of the U_i. Since T is finitely branching, we have $T_{x\restriction n} = \bigcup_{k<m} T_{(x\restriction n)^\frown\langle a_k\rangle}$ for some finite list $a_0, a_1, \dots, a_{m-1}$ of elements of ω. Suppose towards a contradiction that each $[T_{(x\restriction n)^\frown\langle a_k\rangle}]$ can be covered by $\{U_i\}_{i\in F_k}$ for some finite set $F_k \subseteq I$. Then we have $F = \bigcup_{k<m} F_k$ a finite set so that $[T_{x\restriction n}]$ is covered by $\{U_i\}_{i\in F}$, contradicting our inductive hypothesis.

So there must be some $k < m$ so that $[T_{(x\restriction n)^\frown\langle a_k\rangle}]$ cannot be covered by finitely many of the U_i. Set $x(n) = a_k$; by induction, we obtain the desired x. ⊣

There is one more notion we would like to introduce and examine in the context of ω^ω and 2^ω. This notion may be thought of as a purely topological analogue of Lebesgue measure zero. The main idea is to try to isolate some class of sets that is intuitively small, in some robust way. The next definition is our first approximation to this notion.

DEFINITION 3.6. Let X be a topological space. We say that $A \subseteq X$ is **nowhere dense** if for every non-empty open $U \subseteq X$, there is a non-empty open V with $V \subseteq U$ and $V \cap A = \varnothing$.

So a set $A \subseteq \omega^\omega$ is nowhere dense if for every $s \in \omega^{<\omega}$, there is some extension $t \supseteq s$ with $N_t \cap A = \varnothing$. For example, the set U_1 defined earlier is nowhere dense. Also, 2^ω is nowhere dense as a subset of ω^ω (but not as a subset of itself, of course!).

Any nowhere dense set can be enlarged a bit and still be nowhere dense.

PROPOSITION 3.7. *If $A \subseteq X$ is nowhere dense, then so is its closure $\bar{A} = A \cup \{x \in X \mid x$ is a limit point of $A\}$.*

PROOF. This follows immediately from the fact that if V is an open set with $V \cap A = \varnothing$, then $V \cap \bar{A} = \varnothing$. ⊣

The reader should verify that a finite union of nowhere dense sets is nowhere dense. Unfortunately (and in contrast with the Lebesgue measure zero sets), it is not the case that a countable union of nowhere dense sets is nowhere dense. For example, a countable dense set is a countable union of singletons, each of which is nowhere dense. The following notion of smallness is more robust.

DEFINITION 3.8. A set $A \subseteq X$ is **meager** if it is contained in the union of countably many nowhere dense sets.

It is immediate that if $M_0 \subseteq M$ with M meager, M_0 is meager as well. Nowhere dense sets are obviously meager. And since countable unions of countable sets are countable, we have that the countable union of meager sets is meager. In particular, countable sets are meager. This is a good thing: we think of countable sets as intuitively small.

Of course, one set that shouldn't be small is ω^ω itself. This is the content of the following theorem.

THEOREM 3.9 (The Baire Category Theorem). *The space ω^ω is not meager as a subset of itself. (Similarly for 2^ω.)*

PROOF. We have to show that no countable union of nowhere dense subsets of ω^ω is equal to all of ω^ω. Let $\{C_n\}_{n\in\omega}$ be a countable collection of nowhere dense sets. Replacing the sets C_n with their closures $\bar{C_n}$ if necessary, we may assume each C_n is closed. Then $U_n = \omega^\omega \setminus C_n$ is an open dense set.

To show $\bigcup_{n\in\omega} C_n$ is not all of ω^ω, it is sufficient to show $\bigcap_{n\in\omega} U_n$ is non-empty. In fact we can do even better.

CLAIM. *The countable intersection of dense open sets is dense.*

Fix $s \in \omega^{<\omega}$. We need to show $N_s \cap \bigcap_{n\in\omega} U_n$ is non-empty. Set $s_0 = s$. Suppose inductively that s_n has been defined. Since U_n is dense open, the set $U_n \cap N_{s_n}$ is open and non-empty, so there is some proper extension $s_{n+1} \supsetneq s_n$ so that $N_{s_{n+1}} \subseteq U_n$.

Put $x = \bigcup_{n\in\omega} s_n$. Clearly $x \in N_s = N_{s_0}$. And for each n, we have $x \in N_{s_n} \subseteq U_n$. It follows that $x \in N_s \cap \bigcap_{n\in\omega} U_n$. This proves the claim, and the theorem. $\dashv$

A set A in ω^ω is called **comeager** if its complement $\omega^\omega \setminus A$ is meager. A countable intersection of open sets is called a G_δ set. By the proof just given, the countable intersection of open dense sets is always a dense G_δ set. Thus a set is comeager if and only if it contains a dense G_δ.

We close this section by connecting the topology of Baire space with determinacy. The characterization of closed sets in Theorem 3.4 gives us the following.

THEOREM 3.10. *Let $C \subseteq \omega^\omega$ be closed. Then $G(C)$ is determined.*

PROOF. Fix a tree T on ω so that $C = [T]$. Let T' be the tree defined by

$$T' = \{s \in \omega \mid s \in T \text{ or } (\exists n) s = t^\frown\langle n\rangle \text{ for some } t \in T \text{ with } \ell(t) \text{ odd.}\}$$

Note that $[T'] = [T] = C$, and all terminal nodes in T' have even length. By the Gale-Stewart Theorem 1.7, the game $G([T']; T')$ is determined. We show how to produce a winning strategy in $G(C)$ given one in $G([T']; T')$.

Suppose σ is a winning strategy for Player I in $G([T']; T')$. By our definition of T', σ can put no restrictions on Player II's moves; that is, whenever $s \in \sigma$ has odd length, then $s^\frown\langle n\rangle \in \sigma$ for all $n \in \omega$. Furthermore, since all terminal nodes in T' have even length, every play of σ is infinite. Consequently, σ is a strategy for Player I in $\omega^{<\omega}$, and is winning in $G(C)$, since $[\sigma] \subseteq [T'] = C$.

Suppose now that τ is winning for Player II in $G([T']; T')$. We can extend τ to a strategy in $\omega^{<\omega}$ by letting, for all s with even length,

$$\tau'(s) = \begin{cases} \tau(s) & \text{if } s \in \tau; \\ 0 & \text{otherwise.} \end{cases}$$

Then τ' (or to be more precise, the tree of nodes reachable by playing according to τ' for Player II, and putting no restrictions on Player I's moves) is a strategy for Player II. Since τ is winning for Player II in $G([T']; T')$, there are no infinite plays compatible with τ, and it follows that if $x \in [\tau']$, there is some n so that $x \upharpoonright n$ is terminal in T'; hence $x \notin [T'] = C$. This shows τ' is winning for Player II in $G(C)$. ⊣

§4. Polish Spaces and Pointclasses. In the last section, we introduced the spaces ω^ω and 2^ω and isolated some useful topological properties of these. In this section, we abstract these properties into a definition of a class of structures that includes the spaces ω^ω, 2^ω, $\mathbb{R}$, as well as their products, and many others.

Recall that a metric space (X, d) is **complete** if every Cauchy sequence in X converges: that is, whenever $\langle x_n \rangle_{n\in\omega}$ is a sequence of elements of X so that for all ε, there is some N so that $m, n \geq N$ implies $d(x_m, x_n) < \varepsilon$, then $\lim_{n\to\infty} x_n$ exists in X. We leave it as an exercise to verify that Baire space and Cantor space are both complete when endowed with the metric defined in the last section.

Definition 4.1. A topological space $(X, \mathcal{T})$ is a **Polish space** if it is separable, and there exists a metric $d : X \times X \to \mathbb{R}$ that generates the topology $\mathcal{T}$ of X, and so that (X, d) is a complete metric space.

Notice that the definition of a Polish space asserts the existence of some complete metric, but there needn't be a unique such. The point is that if $\mathcal{T}$ is generated by a complete metric it will have nice properties, but we are more interested just in the topology than the particular metric generating it. Nonetheless, we typically suppress mention of the topology $\mathcal{T}$, using the domain set X to denote the Polish space $(X, \mathcal{T})$ when $\mathcal{T}$ is clear.

We have seen that $\mathbb{R}, \omega^\omega$ and 2^ω are Polish spaces. So is ω with the discrete topology, as witnessed by the metric d on ω with $d(m, n) = 1$ for all $m \neq n$. Furthermore, if X, Y are Polish spaces, then so is their product $X \times Y$. This can be seen by setting

$$d_{X\times Y}(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle) = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}$$

where d_X, d_Y are the complete metrics witnessing Polishness of X, Y, respectively.

We remark that the proof of the Baire category theorem in the last section was sufficiently general to go through for arbitrary Polish spaces. We obtain: *If X is Polish, then X is not meager in itself.*

The following theorem gives an indication of the important status of ω^ω among all Polish spaces.

Theorem 4.2. *Let Y be Polish. Then there is a continuous surjection $f : \omega^\omega \to Y$.*

Proof. Let d be a complete metric generating the topology on Y, and fix a countable dense subset $D = \{c_0, c_1, \dots\}$ of Y. We define recursively, for each non-empty $s \in \omega^{<\omega}$, an element y_s of D. We ensure for each $x \in \omega^\omega$ that the sequence $\langle y_{x\restriction n}\rangle_{n\in\omega}$ is Cauchy in Y.

For each $a \in \omega$, set $y_{\langle a\rangle} = c_a$. Now suppose inductively that we have defined c_s for some $s \in \omega^{<\omega}$ with $\ell(s) = n \geq 1$. For each $a \in \omega$, define $y_{s^\frown a}$ as follows: if $d(y_s, c_a) < 2^{-n}$, let $y_{s^\frown\langle a\rangle} = c_a$; otherwise, set $y_{s^\frown\langle a\rangle} = y_s$.

We claim for each x, $\langle y_{x\restriction n}\rangle_{n\in\omega}$ is Cauchy. For we have, for all $1 \leq m \leq n$,

$$\begin{aligned} d(y_{x\restriction m}, y_{x\restriction n}) &\leq d(y_{x\restriction m}, y_{x\restriction m+1}) + d(y_{x\restriction m+1}, y_{x\restriction m+2}) + \cdots + d(y_{x\restriction n-1}, y_{x\restriction n}) \\ &< 2^{-m} + 2^{-(m+1)} + \cdots + 2^{-(n-1)} < 2^{-(m-1)}. \end{aligned}$$

Given $\varepsilon > 0$, take N to be large enough that $2^{-(N-1)} < \varepsilon$; this witnesses Cauchyness of $\langle y_{x\restriction n}\rangle_{n\in\omega}$.

Now by completeness of Y, we may set $f(x) = \lim_{n\to\infty} y_{x\restriction n}$ for each $x \in \omega^\omega$. We claim $f : \omega^\omega \to Y$ is continuous. For suppose $x_0, x_1 \in \omega^\omega$ and $d(x_0, x_1) < 2^{-n}$, $n \geq 0$. This implies $x_0 \restriction n = x_1 \restriction n$, so that in particular, $y_{x_0\restriction n} = y_{x_1\restriction n}$. We then have $d(f(x_0), f(x_1)) \leq d(f(x_0), y_{x_0\restriction n}) + d(y_{x_1\restriction n}, f(x_1)) \leq 2^{-n} + 2^{-n} = 2^{-(n-1)}$. By the ε-δ characterization of continuity, we are done.

Finally, we need to show f is onto. Fix $y \in Y$. Define a sequence of elements of D tending quickly towards y: for all n, let $x(n) \in \omega$ be least so that $d(y, c_{x(n)}) < 2^{-(n+2)}$. We obtain $x \in \omega^\omega$; using the triangle inequality, it's easy to check that $d(c_{x(n)}, c_{x(n+1)}) < 2^{-n+1}$ for all n. Then by induction, we always have $y_{x\restriction n} = a_{x(n)}$, so that $f(x) = \lim_{n\to\infty} a_{x(n)} = y$. ⊣

As remarked above, we regard open sets as the simplest subsets of a Polish space. Shortly we will define larger classes of sets that may contain more complicated sets. We would like to have a way of working with the class of all sets of a particular complexity in arbitrary Polish spaces (not just ω^ω). For this reason, we introduce the following new concept.

Definition 4.3. We call $\mathbf{\Gamma}$ a **pointclass** if it consists of pairs (A, X), where A is a subset of the Polish space X. We say $\mathbf{\Gamma}$ is **closed under continuous substitution** if, whenever X, Y are Polish spaces, $f : X \to Y$ is continuous, and (A, Y) belongs to $\mathbf{\Gamma}$, then also $(f^{-1}[A], X) \in \mathbf{\Gamma}$.

Given a pointclass $\mathbf{\Gamma}$ and a Polish space X, the **restriction of $\mathbf{\Gamma}$ to** X is the collection of sets A so that $(A, X) \in \mathbf{\Gamma}$; that is, $\mathbf{\Gamma}(X) = \mathbf{\Gamma} \cap \mathcal{P}(X)$.

The **dual pointclass of $\mathbf{\Gamma}$**, denoted $\neg\mathbf{\Gamma}$, is the class of complements of elements of $\mathbf{\Gamma}$: that is, $(A, X) \in \neg\mathbf{\Gamma}$ if and only if $(\neg A, X) \in \mathbf{\Gamma}$; here $\neg A = X \setminus A$. A pointclass is **self-dual** if $\mathbf{\Gamma} = \neg\mathbf{\Gamma}$.

Typically, the ambient space X will be understood and we simply write $A \in \mathbf{\Gamma}$ or say "A is $\mathbf{\Gamma}$" to mean that $(A, X) \in \mathbf{\Gamma}$.

Notice that $\mathbf{\Gamma}$ is closed under continuous substitution if and only if $\neg\mathbf{\Gamma}$ is. The class of open sets in Polish spaces is an example of a pointclass closed under continuous substitution; the class of closed sets (and that of clopen sets) is also closed under continuous substitution.

Let us connect the notions we are developing to determinacy.

Definition 4.4. Let $\mathbf{\Gamma}$ be a pointclass. We say **$\mathbf{\Gamma}$ determinacy** holds (and write $\mathbf{\Gamma}$-DET) if whenever $A \in \mathbf{\Gamma}(\omega^\omega)$, the game $G(A)$ is determined.

We saw that $\mathbf{\Gamma}$-DET holds when $\mathbf{\Gamma}$ is the class of closed sets. By the following theorem, we also have $\neg\mathbf{\Gamma}$ (open) determinacy.

THEOREM 4.5. *Suppose $\mathbf{\Gamma}$ is a pointclass closed under continuous substitution. Then $\mathbf{\Gamma}$ determinacy is equivalent to $\neg\mathbf{\Gamma}$ determinacy.*

For contrast, recall that (under the Axiom of Choice) $G(A)$ may be determined while $G(\omega^\omega \setminus A)$ is not.

PROOF. Let $A \in \neg\mathbf{\Gamma}(\omega^\omega)$ and suppose $\mathbf{\Gamma}$ determinacy holds. We wish to show $G(A)$ is determined. The proof illustrates a common technique in proofs of determinacy: the simulation of play in $G(A)$ by that in an auxiliary game.

Define $f : \omega^\omega \to \omega^\omega$ by $f(x)(n) = x(n+1)$ for all $n \in \omega$. Clearly f is continuous. Since $\neg\mathbf{\Gamma}$ is closed under continuous substitution, we have $f^{-1}[A] \in \neg\mathbf{\Gamma}$. Let $B = \neg f^{-1}[A]$. Then $B \in \mathbf{\Gamma}$ is determined by hypothesis.

Suppose Player II wins $G(B)$ with strategy τ. We obtain a strategy σ for Player I to win $G(A)$ by pretending we are Player II in $G(B)$, and that Player I played first move 0. That is, let $\sigma(s) = \tau(\langle 0\rangle^\frown s)$ for all $s \in \omega^{<\omega}$ for which the latter is defined.

Then σ is a strategy for Player II in $\omega^{<\omega}$. Suppose x is a play compatible with σ; then $\langle 0\rangle^\frown x$ is a play compatible with τ. Since τ is winning for Player II, $\langle 0\rangle^\frown x \notin B$, so that $f(\langle 0\rangle^\frown x) = x \in A$. Thus σ is winning for Player I in $G(A)$.

The argument when Player I wins $G(B)$ is similar: If σ is the winning strategy, then use it to play as Player II to win $G(A)$ (now ignoring the first move made by σ).

We have shown $\mathbf{\Gamma}$ determinacy implies $\neg\mathbf{\Gamma}$ determinacy; the converse holds by symmetry. $\dashv$

We have obtained that open sets and closed sets are determined. In order to investigate determinacy for more complicated sets, we first explore a way of producing sets that are more complicated.

The complement of an open set is not, in general, open; and the intersection of countably many open sets may be neither open nor closed. Iterating the operations of complement and countable union gives us a hierarchy of increasingly complicated sets. The next definition is central to our study of sets of reals.

DEFINITION 4.6. Let X be a Polish space. We define a hierarchy of pointclasses $\mathbf{\Sigma}^0_\alpha(X)$, $\mathbf{\Pi}^0_\alpha(X)$, $\mathbf{\Delta}^0_\alpha(X)$ for $1 \leq \alpha < \omega_1$ by transfinite recursion.

1. $U \in \mathbf{\Sigma}^0_1(X)$ iff U is an open set in X.
2. Assuming $\mathbf{\Sigma}^0_\alpha(X)$ is defined, $\mathbf{\Pi}^0_\alpha(X) = \{A \subseteq X \mid X \setminus A \in \mathbf{\Sigma}^0_\alpha(X)\}$.
3. Assuming $\mathbf{\Pi}^0_\beta(X)$ is defined for all $1 \leq \beta < \alpha$, we let $\mathbf{\Sigma}^0_\alpha(X)$ be the set of countable unions of sets in $\bigcup_{\beta<\alpha} \mathbf{\Pi}^0_\beta(X)$. That is, $A \in \mathbf{\Sigma}^0_\alpha(X)$ if and only if $A = \bigcup_{n\in\omega} A_n$ for some sequence $\langle A_n\rangle_{n\in\omega}$ with each $A_n \in \mathbf{\Pi}^0_{\beta_n}$ for some $\beta_n < \alpha$.

We furthermore define the **ambiguous pointclasses** $\mathbf{\Delta}^0_\alpha(X)$ to consist of those sets that are in both $\mathbf{\Sigma}^0_\alpha(X)$ and $\mathbf{\Pi}^0_\alpha(X)$. That is, $\mathbf{\Delta}^0_\alpha(X) = \mathbf{\Sigma}^0_\alpha(X) \cap \mathbf{\Pi}^0_\alpha(X)$.

We define $\mathbf{\Sigma}^0_\alpha$ to be the pointclass consisting of (X, A) with A in $\mathbf{\Sigma}^0_\alpha(X)$ as X ranges over all Polish spaces. We define $\mathbf{\Pi}^0_\alpha$ and $\mathbf{\Delta}^0_\alpha$ similarly.

The classes $\mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha$ are the **Borel pointclasses**. A set A is a **Borel set in** X if $A \in \mathbf{\Sigma}^0_\alpha(X)$ or $A \in \mathbf{\Pi}^0_\alpha(X)$ for some $\alpha < \omega_1$. We set $\mathcal{B}(X) = \bigcup_{\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha(X)$.

Let's look at the first few levels of this hierarchy. Of course, $\mathbf{\Pi}^0_1$ is exactly the collection of closed sets. The collection $\mathbf{\Sigma}^0_2$ consists of all the countable unions of closed sets;

sometimes these are also called F_σ sets. Since singletons are closed, any countable set is $\mathbf{\Sigma}^0_2$.

$\mathbf{\Pi}^0_2$ consists of all countable intersections of open sets; these are also called the G_δ sets. In the proof of the Baire category theorem we showed the countable intersection of dense open sets is dense; such sets are called dense G_δ sets. A set is meager precisely when it is disjoint from a dense G_δ set.

EXAMPLE 4.7. Let $a < b \in \mathbb{R}$. Then the half-open interval $[a, b)$ is $\mathbf{\Delta}^0_2$: It can be written both as the countable union of closed sets and as the countable intersection of open sets.

EXAMPLE 4.8. The set $\mathbb{Q}$ is $\mathbf{\Sigma}^0_2$ as a subset of $\mathbb{R}$. Since $\mathbb{Q}$ is meager, it cannot be the intersection of countably many (necessarily dense) open sets, since this would imply $\mathbb{R}$ is the union of two meager sets, contradicting the Baire category theorem. So Q is not $\mathbf{\Pi}^0_2$ (and so not $\mathbf{\Delta}^0_2$).

We pursue a systematic study of this hierarchy in the next section.

§5. The Borel Hierarchy.

THEOREM 5.1. *Each pointclass $\mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha, \mathbf{\Delta}^0_\alpha$ is closed under continuous substitution.*

PROOF. We proceed by induction, just as the Borel pointclasses were defined. For $\mathbf{\Sigma}^0_1$, this is immediate from the definition of continuity. Having shown $\mathbf{\Sigma}^0_\alpha$ is closed under continuous substitution, suppose $A \subseteq Y$ is in $\mathbf{\Pi}^0_\alpha$ and $f : X \to Y$ is continuous. Then since $Y \setminus A \in \mathbf{\Sigma}^0_\alpha$, we have that $f^{-1}[Y \setminus A] \in \mathbf{\Sigma}^0_\alpha$. It follows that $f^{-1}[A] = X \setminus f^{-1}[Y \setminus A]$ belongs to $\mathbf{\Pi}^0_\alpha$, as needed.

Finally suppose $\mathbf{\Pi}^0_\beta$ is closed under continuous substitution for all $\beta < \alpha$. Let $A \in \mathbf{\Sigma}^0_\alpha(Y)$; then $A = \bigcup_{n\in\omega} A_n$ where each A_n is in $\mathbf{\Pi}^0_{\beta_n}(Y)$ for some $\beta_n < \alpha$. By inductive hypothesis, $f^{-1}[A_n] \in \mathbf{\Pi}^0_{\beta_n}(X)$ for each $n < \omega$, and then $f^{-1}[A] = \bigcup_{n\in\omega} f^{-1}[A_n] \in \mathbf{\Sigma}^0_\alpha(X)$, as needed.

The claim for $\mathbf{\Delta}^0_\alpha$ follows immediately. ⊣

Let us analyze the hierarchy of Borel sets a little further. First, we note that it really is a hierarchy.

PROPOSITION 5.2. *If $1 \le \beta < \alpha < \omega_1$, we have $\mathbf{\Sigma}^0_\beta \subseteq \mathbf{\Delta}^0_\alpha$, $\mathbf{\Pi}^0_\beta \subseteq \mathbf{\Delta}^0_\alpha$, while $\mathbf{\Delta}^0_\beta \subseteq \mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Delta}^0_\beta \subseteq \mathbf{\Pi}^0_\alpha$.*

PROOF. If we can show the former claim, then $\mathbf{\Delta}^0_\beta \subseteq \mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Delta}^0_\beta \subseteq \mathbf{\Sigma}^0_\alpha$ follows from the definition. We prove $\mathbf{\Sigma}^0_\beta \subseteq \mathbf{\Delta}^0_\alpha$; that $\mathbf{\Pi}^0_\beta \subseteq \mathbf{\Delta}^0_\alpha$ is immediate by taking complements. For this it is clearly enough to only deal with successor ordinals and show $\mathbf{\Sigma}^0_\beta \subseteq \mathbf{\Delta}^0_{\beta+1}$ for all $1 \le \beta < \omega_1$.

Showing $\mathbf{\Sigma}^0_\beta \subseteq \mathbf{\Pi}^0_{\beta+1}$ is easy: say $A \in \mathbf{\Sigma}^0_\beta$. Then $B = X \setminus A \in \mathbf{\Pi}^0_\beta$. Setting $B_n = B$ for all $n \in \omega$ we have $B = \bigcup_{n\in\omega} B_n$ belongs to $\mathbf{\Sigma}^0_{\beta+1}$ and thus the complement A belongs to $\mathbf{\Sigma}^0_{\beta+1}$.

