DETERMINACY EXERCISES
WEEKEND 2

Problem 1. This problem fulfills an old promise to prove the Gale-Stewart theorem without choice. For simplicity's sake, we consider a closed game $G(C)$ played on $\omega^{<\omega}$; so there is a tree T with $C = [T]$.

1. Define a function $F : \mathcal{P}(\omega^{<\omega}) \to \mathcal{P}(\omega^{<\omega})$, by

   $$F(X) = \{s \in \omega^{<\omega} \mid \ell(s) \text{ is even, and } (\forall n)(\exists m) s^\frown\langle n, m\rangle \text{ is in } X \text{ or not in } T\}.$$

   Recursively define $X_0 = \varnothing$, and $X_\alpha = F(\bigcup_{\eta<\alpha} X_\eta)$.

   Show there is an ordinal $\Omega < \omega_1$ so that $X_\Omega = X_{\Omega+1}$.
2. Show that if $\varnothing \in X_\Omega$, then Player II has a winning strategy in $G(C)$.
3. Show that if $\varnothing \notin X_\Omega$, then Player I has a winning strategy in $G(C)$.
4. Observe the same argument proves Gale-Stewart for a tree $T \subseteq X^{<\omega}$, assuming only that X can be well-ordered.

Problem 2 (Steel, Van Wesep). Assume AD; then self-dual Wadgewise sets are self-dual Lipschitzwise (that is, if $A \leq_{\mathrm{W}} \neg A$ then $A \leq_{\mathrm{L}} \neg A$).

Hint: Supposing not, imitate the proof that the Wadge hierarchy is well-founded, using an infinite list of Wadge games being played simultaneously. By assumption, we have a strategy τ_1 winning for Player II in $G_{\mathrm{W}}(A, \neg A)$; let τ_0 be the easy winning strategy for Player II in $G_{\mathrm{W}}(A, A)$. To produce moves for Player I, try "padding" with extra Lipschitz boards between the Wadge boards, using the winning strategy σ for Player I in $G_{\mathrm{L}}(A, \neg A)$. (It may be informative to first assume τ_1 is 2-Lipschitz—that is, it passes only on every other move.)

Problem 3. Assume the Steel-Van Wesep theorem. Show:

1. If $\langle A_n\rangle_{n\in\omega}$ is a sequence such that for all n, $A_n \leq_{\mathrm{W}} B$, then also $\bigoplus_{n\in\omega} A_n = \bigcup_{n\in\omega}\{\langle n\rangle^\frown x \mid x \in A_n\} \leq_{\mathrm{W}} B$.
2. A non-self-dual Wadge degree consists of a single non-self-dual Lipschitz degree; its W-successor is self-dual.
3. A self-dual Wadge degree is a union of ω_1 self-dual Lipschitz degrees; its W-successor is non-self-dual.
4. Wadge degrees of countable cofinality are self-dual; Wadge degrees of uncountable cofinality are non-self-dual.