FORCING EXERCISES
DAY 1

PROBLEM 1. Let κ, λ and μ be cardinals.
1. Show that $(\kappa^\lambda)^\mu = \kappa^{(\lambda \cdot \mu)}$.
2. Show that $\kappa^\kappa = 2^\kappa$.

PROBLEM 2. Let A be an infinite set of size κ. Show that the set of all bijections from A to A has size 2^κ.

PROBLEM 3. Let κ and λ be cardinals with $\lambda \leq \kappa$ and κ infinite. We write $[\kappa]^\lambda$ for the collection of all subsets of κ of size λ. Show that the cardinality of $[\kappa]^\lambda$ is κ^λ.

PROBLEM 4. Show that the union of a set of cardinals is a cardinal.

PROBLEM 5. Suppose that $\langle \alpha_i \mid i < \lambda \rangle$ is an increasing sequence of ordinals cofinal in some cardinal κ. Show that $\mathrm{cf}(\kappa) = \mathrm{cf}(\lambda)$.

PROBLEM 6 (*). Working in ZF, show the Axiom of Choice is equivalent to the statement that for every ordinal α, $\mathcal{P}(\alpha)$ can be well-ordered.

PROBLEM 7 (*). Let κ be an infinite cardinal. Show there is a family $\mathcal{F}$ of size κ of functions from κ^+ to κ^+ such that for all $\alpha, \beta < \kappa^+$ there is a function $f \in \mathcal{F}$ such that either $f(\alpha) = \beta$ or $f(\beta) = \alpha$.

PROBLEM 8 (*). Show there exists a sequence of functions $\langle f_\alpha \rangle_{\alpha < \omega_1}$ such that
- each $f_\alpha : \alpha \to \omega$ is one-to-one, and
- for all $\alpha < \beta < \omega_1$, the set $\{\xi < \alpha \mid f_\alpha(\xi) \neq f_\beta(\xi)\}$ is finite.

PROBLEM 9 (*). Let κ be an infinite cardinal and $\prec$ be a well-ordering of κ. Show that there is an $X \subseteq \kappa$ such that $|X| = \kappa$ and $\prec$ and $<$ agree on X.