It remains to show $\mathbf{\Sigma}^0_\beta \subseteq \mathbf{\Sigma}^0_{\beta+1}$. This is a little tougher; we do it by induction on $\beta \ge 1$. If $\beta = 1$, notice that any open set in a Polish space is a countable union of closed sets. Thus $\mathbf{\Sigma}^0_1 \subseteq \mathbf{\Sigma}^0_2$. Now assume inductively that we have $\mathbf{\Sigma}^0_\gamma \subseteq \mathbf{\Sigma}^0_{\gamma+1}$ for $\gamma < \beta$; by taking complements we also have $\mathbf{\Pi}^0_\gamma \subseteq \mathbf{\Pi}^0_{\gamma+1}$. Let $A \in \mathbf{\Sigma}^0_\beta$. Then $A = \bigcup_{n\in\omega} A_n$ where each A_n is $\mathbf{\Pi}^0_{\gamma_n}$ for some $\gamma_n < \beta$; thus each A_n is also $\mathbf{\Pi}^0_{\gamma_n+1}$ where $\gamma_n + 1 < \beta + 1$. Thus indeed A is $\mathbf{\Sigma}^0_{\beta+1}$. ⊣

Next we will be interested in the closure properties that the Borel pointclasses enjoy.

Proposition 5.3. *Let X be a Polish space. Then for all $1 \leq \alpha < \omega_1$,*

1. *$\mathbf{\Sigma}^0_\alpha(X)$ is closed under countable unions, and $\mathbf{\Pi}^0_\alpha(X)$ is closed under countable intersections.*
2. *$\mathbf{\Sigma}^0_\alpha(X), \mathbf{\Pi}^0_\alpha(X), \mathbf{\Delta}^0_\alpha(X)$ are each closed under finite unions and intersections.*
3. *$\mathbf{\Delta}^0_\alpha(X)$ is closed under complements; in particular, each $\mathbf{\Delta}^0_\alpha$ is self-dual.*
4. *$\mathcal{B}(X) = \bigcup_{\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha$ is closed under the operations of countable union, countable intersection, and complementation; that is, $\mathcal{B}(X)$ is a **σ-algebra**, and it is the smallest σ-algebra containing the open sets of X.*

Proof. The first and third items are immediate by definition; we leave the second as an exercise. For the last, closure under countable unions follows from the fact that ω_1 is regular: if $A_n \in \mathbf{\Sigma}^0_{\alpha_n}$ for each n, then $\alpha = \sup_{n\in\omega} \alpha_n < \omega_1$, and $\bigcup_{n\in\omega} A_n \in \mathbf{\Sigma}^0_\alpha$.

For the final claim, suppose $\mathcal{F}$ is a σ-algebra containing the open sets of X. Then $\mathbf{\Sigma}^0_1(X) \subseteq \mathcal{F}$, and whenever $\mathbf{\Sigma}^0_\alpha(X) \subseteq \mathcal{F}$ we must have $\mathbf{\Pi}^0_\alpha(X) \subseteq \mathcal{F}$ by closure of $\mathcal{F}$ under complements; similarly, if $\mathbf{\Pi}^0_\beta(X) \subseteq \mathcal{F}$ for all $1 \leq \beta < \alpha$, then $\mathbf{\Sigma}^0_\alpha(X) \subseteq \mathcal{F}$ by closure of $\mathcal{F}$ under countable unions. Thus by transfinite induction we obtain $\mathcal{B}(X) \subseteq \mathcal{F}$. ⊣

We next define an operation on sets in product Polish spaces of the form $\omega \times X$.

Definition 5.4. Let X be Polish, and $A \subseteq \omega \times X$. We define

$$\exists^\omega A = \{x \in X \mid (\exists n \in \omega)\langle n, x\rangle \in A\}$$

and

$$\forall^\omega A = \{x \in X \mid (\forall n \in \omega)\langle n, x\rangle \in A\}.$$

These operations determine corresponding operations on pointclasses:

$$\exists^\omega \mathbf{\Gamma} = \{(\exists^\omega A, X) \mid (A, \omega \times X) \in \mathbf{\Gamma}\}$$

and similarly for $\forall^\omega \mathbf{\Gamma}$.

Before proceeding, we make a general comment about taking *slices*. If X, Y are Polish and $A \subseteq X \times Y$, the **slice of A at x** is defined to be

$$A_x = \{y \in Y \mid \langle x, y\rangle \in A\}.$$

Note that if $\mathbf{\Gamma}$ is closed under continuous substitution, then $A \in \mathbf{\Gamma}$ implies $A_x \in \mathbf{\Gamma}$ for all $x \in X$. This gives us the following:

Proposition 5.5. *Each $\mathbf{\Sigma}^0_\alpha$ is closed under $\exists^\omega$, and each $\mathbf{\Pi}^0_\alpha$ is closed under $\forall^\omega$. In symbols: $\exists^\omega \mathbf{\Sigma}^0_\alpha \subseteq \mathbf{\Sigma}^0_\alpha$ and $\forall^\omega \mathbf{\Pi}^0_\alpha \subseteq \mathbf{\Pi}^0_\alpha$.*

Proof. It suffices to show $\exists^\omega \mathbf{\Sigma}^0_\alpha \subseteq \mathbf{\Sigma}^0_\alpha$, since $\forall^\omega A = \neg(\exists^\omega \neg A)$. By the previous remarks, each A_n is in $\mathbf{\Sigma}^0_\alpha$. But $\exists^\omega A = \bigcup_{n\in\omega} A_n$, and so we're done. ⊣

There is one more important fact about the Borel hierarchy we would like to show. Namely, we want to show new sets are obtained at each level, so that in particular $\mathbf{\Sigma}^0_\alpha \neq \mathbf{\Sigma}^0_{\alpha+1}$ for all $1 \leq \alpha < \omega_1$. This is accomplished by the next theorem.

Theorem 5.6 (The Hierarchy Theorem). *Let X be an uncountable Polish space. Then for each α there is some $A \subseteq X$ with $A \in \mathbf{\Sigma}^0_\alpha \setminus \mathbf{\Pi}^0_\alpha$.*

Notice that by taking complements we get the existence of a set in $\mathbf{\Pi}^0_\alpha \setminus \mathbf{\Sigma}^0_\alpha$. To prove this theorem the main technical tool we will make use of is the notion of a *universal set.*

DEFINITION 5.7. Let X and Y be Polish spaces, and let $\mathbf{\Gamma}$ be a pointclass. A set $W \subseteq X \times Y$ is $\mathbf{\Gamma}$-universal for Y if $W \in \mathbf{\Gamma}$ and for every $A \subseteq Y$ with $A \in \mathbf{\Gamma}$ there is some $x \in X$ such that $A = W_x$.

THEOREM 5.8. *Let X be a Polish space. For each α with $\mathbf{\Gamma}$ equal to either $\mathbf{\Sigma}^0_\alpha$ or $\mathbf{\Pi}^0_\alpha$, there is a $\mathbf{\Gamma}$-universal set $W \subseteq 2^\omega \times X$ for X.*

We start with the open sets.

PROPOSITION 5.9. *For each Polish space X there is a universal open ($\mathbf{\Sigma}^0_1$) set $W \subseteq 2^\omega \times X$.*

PROOF. The idea is simple: Since the space X is Polish it has a countable basis $\{U_i\}_{i\in\omega}$. Thus the open subsets of X are exactly the countable unions of sets of the form U_i, and since there are only $\mathfrak{c} = |2^\omega|$ of these we can use each $x \in 2^\omega$ to encode the possible unions.

Thus we define $W \subseteq 2^\omega \times X$ by

$$\langle f, x\rangle \in W \text{ if and only if } (\exists n \in \omega) x \in U_n \text{ and } f(n) = 1.$$

We need to see W is open. But this is clear, since

$$W = \bigcup_{n\in\omega} \{f \in 2^\omega \mid f(n) = 1\} \times U_n$$

and each set $\{f \in 2^\omega \mid f(n) = 1\} \times U_n$ is open.

Next note W is universal. For let $A \subseteq X$ be open. Then A can be written as a countable union of the U_n; we let $f \in 2^\omega$ indicate which, so that $A = \bigcup\{U_n \mid f(n) = 1\}$. It follows that $W_f = A$ straight from the definition. $\dashv$

The next proposition is clear.

PROPOSITION 5.10. *If $W \subseteq 2^\omega \times X$ is $\mathbf{\Gamma}$-universal, then $(2^\omega \times X) \setminus W$ is $\neg\mathbf{\Gamma}$-universal.*

The last step in the proof of Theorem 5.8 is the following proposition.

PROPOSITION 5.11. *Let $1 \leq \alpha < \omega_1$, and assume that for each $\beta < \alpha$ there is a $\mathbf{\Pi}^0_\beta$-universal set $W^\beta \subseteq 2^\omega \times X$. Then there is a $\mathbf{\Sigma}^0_\alpha$-universal set $W \subseteq 2^\omega \times X$.*

In order to prove this last proposition we first need to bring up coding. It will be useful for us to have a way of encoding an infinite sequence of elements of 2^ω by a single $f \in 2^\omega$. One way to do this is by defining, for $f \in 2^\omega$, $(f)_n$ by $(f)_n(m) = f(2^m 3^n)$. There are two important things to notice. One: the map sending f to $(f)_n$ is continuous, and two: any countable sequence $\langle g_n\rangle_{n\in\omega}$ of members of 2^ω is coded by some f so that $(f)_n = g_n$ for all n.

PROOF OF PROPOSITION 5.11. Let $\{\gamma_k\}_{k\in\omega}$ be an enumeration of all the ordinals below α, enumerated in such a way that each one repeats infinitely often. Notice that then $A \subseteq X$ belongs to $\mathbf{\Sigma}^0_\alpha$ exactly when there are sets A_k in $\mathbf{\Pi}^0_{\gamma_k}$ with $A = \bigcup_{k\in\omega} A_k$.

Now define W by

$$\langle f, x\rangle \in W \text{ if and only if } (\exists k)\langle (f)_k, x\rangle \in W^{\gamma_k}.$$

We claim W belongs to $\mathbf{\Sigma}^0_\alpha$. To see this, for each $k \in \omega$ let B_k be equal to the collection of $\langle f, x\rangle$ such that $x \in W^{\gamma_k}_{(f)_k}$. Then clearly $W = \bigcup_{k\in\omega} B_k$. Now define $\varphi_k : 2^\omega \times X \to 2^\omega \times X$ by $\varphi_k(f, x) = \langle (f)_k, x\rangle$. This map is continuous, and $\varphi_k^{-1}[W^{\gamma_k}]$ is exactly B_k since $\langle f, x\rangle \in B_k$ if and only if $\langle x, (f)_k\rangle \in W^{\gamma_k}$.

We finish by showing W is $\mathbf{\Sigma}^0_\alpha$-universal. Let $A \subseteq X$ be $\mathbf{\Sigma}^0_\alpha$. Then $A = \bigcup_{k\in\omega} A_k$ where each $A_k \subseteq X$ is in $\mathbf{\Pi}^0_{\gamma_k}$. For each k using the universality of W^{γ_k}, we fix g_k so that $A_k = W^{\gamma_k}_{g_k}$. Let f be such that $(f)_k = g_k$. Then we have that $x \in W_f$ exactly when $\langle f, x\rangle$ belongs to W. This holds exactly when for some k we have $\langle (f)_k, x\rangle \in W^{\gamma_k}$, that is $\langle g_k, x\rangle \in W^{\gamma_k}$ which itself is equivalent to $x \in A_k$. So $W_f = A$ as needed. ⊣

Proof of Theorem 5.6. We just do the case $X = 2^\omega$; we leave greater generality for the exercises. Let $W \subseteq 2^\omega \times 2^\omega$ be a $\mathbf{\Sigma}^0_\alpha$-universal set. Set

$$A = \{x \in 2^\omega \mid x \in W_x\}.$$

Then $A \in \mathbf{\Sigma}^0_\alpha$: for if $f : 2^\omega \to 2^\omega \times 2^\omega$ is given by $f(x) = \langle x, x\rangle$, then f is continuous, and $A = f^{-1}W$.

We claim the set A does not belong to $\mathbf{\Pi}^0_\alpha$. For, supposing for contradiction that A did belong to $\mathbf{\Pi}^0_\alpha$, its complement $\neg A = 2^\omega \setminus A$ would belong to $\mathbf{\Sigma}^0_\alpha$. By universality of W that means there is a $y \in 2^\omega$ such that $\neg A = W_y$. But then by definition of A, we have $y \in A$ if and only if $y \in W_y$ if and only if $y \notin A$, a contradiction. ⊣

§6. Wolfe's Theorem. We have seen that closed determinacy $\mathbf{\Pi}^0_1$-DET and open determinacy $\mathbf{\Sigma}^0_1$-DET both hold. Without too much work, we can extend this to the next level of the Borel hierarchy, $\mathbf{\Sigma}^0_2$.

THEOREM 6.1. *Let $A \subseteq \omega^\omega$ be $\mathbf{\Sigma}^0_2$. Then $G(A)$ is determined.*

PROOF. The proof will be an illustration of a technique that appears in virtually all proofs of determinacy. Namely, we reduce the determinacy of the game we are interested in to that of a certain auxiliary game which is *closed*.

Let A be $\mathbf{\Sigma}^0_2$. Then $A = \bigcup_{n\in\omega} B_n$ with each $U_n \in \mathbf{\Sigma}^0_1$.

In order to win the game $G(A)$, Player I must ensure that the play produced belongs to A, that is, to every open set U_n. Since the sets U_n are open, we have collections $P_n \subseteq \omega^{<\omega}$ of positions so that $U_n = \bigcup_{s\in P_n} N_s$. To win the game $G(A)$ Player I must enter the sets P_n "one at a time." We define an auxiliary game G^* that makes this precise.

The game is played in a tree on $\mathcal{P}(\omega^{<\omega}) \cup \omega^{<\omega}$. Player I must produce strategies σ_n in games on ω; Player II must respond with positions compatible with σ_n.

I	σ_0		σ_1		$\dots$	σ_n		$\dots$
II		s_0		s_1	$\dots$		s_n	$\dots$

FIGURE 2. The auxiliary game G^*.

The rules of the game are as follows: σ_0 must be a strategy in $\omega^{<\omega}$ that is winning for Player I in $G(U_0)$. Player II must respond with a position $s_0 \in \sigma_0$ so that $N_{s_0} \subseteq U_0$ and for all $i < \ell(s)$, $N_{s_0} \not\subseteq U_0$.

At the $n+1$th round of the game, Player I is required to play a strategy winning in $G(U_{n+1}; (\omega^{<\omega})_{s_n})$. Player II must respond with a proper extension $s_{n+1} \supsetneq s_n$ so that $N_{s_{n+1}} \subseteq U_{n+1}$, and whenever $\ell(s_n) < i < \ell(s_{n+1})$, we have $N_{s_{n+1}\restriction i} \not\subseteq U_{n+1}$.

If at any time Player I cannot produce a strategy as required, then the game ends with Player II the winner. Player I wins all infinite plays. Note that the positions of G^* form a tree T in which all terminal nodes have even length; thus G^* has the form $G([T]; T)$.

By the Gale-Stewart Theorem, the game G^* is determined. Suppose first that Player I has a winning strategy σ^*. It is easy to see how to convert σ^* to a winning strategy σ for Player I in $G(A)$: namely, let σ_0 be the first move made by σ^*. Have Player I play according to σ_0 until a position s_0 is reached with $N_{s_0} \subseteq U_0$ (this is bound to happen since σ_0 is winning for Player I in $G(U_0)$). Then attribute the move s_0 to Player II in the game G^*. σ^* produces a strategy $\sigma_1 = \sigma^*(\langle \sigma_0, s_0\rangle)$, which we use to play against II until a position s_1 is reached with $N_{s_1} \subseteq U_1$ which we attribute to II in G^*; and so on.

Any play x compatible with the strategy σ we have described is clearly in A, since for all n we have $s_n \subseteq x$ and $N_{s_n} \subseteq U_n$ for each n. So σ is winning for Player I in $G(A)$.

Suppose now that Player II has a winning strategy τ^* in G^*. It is a bit harder to see how to convert this strategy to a winning τ for II in $G(A)$, since Player II has the daunting task of attributing strategies σ_n to Player I as moves in G^*. We need to play in such a way that if we reach some position s_0 with $N_{s_0} \subseteq U_0$, for example, then $s_0 = \tau^*(\langle \sigma_0\rangle)$ for some legal move σ_0 by Player I—*without* knowing in advance what σ_0 is.

To simplify notation, we work on the case $n = 0$. Let Q be the set of nodes in $\omega^{<\omega}$ that are obtained as a response by τ^* to some legal move by Player I; that is,

$$Q = \{s \in \omega^{<\omega} \mid \tau^*(\langle\sigma\rangle) = s \text{ for some } \sigma \text{ winning for Player I in } G(U_0)\}.$$

Notice that if Q is empty, then there is no winning strategy for Player I in $G(U_0)$. By Gale-Stewart, we can then let τ be a winning strategy for II in $G(U_0)$, which is then automatically winning for II in $G(A)$.

So we can assume Q is non-empty. Consider now the set

$$V = \bigcup\{N_s \mid N_s \subseteq U_0 \text{ and no initial segment of } s \text{ is in } Q\}.$$

Claim. *Player II has a winning strategy τ_0 in $G(V)$.*

Proof of claim. We claim Player I cannot win the game $G(V)$. For if σ_0 were a winning strategy in $G(V)$ for Player I, then it would also be winning in $G(U_0)$, and so be a legal first move in G^*. Consider $\tau^*(\langle\sigma_0\rangle) = s_0$. Then $s_0 \in Q$, and by the rules of the game G^*, $N_{s_0\restriction i} \not\subseteq U_0$ for all $i < \ell(s)$. But then no extension of s_0 can belong to V, by definition, contradicting that σ_0 was winning for Player I in $G(V)$!

V is open. By Gale-Stewart, there is a winning strategy τ_0 for Player II in $G(V)$. ⊣

Let τ_0 be given by the claim. Have τ agree with τ_0 until, if ever, a position s_0 with $N_{s_0} \subseteq U_0$ is reached. (If no such position is reached, then τ_0 produces an infinite play outside of U_0, which is then a win for II in $G(A)$.) By the fact that τ_0 is winning in $G(V)$ for Player II, we must have s_0 in Q. So let σ_0 be some strategy for Player I so that $\langle\sigma_0\rangle \in T$, and $\tau^*(\langle\sigma_0\rangle) = s_0$.

Repeating this argument at s_0 with $\langle\sigma_0, s_0\rangle \in \tau^*$, we have that either Player I has run out of legal moves in G^*, in which case we have a winning strategy for Player II to avoid U_1 and thus get out of A, or we can play so that if s_1 is reached with $N_{s_1} \subseteq U_1$, then there is some σ_1 legal for II in G^* with $\tau^*(\langle\sigma_0, s_0, \sigma_1\rangle) = s_1$. Continuing in this way, because τ^* is winning for Player II in G^*, we must eventually reach a position in G^* at which Player I has no legal moves, and so Player II wins in $G(A)$. ⊣

The main idea of this proof—reducing to a closed game in which Player I plays auxiliary moves which are *strategies*—can be stretched with a great deal of difficulty to give a proof of the following landmark theorem.

Theorem 6.2 (Martin). *All Borel games are determined.*

Later on in the course, we will obtain this (and more) determinacy by assuming the existence of a certain large cardinal (one rather larger than an inaccessible cardinal). Martin's proof, however, goes through in just the standard axioms of set theory, ZF (no choice necessary!).

This sharper proof is a bit beyond the scope of these notes, but we make a few remarks. The idea is essentially to show that the determinacy of a given closed set can be *continuously* reduced to that of a *clopen* set in a larger tree; in fact, this can be done simultaneously for countably many closed sets. If we do this for all of the countably many sets involved in the construction of a $\mathbf{\Sigma}^0_{\alpha+1}$ set, then we reduce determinacy of this set to that of a $\mathbf{\Sigma}^0_\alpha$ set (in a larger tree). By iterating this procedure α many times, we reduce to determinacy of a $\mathbf{\Sigma}^0_1$ set in a tree on $\mathcal{P}^{\alpha+1}(\omega)$.

It is a result due to Harvey Friedman that this use of the Power Set Axiom is necessary. Indeed, $\mathbf{\Sigma}^0_{1+\alpha+3}$-DET cannot be proven without (roughly) $\alpha + 1$ many iterated applications of the Power Set Axiom to ω. This is a somewhat remarkable state of affairs: Borel determinacy, though it is a simple statement about subsets of ω^ω, cannot be proven without appeal to the existence of uncountably many larger infinities than ω^ω!

Borel determinacy is, in a sense, the extent of determinacy provable without recourse to large cardinals: if $\mathbf{\Gamma}$ is a pointclass closed under continuous substitution that *properly* contains $\bigcup_{\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha$, then $\mathbf{\Gamma}$-DET implies ZFC is consistent with the existence of inaccessible cardinals, and more—in particular, $\mathbf{\Gamma}$-DET is not provable in ZFC.

We will see more of this connection between determinacy and large cardinals later on. For now, we explore determinacy principles as a powerful tool for reasoning about sets of reals.

§7. The Baire Property and the Banach-Mazur Game. Recall that a set $A \subseteq X$ is meager in X if it is contained in some countable union of nowhere dense sets; A is comeager in X if its complement is meager. Note that meagerness is a relative notion, in the sense that a set meager in X may not be meager in a subset of X; for example, 2^ω is non-meager in 2^ω by the Baire category theorem, but is meager as a subset of ω^ω. The following proposition shows that meagerness persists between a space and its subsets, provided those subsets are *open*.

PROPOSITION 7.1. *Suppose $X \subseteq Y$ and X is an open set in Y. Then $A \subseteq X$ is meager in X exactly when it is meager in Y.*

PROOF. First suppose A is meager as a subset of X. Then $A \subseteq \bigcup_{n\in\omega} C_n$ where each $C_n \subseteq X$ is nowhere dense as a subset of X. We claim each C_n is also nowhere dense as a subset of Y, from which it follows that A is meager in Y. For if $U \subseteq Y$ is a non-empty open set, then $U \cap X$ is open in X. If it is empty, then we already have $U \cap C_n = \varnothing$; otherwise, since C_n is nowhere dense there is $V_0 \subseteq U \cap X$ which is open in X and disjoint from C_n. Then we have $V_0 = V \cap X$ for some open $V \subseteq Y$, and we easily see that V is disjoint from C_n as needed.

Going the other way, suppose A is meager as a subset of Y. Then $A \subseteq \bigcup_{n\in\omega} C_n$, where each C_n is nowhere dense as a subset of Y. The reader may check that $C_n \cap X$ is nowhere dense as a subset of X, which is enough. ⊣

We think of the meager sets as small, or thin. The next class of sets we define are those which are just a meager set away from being open.

DEFINITION 7.2. Let X be a Polish space. A set $B \subseteq X$ has the **Baire property** if there is some open set $U \subseteq X$ such that $B \triangle U$ is meager. (Here $B \triangle U$ is the symmetric difference $(B \setminus U) \cup (U \setminus B)$.)

PROPOSITION 7.3. *Suppose $B \subseteq \omega^\omega$ has the Baire property. Then either B is meager or there is some s such that $B \cap N_s$ is comeager in the topology on N_s.*

PROOF. Since B has the Baire property there is some open set U such that $(B \setminus U) \cup (U \setminus B)$ is meager. If U is empty then B is meager. So assume U is non-empty. Let $N_s \subseteq U$. Since $U \setminus B$ is meager so is $N_s \setminus B$.

We have $N_s = (B \cap N_s) \cup (N_s \setminus B)$. Since $N_s \setminus B$ is meager in N_s by the last proposition, the former set is comeager in N_s. ⊣

PROPOSITION 7.4. *Every Borel set has the Baire property.*

PROOF. It is enough to show that the open sets have the Baire property, and that the class of sets with the Baire property is closed under intersection and countable unions.

Certainly the open sets have the Baire property; if B is open, take $U = B$ and $B \triangle U = \varnothing$ is meager. For countable unions, suppose $B_0, B_1, \dots$ all have the Baire property. Thus for each B_n there is an open set U_n so that $B_n \triangle U_n$ is meager. It follows that the union $\bigcup_{n<\omega} B_n \triangle U_n$ is meager. Let $B = \bigcup_{n<\omega} B_n$ and let $U = \bigcup_{n<\omega} U_n$. Since $B \triangle U \subseteq \bigcup_{n\in\omega} B_n \triangle U_n$ we see that B has the Baire property.

Now for complements. Suppose B has the Baire property. We want to show $\omega^\omega \setminus B$ has the Baire property. Let U be open with $B \triangle U$ meager. Let C be the closure of U; so $\omega^\omega \setminus C$ is an open set. Notice that $(\omega^\omega \setminus B) \triangle (\omega^\omega \setminus U)$ is equal to $B \triangle U$. Also notice that $(\omega^\omega \setminus B) \triangle (\omega^\omega \setminus C) \subseteq (\omega^\omega \setminus B) \triangle (\omega^\omega \setminus U) \cup (C \setminus U)$. If we can show that $C \setminus U$ is nowhere dense we will be done.

