FORCING EXERCISES
DAY 12

PROBLEM 1. Let $\mathbb{P} \in M$ be a separative partial order, and $\dot{G}$ the canonical name for the generic filter. Show that $q \Vdash \check{p} \in \dot{G}$ if and only if $q \leq p$.

PROBLEM 2. A poset is **almost homogeneous** if for any $p, q \in \mathbb{P}$ there is an automorphism $i : \mathbb{P} \to \mathbb{P}$ such that $i(p)$ and q are compatible.

1. Show that if I, J are sets with I infinite then the poset
$$\mathrm{Fn}(I, J) = \{p \mid \mathrm{dom}(p) \subset I \text{ is finite and } \mathrm{range}(p) \subseteq J\},$$
ordered by reverse inclusion, is almost homogeneous.
2. Suppose that $\mathbb{P}$ is almost homogeneous and G is $\mathbb{P}$-generic. Let $x_1, \dots, x_n \in M$. Show that if $M[G] \vDash \phi(x_1, \dots, x_n)$ then in fact $\mathbb{1} \Vdash \phi(\check{x_1}, \dots, \check{x_n})$.
3. Conclude that if $\mathbb{P}$ is almost homogeneous, then for any $\mathbb{P}$-generic filters G and H, $M[G]$ and $M[H]$ have the same first-order theory (we say $M[G], M[H]$ are **elementarily equivalent**).

PROBLEM 3. Let $\mathbb{P}$ be a poset, and let $\mathbb{Q}$ be a dense set of $\mathbb{P}$, viewed as a subposet. Do the following.

1. If G is $\mathbb{P}$-generic, then if we set $H = G \cap \mathbb{Q}$, we have that H is $\mathbb{Q}$-generic. Further, $G = \{p \in \mathbb{P} : (\exists q \in H) q \leq p\}$.
2. If H is $\mathbb{Q}$-generic, then if we set $G = \{p \in \mathbb{P} : (\exists q \in H) q \leq p\}$ then G is $\mathbb{P}$-generic. Further, $H = G \cap \mathbb{Q}$.
3. If G and H are taken as in either of the two above, then $M[G] = M[H]$.
4. Any $\mathbb{Q}$-name is a $\mathbb{P}$-name, and if $\tau_1, \dots, \tau_n \in M^{\mathbb{Q}}$ then $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \phi(\tau_1, \dots, \tau_n)$ if and only if $\mathbb{1}_{\mathbb{Q}} \Vdash_{\mathbb{Q}} \phi(\tau_1, \dots, \tau_n)$.
5. For any $\tau \in M^{\mathbb{P}}$ there is a $\sigma \in M^{\mathbb{Q}}$ such that $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \tau = \sigma$.

PROBLEM 4. Two forcing notions $\mathbb{P}$ and $\mathbb{Q}$ are **forcing equivalent** if for some $\mathbb{P}$-name σ and $\mathbb{Q}$-name τ, we have

(a) $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}}$ "τ is a $\mathbb{Q}$-generic filter."
(b) $\mathbb{1}_{\mathbb{Q}} \Vdash_{\mathbb{Q}}$ "σ is a $\mathbb{P}$-generic filter."
(c) If G is $\mathbb{P}$-generic then $G = \sigma[\tau[G]]$.
(d) If H is $\mathbb{Q}$-generic then $H = \tau[\sigma[H]]$.

Do the following:

1. Show that if $\mathbb{Q}$ is a dense subset of $\mathbb{P}$, then the two are forcing equivalent.
2. Show that forcing equivalence is an equivalence relation.

PROBLEM 5. Let $\mathbb{P}$ be a separative poset. For antichains $A_1, A_2 \subseteq \mathbb{P}$, we A_1 is a **refinement** of A_2 if for all $a_1 \in A_1$ there is $a_2 \in A_2$ with $a_1 \leq a_2$. Show the following are equivalent, for M a countable transitive model and κ a regular cardinal of M.

1. $M \models$ the intersection of κ-many dense open subsets of $\mathbb{P}$ is dense open.

2. $M \models$ any collection of κ-many maximal antichains of $\mathbb{P}$ has a common refinement.
3. Whenever $f : \kappa \to M$ is a function in $M[G]$, then in fact f belongs to M.

Such a poset is called **κ-distributive**.

Problem 6 (*). Show that if $\mathbb{P} \in M$ is ccc and ω_1-distributive (in M), then $\mathbb{P}$ is trivial: for any $\mathbb{P}$-generic G, we have $M[G] = M$.