DETERMINACY EXERCISES
DAY 5

Problem 1. Prove that regardless of $\alpha < \omega_1$, there is no $W \subseteq 2^\omega \times 2^\omega$ that is $\boldsymbol{\Delta}^0_\alpha$-universal for 2^ω.

Problem 2. Suppose $\boldsymbol{\Gamma}$ is closed under continuous substitution, and that for every Polish space X there is a set $W \subseteq 2^\omega \times X$ which is $\boldsymbol{\Gamma}$-universal for X. Show for every Polish space X that there is $W^* \subseteq 2^\omega \times X$ which is $\exists^\omega\boldsymbol{\Gamma}$-universal for X.

Problem 3. Show $\exists^\omega\boldsymbol{\Pi}^0_\alpha = \boldsymbol{\Sigma}^0_{\alpha+1}$.

Definition. A set $P \subseteq X$ of a Polish space is **perfect** if P is closed and every element of P is a limit point of P. If X itself is perfect, we say X is a **perfect Polish space**.

Problem 4. The following exercises generalize the hierarchy theorem to perfect Polish spaces besides 2^ω.

1. Suppose X is a perfect Polish space. Show there is a continuous injective function $f : 2^\omega \to X$.
2. Prove that if $f : 2^\omega \to X$ is continuous and surjective, then f has the nice property that whenever $U \subseteq 2^\omega$ is open, so is $f[U]$.
3. Prove that if $1 < \alpha < \omega_1$, then for every Polish space Y and perfect Polish space X, there is a $\boldsymbol{\Sigma}^0_\alpha$-universal set $W \subseteq X \times Y$ for Y.
4. Conclude that if $\alpha \geq 1$ and Y is perfect Polish, then there is $A \subseteq Y$ in $\boldsymbol{\Sigma}^0_\alpha \setminus \boldsymbol{\Pi}^0_\alpha$.