Let $V \subseteq \omega^\omega$ be an open set; we want to find an open $W \subseteq V$ disjoint from $C \setminus U$. If $V \cap (C \setminus U)$ is empty there is nothing for us to do. Otherwise, let $x \in V \cap (C \setminus U)$. Then x is a limit point of U; hence by definition of a limit point $V \cap U$ is non-empty. Take $W = V \cap U$; then $W \cap (C \setminus U)$ is empty. ⊣

DEFINITION 7.5. We say that $A \subseteq \omega^\omega$ is a **tail set** if for every $x, y \in \omega^\omega$ if there exists some $k \in \omega$ such that $x(j) = y(j)$ for all $j > k$, then x belongs to A exactly when y belongs to A.

In other words, if we consider the equivalence relation E_0 defined by $x \, E_0 \, y$ if and only if there exists some $k \in \omega$ such that $x(j) = y(j)$ for all $j > k$, then a tail set is one which is a union of E_0 equivalence classes.

THEOREM 7.6. *If A is a tail set with the Baire property, then A is either meager or comeager.*

PROOF. Let us suppose towards a contradiction that A is neither comeager nor meager. Since A has the Baire property, there is by Proposition 7.3 some s so that $A \cap N_s$ is comeager in N_s. By the same reasoning applied to the complement of A, there is some t so that $(\omega^\omega \setminus A) \cap N_t$ is comeager in N_t; that is, $A \cap N_t$ is meager in N_t.

Extending one of s, t if necessary, we may by Proposition 7.1 assume $\ell(s) = \ell(t) = k$. Consider the map $\varphi : N_s \to N_t$ defined by

$$\varphi(x)(i) = \begin{cases} t(i) \text{ if } i < k, \\ x(i) \text{ otherwise.} \end{cases}$$

This is clearly a homeomorphism. Thus the image of $A \cap N_s$ under φ should be comeager in N_t. But in fact $\varphi(x) \in A$ if and only if $x \in A$ because A is a tail set. Then $\varphi[A \cap N_s] = A \cap N_t$ with the latter meager in N_t, a contradiction! ⊣

The same argument of course works for subsets of 2^ω.

This theorem is handy in immediately identifying that certain sets are meager or comeager. For example, the set $\{x \in \omega^\omega : \lim_{n\to\infty} x(n) = \infty\}$ is a tail set and is Borel, so has the Baire property. So without even thinking about it we know it must be either meager or comeager.

Under the Axiom of Choice there are sets which do not have the Baire property (exercise). On the other hand, under AD there are no such examples. This we aim to show next.

Definition 7.7. Let $A \subseteq \omega^\omega$. We define the **Banach-Mazur Game** $G_{\mathrm{BM}}(A)$ to be the game with moves in $\omega^{<\omega}$, played as follows: Player I plays s_0, Player II plays $s_1 \supsetneq s_0$, Player I plays $s_2 \supsetneq s_1$, and so on.

$$\begin{array}{c|cccccccc} \mathrm{I} & s_0 & & s_2 & & \dots & s_{2n} & & \dots \\ \hline \mathrm{II} & & s_1 & & s_3 & \dots & & s_{2n+1} & \dots \end{array}$$

Figure 3. The Banach-Mazur game $G_{\mathrm{BM}}(A)$.

A play of the game is an increasing sequence $\langle s_n \rangle_{n\in\omega}$ of elements of s_n. Set $x = \bigcup_{n\in\omega} s_n$. Then Player I wins if $x \in A$; otherwise, Player II wins.

Claim. *A is meager if and only if Player II has a winning strategy in $G_{\mathrm{BM}}(A)$.*

Proof. First suppose that A is meager. Write $A \subseteq \bigcup_{n\in\omega} C_n$ where each C_n is nowhere dense. Player II's strategy essentially consists in proving the Baire category theorem with the sets C_n. Namely, given s_{2n}, since C_n is nowhere dense, there is some $s_{n+1} \supsetneq s_n$ with $N_{s_{n+1}} \cap C_n = \varnothing$. Playing in this fashion clearly produces a real $x \notin A$, so this strategy is winning for Player II.

Conversely, suppose that Player II has some winning strategy τ. For each position p in the Banach-Mazur game, let $s_p = \bigcup_{i<\ell(p)} p(i)$ denote the node in $\omega^{<\omega}$ reached by p (so if $p = \langle s_0, \dots, s_n\rangle$ then $s_p = s_n$, and $s_\varnothing = \varnothing$). For each even-length position $p \in \tau$, we define a set $D_p \subseteq \omega^\omega$ by

$$D_p = \bigcup \{N_t \mid t \perp s_p \text{ or } (\exists s \supsetneq s_p)\tau(p^\frown\langle s\rangle) = t\}.$$

So x belongs to D_p if and only if $x \perp p$, or there is some move by Player I at p which prompts τ to respond with an initial segment of x.

Clearly D_p is open; we claim it is dense. Fix $s \in \omega^{<\omega}$ with $\ell(s) > \ell(s_p)$. If $s \perp s_n$, then by definition of D_p we have $N_s \subseteq D_p$. Otherwise $s \supsetneq s_p$, so s is a legal move for Player I at p. Set $t = \tau(p^\frown\langle s\rangle)$. Then $t \supsetneq s$ and $N_t \subseteq D_p$ as needed.

Now $\bigcap_p D_p$ is a dense G_δ, so is comeager. Suppose $x \in \bigcap_p D_p$. Then we can inductively construct a play $\langle s_0, s_1, s_2, \dots\rangle$ of $G_{\mathrm{BM}}(A)$ compatible with τ and so that $x = \bigcup_{n\in\omega} s_n$: let s_0 be a move by Player I witnessing $x \in D_\varnothing$, and s_1 τ's response. And inductively, set s_{2n} a witness to membership of x in $D_{\langle s_0,\dots,s_{2n}\rangle}$, and $s_{2n+1} = \tau(\langle s_0, \dots, s_{2n}\rangle) \subseteq x$.

Since τ is winning for Player II, $x \notin A$. So A is disjoint from a comeager set, hence meager. $\dashv$

Claim. *Player I has a winning strategy in $G_{\mathrm{BM}}(A)$ if and only if there is some $s \in \omega^{<\omega}$ with A comeager in N_s.*

Proof. The same arguments in the proof of the previous claim show this. Just note that after Player I plays a first move s_0, the game is essentially $G_{\mathrm{BM}}(N_{s_0} \setminus A)$ with the roles of the players reversed. $\dashv$

Claim. *Given $A \subseteq \omega^\omega$ there is an open set U_A such that if $G_{\mathrm{BM}}(A\setminus U_A)$ is determined, then A has the Baire property.*

Proof. Let $U_A = \bigcup\{N_s \mid A \text{ is comeager in } N_s\}$. We claim that I cannot have a winning strategy in $G_{\mathrm{BM}}(A \setminus U_A)$; supposing otherwise, we have by the last claim that

$(A \setminus U_A)$ is comeager in N_s for some s. But then clearly A is comeager in N_s, so that $N_s \subseteq U_A$, so that $(A \setminus U_A) \cap N_s$ is empty, a contradiction.

Then if the game is determined it must be Player II who has the winning strategy. By the first claim $A \setminus U_A$ is meager. And $U_A \setminus A$ is also meager, being contained in the union of all the $N_s \setminus A$ for which this set is meager. So $A \triangle U_A = (A \setminus U_A) \cup (U_A \setminus A)$ is meager. Thus A has the Baire property. ⊣

THEOREM 7.8. *Assume* AD*. Then every set $A \subseteq \omega^\omega$ has the Baire property.*

PROOF. By the previous claims it is enough to see that $G_{\mathrm{BM}}(A)$ is determined for all $A \subseteq \omega^\omega$. But $G_{\mathrm{BM}}(A)$ can clearly be coded by a game on ω of the form $G(A^*)$, for example by fixing an enumeration $\langle t_i \rangle_{\in\omega}$ of $\omega^{<\omega} \setminus \{\varnothing\}$ and setting $x \in A^*$ if and only if $\varphi(x) = t_{x(0)}{}^\frown t_{x(1)}{}^\frown t_{x(2)}{}^\frown \cdots \in A$. ⊣

This argument yields the following slight refinement which still applies in settings where Choice may hold:

THEOREM 7.9. *Let $\mathbf{\Gamma}$ be a pointclass closed under continuous substitution and finite intersection, with $\mathbf{\Gamma} \supseteq \mathbf{\Pi}^0_1$. If $\mathbf{\Gamma}$-DET holds, then every member of $\mathbf{\Gamma}$ has the Baire property.*

PROOF. Let $A \in \mathbf{\Gamma}$. By assumption $A_0 = A \setminus U_A \in \mathbf{\Gamma}$. Again it is sufficient to show $G_{\mathrm{BM}}(A_0)$ is determined; but by the proof of Theorem 7.8 we have $A_0^* \subseteq \omega^\omega$ and a game $G(A_0^*)$ whose determinacy is clearly equivalent to that of $G_{\mathrm{BM}}(A_0)$. Since φ is a continuous map, we have $A_0^* = \varphi^{-1}[A_0] \in \mathbf{\Gamma}$, and by $\mathbf{\Gamma}$-DET, we are done. ⊣

§8. Wadge and Lipschitz Reducibility. What does is mean for one set of reals $A \subseteq \omega^\omega$ to be simpler than another, B? If A and B both happen to be Borel the answer would be clear: A is simpler than B if it appears in a lower level of the Borel hierarchy. But what if these sets aren't Borel?

Here is one idea. Suppose B is Borel, and α is minimal with $B \in \mathbf{\Sigma}^0_\alpha$. For any continuous function $f : \omega^\omega \to \omega^\omega$, the set $A = f^{-1}[B]$ is also in $\mathbf{\Sigma}^0_\alpha$ by closure under continuous reducibility. But it could also be in $\mathbf{\Sigma}^0_\beta$ for some $\beta < \alpha$, and indeed, all sets in $\mathbf{\Sigma}^0_\beta$ for $\beta < \alpha$ are of the form $f^{-1}[B]$ for some continuous f.

In light of this, A should be thought of as simpler than B precisely when $f^{-1}[B] = A$ for some continuous f. Intuitively, the function f reduces the problem of deciding whether $x \in A$ to that of deciding $f(x) \in B$. Since we think of continuous functions as being "simple", the problem of deciding membership in A is no harder than that for B (compare analogous notions—Turing reducibility or polynomial time reducibility—in the theories of computation and complexity).

DEFINITION 8.1. Let $A, B \subseteq \omega^\omega$. We write $A \leq_{\mathrm{W}} B$ and say A is **Wadge reducible** (or **continuously reducible**) to B if there is a continuous function $f : \omega^\omega \to \omega^\omega$ such that $x \in A$ if and only if $f(x) \in B$.

Of course, this says exactly that $A \leq_{\mathrm{W}} B$ if and only if $f^{-1}[B] = A$ for some continuous $f : \omega^\omega \to \omega^\omega$. We immediately get:

PROPOSITION 8.2. *Suppose $\mathbf{\Gamma}$ is closed under continuous substitution. If $B \in \mathbf{\Gamma}$ and $A \leq_{\mathrm{W}} B$, then $A \in \mathbf{\Gamma}$.*

So for example, each level of the Borel hierarchy is closed downwards under $\leq_W$.
The basic properties of $\leq_W$ are immediate:

Proposition 8.3. *The relation $\leq_W$ is a preorder of $\mathcal{P}(\omega^\omega)$. That is,*

1. $\leq_W$ *is transitive.*
2. $\leq_W$ *is reflexive.*

Note however that $\leq_W$ is not antisymmetric: for example, if $A = \{x \in \omega^\omega \mid (\forall n)x(n)$ is even$\}$ and $B = \{x \in \omega^\omega \mid (\forall n)x(n)$ is odd$\}$, then $A \leq_W B$ and $B \leq_W A$ (with the same f witnessing both directions), but clearly $A \neq B$. However, under our notion of reducibility, these two sets have exactly the same complexity, and we introduce a notion to identify them.

Definition 8.4. Let $A, B \subseteq \omega^\omega$. We say that A and B are **Wadge equivalent**, written $A \equiv_W B$, if $A \leq_W B$ and $B \leq_W A$.

Proposition 8.5. *The relation $A \equiv_W B$ on $\mathcal{P}(\omega^\omega)$ is an equivalence relation.*

For $A \subseteq \omega^\omega$ we define the **Wadge degree of** A to be the equivalence class $[A]_W = \{B \subseteq \omega^\omega \mid B \equiv_W A\}$ of A with respect to $\equiv_W$. We will use the letters a, b, c to denote Wadge degrees.

Each Wadge degree is a pointclass. By definition each Wadge degree is closed under continuous substitution (restricting now to functions $f : \omega^\omega \to \omega^\omega$). They are also the smallest such pointclasses.

Proposition 8.6. *Suppose $\boldsymbol{\Gamma}$ is closed under continuous substitution. Then $\boldsymbol{\Gamma}(\omega^\omega)$ is a disjoint union of Wadge degrees.*

It's easy to see that the Wadge order and the operation of complementation is well-defined on the Wadge degrees:

Proposition 8.7. *Let $A, A', B, B' \subseteq \omega^\omega$ with $A \equiv_W B$ and $A' \equiv_W B'$. Then*

1. $A \leq_W B$ *if and only if* $A' \leq_W B'$, *and*
2. $\neg A \equiv_W \neg A'$.

This justifies our use of the notation $\neg a = [\neg A]_W$ when $A \in a$, and $a \leq_W b$ when $A \leq_W B$ for any $A \in a$ and $B \in b$ (the choice of representative A and B does not matter). Thus $\leq_W$, taken as an order on the collection of Wadge degrees, is antisymmetric, and so $\leq_W$ partially orders the Wadge degrees.

Example 8.8. If $A \leq_W \omega^\omega$, then $A = \omega^\omega$. So $[\omega^\omega]_W = \{\omega^\omega\}$. The dual degree is $[\varnothing]_W = \{\varnothing\}$. If B is any non-empty set with non-empty complement, then $\omega^\omega, \varnothing$ are both $\leq_W$-below B.

Example 8.9. Suppose C is clopen, and B is any non-empty set with non-empty complement. Then the map sending $x \in C$ to some fixed $y_0 \in B$, and $x \notin C$ to some fixed $y_1 \notin B$, is a continuous reduction witnessing $C \leq_W B$.

So we have that $\varnothing, \omega^\omega$ are the unique $\leq_W$-minimal sets, and just above them is the Wadge degree of non-trivial clopen sets.

Before venturing any further up into the world of Wadge degrees, we introduce a game that will be a powerful tool in their analysis.

DEFINITION 8.10. Let $A, B \subseteq \omega^\omega$. The **Wadge Game** $G_{\mathrm{W}}(A,B)$ is played as follows: Players I and II build reals $x, y \in \omega^\omega$, respectively. For ω-many rounds, Player I produces the digits $x(n)$ of x in order, and Player II may either play the next digit of y, or "pass".

$$\begin{array}{c|cccccccccc}
\mathrm{I} & x(0) & x(1) & \dots & x(n_0) & & x(n_0+1) & \dots & x(n_1) & & \dots \\
\hline
\mathrm{II} & & & & & y(0) & & & & y(1) & \dots
\end{array}$$

FIGURE 4. A play of the Wadge game $G_{\mathrm{W}}(A,B)$.

The game concludes after infinitely many rounds. If for some n, $y(n)$ is undefined, then Player II loses. Otherwise, the players have produced $x, y \in \omega^\omega$. Then Player II wins if it is the case that $x \in A \iff y \in B$; that is, either $x \in A$ and $y \in B$, or $x \notin A$ and $y \notin B$.

We remark that though this is not a game on a tree in $\omega^{<\omega}$ as required by Definition 1.3, it is easy enough to encode the Wadge game as one (say, by letting the move 0 by II indicate a pass, and letting $n+1$ code play of $y(i) = n$). Of course, it is simpler to deal with these games directly, with the notions of position, strategy, determined game, etc. adapted appropriately.

PROPOSITION 8.11. *Let $A, B \subseteq \omega^\omega$. Then $A \leq_{\mathrm{W}} B$ if and only if Player II has a winning strategy in $G_{\mathrm{W}}(A,B)$.*

PROOF. The point is that strategies for Player II are essentially the same thing as continuous functions; continuity corresponds to the fact that strategies τ for Player II produce the digits $y(i)$ of y based on a finite amount of information about x.

Suppose first that τ is a winning strategy for Player II. Define $f : \omega^\omega \to \omega^\omega$ in the natural way: Given a real $x \in \omega^\omega$, let $\langle y(n)\rangle_{n\in\omega}$ be the sequence produced by II when Player I plays x. Note that because τ is winning for Player II, we must have $y(n)$ defined for all n, so $y \in \omega^\omega$. Since τ is winning for Player II, we have $x \in A$ if and only if $y \in B$, so that f is a reduction. We just need that f is continuous. For all k, there is some finite stage n_k of the game at which τ produces $y(k)$. Then any extension x' of $x \restriction n_k$ will prompt a response $f(x') = y'$ by τ with $y \restriction k = y' \restriction k$. This shows $f[N_{x\restriction n_k}] \subseteq N_{y\restriction k}$, which proves f is continuous.

Conversely, suppose we have a continuous $f : \omega^\omega \to \omega^\omega$ with $f^{-1}[B] = A$. We define a strategy τ by describing how to play for Player II: First Player II waits until Player I has played naturals $x(0), \dots, x(n_0)$ so that for some $a_0 \in \omega$, we have $f[N_{\langle x(0),\dots,x(n_0)\rangle}] \subseteq f[N_{\langle a_0\rangle}]$; then Player II plays $y(0) = a_0$. Player II then waits until Player I has given $x(0), \dots, x(n_1)$, with $n_1 > n_0$, so that for some a_1, $f[N_{\langle x(0),\dots,x(n_1)\rangle}] \subseteq N_{\langle y(0),a_1\rangle}$; then plays $y(1) = a_1$. And so on.

We claim that against any play by x, this strategy must always produce infinitely many moves by Player II. For otherwise, we obtain a real x, a natural N, and a finite sequence $t = \langle y(0), \dots, y(i-1)\rangle$, so that there do not exist $n > N$ and $a \in \omega$ for which $f[N_{x\restriction n}] \subseteq N_{t^\frown\langle a\rangle}$. But it is clear by our definition of the strategy τ that we should have $f(x) \restriction i = t$. This gives a contradiction, since by continuity of f, we have $N_{x\restriction n} \subseteq f^{-1}[N_{y\restriction i+1}]$ for some $n > N$.

Since the strategy τ responds to x with $f(x)$ and f witnesses $A \leq_{\mathrm{W}} B$, it is immediate that τ is winning for II in the Wadge game $G_{\mathrm{W}}(A,B)$. $\dashv$

The Wadge game is somewhat complicated because we allow Player II to pass. What if we deny Player II this privilege?

DEFINITION 8.12. A function $f : \omega^\omega \to \omega^\omega$ is **Lipschitz** if whenever $f(x) = y$ and $n \in \omega$, $N_{x\restriction n} \subseteq f^{-1}[N_{y\restriction n}]$.

So a function is Lipschitz if the first n digits of $f(x)$ depend only on (at most) the first n digits of x. Clearly Lipschitz functions are continuous, but the reverse is not true: For example, consider the map $\langle x(n)\rangle_{n\in\omega} \mapsto \langle x(2n)\rangle_{n\in\omega}$.

Notice that the composition of Lipschitz functions is Lipschitz. So we feel comfortable making the following definitions.

DEFINITION 8.13. Let $A, B \subseteq \omega^\omega$. We write $A \leq_{\mathrm{L}} B$ and say A is **Lipschitz reducible** to B if there is a Lipschitz function $f : \omega^\omega \to \omega^\omega$ such that for each $x \in \omega^\omega$, we have $x \in A$ if and only if $f(x) \in B$.

If $A \leq_{\mathrm{L}} B$ and $B \leq_{\mathrm{L}} A$, we say A and B are **Lipschitz equivalent**, and write $A \equiv_{\mathrm{L}} B$. The **Lipschitz degrees** are the equivalence classes of $\equiv_{\mathrm{L}}$.

Clearly $A \leq_{\mathrm{L}} B$ implies $B \leq_{\mathrm{L}} A$. It follows that each Wadge degree is a disjoint union of Lipschitz degrees.

Of course, Lipschitz and Wadge reducibility are not the same thing. To see this, notice $N_{\langle 0,0\rangle} \leq_{\mathrm{W}} N_{\langle 0\rangle}$ (as witnessed by the map $x \mapsto \langle x(0)+x(1), x(2), \ldots\rangle$), but $N_{\langle 0,0\rangle} \not\leq_{\mathrm{L}} N_{\langle 0\rangle}$, since any Lipschitz f with $f[N_{\langle 0,0\rangle}] \subseteq N_{\langle 0\rangle}$ would have to satisfy $f[N_{\langle 0,1\rangle}] \subseteq N_{\langle 0\rangle}$, and so fail to be a reduction.

To return to our motivating point: Just as with Wadge reducibility, Lipschitz reducibility admits a characterization in terms of games.

DEFINITION 8.14. Let $A, B \subseteq \omega^\omega$. The **Lipschitz Game** $G_{\mathrm{L}}(A, B)$ is played as follows: Players I and II alternate playing natural numbers to produce reals $x, y \in \omega^\omega$, respectively.

I	$x(0)$		$x(1)$		$x(2)$		$\ldots$	$x(n)$		$\ldots$
II		$y(0)$		$y(1)$		$y(2)$	$\ldots$		$y(n)$	$\ldots$

FIGURE 5. A play of the Lipschitz game $G_{\mathrm{L}}(A, B)$.

Player II wins if it is the case that $x \in A \iff y \in B$; that is, either $x \in A$ and $y \in B$, or $x \notin A$ and $y \notin B$.

The proof of the next proposition is almost identical to that of Proposition 8.11.

PROPOSITION 8.15. *Let $A, B \subseteq \omega^\omega$; then Player II has a winning strategy in $G_{\mathrm{L}}(A, B)$ if and only if $A \leq_{\mathrm{L}} B$.*

When are the Wadge and Lipschitz games determined? The following proposition shows that for many pointclasses $\boldsymbol{\Gamma}$, $\boldsymbol{\Gamma}$-DET is enough to ensure the existence of winning strategies in the Lipschitz game.

PROPOSITION 8.16. *Suppose $\boldsymbol{\Gamma}$-DET holds, where $\boldsymbol{\Gamma}$ is a pointclass closed under continuous substitution, complementation, finite union, and finite intersection. Then for all sets $A, B \subseteq \omega^\omega$ in $\boldsymbol{\Gamma}$, the game $G_{\mathrm{L}}(A, B)$ is determined.*

PROOF. Let π_1, π_2 project ω^ω to even and odd coordinates, respectively; i.e. for $z \in \omega^\omega$, $\pi_1(z) = \langle z(2n)\rangle_{n\in\omega}$ and $\pi_2(z) = \langle z(2n+1)\rangle_{n\in\omega}$. Set

$$C = \{z \in \omega^\omega \mid \pi_1(z) \in A \iff \pi_2(z) \in B\}.$$

It is easy to check, using the closure properties of $\mathbf{\Gamma}$, that $C \in \mathbf{\Gamma}$, and determinacy of $G(C)$ yields a winning strategy for the same player in $G_{\mathrm{L}}(A, B)$. ⊣

So for example, if A, B are both Borel, then $G_{\mathrm{L}}(A, B)$ has Borel payoff, and by Borel determinacy, a winning strategy exists in the Lipschitz game.

Our purpose in introducing Lipschitz reducibility is two-fold. First, the game characterization of Lipschitz reducibility is simpler than that of Wadge reducibility. And second, since Lipschitz reducibility is a refinement of Wadge reducibility, many of the properties of Wadge reduction we prove are true for, and implied by, those for Lipschitz degrees.

The following lemma is the fundamental consequence of determinacy for structure of the Wadge and Lipschitz degrees.

LEMMA 8.17 (Wadge). *Suppose* $\mathbf{\Gamma}$-DET *holds, where* $\mathbf{\Gamma}$ *is a pointclass closed under continuous substitution, complementation, finite union, and finite intersection. For all sets* $A, B \in \omega^\omega$*, we have* $A \leq_{\mathrm{L}} B$ *or* $B \leq_{\mathrm{L}} \omega^\omega \setminus A$*; and so in particular,* $A \leq_{\mathrm{W}} B$ *or* $B \leq_{\mathrm{W}} \omega^\omega \setminus A$.

