

FORCING EXERCISES
DAY 7

PROBLEM 1. Express the axioms for a dense linear order without endpoints as sentences in the language of a single binary relation $<$.

PROBLEM 2. The signature for set theory consists of a single binary relation symbol, \in . Consider the following as τ -structures. Which axioms of ZFC does each satisfy?

1. $(\mathbb{Z}, <)$
2. (ω, \in)
3. (ω_1, \in)

PROBLEM 3 (The Tarski-Vaught test). Show that $\mathcal{M} \prec \mathcal{N}$ if and only if for all formulas $\phi(u, \vec{v})$ and all $\vec{a} \in M^n$, $\mathcal{N} \models \exists x \phi(x, \vec{a})$ if and only if there is $b \in M$, $\mathcal{M} \models \phi(b, \vec{a})$. The following list outlines the statements that you need to prove.

1. Prove the forward direction of the theorem. Use the definition of satisfaction at the key point.
2. For the reverse direction, assume that the right hand side of the ‘if and only if’ statement holds and use the technique of *induction on formulas* outlined below. Formulas were constructed inductively using a certain process. We specified certain *basic* or *atomic* things as formulas. For example, $t_1 = t_2$ where t_1 and t_2 are terms. Then we specified how new formulas are built out of previously defined formulas. For example if ϕ and ψ are formulas, then $\phi \wedge \psi$ is also a formula. In the following order prove
 - (a) If $\phi(\vec{v})$ is an atomic formula and $\vec{a} \in M^n$, then $\mathcal{M} \models \phi(\vec{a})$ if and only if $\mathcal{N} \models \phi(\vec{a})$.
 - (b) For each of the methods of constructing a new formula from an existing formula, show that if elementarity ($\mathcal{M} \models \phi$ if and only if $\mathcal{N} \models \phi$) holds for the old formula(s), then it holds for the new formula. Hints for this step: The definition of satisfaction of formulas is key. The only place the assumption is used is for formulas $\exists x \phi(x, \vec{a})$.

PROBLEM 4 (Lowenheim-Skolem theorem). Show that for every structure \mathcal{N} and every subset $N_0 \subseteq N$, there is an elementary submodel $\mathcal{M} \prec \mathcal{N}$, such that $N_0 \subseteq M$ and $|M| \leq |N_0| \cdot \aleph_0 \cdot |\tau|$. Hints: Use the Tarski-Vaught test. Use the axiom of choice to define a function for each existential formula, which given a sequence of parameters returns a witness to the formula if it exists.

PROBLEM 5. Can the *completeness* of the real line be expressed as a first order sentence about the structure $(\mathbb{R}, +, \cdot, 0, 1, <)$? Give a proof of your answer.

THEOREM 1 (Compactness Theorem). *Let T be a theory. If every finite subset T' of T has a model, then T has a model.*

PROBLEM 6. Use the Completeness theorem to prove the Compactness theorem.

PROBLEM 7. Use the compactness theorem to construct a non-Archimedean field. Recall that a field is Archimedean if there is no $x > 0$ such that for all $n \geq 1$, $x < \frac{1}{n}$.

PROBLEM 8. Use the compactness theorem to prove that there is a non-standard model of Peano Arithmetic. A non-standard model of Peano Arithmetic is not isomorphic to the standard model.

PROBLEM 9 (*). What are the possible order types of a non-standard model of Peano Arithmetic?

PROBLEM 10 (Upward Lowenheim-Skolem theorem). Suppose that T is a theory in a signature τ with an infinite model. Show that T has models of any cardinality κ where $\kappa \geq |\tau| \cdot \aleph_0$. Hint: Create a signature τ' which contains τ and also has many constant symbols. Show there is a κ sized model of a suitable τ' -theory.