FORCING EXERCISES
DAY 2

Problem 1. Let κ be an infinite cardinal. Show that κ^+ is a regular cardinal.

Problem 2. Given a linear order $(L, <)$, consider the topology on L whose basis consists of sets of the form

$$\{x \in L \mid a < x\}, \{x \in L \mid x < b\}, \{x \in L \mid a < x < b\}$$

for any $a, b \in L$. This is called the **order topology** on $(L, <)$.

(a) The order topology on $(\mathbb{R}, <)$ coincides with the usual one.
(b) Which ordinals α are compact with respect to the order topology on $(\alpha, \in)$?
(c) (*) Suppose $f : \omega_1 \to \mathbb{R}$ is continuous. Show f is eventually constant.

Problem 3. A family $\mathcal{F} \subseteq \mathcal{P}(\omega)$ is called a **splitting family** if for every infinite $X \subseteq \omega$ there is a set $A \in \mathcal{F}$ such that $|X \cap A| = |X \setminus A| = \omega$. The **splitting number** $\mathfrak{s}$ is the minimal cardinality of a splitting family. Show that $\omega < \mathfrak{s} \leq 2^\omega$.

Problem 4. Let A, B be subsets of ω. We write $A \subseteq^* B$ when $A \setminus B$ is finite. A sequence $\langle A_\alpha \mid \alpha < \kappa \rangle$ of distinct infinite subsets of ω is a **tower** if $A_\beta \subseteq^* A_\alpha$ whenever $\alpha < \beta$. The **tower number** $\mathfrak{t}$ is the minimal length of a maximal tower (a maximal tower is one for which no further set is almost contained in every member of the tower).

1. Show that $\omega < \mathfrak{t} \leq 2^\omega$.
2. Show that $\mathfrak{t}$ is a regular cardinal.

Problem 5. Show that $\mathfrak{b}$ is a regular cardinal.

Problem 6. Show that there is a MAD family of cardinality $2^{\aleph_0}$.

Problem 7 (*). Define a set $S \subseteq \omega_1 \times \omega$ to be a "large rectangle" if $S = A \times B$ where A is uncountable and B is infinite. Show that CH implies that there is a set $T \subseteq \omega_1 \times \omega$ such that every large rectangle S intersects both T and $(\omega_1 \times \omega) \setminus T$.