PROOF. If Player II has a winning strategy in the Wadge game $G_{\mathrm{L}}(A, B)$, then $A \leq_{\mathrm{L}} B$ by Proposition 8.15. Suppose now that Player I has a winning strategy σ in $G_{\mathrm{L}}(A, B)$. Then define $g : \omega^\omega \to \omega^\omega$ by letting $g(y)$ be that unique x which σ produces when Player II plays y. Then g is Lipschitz (in fact, better than Lipschitz, since $g(y) \restriction n + 1$ depends only on $y \restriction n$). Because σ is a winning strategy for Player I, we have that it is not the case that $y \in B \iff g(y) \in A$. Equivalently, $y \in B \iff g(y) \notin A$, and so g witnesses $B \leq_{\mathrm{L}} \omega^\omega \setminus A$. ⊣

Assuming AD imposes a great deal of structure on the Wadge degrees. Even without this assumption, Borel determinacy gives us these nice consequences for the Borel Wadge degrees.

The following definition captures what it means for a set B in $\mathbf{\Gamma}$ to be as complicated as possible; note the analogy with the notion of completeness in (e.g.) complexity theory.

DEFINITION 8.18. Let $\mathbf{\Gamma}$ be a pointclass and let $B \subseteq \omega^\omega$. We say B is **$\mathbf{\Gamma}$-complete** if $B \in \mathbf{\Gamma}$, and for all $A \in \mathbf{\Gamma}(\omega^\omega)$, we have $A \leq_{\mathrm{W}} B$.

For many pointclasses, we have the simplest possible characterization of completeness:

THEOREM 8.19. *Let* $\mathbf{\Gamma}$ *be a pointclass satisfying the hypotheses of Lemma 8.17 that is not self-dual, and so that* $\mathbf{\Gamma}$-DET *holds. Let* $A \subseteq \omega^\omega$*. Then* A *is* $\mathbf{\Gamma}$*-complete if and only if* $A \in \mathbf{\Gamma} \setminus \neg\mathbf{\Gamma}$.

In particular, A *is* $\mathbf{\Sigma}^0_\alpha$*-complete if and only if* $A \in \mathbf{\Sigma}^0_\alpha \setminus \mathbf{\Pi}^0_\alpha$.

PROOF. Suppose A were $\mathbf{\Gamma}$-complete. Suppose towards a contradiction that $A \in \neg\mathbf{\Gamma}$. Let $B \in \mathbf{\Gamma}$ be arbitrary; then $B \leq_{\mathrm{L}} A$ by $\mathbf{\Gamma}$-completeness of A. By closure under continuous substitution, we obtain $B \in \neg\mathbf{\Gamma}$. So $\mathbf{\Gamma} \subseteq \neg\mathbf{\Gamma}$. But this contradicts the assumption that $\mathbf{\Gamma}$ is not self-dual.

Conversely, suppose $A \in \mathbf{\Gamma} \setminus \neg\mathbf{\Gamma}$. We want to show $B \leq_{\mathrm{W}} A$, for all $B \in \mathbf{\Gamma}$. Otherwise, by $\mathbf{\Gamma}$-DET and by the Wadge Lemma 8.17, we have $A \leq_{\mathrm{W}} \neg B$. But then $A \in \neg\mathbf{\Gamma}$ by closure under continuous substitution, a contradiction. ⊣

Notice that the same proof works if instead of considering Wadge reduction, we consider sets A which are complete with respect to Lipschitz reduction. In particular, we obtain that A is $\mathbf{\Sigma}^0_\alpha$-complete precisely when $B \leq_{\mathrm{L}} A$ for all $B \in \mathbf{\Sigma}^0_\alpha$. So $\mathbf{\Sigma}^0_\alpha \setminus \mathbf{\Pi}^0_\alpha$ consists of just a single Wadge (Lipschitz) degree.

Next we show that the Wadge Lemma implies that the Wadge degrees are "almost linearly ordered" by $\leq_{\mathrm{W}}$; the only difficulty is the non-self-dual degrees. Let us write $a <_{\mathrm{W}} b$ if $a \leq_{\mathrm{W}} b$ and $b \not\leq_{\mathrm{W}} a$. Notice that under AD, $\leq_{\mathrm{W}}$ is a partial order, so that $a <_{\mathrm{W}} b$ if any only if $a \leq_{\mathrm{W}} b$ and $a \neq b$.

Similarly define $a <_{\mathrm{L}} b$.

THEOREM 8.20. *Assume* AD. *Then the Wadge degrees are **semilinearly ordered** by* $<_{\mathrm{W}}$. *Precisely: Let a be a Wadge degree. Assume $a = \neg a$. Then for any Wadge degree b, exactly one of the following holds:*

1. $a = b$.
2. $a <_{\mathrm{W}} b$.
3. $b <_{\mathrm{W}} a$.

If instead $a \neq \neg a$, then for any Wadge degree b, we have exactly one of:

1. $a = b$.
2. $\neg a = b$.
3. $a, \neg a <_{\mathrm{W}} b$.
4. $b <_{\mathrm{W}} a, \neg a$.

The same holds for the Lipschitz degrees.

Dropping the AD assumption, the theorem remains true for the Borel Wadge degrees.

PROOF. First we deal with the case $a = \neg a$. Suppose $a \neq b$ and $a \not<_{\mathrm{W}} b$. Then by Wadge's lemma, $b \leq_{\mathrm{W}} \neg a = a$. So $b <_{\mathrm{W}} a$.

Suppose now $a \neq \neg a$. If the first three items fail, we have $a \neq b$, $\neg a \neq b$, and either $a \not<_{\mathrm{W}} b$ or $\neg a <_{\mathrm{W}} b$. Say $\neg a \not<_{\mathrm{W}} b$. Then by Wadge's lemma we get $b \leq_{\mathrm{W}} a$. Since $a \neq b$ it must be the case that $a \not\leq_{\mathrm{W}} b$, so that $b \leq_{\mathrm{W}} \neg a$. Since b does not equal either a or $\neg a$, we have $b <_{\mathrm{W}} a$ and $b <_{\mathrm{W}} \neg a$. The case $a \not<_{\mathrm{W}} b$ follows by symmetry. ⊣

So assuming determinacy, we obtain an almost linear order of the Wadge (Lipschitz) degrees. If we collapse these classes further by identifying the non-self-dual degrees, then we have a way of linearly ordering all sets of reals according to their complexity.

§9. Well-foundedness of the Wadge Hierarchy and the ordinal Θ. The Borel hierarchy provided a nice stratification of the class of Borel sets, and we have seen that under AD, the Wadge order not only refines this picture, but extends it to an almost linear hierarchy on *all* sets of reals. In fact, the picture is even nicer: the Wadge hierarchy is well-founded!

THEOREM 9.1 (Martin). *Assume* $\mathsf{AD} + \mathsf{DC}$. *Then $\leq_{\mathrm{W}}$ is well-founded, i.e. for any non-empty collection $\mathcal{A}$ of sets of reals, there some $A \in \mathcal{A}$ that is $\leq_{\mathrm{W}}$-minimal in A.*

PROOF. We note first that assuming AD, $A <_{\mathrm{W}} B$ implies $A <_{\mathrm{L}} B$ (exercise).

Assume towards a contradiction that $\leq_{\mathrm{W}}$ is not well-founded. Using DC, we obtain an infinite sequence $\langle A_n \rangle_{n \in \omega}$ of sets $A_n \subseteq \omega_\omega$ so that $A_{n+1} <_{\mathrm{W}} A_n$ for all $n \in \omega$. (We remark that without assuming DC, our proof will show just with AD that there is no infinite $<_{\mathrm{W}}$-descending sequence of sets of reals.)

By our first observation, we have $A_{n+1} <_{\mathrm{L}} A_n$ for all n. By definition, $A_n \not\leq_{\mathrm{L}} A_{n+1}$ for all n. We also have $A_n \not\leq_{\mathrm{L}} \neg A_{n+1}$ for all n; since otherwise, we would have

$$A_n \leq_{\mathrm{L}} \neg A_{n+1} \leq_{\mathrm{L}} \neg A_n \leq_{\mathrm{L}} A_{n+1} \leq_{\mathrm{L}} A_n,$$

implying $A_n \equiv_{\mathrm{L}} A_{n+1}$, contrary to our assumption.

For $n \in \omega$, let G_n^0 denote the game $G_{\mathrm{L}}(A_n, \neg A_{n+1})$, and G_n^1 the game $G_{\mathrm{L}}(A_n, A_{n+1})$. By Proposition 8.15, Player I has a winning strategy in each of the games G_n^0, G_n^1. Let σ_n^0, σ_n^1 be winning strategies for Player I in these respective games (note this uses $\mathsf{AC}_\omega(\mathbb{R})$, which follows from AD).

We define a map on 2^ω so that each $u \in 2^\omega$ produces an infinite sequence $\langle x_n^u \rangle_{u\in\omega}$ of elements of ω^ω. This sequence will be obtained by simultaneously playing the Lipschitz games $G_n^{u(n)}$ for $n \in \omega$. In each game $G_n^{u(n)}$, Player I will always play according to $\sigma_n^{u(n)}$, and Player II's moves are obtained by copying Player I's moves from the next game $G_{n+1}^{u(n)}$.

$$
\begin{array}{cc|cccccccc}
 & \mathrm{I} & x_0^u(0) & & x_0^u(1) & & \dots & x_0^u(k) & & \dots \\
\cline{2-10}
G_0^{u(0)} & \mathrm{II} & & x_1^u(0) & & x_1^u(1) & \dots & & x_1^u(k) & \dots \\
 & & & \nearrow & & \nearrow & & & \nearrow & \\
 & \mathrm{I} & x_1^u(0) & & x_1^u(1) & & \dots & x_1^u(k) & & \dots \\
\cline{2-10}
G_1^{u(1)} & \mathrm{II} & & x_2^u(0) & & x_2^u(1) & \dots & & x_2^u(k) & \dots \\
 & & & \nearrow & & \nearrow & & & \nearrow & \\
 & \mathrm{I} & x_2^u(0) & & x_2^u(1) & & \dots & x_2^u(k) & & \dots \\
\cline{2-10}
G_2^{u(2)} & \mathrm{II} & & x_3^u(0) & & x_3^u(1) & \dots & & x_3^u(k) & \dots \\
 & & & \nearrow & & \nearrow & & & \nearrow & \\
 & & & \vdots & & \vdots & & & \vdots & \\
 & \mathrm{I} & x_n^u(0) & & x_n^u(1) & & \dots & x_n^u(k) & & \dots \\
\cline{2-10}
G_n^{u(n)} & \mathrm{II} & & x_{n+1}^u(0) & & x_{n+1}^u(1) & \dots & & x_{n+1}^u(k) & \dots \\
 & & & \nearrow & & \nearrow & & & \nearrow & \\
 & & & \vdots & & \vdots & & & \vdots &
\end{array}
$$

FIGURE 6. Obtaining the sequence $\langle x_n^u \rangle_{n\in\omega}$ using the strategies $\sigma_n^{u(n)}$.

A bit more formally, we define simultaneously for all n the values $x_n^u(k)$ for each k, by induction on k. For all n, let $x_n^u(0)$ be $\sigma_n^{u(n)}$'s first move. Supposing inductively that we have defined $x_n^u(i)$ for all $i \leq k$ and all n, we let $x_n^u(k+1)$ be $\sigma_n^{u(n)}$'s move in the game $G_n^{u(n)}$ at the position where Player I has played $x_n(0), \dots, x_n(k)$, and Player II has played $x_{n+1}(0), \dots, x_{n+1}(k)$.

The $\langle x_n^u \rangle_{n\in\omega}$ are defined so that the pair $\langle x_n^u, x_{n+1}^u \rangle$ is the outcome of a play of the game $G_n^{u(n)}$ where Player I plays according to the strategy $\sigma_n^{u(n)}$. Since each $\sigma_n^{u(n)}$ is winning for Player I in the game $G_n^{u(n)}$, we have that whenever $u(n) = 0$, $x_n^u \in A_n$ if and only if $x_{n+1}^u \in A_{n+1}$; and if $u(n) = 1$, then $x_n^u \in A_n$ if and only if $x_{n+1}^u \notin A_{n+1}$.

Notice that the play of each game G_n^u does not affect that of any of the subsequent games G_m^u with $m > n$. In particular, this tells us that x_n^u does not depend on the values

$u(i)$ for $i < n$; thus whenever $u, v \in 2^\omega$ are sequences that agree beyond $n-1$ (that is, $u(m) = v(m)$ for all $m \geq n$), we have $x^u_m = x^v_m$ for all $m \geq n$.

Note also that by our definition of the games $G^{u(n)}_n$, if we toggle a single coordinate $u(n)$ of u, then we "flip" membership of x^u_m in A_m for all $m \leq n$; in particular, if u, v disagree on exactly one coordinate, that we have $x^u_0 \in A_0$ if and only if $x^v_0 \notin A_0$. This rather strange property of the map $u \mapsto x^u_0$ will be enough to reach a contradiction: We will obtain a set which does not have the Baire property, and so by the results of Section 7, violates AD!

So consider now the set

$$B = \{s \in 2^\omega \mid x^u_0 \in A_0\}.$$

We claim this set cannot be meager. For then B would be meager in both $N_{\langle 0\rangle}$ and $N_{\langle 1\rangle}$. But by our previous discussion, $\langle 0\rangle^\frown u \in B$ if and only if $\langle 1\rangle^\frown u \notin B$. In particular, the homeomorphism of 2^ω sending u to the sequence $\langle 1 - u(0), u(1), u(2), \ldots\rangle$ sends the meager set $B \cap N_{\langle 0\rangle}$ to the comeager set $N_{\langle 1\rangle} \setminus B$, a contradiction!

Since we are assuming AD, we have from Theorem 7.8 that every set in 2^ω has the Baire property. Since B is not meager, we obtain by Proposition 7.3 some $s \in 2^{<\omega}$ so that B is comeager in N_s. But now we have, for all $u \in 2^\omega$,

$$s^\frown\langle 0\rangle^\frown u \in B \iff x^{s^\frown\langle 0\rangle^\frown u} \in A_0 \iff x^{s^\frown\langle 1\rangle^\frown u} \notin A_0 \iff s^\frown\langle 1\rangle^\frown u \notin B.$$

We again have a contradiction: the homeomorphism toggling the $\ell(s)$th coordinate of u maps the comeager-in-$N_{s^\frown\langle 0\rangle}$ set B to the meager-in-$N_{s^\frown\langle 1\rangle}$ set $\neg B$. ⊣

In the event that AD fails, we still obtain this nice structure of the Wadge hierarchy when we restrict to Wadge degrees contained in pointclasses $\boldsymbol{\Gamma}$ for which we have determinacy.

Theorem 9.2. *Assume $\boldsymbol{\Gamma}$-DET, where $\boldsymbol{\Gamma}$ is a pointclass closed under finite union and intersection, complementation, and continuous substitution. Then $<_{\mathrm{L}}$ (and hence $<_{\mathrm{W}}$) restricted to the set of degrees a with $a \subseteq \boldsymbol{\Gamma}(\omega^\omega)$ is well-founded.*

In particular, $<_{\mathrm{L}}$ and $<_{\mathrm{W}}$ are well-founded on the Borel Wadge degrees.

The so-called "reduced" Wadge degrees are obtained by identifying each non-self-dual pointclass a with its dual $\neg a$. Defining $<_{\mathrm{W}}$ on these degrees in the natural way, we have that $<_{\mathrm{W}}$ is a well-order on the reduced Wadge hierarchy. To each set $A \subseteq \omega^\omega$ we assign an ordinal, its **Wadge rank**, defined to be the order-type of the reduced Wadge degrees below A under $<_{\mathrm{W}}$.

How tall is the Wadge hierarchy?

Definition 9.3. The ordinal Θ ("Big Theta") is the least ordinal that is not the surjective image of the reals. That is,

$$\Theta = \{\alpha \mid \text{there exists a surjection } \varphi : \omega^\omega \to \alpha\}.$$

It's easy to see Θ is a cardinal, and by Hartogs' trick, $\omega_1 < \Theta$. Assuming the Axiom of Choice, Θ is simply $\mathfrak{c}^+$, and the Continuum Hypothesis is equivalent to the statement $\Theta = \omega_2$. When choice fails, it still makes sense to talk about Θ, and the statement $\Theta = \omega_2$ can be thought of as a "choice-free" version of the Continuum Hypothesis.

The next theorem is our reason for introducing Θ.

Theorem 9.4. *Assume $\mathsf{AD} + \mathsf{DC}$. Let λ be the set of Wadge ranks of sets $A \subseteq \omega^\omega$. Then $\lambda = \Theta$.*

Proof. Fix an indexing $\langle f_x \rangle_{x\in\omega^\omega}$ of all continuous functions $f : \omega^\omega \to \omega^\omega$. If $\alpha < \lambda$, then there is some $A \subseteq \omega^\omega$ so that the Wadge rank of A is α. Define a map $\varphi : \omega^\omega \to \alpha$ by letting $\varphi(x)$ be the Wadge rank of $f_x^{-1}[A]$. Then φ is a surjection onto α.

For the converse, we first observe that given a set $A \subseteq \omega^\omega$, there is a uniform way to define a set $A' \subseteq \omega^\omega$ so that $A, \neg A <_{\mathrm{W}} A'$ by diagonalizing against all continuous functions. Namely,

$$A' = \{\langle 0\rangle^\frown x \mid f_x(\langle 0\rangle^\frown x) \notin A\} \cup \{\langle 1\rangle^\frown x \mid f_x(\langle 1\rangle^\frown x) \in A\}.$$

Then there can be no continuous reduction $f : A' \to A$, for then $f = f_x$ for some x, and we have $\langle 0\rangle^\frown x \in A'$ if and only if $f(\langle 0\rangle^\frown x) \notin A$. Similarly, $A' \not\leq_{\mathrm{W}} \neg A$.

Now suppose $\alpha < \Theta$. Then there is some surjection $\varphi : \omega^\omega \to \alpha$. We use this surjection to define a $<_{\mathrm{W}}$-increasing sequence of sets $\langle A_\xi \rangle_{\xi<\alpha}$ by induction on ξ.

Suppose we have defined A_ξ for all $\xi < \eta$. Put

$$B_\eta = \{x \in \omega^\omega \mid \varphi(\pi_1(x)) < \eta \text{ and } \pi_2(x) \in A_{\varphi(\pi_1(x))}\}.$$

where here $\pi_1 : x \mapsto \langle x(0), x(2), \ldots\rangle$ and $\pi_2 : x \mapsto \langle x(1), x(3), \ldots\rangle$. Put $A_\eta = B'_\eta$. It is easy to see $A_\xi \leq_{\mathrm{W}} B_\eta$ for each $\xi < \eta$: If y satisfies $\varphi(y) = \xi$, then the map $x \mapsto \langle z(0), x(0), z(1), x(1), \ldots\rangle$ is a continuous reduction of A_ξ to B_η. Then by the above remarks, we have $A_\xi <_{\mathrm{W}} B'_\eta = A_\eta$, as needed. ⊣

We remarked above that $\Theta = \omega_2$ can be regarded as a choice-free version of the Continuum Hypothesis. It turns out that under AD, this hypothesis fails rather badly: If AD holds, then Θ is a limit cardinal; and in the presence of some weak choice, Θ is even inaccessible! We will investigate the connection between determinacy and large cardinals later.

So far we know the Axiom of Determinacy banishes pathological consequences of the Axiom of Choice, and it gives us a greal deal of information about the structure of sets of reals. But which sets can we actually prove are determined? And what sort of assumptions will we need?

Our next immediate goal is to define a hierarchy of sets beyond the Borel sets. We will develop a structure theory for the lowest sets in this hierarchy, and, from strong hypotheses, prove these are determined.

§10. The Projective Hierarchy. Our basic operations for generating the Borel sets were negation and countable union, and we saw how the latter could be realized as quantification $\exists^\omega$ over the set of natural numbers. What if we allow ourselves to quantify over a bigger set?

Definition 10.1. Let X be a Polish space, and $A \subseteq \omega^\omega \times X$. We define

$$\exists^{\omega^\omega} A = \{y \in X \mid (\exists x \in \omega^\omega)\langle x, y\rangle \in A\}.$$

For a pointclass $\mathbf{\Gamma}$, we let

$$\exists^{\omega^\omega}\mathbf{\Gamma} = \{(A, X) \mid A = \exists^{\omega^\omega} B \text{ for some } (B, \omega^\omega \times X) \in \mathbf{\Gamma}\}.$$

Similarly define operations $\forall^{\omega^\omega} A$ and $\forall^{\omega^\omega}\mathbf{\Gamma}$ for sets and pointclasses.

Definition 10.2. For $n \in \omega$ we define the **projective pointclasses** $\mathbf{\Sigma}^1_n, \mathbf{\Pi}^1_n$ by induction, as follows:

1. $\mathbf{\Pi}^1_0 = \mathbf{\Pi}^0_1$, and $\mathbf{\Sigma}^1_0 = \mathbf{\Sigma}^0_1$.

2. Given $\mathbf{\Sigma}^1_n$, let $\mathbf{\Pi}^1_n$ be the dual pointclass, $\mathbf{\Pi}^1_n = \neg\mathbf{\Sigma}^1_n$.
3. Given $\mathbf{\Pi}^1_n$, put $\mathbf{\Sigma}^1_{n+1} = \exists^{\omega^\omega}\mathbf{\Pi}^1_n$.

The **ambiguous projective pointclasses** are defined to be $\mathbf{\Delta}^1_n = \mathbf{\Sigma}^1_n \cap \mathbf{\Pi}^1_n$.

The sets obtained in the projective hierarchy are the **projective sets**. We call the members of $\mathbf{\Sigma}^1_1$ the **analytic sets**; those of $\mathbf{\Pi}^1_1$ are the **coanalytic sets**.

Proposition 10.3. *Each projective pointclass is closed under continuous substitution.*

Proof. We show something a bit more general: if $\mathbf{\Gamma}$ is closed under continuous substitution, then so is $\exists^{\omega^\omega}\mathbf{\Gamma}$. Let $f : X \to Y$ be continuous, and suppose $A \in \exists^{\omega^\omega}\mathbf{\Gamma}$ with $A \subseteq Y$; so $A = \exists^{\omega^\omega} B$ with $B \subseteq \omega^\omega \times Y$. Define $g : \omega^\omega \times X \to \omega^\omega \times Y$ by $g(\langle z, x\rangle) = \langle z, f(x)\rangle$. Then

$$\begin{aligned} f^{-1}[A] &= \{x \in X \mid f(x) \in A\} \\ &= \{x \in X \mid (\exists z \in \omega^\omega)\langle z, f(x)\rangle \in B\} \\ &= \exists^{\omega^\omega} g^{-1}[B], \end{aligned}$$

And this last set belongs to $\exists^{\omega^\omega}\mathbf{\Gamma}$ by closure of $\mathbf{\Gamma}$ under continuous substitution.

The proposition now follows by induction on the levels of the projective hierarchy (using the fact that $\mathbf{\Gamma}$ is closed under continuous substitution if and only if $\neg\mathbf{\Gamma}$ is). $\dashv$

Proposition 10.4. *Each $\mathbf{\Sigma}^1_n$ is closed under $\exists^{\omega^\omega}$; each $\mathbf{\Pi}^1_n$ is closed under $\forall^{\omega^\omega}$.*

Proof. Notice that the second claim follows from the first and De Morgan's law $\forall^{\omega^\omega} A = \neg\exists^{\omega^\omega}\neg A$. Then we have it for $n = 0$, since the projection of an open set is open: If $B \subseteq \omega^\omega \times X$ is open and $x \in \exists^{\omega^\omega} B$, we have $\langle z, x\rangle \in B$; taking $N_s \times U \subseteq B$ with $\langle z, x\rangle \in N_s \times U$, and $U \subseteq \exists^{\omega^\omega} B$.

Suppose now that $A \subseteq \omega^\omega \times X$ is in $\mathbf{\Sigma}^1_{n+1}$ for some $n \in \omega$. By definition $A = \exists^{\omega^\omega} B$ with $B \subseteq \omega^\omega \times \omega^\omega \times X$, $B \in \mathbf{\Pi}^1_n$. We need to show $\exists^{\omega^\omega} A \in \mathbf{\Sigma}^1_{n+1}$. Let $\phi : \omega^\omega \to \omega^\omega \times \omega^\omega$ be a homeomorphism, with $\phi(w) = \langle \phi_0(w), \phi_1(w)\rangle$ for all $w \in \omega^\omega$. Now the set

$$C = \{\langle w, x\rangle \in \omega^\omega \mid \langle \phi_0(w), \phi_1(w), x\rangle \in B\}$$

belongs to $\mathbf{\Pi}^1_n$ by closure under continuous substitution, and $\exists^{\omega^\omega} C = \exists^{\omega^\omega} A$ is in $\mathbf{\Sigma}^1_{n+1}$ as needed. $\dashv$

Just as with the Borel hierarchy, we have universal sets at each level of the projective hierarchy.

