DETERMINACY EXERCISES
DAY 6

Problem 1. Show $\mathsf{AD}_{\mathbb{R}}$ implies $\mathsf{AC}_{\mathbb{R}}(\mathbb{R})$. (This latter choice principle is equivalent to the statement that for all $A \subseteq \mathbb{R} \times \mathbb{R}$, there is a **uniformizing function for** A, i.e., a function $f : \mathbb{R} \to \mathbb{R}$ such that $\langle x, f(x) \rangle \in A$ whenever A_x is non-empty.)

Definition. Let $\alpha > \omega$ be an ordinal. For each $A \subseteq \omega^\alpha$, we define **the long game of length** α, $G^\alpha(A)$, by letting two players alternate choosing natural numbers x_ξ, for $\xi < \alpha$, in increasing order; Player I chooses x_λ for limit $\lambda < \alpha$.

After α many moves, the players have chosen a sequence $x = \langle x_\xi \rangle_{\xi < \alpha} \in \omega^\alpha$. Player I wins if $x \in A$. Player II wins if $x \notin A$. We say the game is determined if one of the players has a winning strategy (with strategy defined appropriately).

AD^α is the statement that for all $A \subseteq A^\alpha$, the game $G^\alpha(A)$ is determined.

Problem 2. Recall (from the weekend problems) that AD implies there is no injection $f : \omega_1 \to \omega^\omega$. Show (without choice) that AD^{ω_1} is false.

Problem 3 (Blass). Show $\mathsf{AD}_{\mathbb{R}}$ is equivalent to AD^{ω^2}.

Problem 4. Let $C \subseteq \omega^\omega$ be clopen, and let $A \subseteq \omega^\omega$ with $A \neq \emptyset$ and $A \neq \omega^\omega$. Then there is a continuous function $f : \omega^\omega \to \omega^\omega$ such that $f^{-1}[A] = C$.

Problem 5. Do the following.

1. Prove that if $A \subseteq \omega^\omega$ is comeager then it contains a non-empty perfect set.
2. Prove that if A has the Baire property then either A or its complement contains a non-empty perfect set.
3. Conclude (using Problem 3 from the weekend) that under the Axiom of Choice, there is a set without the Baire property.

Problem 6. Show the following.

1. (Kuratowski-Ulam). Suppose $A \subseteq \omega^\omega \times \omega^\omega$ has the Baire property. Then A is meager if and only all but a meager set of slices A_x is meager; that is, A is meager iff $\{x \in \omega^\omega \mid A_x \text{ is non-meager}\}$ is meager. (This is a Baire category analogue of Fubini's Theorem.)
2. Let $\prec$ be a well-order of ω^ω with order type ω_1. Then $\{\langle x, y \rangle \in \omega^\omega \times \omega^\omega \mid x \prec y\}$ does not have the Baire property.
3. The same holds when $\prec$ has order type $\mathfrak{c}$.

Definition. Let X, Y be Polish spaces. A function $f : X \to Y$ is **Baire measurable** if $f^{-1}[U]$ has the Baire property whenever $U \subseteq Y$ is open.

Note that Borel functions are Baire measurable.

Problem 7. Show that if $f : \omega^\omega \to \omega^\omega$ is Baire measurable then there is a comeager $A \subseteq \omega^\omega$ so that $f : A \to \omega^\omega$ is continuous on A (with respect to the subspace topology inherited from X).