DETERMINACY EXERCISES
DAY 11

Problem 1. Prove Tarski's ultrafilter lemma: A filter $\mathcal{F} \subseteq \mathcal{P}(X)$ is an ultrafilter if and only if it is $\subseteq$-maximal among filters on X.

Problem 2. Assume AD. Show there is no non-principal ultrafilter on ω.

Definition. Let $x, y \in \omega^\omega$. We say x is **Turing reducible** to y, written $x \leq_T y$, if there is a program that computes the values of x when calls to y as an oracle are permitted; equivalently, there are total recursive functions $f, g : \omega \to \omega$ so that $x(n) = f(\ulcorner \langle n \rangle ^\frown y \restriction g(n) \urcorner)$ for all n (where $s \mapsto \ulcorner s \urcorner \in \omega$ is some fixed recursive coding of finite sequences).

Reals x, y are **Turing equivalent**, written $x =_T y$, if $x \leq_T y$ and $y \leq_T x$. The equivalence classes under Turing equivalence are the **Turing degrees**, $[x]_T$. The set of Turing degrees is denoted $\mathcal{D}_T$.

We say a set X of Turing degrees is a **cone** if there is some $x \in \omega^\omega$ so that whenever $x \leq_T y$, we have $[y]_T \in X$.

Problem 3. Let $M_T = \{X \subseteq \mathcal{D}_T \mid X \text{ contains a cone}\}$. Show M_T is a countably complete filter.

Problem 4. Let $\mathcal{U}$ be a non-principal κ-complete ultrafilter on κ. Show $\mathcal{U}$ is a normal measure if and only if $\mathcal{U}$ is closed under diagonal intersections.

Problem 5. Show that if $\mathcal{U}$ is a normal measure on κ, then $\mathcal{U}$ extends the club filter. That is, if $C \subseteq \kappa$ is a club, we have $C \in \mathcal{U}$.

Problem 6. (Assuming choice.) Suppose $\mathcal{U}$ is a normal measure on κ. Show

1. κ is a strong limit cardinal, that is, $|2^\lambda| < \kappa$ for all $\lambda < \kappa$.
2. Show the set $\{\alpha < \kappa \mid \alpha \text{ is a strongly inaccessible cardinal}\}$ is in $\mathcal{U}$.

Problem 7. (Assuming choice.) Let κ be a regular cardinal. Recall a set $S \subseteq \kappa$ is **stationary** if $S \cap C \neq \varnothing$ for every club C in κ. Show there exist stationary sets $S, T \subseteq \kappa$ that are disjoint.

Problem 8. Let $\mathcal{F}$ be a filter on ω. For a sequence $\langle a_n \rangle_{n \in \omega}$ of real numbers, we say $\lim_{\mathcal{F}} a_n = L$ if for all $\varepsilon > 0$, the set $\{n \in \omega \mid |a_n - L| < \varepsilon\}$ belongs to $\mathcal{F}$; we say L is the **$\mathcal{F}$-limit** of the sequence $\langle a_n \rangle_{n \in \omega}$.

1. Show that if the $\mathcal{F}$-limit of $\langle a_n \rangle_{n \in \omega}$ exists, then it is unique.
2. Is there a filter so that the notion of $\mathcal{F}$-limit coincides with the usual notion?
3. Show that if $\mathcal{U}$ is an ultrafilter, then every bounded sequence has a $\mathcal{U}$-limit.
4. Show that if every bounded sequence has a $\mathcal{U}$-limit, then $\mathcal{U}$ is an ultrafilter.

Problem 9. (Assuming Choice.) The **Stone-Čech compactification** of ω is the space $\beta\omega$ of ultrafilters on ω, with basis of sets $U_A = \{\mathcal{F} \in \beta\omega \mid A \in \mathcal{F}\}$, for $A \subseteq \omega$.

Show $\beta\omega$ is a compact separable topological space, but is not metrizable (and therefore not Polish).