Theorem 10.5. *Suppose $W \subseteq 2^\omega \times \omega^\omega \times X$ is $\mathbf{\Gamma}$-universal for $\omega^\omega \times X$ where $\mathbf{\Gamma}$ is closed under continuous substitution. Then there is $W^* \subseteq 2^\omega \times X$ which is $\exists^{\omega^\omega}\mathbf{\Gamma}$-universal for X.*

Proof. Fix a $\mathbf{\Gamma}$-universal set W for $\omega^\omega \times X$. The $\exists^{\omega^\omega}\mathbf{\Gamma}$-universal set will be the obvious one, obtained by projecting along the ω^ω coordinate; that is

$$W^* = \{\langle u, x\rangle \in 2^\omega \times X \mid (\exists w \in \omega^\omega)\langle u, w, x\rangle \in W\} = \exists^{\omega^\omega}\{\langle w, u, x\rangle \mid \langle u, w, x\rangle \in W\}.$$

Closure of $\mathbf{\Gamma}$ under continuous substitution implies the set on the inside of the $\exists^{\omega^\omega}$ is in $\mathbf{\Gamma}$; so W^* is in $\exists^{\omega^\omega}\mathbf{\Gamma}$.

Now if $A \in \exists^{\omega^\omega}\mathbf{\Gamma}$, we have $A = \exists^{\omega^\omega} B$ with $B \subseteq \omega^\omega \times X$ in $\mathbf{\Gamma}$. Say $B = W_u$ with $u \in 2^\omega$. It is now easy to check that $A = W^*_u$. $\dashv$

COROLLARY 10.6. *For every Polish space X and $n \in \omega$, there exist $\mathbf{\Sigma}^1_n$-universal and $\mathbf{\Pi}^1_n$-universal sets for X.*

With the same diagonalization argument we used on the Borel hierarchy, we have

COROLLARY 10.7. *For ever $n \in \omega$ there is a set $A \subseteq 2^\omega$ in $\mathbf{\Sigma}^1_n \setminus \mathbf{\Pi}^1_n$ (and so $\neg A \in \mathbf{\Pi}^1_n \setminus \mathbf{\Sigma}^1_n$).*

Let us mention some more closure properties. First we note that our existential quantifier can range over any Polish space.

DEFINITION 10.8. Let X, Y be Polish spaces, with $B \subseteq X \times Y$. Then $\exists^X B$ is the set $\{y \in Y \mid \langle x, y\rangle \in B\}$.

PROPOSITION 10.9. *Each $\mathbf{\Sigma}^1_n$ is closed under $\exists^X$, for all Polish spaces X.*

PROOF. Exercise. $\dashv$

PROPOSITION 10.10. *Each $\mathbf{\Sigma}^1_n, \mathbf{\Pi}^1_n, \mathbf{\Delta}^1_n$ for $n > 0$ is closed under $\exists^\omega, \forall^\omega$, countable unions, and countable intersections.*

PROOF. We first show $\mathbf{\Sigma}^1_n$ is closed under $\forall^\omega$ and $\exists^\omega$; then closure for $\mathbf{\Pi}^1_n$ follows from De Morgan's laws, and for the $\mathbf{\Delta}^1_n$ by definition.

Closure under $\exists^\omega$ follows from the last proposition, since ω is a Polish space. So suppose $B \subseteq \omega \times X$ with $B \in \mathbf{\Sigma}^1_n$. We need to show $\forall^\omega B \in \mathbf{\Sigma}^1_n$. We have by definition of $\mathbf{\Sigma}^1_n$ that there is a set $C \in \mathbf{\Pi}^1_{n-1}$ so that $B = \exists^{\omega^\omega} C$. Now

$$x \in \forall^\omega B \iff (\forall n \in \omega)\langle n, x\rangle \in B \iff (\forall n \in \omega)(\exists w \in \omega^\omega)\langle w, n, x\rangle \in C.$$

We require a way of reversing the order of quantifiers $\forall^n, \exists^{\omega^\omega}$. Let $w \mapsto \langle (w)_n\rangle_{n\in\omega}$ be a homeomorphism of ω^ω with $(\omega^\omega)^\omega$, so that each ω-sequence of elements of ω^ω is coded by a single w. We have, for each $x \in X$,

$$(\forall n \in \omega)(\exists w \in \omega^\omega)\langle w, n, x\rangle \in C \iff (\exists w \in \omega^\omega)(\forall n)\langle (w)_n, n, x\rangle \in C.$$

The right to left direction is clear; for the reverse, suppose for each n there is some $u_n \in \omega^\omega$ so that $\langle u_n, n, x\rangle \in C$, and (by countable choice) let w be a real with $(w)_n = u_n$ for all n.

Now the set $D = \{\langle w, x\rangle \in \omega^\omega \times X \mid (\forall n)\langle (w)_n, n, x\rangle \in C\}$ belongs to $\mathbf{\Pi}^1_{n-1}$, since $C \in \mathbf{\Pi}^1_{n-1}$ and this pointclass is closed under continuous substitution and $\forall^\omega$. Since $\forall^\omega B = \exists^{\omega^\omega} D$, we have $\forall^\omega B \in \mathbf{\Sigma}^1_n$ as needed.

Finally, we need to show $\mathbf{\Sigma}^1_n$ is closed under countable unions and intersections. Suppose $\langle A_n\rangle_{n\in\omega}$ is a sequence of members of $\mathbf{\Sigma}^1_n(X)$. Let $W \subseteq 2^\omega \times X$ be $\mathbf{\Sigma}^1_n$-universal for X. For each $n \in \omega$, pick some $y_n \in 2^\omega$ with $A_n = W_{y_n}$. By closure under continuous substitution, the set $C = \{\langle n, x\rangle \mid \langle y_n, x\rangle \in W\} = \{\langle n, x\rangle \mid x \in A_n\}$ is in $\mathbf{\Sigma}^1_n$ (since the map $n \mapsto y_n$ is automatically continuous). But $\exists^\omega C = \bigcup_{n\in\omega} A_n$ and $\forall^\omega C = \bigcap_{n\in\omega} A_n$ then both belong to $\mathbf{\Sigma}^1_n$. $\dashv$

COROLLARY 10.11. *Every Borel set belongs to $\mathbf{\Delta}^1_1$.*

PROOF. Every closed $F \subseteq X$ belongs to $\mathbf{\Sigma}^1_1$, since $F \times F$ is closed in $X \times X$, and so $F = \exists^X F \times F \in \mathbf{\Sigma}^1_1$. Every open set is the countable union of closed sets, and so is in $\mathbf{\Sigma}^1_1$ by the previous proposition. Then the closed and open sets are in $\mathbf{\Pi}^1_1$ as well.

Now $\mathbf{\Delta}^1_1$ contains the open and closed sets, and is closed under complement, countable union, and countable intersection. It follows that $\mathbf{\Delta}^1_1$ contains all the Borel sets. $\dashv$

We have seen that the analytic sets contain the Borel sets, and that there is a set that is in $\mathbf{\Sigma}^1_1$ but not in $\mathbf{\Delta}^1_1$—in particular, this set is not Borel. That the projection of a Borel set in the plane is Borel was incorrectly asserted by Lebesgue; the existence of a counterexample was discovered by Suslin, a graduate student at the time. And so descriptive set theory was born.

In a surprising and useful turn of events, the converse of the previous corollary holds: $\mathbf{\Delta}^1_1$ consists of exactly the Borel sets! In order to show this, we need some tools to help us analyze $\mathbf{\Sigma}^1_1$. Recall that closed sets in Baire space were precisely the sets of branches through trees $T \subseteq \omega^{<\omega}$. Since sets in $\mathbf{\Sigma}^1_1$ are projections of closed sets in $\omega^\omega \times \omega^\omega$, it will be useful to introduce a system of notation to study trees

Definition 10.12. We say a non-empty set $T \subseteq \omega^{<\omega} \times \omega^{<\omega}$ is a **tree** if

1. For all $\langle s, t\rangle \in T$, we have $\ell(s) = \ell(t)$.
2. If $s \subseteq s'$, $t \subseteq t'$, $\ell(s) = \ell(t)$ and $\langle s', t'\rangle \in T$, then $\langle s, t\rangle \in T$.

We say that $\langle x, y\rangle \in \omega^\omega \times \omega^\omega$ is a **branch** through the tree T if for all n, $\langle x \restriction n, y \restriction n\rangle \in T$, and write $[T] \subseteq \omega^\omega \times \omega^\omega$ for the set of branches.

Similar definitions are made for the higher products $\omega^{<\omega} \times \omega^{<\omega} \times \omega^{<\omega}$ and so forth.

Of course, there is an obvious correspondence between trees T on $\omega \times \omega$ and trees in $\omega^{<\omega} \times \omega^{<\omega}$ as defined here. This new definition essentially introduces a systematic abuse of notation, identifying the sequence of pairs $\langle\langle s(0), t(0)\rangle, \dots, \langle s(n-1), t(n-1)\rangle\rangle \in T$ with the pair of sequences $\langle s, t\rangle$.

Proposition 10.13. *A set $C \subseteq \omega^\omega \times \omega^\omega$ is closed if and only if $C = [T]$ for a tree $T \subseteq \omega^{<\omega} \times \omega^{<\omega}$.*

Proof. Set $T = \{\langle x \restriction n, y \restriction n\rangle \mid \langle x, y\rangle \in C\}$; the proof that $[T]$ is closed when T is a tree is the same as before. $\dashv$

Corollary 10.14. *A set $A = \omega^\omega \times \omega^\omega$ is $\mathbf{\Sigma}^1_1$ if and only if $A = \exists^{\omega^\omega}[T]$ for some tree $T \subseteq \omega^{<\omega} \times \omega^{<\omega}$.*

As expected, for $\langle s, t\rangle \in T$ we denote

$$T_{s,t} = \{\langle s', t'\rangle \in T \mid s \subseteq s' \text{ and } t \subseteq t', \text{ or } s' \subseteq s \text{ and } t' \subseteq t\}.$$

Observe we have the equality

$$T_{s,t} = \bigcup_{m,n\in\omega} T_{s^\frown\langle m\rangle, t^\frown\langle n\rangle}.$$

We are just about ready to prove that all $\mathbf{\Delta}^1_1$ sets are Borel. First, one more definition.

Definition 10.15. Suppose A, B are disjoint sets. We say C **separates** A from B if $A \subseteq C$ and $B \cap C = \varnothing$.

The key fact is the following theorem.

Theorem 10.16 (Lusin). *Suppose $A, B \in \mathbf{\Sigma}^1_1(\omega^\omega)$ are disjoint. Then there is a Borel set $C \subseteq \omega^\omega$ that separates A from B.*

Proof. We take advantage of the following simple fact.

Claim. *Suppose $A = \bigcup_{i\in I}$ and $B = \bigcup_{j\in J} B_j$, and suppose for each $i \in I$ and $j \in J$ there is a set $C_{i,j}$ which separates A_i from B_j. Then the set $C = \bigcup_{i\in I} \bigcap_{j\in J} C_{i,j}$ separates A and B.*

Proof of claim. Suppose $x \in A$; then $x \in A_i$ for some i. Since $A \subseteq C_{i,j}$ for all $j \in J$, we have $x \in C$. So $A \subseteq C$.

Now suppose $x \in B$. Then $x \in B_j$ for some $j \in J$. For every i, we have $C_{i,j} \cap B_j = \varnothing$. In particular, $x \notin \bigcap_{j\in J} C_{i,j}$ for each i; so $x \notin C$, and $B \cap C = \varnothing$. ⊣

Now let A, B be disjoint in $\mathbf{\Sigma}^1_1$. Let $S, T \subseteq \omega^{<\omega} \times \omega^{<\omega}$ be trees with $A = \exists^{\omega^\omega}[S]$ and $B = \exists^{\omega^\omega}[T]$. We proceed by contradiction: Suppose A, B cannot be separated by a Borel set.

Now we have

$$A = \exists^{\omega^\omega}[S] = \bigcup_{k,l\in\omega} \exists^{\omega^\omega}[S_{\langle k\rangle,\langle l\rangle}], \qquad B = \exists^{\omega^\omega}[T] = \bigcup_{m,n\in\omega} \exists^{\omega^\omega}[T_{\langle m\rangle,\langle n\rangle}].$$

By (the contrapositive of) the claim, there must exist some $k_0, l_0, m_0, n_0 \in \omega$ so that $\exists^{\omega^\omega}[S_{\langle k_0\rangle,\langle l_0\rangle}], \exists^{\omega^\omega}[T_{\langle m_0,n_0\rangle}]$ cannot be separated by a Borel set. Clearly then $\langle\langle k_0\rangle, \langle l_0\rangle\rangle \in S$ and $\langle\langle m_0\rangle, \langle n_0\rangle\rangle \in T$, since otherwise one of these sets would be empty and so easily separated by a Borel set. Notice also that we must have $l_0 = n_0$, for clearly $\exists^{\omega^\omega}[S_{\langle k_0\rangle,\langle l_0\rangle}] \subseteq N_{\langle l_0\rangle}$ and $\exists^{\omega^\omega}[T_{\langle m_0,n_0\rangle}] \subseteq N_{\langle n_0\rangle}$; if these were distinct, then $N_{\langle l_0\rangle}$ would separate A from B.

Now suppose inductively that we have sequences $s = \langle k_0, \dots, k_{i-1}\rangle$, $t = \langle m_0, \dots, m_{i-1}\rangle$, and $u = \langle n_0, \dots, n_{i-1}\rangle$, so that the sets $\exists^{\omega^\omega}[S_{s,u}]$ and $\exists^{\omega^\omega}[T_{t,u}]$ cannot be separated by a Borel set. By the same argument, we have some k_i, m_i, n_i so that $\langle s^\frown\langle k_i\rangle, u^\frown\langle n_i\rangle\rangle \in S$, $\langle t^\frown\langle m_i\rangle, u^\frown\langle n_i\rangle\rangle \in T$, and the sets $\exists^{\omega^\omega}[S_{s^\frown\langle k_i\rangle,u^\frown\langle n_i\rangle}]$ and $\exists^{\omega^\omega}[T_{\langle t^\frown\langle m_i\rangle,u^\frown\langle n_i\rangle}]$ cannot be separated by a Borel set.

By induction we obtain $x = \langle k_i\rangle_{i\in\omega}$, $y = \langle m_i\rangle_{i\in\omega}$ and $z = \langle n_i\rangle_{i\in\omega}$. By construction we have $\langle x, z\rangle \in S$ and $\langle y, z\rangle \in T$. But then $z \in \exists^{\omega^\omega}[S] \cap \exists^{\omega^\omega}[T] = A \cap B$, contradicting our assumption that A, B were disjoint. ⊣

Corollary 10.17 (Suslin). *The Borel subsets of ω^ω are exactly those in the class $\mathbf{\Delta}^1_1$.*

Proof. We already saw that every Borel set is $\mathbf{\Delta}^1_1$. Suppose that $A \subseteq \omega^\omega$ is in $\mathbf{\Delta}^1_1 = \mathbf{\Sigma}^1_1 \cap \mathbf{\Pi}^1_1$. Then both A and $\neg A$ are in $\mathbf{\Sigma}^1_1$; by the theorem, we have a Borel set C in ω^ω with $A \subseteq C$ and $\neg A \cap C = \varnothing$. But the only possibility for such a C is $C = A$! ⊣

§11. Analyzing Co-analytic Sets. We now restrict our attention to the first level of the projective hierarchy, that of $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$. Our analysis will hinge on the fact that the sets in $\mathbf{\Sigma}^1_1$ are the projections of trees. We start off by defining a notion that lets us convert trees on ω into linear orders.

Definition 11.1. The **Kleene-Brouwer order** is the order $<_{\mathrm{KB}}$ defined on $\omega^{<\omega}$ as follows. We say $s <_{\mathrm{KB}} t$ if and only if

1. $s \supsetneq t$, or
2. $s(n) < t(n)$, where n is least such that $s(n) \neq t(n)$.

Proposition 11.2. $<_{\mathrm{KB}}$ *is a linear order.*

Any linear order restricted to a subset of its domain is again a linear order; in particular, $<_{\mathrm{KB}}$ is a linear order on any tree $T \subseteq \omega^{<\omega}$. The following proposition is our reason for introducing $<_{\mathrm{KB}}$.

PROPOSITION 11.3. *Suppose T is a tree on ω, and that $\langle s_n \rangle_{n\in\omega}$ is an infinite sequence of nodes in T with $s_{n+1} <_{\mathrm{KB}} s_n$ for all n. Then there is an infinite branch through T.*

PROOF. By our definition of $<_{\mathrm{KB}}$, we have $s_{n+1}(0) \leq s_n(0)$ for all n. In particular, the sequence $\langle s_n(0)\rangle_{n\in\omega}$ is eventually constant, so there must be some k_0 so that $s_n(0) = k_0$ for all but finitely many n.

Now suppose inductively that we have found $t = \langle k_0, \dots, k_{i-1}\rangle$ so that $t \subseteq s_n$ for all but finitely many n. For each such n, we again have $s_{n+1}(i) \leq s_n(i)$, and so there exists k_i so that $s_n(i) = k_i$ for all but finitely many n; thus eventually $t^\frown\langle k_i\rangle \subseteq s_n$.

By construction each finite string $\langle k_0, \dots, k_i\rangle$ is an initial segment of some s_n, and so the sequence $\langle k_n\rangle_{n\in\omega}$ is a branch through T. ⊣

COROLLARY 11.4. *Let T be a tree on ω. Then $[T] = \varnothing$ if and only if $<_{\mathrm{KB}}$ restricted to T is a well-order.*

PROOF. We have just shown that if $<_{\mathrm{KB}}$ is not a well-order on T, then it $[T] \neq \varnothing$. Conversely, suppose $x \in [T]$; then $x \restriction n+1 <_{\mathrm{KB}} x \restriction n$ for all n, so $<_{\mathrm{KB}}$ is ill-founded on T. ⊣

This justifies the following terminology: A tree T is **well-founded** if it has no infinite branches.

We can now give a useful characterization of $\mathbf{\Pi}^1_1$ sets. We need one more piece of notation.

Let $T \subseteq \omega^{<\omega} \times \omega^{<\omega}$ be a tree. For $y \in \omega^\omega$, we let $T(y) \subseteq \omega^{<\omega}$ be the set

$$T(y) = \{s \in \omega^{<\omega} \mid \langle s, y \restriction \ell(s)\rangle \in T\}.$$

Then $T(y)$ is a tree.

PROPOSITION 11.5. *For all $x, y \in \omega^\omega$ and trees $T \subseteq \omega^{<\omega} \times \omega^{<\omega}$, we have $\langle x, y\rangle \in [T]$ if and only if $x \in [T(y)]$.*

PROOF. This falls right out of the definitions: $x \in [T(y)]$ iff $(\forall n \in \omega) x \restriction n \in T(y)$ iff $(\forall n \in \omega)\langle x \restriction n, y \restriction n\rangle \in T$ iff $\langle x, y\rangle \in [T]$. ⊣

COROLLARY 11.6. *A set B is $\mathbf{\Pi}^1_1$ if and only if there is some tree $T \subseteq \omega^{<\omega} \times \omega^{<\omega}$ such that*

$$B = \{y \in \omega^\omega \mid T(y) \textit{ is well-founded}\}.$$

PROOF. If B is $\mathbf{\Pi}^1_1$, then $\neg B$ is $\mathbf{\Sigma}^1_1$ and so $\neg B = \exists^{\omega^\omega}[T]$ for some tree T. Then $y \in B$ iff $y \notin \exists^{\omega^\omega}[T]$ iff $(\forall x)\langle x, y\rangle \notin [T]$ iff $(\forall x) x \notin [T(y)]$ iff $T(y)$ is well-founded. ⊣

With this characterization down, we can isolate a particularly interesting $\mathbf{\Pi}^1_1$ set. But first, let's talk about coding. We want to regard elements of 2^ω as coding binary relations on ω, that is, members of $\mathcal{P}(\omega \times \omega)$. For this, we set down a canonical way of identifying ω and $\omega \times \omega$: Set $\ulcorner i, j \urcorner = 2^i(2j+1)$ for $i, j \in \omega$. As the reader can check, this is a bijection from $\omega \times \omega$ to ω.

Now, given $x \in 2^\omega$, define R_x to be the relation on ω obtained by setting

$$i \, R_x \, j \iff x(\ulcorner i, j \urcorner) = 1.$$

We can now encode classes of countable mathematical structures as sets of reals, and talk about the complexity of these in terms of descriptive set theory. For example, let LO be the set of all x encoding a linear order:

$$\mathrm{LO} = \{x \in 2^\omega \mid R_x \text{ is a linear order of some subset of } \omega\}.$$

PROPOSITION 11.7. LO *is Borel.*

PROOF. This amounts to writing down the definition of a linear order and observing that the only quantifiers we use are first-order—that is, we only quantify over elements of the linear order (as opposed to its subsets). We have

$$\begin{aligned} x \in \mathrm{LO} \iff & (\forall i \in \omega)(i \, R_x \, i) \\ & \wedge (\forall i, j, k \in \omega)(i \, R_x \, j \wedge j \, R_x \, k \to i \, R_x \, k) \\ & \wedge (\forall i, j \in \omega)(i \, R_x \, j \wedge j \, R_x \, i \to i = j) \\ & \wedge (\forall i, j \in \omega)(i \, R_x \, i \wedge j \, R_x \, j \to i \, R_x \, j \vee j \, R_x \, i). \end{aligned}$$

Since for any fixed $i, j \in \omega$, the set of $\langle x, i, j\rangle$ satisfying $i \, R_x \, j$ (equivalently, $x(\ulcorner i, j \urcorner) = 1$) is clopen, we have a Borel definition of LO. ⊣

We obtain a more complicated class of structures by restricting to *well-orders*:

$$\mathrm{WO} = \{x \in 2^\omega \mid R_x \text{ is a well-order}\}.$$

PROPOSITION 11.8. WO *is* $\mathbf{\Pi}^1_1$.

PROOF. Notice that $x \in \mathrm{WO}$ if and only if $x \in \mathrm{LO}$ and R_x has no infinite descending chains. This last condition is the same as saying there is no infinite sequence $i_0, i_1, \dots$ such that $i_{n+1} \, R_x \, i_n$ and $i_n \neq i_{n+1}$ for all $n \in \omega$. Thus $x \in \mathrm{WO}$ if and only if

$$x \in \mathrm{LO} \wedge \neg(\exists y \in \omega^\omega)(\forall n \in \omega)\neg(y(n+1) \neq y(n) \wedge x(\ulcorner y(n+1), y(n) \urcorner) = 1).$$

Now the set of $\langle x, y, n\rangle$ such that $y(n+1) \neq y(n) \wedge x(\ulcorner y(n+1), y(n)\urcorner) = 1$ is clearly open. So by the closure properties of $\mathbf{\Sigma}^1_1$, we have a $\mathbf{\Pi}^1_1$ definition of WO. ⊣

Now if $x \in \mathrm{WO}$ then its associated well-ordering of ω is isomorphic to some countable ordinal, the **order-type of** x. Let us write $\mathrm{ot}(x) = \gamma$ if and only if (ω, R_x) is isomorphic to $(\gamma, \in)$.

PROPOSITION 11.9. *For each* γ*, let* $\mathrm{WO}_\gamma = \{x \in 2^\omega \mid \mathrm{ot}(x) = \gamma\}$*. Then* WO_γ *is* $\mathbf{\Sigma}^1_1$.

PROOF. Exercise. ⊣

Notice that this gives us (without using choice) an equivalence relation on 2^ω with precisely ω_1 equivalence classes, each of which is $\mathbf{\Sigma}^1_1$. Under AD, there is no selector for this relation.

Let's now see that WO is as complicated as a $\mathbf{\Pi}^1_1$ set can get.

THEOREM 11.10. *Let* $A \subseteq \omega^\omega$*. Then* A *is* $\mathbf{\Pi}^1_1$ *if and only if there is a continuous function* $f : \omega^\omega \to 2^\omega$ *such that for all* $x \in \omega^\omega$*,* $f(x) \in \mathrm{LO}$*, and* f *satisfies the equivalence*

$$x \in A \iff f(x) \in \mathrm{WO}.$$

PROOF. If we have such a function f, then $A = f^{-1}[\mathrm{WO}]$. That $A \in \mathbf{\Pi}^1_1$ follows from closure of this pointclass under continuous preimages.

