FORCING EXERCISES
DAY 13

Definition 1. A map $\pi : \mathbb{P} \to \mathbb{Q}$ is a **projection** if

1. $\pi(\mathbb{1}_{\mathbb{P}}) = \mathbb{1}_{\mathbb{Q}}$,
2. for all $p_1, p_2 \in \mathbb{P}$, $p_1 \leq p_2$ implies that $\pi(p_1) \leq \pi(p_2)$ and
3. for all $p \in \mathbb{P}$ and all $q \leq \pi(p)$, there is $p' \in \mathbb{P}$ such that $p' \leq p$ and $\pi(p') \leq q$.

Let $\pi : \mathbb{P} \to \mathbb{Q}$ be a projection.

Problem 1. If G is $\mathbb{P}$-generic, then the upwards closure of $\{\pi(p) \mid p \in G\}$ is $\mathbb{Q}$-generic.

Definition 2. Let κ be a regular cardinal. $\mathbb{P}$ has the **κ-chain condition** (κ-cc) if every antichain of $\mathbb{P}$ has size less than κ.

Problem 2. If $\mathbb{P}$ is κ-cc, then $\mathbb{Q}$ is κ-cc.

Problem 3. Let H be $\mathbb{Q}$-generic. In $M[H]$ define a poset $\mathbb{P}/H$ whose underlying set is $\{p \in \mathbb{P} \mid \pi(p) \in H\}$ and is ordered as a suborder of $\mathbb{P}$. (Note that $M[H]$ is a perfectly good transitive model of ZFC, so we can force over it.) Let G be $\mathbb{P}/H$-generic over $M[H]$. Show that G is a $\mathbb{P}$-generic filter over M.

Problem 4. Let $\mathbb{C}$ be Cohen forcing. Define $\pi : \mathbb{C} \to \mathbb{C}$ by $\pi(p)(n) = p(2n)$ whenever $2n$ is in the domain of p. Show that π is a projection. (Notice that this solves Problem 5 part 2 from Day 11.)

Problem 5. Let G be $\mathbb{C}$-generic. In $M[G]$ define $\mathbb{C}/G$ as above using π as in the previous problem. What is this poset? Can you give a concrete characterization?

Problem 6. Let $\mathbb{P}$ be countably closed forcing and let T be a tree of height ω_1 with no cofinal branch. Show that for all M-generic G, T has no cofinal branch in $M[G]$.

Problem 7. Let $\mathbb{P}$ be countably closed forcing and $2^\omega = \omega_2$. Suppose that T is an ω_2-tree. Show that if $b \in M[G]$ is a cofinal branch through T, then $b \in M$.

Problem 8. Let $\mathbb{P}$ be ω_1-Knaster. Show that if T is a tree of height ω_1 with no cofinal branch, then for all M-generic G, T has no cofinal branch in $M[G]$.

Problem 9 (*). Suppose c is a Cohen real over M. Show in $M[c]$ there exists a Suslin tree.