# FORCING EXERCISES
# DAY 10

PROBLEM 1. A set $E \subseteq \mathbb{P}$ is **predense** if for every $p \in \mathbb{P}$, there is some $q \in E$ so that $p, q$ are compatible. Show the following are equivalent for a filter $G$:

1. $G$ is $\mathbb{P}$-generic over $M$.
2. $G \cap A \neq \emptyset$ for every maximal antichain $A \in M$.
3. $G \cap E \neq \emptyset$ for every predense $E \in M$.

PROBLEM 2. Let $\mathbb{P}$ be the poset of functions $p : n \to \omega$ for all $n \in \omega$, ordered under reverse inclusion. For $n \in \omega$, let $D_n = \{p \mid n \in \mathrm{dom}(p)\}$, and put $\mathcal{D} = \{D_n : n \in \omega\}$. By $\mathrm{MA}(\omega)$, we know there is a $\mathcal{D}$-generic filter $G$, and $g = \bigcup G$ is a function from $\omega \to \omega$. For the following, give a $\mathbb{P}$-name $\tau$ that satisfies the given conditions for *all* $\mathcal{D}$-generic filters $G$.

1. $\tau[G] = 1$ if $g(0) = 0$ and $\tau[G] = 0$ otherwise.
2. $\tau[G] = 1$ if $6 \in \mathrm{range}(g)$ and $\tau[G] = 0$ otherwise.
3. $\tau[G] = \omega$ if and only if $g$ is surjective.
4. $\tau[G] = g(0)$.
5. $\tau[G] = \{g(0)\}$.
6. $\tau[G] = g^{-1}[\{0\}]$.

Also show that for no name $\tau$ do we have $\tau[G] = 1$ if and only if $g$ is surjective.

PROBLEM 3. Let $\sigma, \tau \in M^{\mathbb{P}}$. Construct a name $\pi \in M^{\mathbb{P}}$ so that regardless of $G$, $\pi[G] = \sigma[G] \cup \tau[G]$.

PROBLEM 4. Let $\sigma \in M^{\mathbb{P}}$. Construct a name $\pi \in M^{\mathbb{P}}$ so that regardless of the filter $G$, $\pi[G] = \bigcup \sigma[G]$.

PROBLEM 5. Let $\tau \in M^{\mathbb{P}}$ with $\mathrm{dom}(\tau) \subseteq \{\check{n} : n \in \omega\}$. Construct a name $\sigma \in \mathbb{P}$ so that regardless of the generic filter $G$, $\sigma[G] = \omega \setminus \tau[G]$.

PROBLEM 6. Suppose $\mathbb{P}$ is a separative poset in $M$. Show that the set

$$\{\tau \in M^{\mathbb{P}} : \tau[G] = \emptyset \text{ for every } M\text{-generic filter } G\}$$

is an element of $M$, but the set

$$\{\tau \in M^{\mathbb{P}} : \tau[G] = \emptyset \text{ for some } M\text{-generic filter } G\}$$

is not.

PROBLEM 7 (*). Assume $\mathbb{P}$ is separative. Show that we obtain the same models $M[G]$ if we require all $\mathbb{P}$-names to be *functions*. That is, consider $M^{\mathrm{fn}(\mathbb{P})}$ to consist of functions $\tau$ with range in $\mathbb{P}$ and domain assumed (inductively) to be a set of $\mathbb{P}$-names in $M^{\mathrm{fn}(\mathbb{P})}$. Show $M[G] = \{\tau[G] : \tau \in M^{\mathrm{fn}(\mathbb{P})}\}$.

Indeed: show that for any $\mathbb{P}$-name $\tau$ there is $\tau_f \in M^{\mathrm{fn}(\mathbb{P})}$ so that $\tau[G] = \tau_f[G]$ for *every* $\mathbb{P}$-generic $G$ over $M$.