Now, suppose $A \in \mathbf{\Pi}^1_1$. We have a tree $T \subseteq \omega^{<\omega} \times \omega^{<\omega}$ so that $x \in A$ precisely when $x \notin \exists^{\omega^\omega}[T]$; equivalently, $x \in A$ if and only if $T(x)$ is well-founded if and only if $<_{\mathrm{KB}}$ restricted to $T(x)$ is a well-order.

The trick, then, is to try to define f so that $f(x)$ will encode the Kleene-Brouwer order on $T(x)$. Let's fix an enumeration $\langle s_i \rangle_{i\in\omega}$ of $\omega^{<\omega}$; let's also require that our enumeration has the property that $i \leq j$ whenever $s_i \subseteq s_j$ (that is, we list all proper initial segments of s_i before we list s_i). We will define $f(x)$ so that $i \; R_{f(x)} \; j$ when $s_i, s_j \in T(x)$ and $s_i <_{\mathrm{KB}} s_j$. For those i for which $s_i \notin T(x)$, we simply put i on the top of $R_{f(x)}$ in the usual order; this ensures that $R_{f(x)}$ has domain all of ω.

Formally, we define

$$f(x)(\ulcorner i, j \urcorner) = \begin{cases} 1 & \text{if } s_i, s_j \in T(x), \text{ and } s_i <_{\mathrm{KB}} s_j, \\ 1 & \text{if } i < j \text{ and } s_i, s_j \notin T(x), \\ 1 & \text{if } s_i \in T(x) \text{ and } s_j \notin T_x, \\ 0 & \text{otherwise.} \end{cases}$$

Now for any x, $f(x)$ is a linear order because $<_{\mathrm{KB}}$ is; and $f(x)$ is well-founded exactly when $<_{\mathrm{KB}}$ is, that is, when $x \in A$.

Finally, notice that f is continuous: given $x \restriction n$, we know exactly which elements of $\omega^{<\omega}$ of length at most n are in $T(x)$, and so know the values of $f(x)(\ulcorner i, j \urcorner)$ whenever $\ulcorner i, j \urcorner = 2^i(2j+1) \leq n$—in fact, because of how we enumerated the s_i, this guarantees f is Lipschitz. ⊣

Corollary 11.11. *The set* WO *is* $\mathbf{\Pi}^1_1$ *and not* $\mathbf{\Sigma}^1_1$*.*

Proof. Because by closure under continuous substitution WO $\in \mathbf{\Sigma}^1_1$ would imply $\mathbf{\Pi}^1_1 \subseteq \mathbf{\Sigma}^1_1$! ⊣

The following theorem is usually invoked as "$\mathbf{\Sigma}^1_1$ Boundedness".

Theorem 11.12. *Suppose $B \in \mathbf{\Sigma}^1_1$ and $B \subseteq$ WO. Then there is some $\gamma < \omega_1$ so that for all $x \in B$, we have* $\mathrm{ot}(x) < \gamma$*.*

Proof. Suppose otherwise towards a contradiction, so members of B achieve arbitrarily high countable order-type. We'll show that WO would then be a member of $\mathbf{\Sigma}^1_1$.

For each $x \in$ WO, we have by assumption some $y \in B$ with $\mathrm{ot}(x) \leq \mathrm{ot}(y)$. In particular, we have an injective map $f : \omega \to \omega$ which embeds the linear order coded by x into that coded by y; that is to say, $x(\ulcorner i, j \urcorner) = 1$ if and only if $y(\ulcorner f(i), f(j) \urcorner) = 1$. Conversely, given $x \in$ LO and such a map f and $y \in B$, we have $x \in$ WO. That is,

$$x \in \mathrm{WO} \iff (x \in \mathrm{LO}) \wedge (\exists y \in B)(\exists f : \omega \to \omega) f \text{ embeds } R_x \text{ into } R_y.$$

We claim this definition is $\mathbf{\Sigma}^1_1$. Since LO is Borel and B is $\mathbf{\Sigma}^1_1$, this is guaranteed by the closure properties of $\mathbf{\Sigma}^1_1$, provided the set $\{\langle x, y \rangle \mid (\exists f : \omega \to \omega) f \text{ embeds } R_x \text{ into } R_y\}$ is shown to be $\mathbf{\Sigma}^1_1$. This sacred task we entrust to the reader. ⊣

§12. Ultrafilters and Measurable Cardinals. Our next task is to connect determinacy with large cardinals. For this purpose, we introduce an abstract notion of *size* for subsets of a fixed set X.

DEFINITION 12.1. Let X be a set. A **filter** on X is a collection $\mathcal{F} \subseteq \mathcal{P}(X)$ satisfying
1. $X \in \mathcal{F}$, and $\varnothing \notin \mathcal{F}$.
2. If $A \in \mathcal{F}$ and $A \subseteq B$, then $B \in \mathcal{F}$.
3. For all $A, B \in \mathcal{F}$, we have $A \cap B \in \mathcal{F}$.

A filter is **κ-complete** for κ a cardinal if for all $\alpha < \kappa$ and sequences $\langle A_\alpha \rangle_{\xi<\alpha}$ of sets in $\mathcal{F}$, we have $\bigcap_{\xi<\alpha} A_\xi \in \mathcal{F}$. A filter is **countably complete** if and only if it is ω_1-complete (closed under countable intersections).

EXAMPLE 12.2. For X infinite, $\mathcal{F} = \{A \subseteq X \mid X \setminus A \text{ is finite}\}$ is a filter on X; it is called the **Fréchet filter** on X.

EXAMPLE 12.3. $\mathcal{F} = \{A \subseteq \omega^\omega \mid A \text{ is comeager}\}$ is a countably complete filter on ω^ω.

EXAMPLE 12.4. Suppose κ is regular; then $\mathcal{F} = \{A \subseteq \kappa \mid A \text{ is club in } \kappa\}$ is a κ-complete filter on κ.

We think of sets in a filter as being "large"; complements of sets in a filter are "small". We will be particularly concerned with those filters which assign every subset of X a value of either "large" or "small".

DEFINITION 12.5. A filter $\mathcal{U}$ on X is an **ultrafilter** if for every $A \subseteq X$, either $A \in \mathcal{U}$ or $X \setminus A \in \mathcal{U}$.

We have one trivial example of an ultrafilter: given a non-empty set X with $a \in X$, let $\mathcal{U} = \{A \subseteq X \mid a \in X\}$. Then $\mathcal{U}$ is an ultrafilter on X, called the **principal ultrafilter** generated by a.

More interesting are non-principal ultrafilters. Notice that if an ultrafilter $\mathcal{U}$ on X is non-principal, then it can contain no sets of size 2: If $\{a, b\} \in \mathcal{U}$, then either $\{a\} \in \mathcal{U}$, or else $X \setminus \{a\} \in U$, so that intersecting with $\{a, b\}$ yields $\{b\} \in \mathcal{U}$. Proceeding inductively, we can show $\mathcal{U}$ contains no finite sets. In particular, a non-principal ultrafilter must extend the Fréchet filter on X.

THEOREM 12.6 (Tarski). *A filter $\mathcal{U}$ on X is an ultrafilter if and only if it is a maximal filter: that is, whenever $\mathcal{F}$ is another filter on X with $\mathcal{U} \subseteq \mathcal{F}$, we have $\mathcal{F} = \mathcal{U}$.*

PROOF. Exercise. ⊣

COROLLARY 12.7. *Assume the Axiom of Choice. Then for every infinite set X, there is a non-principal ultrafilter $\mathcal{U}$ on X.*

PROOF. Let $\mathcal{F} \subseteq \mathcal{P}(X)$ be the Fréchet filter on X. By Zorn's Lemma, there is a maximal filter $\mathcal{U} \supseteq \mathcal{F}$; then by Theorem 12.6, $\mathcal{U}$ is an ultrafilter, and since $\mathcal{U}$ extends $\mathcal{F}$, it is non-principal. ⊣

We remark that choice is required to produce non-principal ultrafilters in general, and AD implies there are no non-principal ultrafilters on ω.

We next isolate an additional property of ultrafilters on ordinals that entails a number of nice properties.

Definition 12.8. Let $\mathcal{U}$ be an ultrafilter on κ. We say $\mathcal{U}$ is a **normal measure** if it is non-principal and κ-complete, and whenever $A \in \mathcal{U}$ and $f : A \to \kappa$ satisfies $f(\alpha) < \alpha$ for all $\alpha \in A$, we have some fixed $\xi < \kappa$ so that

$$\{\alpha < \kappa \mid f(\alpha) = \xi\} \in \mathcal{U}.$$

A cardinal κ is **measurable** if there is a normal measure on κ.

Normality is a form of *reflection*: if there is a normal measure $\mathcal{U}$ on κ, then many first-order properties of κ also hold for a $\mathcal{U}$-large set of ordinals $\alpha < \kappa$. Let's first see how measurability of κ entails largeness.

Theorem 12.9 (Ulam). *(Using the Axiom of Choice.) Suppose κ is a measurable cardinal. Then κ is strongly inaccessible.*

Proof. Fix a normal measure $\mathcal{U}$ on κ. We first show κ is regular. Suppose $f : \lambda \to \kappa$ is an increasing map with $\lambda < \kappa$, and, towards a contradiction, that $f[\lambda]$ is unbounded in κ. For each $\alpha < \kappa$, let $G(\alpha)$ be the least $\xi < \lambda$ with $\alpha < f(\xi)$. Since $\mathcal{U}$ is κ-complete and non-principal, we have that $A = \kappa \setminus \lambda$ is in $\mathcal{U}$, and $G(\alpha) < \alpha$ for all $\alpha \in A$. It follows from normality that we have some fixed $\xi < \lambda$ and $B \in \mathcal{U}$ with $G(\alpha) = \xi$ for all $\alpha \in B$; but since B is in $\mathcal{U}$, it is unbounded in κ. But this contradicts our assumption that $f[\lambda]$ was unbounded in κ.

We next need to show κ is a limit cardinal. Suppose not; then $\kappa = \lambda^+$ for some $\lambda < \kappa$. We will reach a contradiction through the use of an *Ulam matrix*.

Lemma 12.10. *(Using the Axiom of Choice.) Let λ be a cardinal. Then there is an array $\langle A^\alpha_\xi \mid \xi < \lambda, \alpha < \lambda^+\rangle$ of subsets of λ^+ so that*

1. *For each $\xi < \lambda$ and $\alpha \neq \beta < \lambda^+$, $A^\alpha_\xi \cap A^\beta_\xi = \varnothing$.*
2. *For $\alpha < \lambda^+$, we have $|\lambda^+ \setminus \bigcup_{\xi<\lambda} A^\alpha_\xi| < \lambda^+$.*

Proof of Lemma. For each $\eta < \lambda^+$, fix a surjection $f_\eta : \lambda \to \eta + 1$. Let

$$A^\alpha_\xi = \{\eta < \lambda^+ \mid f_\eta(\xi) = \alpha\}.$$

That (1) holds is immediate. And for each $\alpha < \lambda^+$, we have for each η with $\alpha \leq \eta < \lambda^+$ that $f_\eta(\xi) = \alpha$ for some ξ; that is, $\lambda^+ \setminus \alpha \subseteq \bigcup_{\xi<\lambda} A^\alpha_\xi$, and this gives (2). ⊣

Let $\langle A^\alpha_\xi \mid \xi < \lambda, \alpha < \lambda^+\rangle$ be as in the lemma. Now for each $\alpha < \lambda^+$, we have $\bigcup_{\xi<\lambda} A^\alpha_\xi \in \mathcal{U}$; that is, $\bigcap_{\xi<\lambda}(\lambda^+ \setminus A^\alpha_\xi) \notin \mathcal{U}$. So by κ-completeness of $\mathcal{U}$, there must be some ξ so that $A^\alpha_\xi \in \mathcal{U}$.

For each $\alpha < \kappa = \lambda^+$, let $F(\alpha) = \xi$ least so that $A^\alpha_\xi \in \mathcal{U}$. Notice that the set $\{\alpha < \kappa \mid F(\alpha) < \alpha\}$ belongs to $\mathcal{U}$, so that by normality, we have some fixed ξ so that $\{\alpha < \kappa \mid A^\alpha_\xi = A^\alpha_{F(\alpha)} \in \mathcal{U}\}$ is in $\mathcal{U}$. But this is a clear contradiction to condition (1) on our matrix $\langle A^\alpha_\xi\rangle$.

We leave the proof that κ is strong limit as an exercise. ⊣

The main property of normal measures we need is an analogue of Ramsey's theorem. For the following proposition, recall $[X]^n$ denotes the collection of n-element subsets of X.

Proposition 12.11. *(Using the Axiom of Choice.) Ramsey's theorem fails for ω_1. That is, there is a coloring $c : [\omega_1]^2 \to 2$ with no homogeneous subset of size ω_1, i.e. if $H \subseteq \omega_1$ with $|H| = \omega_1$, then $c \restriction [H]^2$ is non-constant.*

PROOF. Using choice, let $\langle x_\alpha \mid \alpha < \omega_1 \rangle$ be a sequence of real numbers. Color $[\omega_1]^2$ by setting

$$c(\{\alpha, \beta\}) = 0 \iff (\alpha < \beta \iff x_\alpha < x_\beta).$$

Suppose towards a contradiction that H was a monochromatic subset of size ω_1, say $H = \{\xi_\alpha \mid \alpha < \omega_1\}$ enumerates H in increasing order. If c is constant on $[H]^2$ with value 0, then for each $\alpha < \omega_1$ let q_α be a rational number with $x_\alpha < q_\alpha < x_{\alpha+1}$. But then $\alpha \mapsto q_\alpha$ is an injection of ω_1 into $\mathbb{Q}$, a contradiction! Similarly argue when c is constant with value 1. ⊣

On the other hand, we do have a version of Ramsey's theorem for measurable cardinals.

THEOREM 12.12 (Rowbottom). *Let $\mathcal{U}$ be a normal measure on κ, and let $c : [\kappa]^n \to \lambda$ be a coloring of n-element subsets of κ. Then there is a set $H \in \mathcal{U}$ which is homogeneous for c; that is, $c \restriction [H]^n$ is constant.*

PROOF. The case $n = 1$ is immediate by κ-completeness. So suppose we have the theorem for some fixed n; let $c : [\kappa]^n \to \lambda$ be a coloring. We define a sequence of colorings $c_\alpha : [\kappa \setminus (\alpha + 1)]^n \to \lambda$ for $\alpha < \kappa$, by setting

$$c_\alpha(a) = c(a \cup \{\alpha\})$$

for all $a \subseteq \kappa \setminus (\alpha+1)$ of size n. For each α, we have by inductive hypothesis some $H_\alpha \in \mathcal{U}$ (with $H_\alpha \subseteq \kappa \setminus (\alpha + 1)$) so that H_α is homogeneous for c_α, say with constant value ξ_α. Set H' to be the diagonal intersection of the H_α, that is

$$H' = \triangle_{\alpha<\kappa} H_\alpha = \{\alpha < \kappa \mid \alpha \in \textstyle\bigcap_{\beta<\alpha} H_\beta\}.$$

Finally, we have by κ-completeness some $\xi < \lambda$ and a set $A \in \mathcal{U}$ so that $\xi_\alpha = \xi$ for all $\alpha \in A$. Put $H = H' \cap A$.

Now let $a \in [H]^{n+1}$, say $a = \{\alpha_0, \dots, \alpha_n\}$, with the elements listed in increasing order. Then $H' \setminus (\alpha_0 + 1) \subseteq H_{\alpha_0}$ by definition, so that in particular we have $\{\alpha_1, \dots, \alpha_n\} \in [H_{\alpha_0}]^n$. Thus

$$c(a) = c_{\alpha_0}(\{\alpha_1, \dots, \alpha_n\}) = \xi_{\alpha_0} = \xi,$$

with the last equality holding since $\alpha_0 \in A$. Thus H is homogeneous for c as needed. ⊣

We remark that although most of the combinatorial content of measurable cardinals comes through normality, the existence of measurables follows from just the existence of a non-principal countably complete ultrafilter.

§13. Analytic Determinacy from a Measurable. With the results of the last two sections, we are ready to prove determinacy of $\mathbf{\Pi}^1_1$ games from the existence of a measurable cardinal.

THEOREM 13.1 (Martin). *Assume there is a measurable cardinal. Then* $\mathbf{\Pi}^1_1$-DET *holds.*

PROOF. Let $A \subseteq \omega^\omega$ be $\mathbf{\Pi}^1_1$. The idea is to simulate the game $G(A)$ by an auxiliary closed game in the players collaborate to form a real x as usual, but Player I is also required to produce a witness to membership of x in A.

By Theorem 11.10, we have a Lipschitz reduction $f : \omega^\omega \to \mathrm{LO}$ so that for all x, $x \in A$ iff $f(x) \in \mathrm{WO}$. From that proof, we have that the order $R_{f(x)}$ restricted to n only depends on $x \restriction n$; so we let $<_s$ denote the linear order of $\ell(s)$ that results from restricting $R_{f(x)}$, for any $x \supseteq s$; furthermore set $<_x = \bigcup_{n\in\omega} <_{x\restriction n} = R_{f(x)}$. So for all $x \in \omega^\omega$, we have $x \in A$ iff $<_x$ is a well-ordering.

Now consider an auxiliary game $G^*(A)$, played as follows: The players take turns producing naturals $x(0), x(1), \ldots$, and Player I also plays ordinals $\xi_n < \kappa$ for $n \in \omega$.

I	$x(0), \xi_0$		$\xi_1, x(2), \xi_2$		$\ldots$	$\xi_{2n-1}, x(2n), \xi_{2n}$		$\ldots$
II		$x(1)$		$x(3)$	$\ldots$		$x(2n+1)$	$\ldots$

FIGURE 7. A play of the auxiliary game $G^*(A)$.

At each stage of the game, $\langle \xi_i \rangle_{i<n}$ is required to be an embedding of $(n, <_{x\restriction n})$ into κ; that is, Player I must ensure $i <_{x\restriction n} j$ if and only if $\xi_i < \xi_j$. Player I wins all infinite plays of $G^*(A)$.

Now the auxiliary game $G^*(A)$ is closed, and so by Gale-Stewart, is determined. If Player I has a winning strategy σ^*, then Player I easily converts this to a winning strategy σ in $G(A)$, simply by ignoring the extra moves ξ_n. This is winning since any play x according to σ lifts uniquely to a play of σ^* with the same natural number moves $x(n)$; then the sequence $\langle \xi_n \rangle_{n\in\omega}$ is a witness to well-foundedness of $<_x$, so that $x \in A$.

Suppose now that Player II has a winning strategy τ^* in $G^*(A)$. In order to convert this to a winning strategy in $G(A)$, we need some way of attributing ordinal moves ξ_n to Player I in $G^*(A)$ so that partial plays in τ lift to partial plays in $G^*(A)$. We describe how to play for Player II, inductively defining a sequence of sets $H_n \in \mathcal{U}$ so that τ^* always plays the same natural $x(2n+1)$, regardless of which ordinals $\langle \xi_i \rangle_{i<n}$ Player I has played so far, as long as they come from H_n.

First consider $n = 0$. Player I plays $x(0)$ in $G(A)$, and we need to attribute ξ_0 to Player I in $G^*(A)$. Let $c : \kappa \to \omega$ be the map obtained by setting $c(\xi) = \tau^*(\langle x(0), \xi \rangle)$. By κ-completeness of $\mathcal{U}$, there is a set $H_0 \in \mathcal{U}$ so that $c \restriction H_0$ is constant. Let $\tau(\langle x(0) \rangle)$ be this constant value.

Now suppose inductively that we have $H_n \in \mathcal{U}$, and that we have produced a play $\langle x(0), \ldots, x(2n)$, so that for all i with $2i + 1 < 2n$,

$$\tau^*(\langle x(0), \xi_1, x(2), \ldots, x(2i), \xi_{2i} \rangle) = x(2i+1),$$

as long we have all the ξ_j in H_n and $j \mapsto \xi_j$ preserving the order $<_{x\restriction 2i}$.

Notice that for each $2n+1$-element subset a of H_n, there is a unique way to embed $2n+1$ into a in a $<_{x\restriction 2n+1}$-preserving way. We thus obtain an ω-coloring of $[H]^{2n+1}$, setting

$$c(\{\xi_0, \ldots, \xi_{2n}\}) = \tau^*(\langle x(0), \xi_{i_0}, \ldots, x(2i), \xi_{i_{2i}} \rangle),$$

where $i_0, \ldots, i_{2n}$ reindexes the ordinals ξ_i in such a way as to preserve the order $<_{x\restriction 2n+1}$. Then by Rowbottom's Theorem 12.12, we have $H_{n+1} \in \mathcal{U}$ homogeneous for c. Let $\tau(\langle x(0), \ldots, x(2i) \rangle)$ be this constant value. Then we have preserved the condition of our inductive definition of the strategy τ.

We have thus described a strategy τ for Player II in $G(A)$; we claim it is a winning strategy. Suppose $x \in [\tau]$ is an infinite play according to σ. Let H_n be the homogeneous sets obtained as in the above construction as the real x is played, and put $H = \bigcap_{n\in\omega} H_n$; then $H \in \mathcal{U}$ by κ-completeness of $\mathcal{U}$.

Suppose for contradiction $x \in A$. Then $<_x$ is a well-order, so we have some sequence $\langle \xi_n \rangle_{n\in\omega}$ so that each $\xi_n \in H$ and $i <_x j$ iff $\xi_i < \xi_j$. By our construction of τ, we have $x(2n) = \tau^*(\langle x(0), \xi_0, x(1), \dots, x(2n-1), \xi_{2n-1}\rangle)$ for all $n \in \omega$. Then $x, \langle \xi_n \rangle_{n\in\omega}$ is an infinite play in $G^*(A)$ according to τ^*. But this contradicts our assumption that τ^* was winning for Player II in $G^*(A)$! ⊣

We remark that a measurable cardinal is a bit stronger than what is strictly necessary to prove $\mathbf{\Pi}^1_1$-DET. By joint results of Martin and Harrington, $\mathbf{\Pi}^1_1$-DET is equivalent to the statement that for all reals x, there exists another real, $x^\#$ ("x sharp"), which measures the distance between the set theoretic universe and the universe of sets constructible from x. In a sense, $x^\#$ is the least complicated real that cannot be seen by the smallest model of set theory containing x; the sharp operator may be thought of as a "models of set theory" analogue of the jump operator on Turing degrees.

§14. Large Cardinals from the Axiom of Determinacy. We have just seen that we can obtain determinacy from certain large cardinal assumptions. In this section we see how to go the other way. We first present Solovay's seminal proof that AD implies the existence of a measurable cardinal. This result is already striking, but what is more surprising is the identity of the cardinal of interest: it is none other than ω_1!

To motivate the game, we will return to a game that was first introduced in the exercises. First, let's visit another old friend, the club filter. Recall a subset A of ω_1 was called a club if it was closed and unbounded in ω_1.

Proposition 14.1. *Let $F : \omega_1 \to \omega_1$ be a function, and let*

$$C_F = \{\alpha < \omega_1 \mid F[\alpha] \subseteq \alpha\}$$

be the set of closure points of F. Then C_F is a club.

Proof. It's easy to see C_F is closed, since if $\langle \alpha_n \rangle_{n\in\omega}$ is a sequence of closure points of F, then $F[\sup_{n\in\omega} \alpha_n] = \bigcup_{n\in\omega} F[\alpha_n] \subseteq \sup_{n\in\omega} \alpha_n$. So we just need to see C_F is unbounded.

Let α_0 be arbitrary, and inductively define $\alpha_{n+1} = \sup F[\alpha_n] + 1$, for each $n \in \omega$. Note that regularity of ω_1 implies $\alpha_n < \omega_1$ for all n, and this sequence is increasing. If $\alpha = \sup_{n\in\omega} \alpha_n$, then for any $\beta < \alpha$, we have $\beta < \alpha_n$ for some n; but then $F(\beta) \in F[\alpha_n] \subset \alpha_{n+1} < \alpha$, so that α is a closure point of F, as needed. ⊣

Proposition 14.2. *Suppose C_α is club in ω_1 for each $\alpha < \omega_1$. Then the diagonal intersection $\triangle_{\alpha<\omega_1} C_\alpha = \{\alpha < \omega_1 \mid \alpha \in \bigcap_{\beta<\alpha} C_\beta\}$ contains a club.*

Proof. Let $F : \omega_1 \to \omega_1$ be defined by letting $F(\alpha) = \min \bigcap_{\xi<\alpha} C_\xi \setminus \alpha$; note that $F(\alpha)$ is always defined, since the intersection of countably many clubs in ω_1 is a club (exercise!). We claim the diagonal intersection $\triangle_{\alpha<\omega_1} C_\alpha$ contains the set C_F of closure points of F.

