DETERMINACY EXERCISES
DAY 4

Problem 1. Show a Polish space is **second countable**: that is, if X is Polish, then X has a countable basis.

Problem 2. Let X, Y be Polish spaces. Define $C(X,Y) = \{f \in Y^X \mid f : X \to Y \text{ is continuous}\}$. Show $|C(X,Y)| \leq \mathfrak{c}$.

Problem 3. If X, Y are Polish and $A \subseteq X \times Y$, the **slice of** A **at** x is defined to be

$$A_x = \{y \in Y \mid \langle x, y \rangle \in A\}.$$

Show that if $A \in \mathbf{\Gamma}$ with $\mathbf{\Gamma}$ a pointclass closed under continuous substitution, then for each $x \in X$, A_x is in $\mathbf{\Gamma}$ as well.

Problem 4. Let X be Polish. Is there always a continuous surjection $f : \mathbb{R} \to X$?

Problem 5. Suppose X is a Polish space, and $\alpha < \omega_1$. Show that is $A_0, A_1, \dots A_n$ are in $\mathbf{\Sigma}^0_\alpha(X)$, then $A_0 \cap A_1 \cap \cdots \cap A_n \in \mathbf{\Sigma}^0_\alpha(X)$.

Problem 6. Suppose $X \subseteq Y$ are Polish spaces (where X has the subspace topology inherited from Y). Show that a set $A \subseteq X$ is Borel exactly when $A = X \cap B$ for some Borel $B \subseteq Y$.

Definition. Let X, Y be Polish spaces. A function $f : X \to Y$ is **Borel** (or **Borel measurable**) if whenever $B \subseteq Y$ is Borel, so is $f^{-1}[B]$. (So for example, continuous functions are Borel.)

A **Borel isomorphism** is a Borel bijection $f : X \to Y$ with Borel inverse; equivalently, a Borel bijection that maps Borel sets to Borel sets. If such a map exists we say X and Y are **Borel isomorphic**.

Problem 7. Suppose X, Y are Polish spaces. Let $C_0 \subseteq X$ and $C_1 \subseteq Y$ be countable, and let $X' = X \setminus C_0$, $Y' = Y \setminus C_1$. Suppose $f : X' \to Y'$ is a homeomorphism, and $g : C_0 \to C_1$ is a bijection. Then $h = f \cup g$ defined by

$$h(x) = \begin{cases} f(x) \text{ if } x \in X', \\ g(x) \text{ if } x \in C_0 \end{cases}$$

is a Borel isomorphism between X and Y.

Problem 8. Show ω^ω, 2^ω and $\mathbb{R}$ are pairwise Borel isomorphic.

Problem 9. Show that if $f : X \to Y$ is Borel with X, Y Polish, then the graph

$$\text{graph}(f) = \{\langle x, y \rangle \mid f(x) = y\}$$

is Borel as a subset of $X \times Y$.