Suppose $\alpha \in C_F$. Let $\xi < \alpha$; then for each β with $\xi < \alpha$, we have $\beta \leq F(\beta) < \alpha$, and $F(\beta) \in C_\xi$. In particular, we have unboundedly many points of C_ξ below α, so that $\alpha \in C_\xi$. This shows $\alpha \in \triangle_{\alpha<\omega_1} C_\alpha$. ⊣

Definition 14.3. Let $A \subseteq \omega_1$. The **club game** $G_c(A)$ is played on ω_1 as follows: Players I and II take turns to produce a sequence of ordinals $\xi_0 < \xi_1 < \xi_2 < \dots$ with each $\xi_n < \omega_1$.

$$\begin{array}{c|cccccccc} \mathrm{I} & \xi_0 & & \xi_2 & & \dots & \xi_{2n} & & \dots \\ \hline \mathrm{II} & & \xi_1 & & \xi_3 & \dots & & \xi_{2n+1} & \dots \end{array}$$

FIGURE 8. A play of the club game $G_c(A)$.

Player I wins if and only if $\sup_{n<\omega} \xi_n \in A$.

PROPOSITION 14.4. *Player I has a winning strategy in the club game $G_c(A)$ if and only if A contains a club.*

PROOF. Suppose A contains a club;

Conversely, suppose σ is a winning strategy for Player I in $G_c(A)$. Define a function $F : \omega_1 \to \omega_1$ by taking $F(\alpha)$ to be the supremum of moves made by σ in response to sequences in α. That is,

$$F(\alpha) = \sup\{\sigma(\langle \xi_0, \xi_1, \dots, \xi_{2n+1}\rangle) \mid \xi_0 < \xi_1 < \dots < \xi_n < \alpha, \text{ and } \langle \xi_0, \dots, \xi_{2n+1}\rangle \in \sigma\}.$$

Notice that $\alpha^{<\omega}$ is countable for each α, so $F(\alpha)$ is defined and less than ω_1 for each α. Let C_F be the set of closure points of F above $\varnothing$. Then C_F is a club; we claim $C_F \subseteq A$.

For suppose $\alpha \in C_F$. Fix a countable increasing sequence $\langle \eta_k \rangle_{k\in\omega}$ with $\sup_{k\in\omega} \eta_k = \alpha$. Define a play against σ by induction: suppose we are given $\langle \xi_0, \dots, \xi_{2n}\rangle \in \sigma$, with each $\xi_i < \alpha$. Let $\xi_{2n+1} = \eta_k$, where k is least so that $\eta_k > \xi_{2n}$. Since α is a closure point of F, we have $\xi_{2n+2} = \sigma(\langle \xi_0, \dots, \xi_{2n+1}\rangle) \in \alpha$. So the induction proceeds.

The sequence produced is defined so that $\eta_n \le \xi_{2n+1} < \alpha$ for all $n \in \omega$; in particular, $\sup_{n\in\omega} \xi_n = \alpha$, so we have $\alpha \in A$ as required. ⊣

The same proof gives

PROPOSITION 14.5. *Player II has a winning strategy in $G_c(A)$ if and only if $\omega_1 \setminus A$ contains a club.*

Now, if we had determinacy of all the games $G_c(A)$, we'd be done: since then every set $A \subseteq \omega_1$ either contains or is disjoint from a club, and we have already seen the club filter is closed under diagonal intersections, so is normal!

Of course, we are only assuming AD, determinacy for games with natural number moves. Nonetheless, we obtain determinacy of the games $G_c(A)$, and in the obvious way: by having the players play codes for reals coding countable well-orders.

For the following definition, recall we have a canonical map $x \mapsto \langle (x)_n \rangle_{n\in\omega}$ whereby each $x \in \omega^\omega$ codes a countable sequence of reals $(x)_n$.

DEFINITION 14.6. Let $A \subseteq \omega_1$. The **coded club game** $G_{cc}(A)$ is played on ω: Player I plays a real x, and Player II plays a real y.

$$\begin{array}{c|cccccccccc} \mathrm{I} & x(0) & & x(1) & & x(2) & & \dots & x(n) & & \dots \\ \hline \mathrm{II} & & y(0) & & y(1) & & y(2) & \dots & & y(n) & \dots \end{array}$$

FIGURE 9. A play of the coded club game $G_{cc}(A)$.

The winner is determined as follows: we regard the reals x, y as coding sequences $\langle (x)_n \rangle_{n\in\omega}, \langle (y)_n \rangle_{n\in\omega}$, respectively. If there is some n so that one of $(x)_n, (y)_n$ is not in

WO, then, letting n be the least such, if $(x)_n \notin$ WO, Player I loses; if $(x)_n \in$ WO and $(y)_n \notin$ WO, then Player II loses.

If neither player has yet lost, then we have sequences of ordinals $\xi_n = \mathrm{ot}((x)_n)$ and $\zeta_n = \mathrm{ot}((y)_n)$. Let $\alpha = \sup\{\xi_n\} \cup \{\zeta_n\}$. Then Player I wins if $\alpha \in A$, and Player II wins otherwise.

THEOREM 14.7 (Solovay). *Suppose* AD *holds. Then the club filter on ω_1 is an ultrafilter; in particular, ω_1 is a measurable cardinal.*

PROOF. Let $A \subseteq \omega_1$. Then by AD, the game $G_{cc}(A)$ is determined. Suppose Player I is the winner, as witnessed by the strategy σ. We show A contains a club; the case when Player II wins is similar.

As in the club game $G_c(A)$, we wish to define a function $F : \omega_1 \to \omega_1$ so that $F(\alpha)$ will be an upper bound on order-types of possible moves by σ in response to sequences of reals coding well-orders of order-type $< \alpha$. However, this has grown hairier, since there are uncountably many possible such reals, and so it's not immediately obvious that such an $F(\alpha) < \omega_1$ will exist.

We define, for each $\alpha < \omega_1$ and $n < \omega$,

$$X_\alpha^n = \{(\sigma * y)_n \in \omega^\omega \mid (\forall m < n)(y)_m \in \mathrm{WO} \text{ and } \mathrm{ot}((y)_m) < \alpha\}.$$

Here $\sigma * y$ denotes the real x played by σ in response to Player II's play y. Notice X_α^n is $\mathbf{\Sigma}_1^1$, since for each $\gamma < \alpha$, the set of $y \in \mathrm{WO}_\gamma$ is $\mathbf{\Sigma}_1^1$, and then the set of y with $(\forall m < n)(y)_m \in \mathrm{WO}_\gamma$, for some $\gamma < \alpha$, is obtained by countable unions and intersections and continuous preimages. X_α^n is then just the image of this set of y under the continuous map $y \mapsto (\sigma * y)_n$.

Now since σ is winning for Player I, we have by the rules of the game $G_{cc}(A)$ that $X_\alpha^n \subseteq$ WO. Then by $\mathbf{\Sigma}_1^1$ Boundedness, we have that $\{\mathrm{ot}(x) \mid x \in X_\alpha^n\}$ is bounded in ω_1; let $\beta_n(\alpha)$ be the supremum of this set, and put $F(\alpha) = \sup_{n\in\omega} \beta_n(\alpha)$.

We again claim that the set C_F of closure points of F is contained in A. For suppose $\gamma \in A$; fix $y \in \omega^\omega$ so that $\sup_{n\in\omega} \mathrm{ot}((y)_n) = \gamma$, and let $x = \sigma * y$. Suppose we have $\alpha = \sup_{m<n} \mathrm{ot}((y)_m) < \gamma$; then $\mathrm{ot}(x)_n \leq \beta_n(\alpha) < F(\alpha) < \gamma$. Thus $\mathrm{ot}((x)_n) < \gamma$ for all n, and we obtain $\gamma = \sup\{\mathrm{ot}((x)_n)\} \cup \{\mathrm{ot}((y)_n\}$; since σ is winning for Player I in $G_{cc}(A)$, we have $\gamma \in A$ as needed. ⊣

ω_1 isn't the only small cardinal that finds itself with large cardinal features under AD; following an alternate proof of Martin's that ω_1 is measurable, Solovay showed ω_2 is measurable; and though each of the ω_n's for $2 < n \leq \omega$ is singular under AD, Kleinberg showed these each possess certain combinatorial properties characteristic of large cardinals.

Recall Θ was defined to be the least ordinal not the surjective image of the reals. We remarked above that $\Theta = \omega_2$ can be regarded as a choice-free version of the Continuum Hypothesis. We now show this hypothesis fails—rather badly—under AD.

THEOREM 14.8 (Moschovakis). *Assume* AD*, and suppose $\alpha < \Theta$. Then there is a surjection $\psi : \omega^\omega \to \mathcal{P}(\alpha)$.*

PROOF. Fix a surjection $\varphi : \omega^\omega \to \alpha$ witnessing $\alpha < \Theta$. We prove inductively (without choice!) that there is a sequence $\langle g_\xi \rangle_{\xi \leq \alpha}$ of surjections $g_\xi : \omega^\omega \to \xi$. Set $g_0 : \omega^\omega \to \mathcal{P}(0)$

the constant 0 map. At successor steps, we set

$$g_{\xi+1}(x) = \begin{cases} g_\xi(\langle x(1), x(2), \ldots\rangle) & \text{if } x(0) \text{ is even;} \\ g_\xi(\langle x(1), x(2), \ldots\rangle) \cup \{\xi\} & \text{if } x(0) \text{ is odd.} \end{cases}$$

Suppose we have reached some limit stage λ. Using φ and the surjections g_ξ so far defined, we may regard each real x as a code for a pair $\langle \eta_x, A_x\rangle$ with $\eta_x < \lambda$ and $A_x \subseteq \eta_x$: Just set $\eta_x = \varphi(\pi_1(x))$ if this is $< \lambda$ (and 0 otherwise), and $A_x = g_\xi(\pi_2(x))$.

With this in mind, we define for each $Z \subseteq \lambda$ a game G_Z on ω in which the players compete to play reals coding the largest possible initial segment of Z. Namely, Player I produces $x \in \omega^\omega$, and II produces $y \in \omega^\omega$ in the usual way. These reals code pairs $\langle \eta_x, A_x\rangle, \langle \eta_y, A_y\rangle$.

Player II wins if either $A_x \neq Z \cap \eta_x$, or if $\eta_x < \eta_y$ and $A_y = Z \cap \eta_y$. Otherwise, Player I wins.

By AD, one of the players has a winning strategy in this game. We claim that the set Z can be reconstructed from a winning strategy (for either player).

Suppose first that σ is a winning strategy for Player I. Then let

$$Z_\sigma = \bigcup\{A_x \mid (\exists y \in \omega^\omega)\sigma \text{ responds to Player II's play } y \text{ with } x\}.$$

We claim $Z_\sigma = Z$. By the rules of the game, any play by σ produces x with $A_x = Z \cap \eta_x$. So it is sufficient to show η_x can be arbitrarily large below λ. Suppose $\xi < \lambda$; let y be a real with $\eta_y = \xi$ and $A_y = Z \cap \xi$. Then σ must respond to y with x coding $\langle \eta_x, A_x\rangle$, and since σ is a winning strategy for Player I, we must have $\eta_x \geq \eta_y = \xi$.

Next suppose Player II has a winning strategy τ. By the definition of the game G_Z, we have that whenever x codes $\langle \eta_x, Z \cap \eta_x\rangle$, τ's response y codes $\langle \eta_y, Z \cap \eta_y\rangle$ with $\eta_y > \eta_x$. We claim Z is the unique set with this property. Suppose Y was another such set; let α be the least element of $Z \triangle Y$. Then, fixing a code x with $\eta_x = \alpha$ and $A_x = Y \cap \alpha = Z \cap \alpha$, let τ respond to x with y. But then $\alpha \in Z$ if and only if $\alpha \in A_y$ if and only if $\alpha \in Y$, contrary to our choice of α.

We now obtain our surjection $g_\lambda : \omega^\omega \to \lambda$, by letting $g_\lambda(x) = Z$ if x codes a winning strategy for either player in the game G_Z. The above argument shows this Z is unique, so our function g_λ is well-defined, and surjective. $\dashv$

It follows from this theorem that under AD, Θ is a limit cardinal. In fact, with some additional assumptions, Θ is regular, and therefore weakly inaccessible.

§15. An Application of Borel Determinacy to Definable Graphs. In this section we introduce some ideas in *Borel combinatorics*, focusing in particular on Borel graphs and colorings. Let's begin by recalling some definitions from the theory of graphs.

Definition 15.1. Let X be a set. Recall that a **graph** G **on** X is a binary relation $G \subseteq X \times X$ that is irreflexive and symmetric; X is the **vertex set** and G is the set of **edges**.

The **degree** of a vertex $x \in X$ is $|\{y \in X \mid \langle x, y\rangle \in G\}|$. If every vertex of X has finite (countable) degree, we say G is **locally finite (countable)**. If every vertex of G has degree exactly d, we say G is d**-regular**.

A **path in** G is a sequence $\langle x_0, \ldots, x_{n-1}\rangle$ of elements of X so that for all $i < n$, $\langle x_i, x_{i+1}\rangle \in G$, and $x_i \neq x_j$ for distinct $i, j < n$, except possibly if $i, j \in \{0, n-1\}$. If $x_0 \neq x_{n-1}$, we call this a **path from** x_0 **to** x_{n-1}. The **connected component** of x in

G is the set of all $y \in X$ for which $y = x$, or there exists a path from x to y. A path is a **cycle** if $x_0 = x_{n-1}$ and $n > 3$; a graph is **acyclic** if it contains no cycles.

For cardinals κ, a **κ-coloring** of G is a function $c : X \to \kappa$ so that adjacent vertices are sent to different colors: that is, if $\langle x, y\rangle \in G$ then $\chi(x) \neq \chi(y)$. The least κ for which there exists a κ-coloring is the **chromatic number** of G, written $\chi(G)$.

The study of arbitrary infinite graphs is a vast departure from the finite setting, and many natural questions turn out to be undecidable in ZFC alone. Instead, we here restrict to a definable setting, so retaining certain similarities with finite graph theory and avoiding the spectre of independence. As we will see, though, these graphs differ from finite graphs in some surprising ways.

Definition 15.2. Let X be a Polish space. A graph G on X is called a **Borel graph** if G is Borel as a subset of $X \times X$.

Example 15.3. Let $X = 2^\omega$, and set $x \, G \, y$ for $x, y \in X$ iff $|\{n \mid x(n) \neq y(n)\}| = 1$. Then G is a locally countable Borel graph whose connected components are precisely the E_0-equivalence classes.

Example 15.4. Let Γ be a countable group with identity element e. $\omega^\Gamma = \{f : \Gamma \to \omega\}$ with the product topology is then a Polish space. Define the **left shift action** of Γ on ω^Γ by setting, for $x \in \omega^\Gamma$ and all $g, h \in \Gamma$,

$$g \cdot x(h) = x(g^{-1}h).$$

Define the **free part** of the action on ω^Γ to be

$$X = \{x \in \omega^\Gamma \mid (\forall g \in \Gamma) g \neq e \text{ implies } g \cdot x \neq x\}.$$

Let Δ be a set of generators for Γ. Then we define a graph G on X by setting

$$x \, G \, y \iff (\exists g \in \Delta) g \cdot x = y \vee g \cdot y = x.$$

Then G is a Borel graph on X, and each connected component is isomorphic to the Cayley graph of Γ generated by Δ.

Example 15.5. Let X be a Polish space, and let $\mathcal{F} = \{F_i\}_{i\in\omega}$ be a countable family of Borel functions $F_i : X \to X$. Then set

$$x \, G_{\mathcal{F}} \, y \iff x \neq y \text{ and } (\exists i) F_i(x) = y \text{ or } F_i(y) = x.$$

Then $G_{\mathcal{F}}$ is a Borel graph. If each F_i is countable-to-one, then $G_{\mathcal{F}}$ is locally countable.

In fact, every locally countable graph has this form:

Theorem 15.6. *Suppose (X, G) is a locally countable Borel graph. Then there is a family $\mathcal{F} = \{F_i\}_{i\in X}$ of Borel functions $F_i : X \to X$ so that $G = G_{\mathcal{F}}$. If every vertex of G has degree at most k, then we can furthermore assume $\mathcal{F}$ contains exactly k functions.*

A proof of this theorem is beyond the scope of these notes, but we will need the theorem to produce definable colorings. As all examples of Borel graphs we consider will be obviously of this form, the reader who wishes to take this characterization as the definition of locally countable Borel graph will lose nothing in doing so.

Definition 15.7. Let (X, G) be a Borel graph. A map $c : X \to \omega$ is a **Borel k-coloring** of G ($k \leq \omega$) if c is a k-coloring which is Borel. The **Borel chromatic number** of G, $\chi_B(G)$, is the least $k \leq \omega$ for which there exists a Borel k-coloring; if there is no such, we say G has uncountable Borel chromatic number, and write $\chi_B(G) > \omega$.

We first observe that these chromatic numbers need not coincide: for example, if G is the graph in Example 15.3, then $\chi(G) = 2$, but it is easy to check that for any 2-coloring $c : 2^\omega \to 2$, that $c^{-1}[\{0\}]$ cannot have the Baire property—in particular, c is not Borel.

This example seems to indicate that our ability to produce Borel colorings hinges in part on whether we can make choices in a definable way. The following is a dramatic example of just how much the Borel picture departs from the finite one:

Proposition 15.8. *There is an acyclic Borel graph G so that $\chi_B(G) > \omega$.*

For contrast, notice that $\chi(G) \leq 2$ for any acyclic G: just choose a vertex from each connected component, color it 0, and alternate colors to color all of G.

To give this example, we need to recall a basic structure from group theory.

Definition 15.9. Let X be a set; let $X^{-1} = \{x^{-1} \mid x \in X\}$ be disjoint from X. The **words in** X are finite sequences of elements of $X \cup X^{<\omega}$, which we denote as $w = x_0^{i_0} x_1^{i_1} \dots x_{n-1}^{i_{n-1}}$, where each $x_k \in X$ and $i_k = \pm 1$. For words u, v, let uv denote the concatenation of u and v.

A **reduced word** is a word that contains no occurrences of xx^{-1} or $x^{-1}x$, for any $x \in X$. By an annoying inductive argument, there exists for each word w a unique reduced word $\overline{w}$ obtained by iteratively deleting occurrences of xx^{-1} or $x^{-1}x$ in w, for $x \in X$. Let $\mathrm{FG}(X)$ be the set of reduced words.

Define a multiplication operation on $\mathrm{FG}(X)$ by setting $u \cdot v = \overline{uv}$. Then $\mathrm{FG}(X)$ is a group, with identity element the empty word; $\mathrm{FG}(X)$ is called the **free group on** X.

Proof of Proposition 15.8. Let $S_\infty = \{f \in \omega^\omega \mid f \text{ is a bijection}\}$. Note S_∞ (considered as a subspace of ω^ω) is Polish. Let $\{g_n\}_{n\in\omega}$ be a dense sequence of elements of S_∞ so that the subgroup generated by the g_n is isomorphic to $\mathrm{FG}(\omega)$ under the map $g_n \mapsto n$.

Let G be the graph on S_∞ obtained by left-multiplication by the g_n; that is, for $x, y \in S_\infty$,

$$x \, G \, y \iff (\exists n) g_n x = y \vee g_n y = x.$$

Since $\langle g_n \rangle_{n\in\omega}$ is a free subgroup of S_∞, we have that G is an acyclic graph. We claim $\chi_B(G) > \omega$; that is, there is no Borel coloring $c : S_\infty \to \omega$. Suppose towards a contradiction that we have such a map.

Since c is Borel, each $C_i = c^{-1}(\{i\})$ has the Baire property; therefore (by Baire category), we have some C_i non-meager, and so comeager in $N_s \cap S_\infty$ for some $s \in \omega^{<\omega}$. Extending s if necessary, we may assume $s \restriction \ell(s)$ is a bijection from $\ell(s)$ to itself.

Now, since $\{g_n\}_{n\in\omega}$ is dense, we have $g_n \restriction \ell(s)$ is the identity map on $\ell(s)$, for some $n \in \omega$. The map $x \mapsto g_n x$ is a homeomorphism of $N_s \cap S_\infty$ to itself; since C_i is comeager in $N_s \cap S_\infty$, the same is true of $g_n C_i$. So there must exist $x, y \in C_i$ so that $g_n x = y$; but then $x \, G \, y$, contradiction our assumption that c was a coloring. $\dashv$

We have seen that the Borel notion of chromatic number diverges from the usual one. We now show that for Borel graphs with finite bounded degree, we at least have a similar bound on $\chi_B(G)$ as we do on $\chi(G)$.

Theorem 15.10 (Kechris-Solecki-Todorcevic). *Suppose (X, G) is a locally finite graph so that every vertex has degree at most k, where $k \leq \omega$. Then $\chi_B(G) \leq k + 1$.*

Proof. Recall the proof for a finite graph: color the vertices in order, giving each vertex the least color $< k+1$ that hasn't already been used by an adjacent vertex. We would like to give the same proof, but need a lemma to let us color the vertices "in order."

Lemma 15.11. *Let G be a locally finite Borel graph on a Polish space X. Then G has Borel chromatic number $\leq \omega$.*

Proof. Fix a family of Borel functions $\mathcal{F} = \{F_i\}_{i\in\omega}$ with $G = G_{\mathcal{F}}$. Let $\{U_n\}_{n\in\omega}$ be a fixed countable basis for X. Define, for each $n \in \omega$,

$$A_n = \{x \in X \mid (\forall i) x \notin F_i^{-1}[U_n] \wedge F_i(x) \notin U_n\}.$$

That is, A_n is the set of x that belong to U_n, but are not G-adjacent to any element of U_n. Note that since G is locally finite, every $x \in X$ belongs to some A_n. We may then set $c(x) = \min\{n \mid x \in A_n\}$. Then c is a Borel ω-coloring. ⊣

Now fix a Borel ω-coloring $c : X \to \omega$ of G. Inductively define a coloring d, by coloring the sets $c^{-1}[\{n\}]$ in order: if $c(x) = 0$, set $d(x) = 0$. If d is defined on $c^{-1}[\{0, 1, \ldots, n-1\}]$, define d on $c^{-1}[\{n\}]$ by setting

$$d(x) = i \text{ least such that } d(y) \neq i \text{ for all } y \text{ with } c(y) < n \text{ and } \langle x, y\rangle \in G.$$

It's easy to check that d so defined is a Borel map, and since every vertex has degree $\leq k$, d is a $k+1$-coloring. ⊣

So we have one similarity between the Borel chromatic number $\chi_B(G)$ and the usual chromatic number $\chi(G)$. A new question arises: What finite values are possible? It turns out every finite $n \geq 2$ can be realized as the chromatic number of an acyclic Borel graph, but the earliest discovered examples all had infinite degree. It was a long-standing open question whether, for all $n \in \omega$, there exists an acyclic Borel graph with degree at most n and Borel chromatic number $n+1$. We now show this is the case.

Definition 15.12. The **n-fold free product of Z_2**, denoted $\mathbb{Z}_2^{*n}$, is the group with n generators, $\gamma_0, \gamma_1, \ldots, \gamma_{n-1}$ satisfying the relations $\gamma_i^2 = e$ for $i < n$; that is, $\mathbb{Z}_2^{*n} = \langle\gamma_0, \ldots, \gamma_{n-1} \mid \gamma_i^2 = e\langle$. It can be obtained by taking the quotient of the free group $\mathrm{FG}(\{\gamma_0, \ldots, \gamma_{n-1}\})$ by the normal subgroup generated by $\{\gamma_i^2 \mid i < n\}$.

The graph we define will be that given by the left shift action of the generators on a certain subset X of $\omega^{\mathbb{Z}_2^{*n}}$. For each map $c : X \to n$, we will define a finite collection of games in which winning strategies will ensure the existence of adjacent $x, y \in X$ with $c(x) = c(y)$. The games defined will be Borel, and so non-n-colorability will follow from Borel determinacy.

Theorem 15.13 (Marks). *Let $n \geq 1$. Then there is a n-regular acyclic Borel graph G with $\chi_B(G) = n+1$.*

Proof. We let $\mathbb{Z}_2^{*n}$ act by left shift on a subset X of $\omega^{\mathbb{Z}_2^{*n}}$. Let

$$X = \{x \in \omega^{\mathbb{Z}_2^{*n}} \mid (\forall \alpha \in \omega^{\mathbb{Z}_2^{*n}})(\forall i < n) x(\alpha) \neq x(\alpha\gamma_i)\}.$$

In particular, we have $\gamma_i \cdot x \neq x$ for all $x \in X$. Let G be the graph on X with $x \mathrel{G} y$ if and only if $\gamma_i \cdot x = y$ for some $i < n$. Notice that G is n-regular, but not acyclic.

We first show that G does not have a Borel n-coloring; we conclude by showing that the same is true of the restriction of G to the free part of the action on X.

Claim. *Suppose $c : X \to n$ is Borel. Then there exists some $x \in X$ and $i < n$ so that $c(x) = c(\gamma_i \cdot x) = i$.*

Proof of claim. We define a game $G_{i,j}$, for each $j \in \omega$ and $i < n$. The players cooperate to define an element x of X with $x(e) = j$. Player I chooses $x(\gamma_i)$; Player II then chooses $x(\gamma_k)$ for $i \neq k < n$. At the nth round of the game, Player I defines $x(\alpha)$ for all reduced words α of length n that start with γ_i; Player II then defines x on reduced words of length n that start with any other γ_k. At the end of play, Player I is the winner if $c(x) \neq i$; otherwise Player II wins.

Notice that the winning condition of the game is Borel, so by Borel determinacy, each $G_{i,j}$ is determined. We claim that for all $j \in \omega$, there is some i so that Player II wins $G_{i,j}$. Otherwise, we would have some j so that Player I has winning strategies σ_i in $G_{i,j}$, for each $i < n$. But then we could play the games $G_{i,j}$ for $i < n$ simultaneously, so that Player I's moves in each $G_{i,j}$ are given by σ_i, and Player II's moves in $G_{i,j}$ are copied from Player I's moves in $G_{k,j}$ for $k \neq i$. We thus obtain a single real x that is produced by all of the strategies σ_i; in particular, we must have $c(x) \neq i$ for all $i < n$. But this is a contradiction.

So we have for every $j \in \omega$ some $i < n$ so that Player II has a winning strategy in $G_{i,j}$. By the Pigeonhole principle, there must be some $j_0 \neq j_1$ so that Player II wins both of the games G_{i,j_0}, G_{i,j_1}. Let τ_0, τ_1 be the respective winning strategies.

We now consider simultaneous play of the games G_{i,j_0}, G_{i,j_1}. We think of G_{i,j_0} as producing $x \in \omega^{\mathbb{Z}_2^{*n}}$; play in G_{i,j_1} is producing $\gamma_i \cdot x$. Copying moves by II in each game to moves for I in the other, we successfully define $x \in X$ with $x(e) = j_0$, $x(\gamma_i) = j_1$, and $c(x) = c(\gamma_i \cdot x) = i$ (since the strategies τ_0, τ_1 are winning for Player II). ⊣

In particular, we have that there is no n-coloring of G; let Y be the free part of X,

$$Y = \{x \in X \mid (\forall \alpha \in \mathbb{Z}_2^{*n})\alpha \cdot x \neq x\}.$$

Then $G \restriction Y$ is n-regular and acyclic. So we need to show $\chi_B(G \restriction Y) = n + 1$.

Lemma 15.14. *There is a Borel function $c^* : X \setminus Y \to n$ such that for all x and $i < n$, either $c^*(x) \neq i$ or $c^*(\gamma_i \cdot x) \neq i$.*

Once we have the lemma, we will be done: for suppose $d : Y \to n$ is a Borel map. Then $c = c^* \cup d$ is also Borel. By what was already shown, we have some $x \in X$ and $i < n$ so that $c(\gamma_i \cdot x) = c(x) = i$. By the lemma, we cannot have $x \notin Y$; so we have $x, \gamma_i \cdot x \in X$ with $x \, G \, \gamma_i \cdot x$; in particular, $d(x) = c(x) = c(\gamma_i \cdot x) = d(\gamma_i \cdot x)$, so d is not a coloring.

Proof of Lemma. Since we are in $X \setminus Y$, we have that every vertex x belongs to a cycle, say $\langle x_0, x_1, \ldots, x_{m+1}\rangle$ with $x = x_0 = x_{m+1}$; this is witnessed by some sequence of generators $\langle \gamma_{i_0}, \gamma_{i_1}, \ldots, \gamma_{i_m}\rangle$ so that $\gamma_{i_k} \neq \gamma_{i_{k+1}}$ for all $k \leq m$. The idea is to pick such a cycle from each connected component, and set $c^*(x_k) = i_k$ for $k \leq m$. We then work our way out from there, defining c^* by induction by cycling through the $i < n$, and setting $c^*(\gamma_i \cdot x) = i$ whenever $c^*(x)$ has already been defined.

The hard part is choosing a cycle from each connected component in a Borel way. For this, first fix an open basis $\langle U_j\rangle_{j\in\omega}$ for X. Define B to be the set of triples

$$(\langle x_0, \ldots, x_{n+1}\rangle, \langle \gamma_{i_0}, \ldots, \gamma_{i_m}\rangle, \langle U_{j_0}, \ldots, U_{j_m}\rangle)$$

satisfying

1. $m \geq 1$.
2. $x_0 = x_{m+1}$ and $x_k \neq x_l$ for $k < l \leq m$.
3. $\gamma_{i_k} \cdot x_k = x_{k+1}$ and $\gamma_{i_k} \neq \gamma_{i_{k+1}}$ for all $k \leq m$.
4. $U_{j_k} \cap U_{j_l} = \varnothing$ for all distinct $k, l \leq n$.
5. $x_i \in U_{m_i}$ for all $i \leq n$.

So the x_k form a cycle, with this witnessed by the γ_{i_k}, and the U_{j_k} separate them.

Given $\boldsymbol{x} = \langle \vec{x_k}, \vec{\gamma_{i_k}}, \vec{U_{j_k}} \rangle \in B$, let $\pi(\boldsymbol{x})$ denote the projection to the second two coordinates of $\boldsymbol{x}$. Suppose $\pi(\boldsymbol{x}) = \pi(\boldsymbol{y})$ for some $\boldsymbol{x}, \boldsymbol{y} \in B$. We claim that then either $x = y$, or the sets $\{x_0, \ldots, x_{m+1}\}$ and $\{y_0, \ldots, y_{m+1}\}$ are disjoint. Firstly we can't have $x_k = y_l$ with $k \neq l$, because these are separated by U_{j_k} and U_{j_l}. And if $x_k = y_k$ for some k, then we obtain $x_l = y_l$ for all $l \leq m$, since both cycles are obtained by application of the same sequence of γ_i's.

Now we can fix some well-order $\prec$ of the countable set $\{\gamma_i\}^{<\omega} \times \{U_j\}^{<\omega}$; we let A be the set of $\boldsymbol{x} \in B$ so that $\pi(\boldsymbol{x}) \preceq \pi(\boldsymbol{y})$ whenever $\boldsymbol{y} \in B$ with y_0 in the same connected component as x_0. Note that this definition only requires quantification over sequences of generators for $\mathbb{Z}_2^{*n}$ and of basis elements U_j, so A is Borel; and if $\langle x_0, \ldots, x_{m+1} \rangle$ is the first coordinate of some $\boldsymbol{x} \in A$, then it is the unique cycle in its connected component belonging to A.

Now, for $(\langle x_0, \ldots, x_{m+1} \rangle, \langle \gamma_{i_0}, \ldots, \gamma_{i_m} \rangle, \langle U_{j_0}, \ldots, U_{j_m} \rangle) \in A$, set $c^*(x_k) = i_k$. Note that then for any x and i, if c^* is defined on both x and $\gamma_i \cdot x$, then $c^*(x) = i$ implies $c^*(\gamma_i \cdot x) \neq i$.

We define c^* in ω many stages, preserving this last condition at each step. Let $i_0, i_1, \ldots$ be a listing of $1, \ldots, n$ so that each $i \leq n$ appears infinitely often. At stage k, if $c^*(x)$ has been defined and $c^*(\gamma_{i_k} \cdot x)$ has not, set $c^*(\gamma_{i_k} \cdot x) = i_k$.

Then c^* is Borel, and it is easy to check it has the property stated in the lemma. $\dashv$

This completes the proof of the lemma, and so the theorem. $\dashv$

§16. Appendix: An Inner Model of the Axiom of Choice. We have seen there is a connection between the axiom AD and the existence of certain large cardinals; we have also made reference to the fact that AD is not provable in, and is fact significantly stronger than, the axioms of ZFC. But we have yet to see why this is so. Indeed, although we have characterized the measurability of ω_1 as a large cardinal property, it is not clear this largeness will carry over to a context in which the Axiom of Choice holds.

In this section, we introduce some ideas to help us study the relative strength of set theoretic hypotheses. The main construction shows that inside any model of ZF, there is a (possibly) smaller submodel which satisfies ZFC. As a consequence, we have that the consistency of ZF implies that of ZFC; but we will also see that large cardinal notions transfer from the larger model to the smaller, so obtaining (e.g.) that AD implies the consistency of ZFC with the existence of strongly inaccessible cardinals.

First, for easy reference, here are the axioms of ZFC.

- *Extensionality*: Two sets are equal if and only if they have the same elements.
- *Pairing*: If a and b are sets, then so is the pair $\{a, b\}$.
- *Comprehension Scheme*: For any definable property $\phi(u)$ and set z, the collection of $x \in z$ such that $\phi(x)$ holds, is a set.
- *Union*: If $\{A_i\}_{i\in I}$ is a set, then so is its union, $\bigcup_{i\in I} A_i$.
- *Power Set*: If X is a set, then so is $\mathcal{P}(X)$, the collection of subsets of X.
- *Infinity*: There is an infinite set.
- *Replacement Scheme*: For any definable property $\phi(u, v)$, if ϕ defines a function on a set a, then the pointwise image of a by ϕ is a set.
- *Foundation*: The membership relation, $\in$, is well-founded; i.e., every non-empty set contains a $\in$-minimal element.
- *Choice*: If $\{A_i\}_{i\in I}$ is a collection of nonempty sets, then there exists a choice function f with domain I, so that $f(i) \in A_i$ for all $i \in I$.

ZFC without the Axiom of Choice is called ZF.

The **language of set theory** is the first-order language whose only non-logical symbol is a binary relation symbol, $\in$. It's worth noting that each axiom of ZFC can be written as a formula in this language. For example, we can formalize the Axiom of Foundation as

$$\forall x(\exists y(y \in x) \to \exists z(z \in x \wedge \forall y(y \in x \to \neg(y \in z)))),$$

and for each formula $\phi(u, v_1, \ldots, v_n)$, we have an instance of the Axiom Scheme of Comprehension,

$$\forall a_1 \ldots \forall a_n \forall x \exists z \forall y(y \in z \longleftrightarrow (y \in x \wedge \phi(y, a_1, \ldots, a_n))).$$

In particular, note that the "definable properties" of the Axiom Schema of Comprehension and Replacement both allow the use of set parameters a_i.

Models of set theory will be our main focus in this section. A model in the language of set theory is a pair (M, E), where M is a set and E is a binary relation on M. Of course, E need not resemble the true membership relation $\in$, but we would like to restrict to those models $\mathcal{M}$ whose interpretation $\in^{\mathcal{M}}$ agrees with the true membership relation; that is, models of the form $(M, \in)$. It will also be important that our models are *transitive*. Recall a set z is transitive if for every $y \in z$, $y \subseteq z$. We will say a model of set theory $(M, \in)$ is transitive if M is.

Transitive models are important because they reflect basic facts about the universe of sets. A stockpile of transitive models in the language of set theory is furnished by the next definition.

Definition 16.1. The **cumulative hierarchy** (or **von Neumann hierarchy**) of sets is defined by transfinite induction. We set

1. $V_0 = \varnothing$.
2. $V_{\alpha+1} = \mathcal{P}(V_\alpha)$.
3. For limit λ, $V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha$.

We set $V = \bigcup_{\alpha\in\mathrm{ON}} V_\alpha$. For $x \in V$, the **von Neumann rank** of x is the least ordinal α so that $x \in V_{\alpha+1}$.

The reader should verify (by induction) that each V_α is transitive, that $\alpha < \beta$ implies $V_\alpha \subseteq V_\beta$, and $V_\alpha \cap \mathrm{ON} = \alpha$ for all ordinals α.

One axiom of ZFC that might seem somewhat less well-motivated than the rest is the Axiom of Foundation. Let's look at this axiom in a little more detail.

Proposition 16.2. *The Axiom of Foundation is equivalent to the statement: For all sets x, there exists an α so that $x \in V_\alpha$.*

Proof. Suppose the Axiom of Foundation holds. Let x be a set. Suppose $x \notin V_\alpha$; by Foundation, we can assume x is $\in$-minimal, that is, every element of x belongs to some V_α. Let $F(y)$ for $y \in x$ be the least ordinal β so that $x \in V_{\beta+1}$. By the Axiom of Replacement, there is an ordinal γ so that $F(y) < \gamma$ for all $y \in x$. Then each $y \in x$ belongs to V_γ; so $x \subseteq V_\gamma$, and by definition, $x \in V_{\alpha+1}$, a contradiction.

Now suppose every x belongs to some V_α. Fix a non-empty set x, and let y be an element of x so that the von Neumann rank α of y is minimal among elements of x. We claim y is $\in$-minimal in x. For otherwise there is some $z \in y \cap x$, and $y \subseteq V_\alpha$ by definition, so that the von Neumann rank of z is less than α, contradicting our choice of y. This proves the Axiom of Foundation. $\dashv$

Now is also a good time to see why strongly inaccessible cardinals imply the consistency of ZFC.

Proposition 16.3. *Working in* ZFC*, suppose κ is a strongly inaccessible cardinal. Then $V_\kappa \models$* ZFC.

Proof. We check Replacement, leaving the rest as an exercise. Suppose ϕ is a formula so that for some fixed $p \in V_\kappa$,

$$V_\kappa \models (\forall x)(\exists! y)\phi(x, y, p).$$

That is, ϕ defines a function $G : V_\kappa \to V_\kappa$. Let $a \in V_\kappa$. Now for each $x \in a$, define $F(x)$ to be the least ordinal $\alpha < \kappa$ so that there is $y \in V_{\alpha+1}$ with $\phi(x, y, p)^{V_\kappa}$. Since κ is strongly inaccessible, we have that the range of F is bounded in κ. In particular, letting $\lambda = \sup F[x]$, we have that the pointwise image of x by G is a subset of V_λ. We obtain precisely the pointwise image in V_κ via an application of Comprehension in V_κ. $\dashv$

Exercise 16.4. Is the converse of this proposition true? That is, if $V_\kappa \models$ ZFC, is κ inaccessible?

There is one other tool that will be essential in our study of inner models.

THEOREM 16.5 (The Reflection Theorem). *Let $\phi(x_1, \dots x_n)$ be a formula. The following is provable in* ZF*: For any ordinal ξ, there is an $\alpha > \xi$ such that for all $\vec{a} \in (V_\alpha)^n$, $\phi^{V_\alpha}(\vec{a})$ if and only if $\phi(\vec{a})$.*

PROOF. Let $\phi_1, \dots \phi_n$ be an enumeration of all subformulas of ϕ. We can assume that $\forall$ does not appear in any of the ϕ_j, since $\forall$ can be replaced with $\neg\exists\neg$. Let ξ be given.

We define by induction an increasing sequence of sets α_i for $i < \omega$. Suppose that α_i has be defined for some $i < \omega$. We choose α_{i+1} with the following property, for all $j \leq n$ and all tuples $\vec{a}$ from V_{α_i}:

$$\text{If } \exists x \phi_j(x, \vec{a}), \text{ then there is } b \in V_{\alpha_{i+1}} \text{ such that } \phi_j(b, \vec{a}).$$

Let $\alpha = \sup_{i<\omega} \alpha_i$. Now we prove that ϕ is absolute to V_α by induction on the complexity of formulas appearing in $\phi_1, \dots \phi_n$. The atomic formula case follows from transitivity of V_α; the conjunction, disjunction, negation and implication steps are straightforward. The existential quantifier step follows from our construction of the α_i: Given a tuple $\vec{a}$ from V_α and a formula ϕ_j for which $\exists x \phi_j(x, \vec{a})$ holds, all of the tuple's elements appear in some $V\alpha_i$ and therefore there is a witness to $\exists x \phi_j(x, \vec{a})$ in $V_{\alpha_{i+1}}$. ⊣

The reader may have noticed the odd way in which the Reflection Theorem is stated: "For each formula ϕ, there is a proof in ZF that...". Looking into our proof, we indeed see that proving an instance of reflection requires an application of the Axiom of Replacement tailored to the formula ϕ. And indeed, the order of quantifiers here can't be reversed: It is an easy consequence of the Reflection Theorem that if T is any finite subset of ZF, then ZF proves the existence of a model of T. If ZF could prove this for all finite subsets T simultaneously, then by compactness we would have proved consistency of ZF in ZF, contradicting Gödel's Second Incompleteness Theorem.

Our next goal is to work in ZF, and give a definition for a class of sets that will constitute a model of ZFC. This definition for membership in this class will simply be a formula in the language of set theory, say θ; then the model will have the form

$$N = \{x \mid \theta(x)\}.$$

Of course, N may then be a proper class; and we would like to have a shorthand way of saying that some formula ϕ in the language of set theory holds in N. We therefore introduce the **relativization** of ϕ to N, written ϕ^N, by induction on formula complexity:

- For atomic ϕ, ϕ^N is just ϕ.
- If ϕ is of the form $\neg\psi$, then ϕ^N is the formula $\neg\psi^N$; similarly for $\vee$ and $\wedge$.
- If ϕ is of the form $\exists x\psi$, then ϕ^N is the formula $(\exists x)\theta(x) \wedge \psi^N$; similarly for $\forall$.

DEFINITION 16.6. A set x is **ordinal definable**, and we say $x \in \mathrm{OD}$, if there is some formula ϕ in the language of set theory and some finite sequence $\alpha_0, \dots, \alpha_n$ of ordinals, so that

$$x = \{y \mid \phi(y, \alpha_0, \dots, \alpha_n)\}.$$

In light of Tarski's theorem on the non-definability of truth, it isn't obvious from our definition that OD should even be a definable class. We show that it is, via the Reflection Theorem.

PROPOSITION 16.7. *A set x belongs to* OD *if and only if there exists a formula ϕ and ordinals β and $\alpha_0, \dots, \alpha_n < \beta$, so that*

$$x = \{y \in V_\beta \mid V_\beta \models \phi(y, \alpha_0, \dots, \alpha_n)\}.$$

PROOF. It is immediate that any x of the form described will be in OD, as we have just given the definition (and using the definability of the satisfaction relation for *sets* V_β). So suppose $x \in \mathrm{OD}$. Let ϕ be the defining formula, and $\alpha_0, \ldots, \alpha_n$ the ordinal parameters witnessing this. By reflection, there is an ordinal β so that ϕ is absolute to V_β; we can assume $x \subseteq V_\beta$. Then for all $\xi_0, \ldots, \xi_n < \beta$ and $y \in V_\beta$, we have

$$\phi(y, \xi_0, \ldots, \xi_n) \iff V_\beta \models \phi(y, \xi_0, \ldots, \xi_n).$$

In particular, plugging in the α_i's for the ξ_i's gives the proposition. $\dashv$

So OD is definable, and it comes with a ready-made well-ordering, by associating each set x in OD with the lexicographically least tuple $\beta, \phi, \alpha_0, \ldots, \alpha_n$ that defines x. But OD suffers from a defect: It need not be transitive! Indeed, each V_α is in OD, and so we will have OD transitive if and only if $V = \mathrm{OD}$. We obtain a transitive model by just keeping those OD sets whose transitive closures are in OD.

Recall the **transitive closure** of a set x, denoted $\mathrm{TC}(x)$, is the least transitive set containing x as a subset; if we set $x_0 = x$ and $x_{n+1} = \bigcup x_n = \{y \mid (\exists z \in x) y \in z\}$, then $\mathrm{TC}(x) = \bigcup_{n\in\omega} x_n$.

DEFINITION 16.8. A set x is **hereditarily ordinal definable**, and we write $x \in \mathrm{HOD}$, if $\mathrm{TC}(\{x\}) \subseteq \mathrm{OD}$.

Then it is immediate that HOD is a definable transitive class; it is also easy to see that HOD contains all the ordinals.

THEOREM 16.9. *Work in* ZF. *Then* HOD *is a transitive proper class model of* ZFC.

PROOF. We check the more non-trivial ZFC axioms. First, Comprehension: suppose ϕ is a formula and that $z, p \in \mathrm{HOD}$. We need to show that HOD satisfies the corresponding instance of Comprehension, that is, we need

$$A = \{x \in z \mid \phi(x, p)^{\mathrm{HOD}}\}$$

to be an element of HOD. Now z, p are in OD, so we can fix formulae ψ, π and ordinals ζ, η that witness this. By Reflection, let β be an ordinal above ζ, η so that $z \subset V_\beta$, and ϕ^{HOD}, ψ, and π are all absolute for V_β. Then we have, for all $x \in V_\beta$,

$$x \in A \iff x \in z \wedge \phi(x,p)^{\mathrm{HOD}} \iff V_\beta \models x \in z \wedge \phi(x,p)^{\mathrm{HOD}}.$$

Then we can substitute in the OD definitions for z and p, and obtain

$$\begin{aligned} A = \{x \in V_\beta \mid V_\beta \models & (\exists z)(\forall a)(a \in z \leftrightarrow \psi(a, \zeta)) \\ & (\exists p)(\forall b)(b \in p \leftrightarrow \pi(b, \eta))(x \in z \wedge \phi(x,p)^{\mathrm{HOD}}\}. \end{aligned}$$

We have a definition of A using only ordinals β, ζ, η as parameters, so $A \in \mathrm{OD}$. That $\mathrm{TC}(\{A\}) \subseteq \mathrm{OD}$ is immediate since $A \subseteq z \in \mathrm{HOD}$.

Now let's check Power Set. We need to show, for each $X \in \mathrm{HOD}$, that $\mathcal{P}(X) \cap \mathrm{HOD} \in \mathrm{HOD}$, as this will witness the Power Set Axiom for X. But

$$a \in P(X) \cap \mathrm{HOD} \iff a \subseteq X \wedge a \in \mathrm{OD}.$$

Now the claim follows, by plugging in the OD definition of X and using definability of membership in OD.

Finally, let us verify that the main axiom of interest, the Axiom of Choice, holds in HOD. For each $x \in \mathrm{OD}$, let $F(x)$ be the lexicographically least tuple $\beta, k, \alpha_0, \ldots, \alpha_n$ so that

$$x = \{y \in V_\beta \mid V_\beta \models \phi_k(y, \alpha_0, \ldots, \alpha_n)\}.$$

Note that the relation $F(x) = \langle \beta, k, \alpha_0, \ldots, \alpha_n$ is an ordinal definable one. So for each $A \in \mathrm{HOD}$, the set $\{\langle x, F(x)\rangle \mid x \in A\}$ is ordinal definable (expanding the OD definition of A); and it is easy to see that each pair $\langle x, F(x)\rangle$ is in HOD. From this set, we obtain a well-order of A, using some canonical well-order of tuples of ordinals. Thus the Axiom of Choice holds in HOD. ⊣

All that is left is to show is that the large cardinals of the AD world entail large cardinal strength in HOD.

Theorem 16.10. *Assume the club filter on κ is an ultrafilter. Then κ is a measurable cardinal in* HOD*.*

Proof. Notice that the club filter is ordinal definable: If $\mathcal{C}_\kappa$ is the club filter on κ, we have

$$X \in \mathcal{C}_\kappa \iff (\exists C \subseteq \kappa) C \subseteq X \wedge C \text{ is a club in } \kappa.$$

Now since every ordinal is in OD, any set of ordinals in OD is automatically in HOD. Put

$$\mathcal{U} = \mathcal{C}_\kappa \cap \mathrm{OD}\,.$$

Then $\mathcal{U} \in \mathrm{HOD}$. It is easy to check that $\mathcal{U}$ is a normal measure in HOD. ⊣

Corollary 16.11. *Assume* AD*. Then there is a measurable cardinal in* HOD*; in particular, we have that there exists a model of* ZFC *plus the existence of a proper class of strongly inaccessible cardinals.*

Proof. The cardinal is $\kappa = \omega_1$. Since HOD is a model of choice, we have in HOD that κ is inaccessible, and there are unboundedly many inaccessible cardinals below κ. So the model $V_\kappa^{\mathrm{HOD}} = V_\kappa \cap \mathrm{HOD}$ witnesses the final clause of the corollary. ⊣