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Vanishing theorems and singularities in birational geometry

– Monograph –

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Preface

This is a preliminary draft of monograph. It builds on lectures notes on a course that Lawrence Ein gave at the University of Catania in Summer 1998, and later again at Hong Kong University in Fall 1999, on lecture notes from the courses that Tommaso de Fernex taught at the University of Utah in Fall 2006, Spring 2010, and Spring 2012, and on lecture notes for the courses taught by Mircea Mustaţă in Winter and Fall 2013 at University of Michigan.

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Contents

Notation and conventions

This is a somewhat random list of conventions and notation, that will probably be adjusted in time.

All schemes are assumed to be separated. With the exception of a few sections, we work with schemes of finite type over a ground field k. In Chapter 1, we only require k to be infinite[1], but starting with Chapter 2, for the sake of simplicity, we assume most of the time that k is algebraically closed. For the same reason, we only consider projective schemes and morphisms, instead of arbitrary complete schemes and proper morphisms. When we start making use of resolutions of singularities and vanishing theorems, we will assume, in addition, that the characteristic of k is zero. If X is a projective scheme over k and $\mathscr{F}$ is a coherent sheaf on X, we put $h^i(X,\mathscr{F}) = \dim_k H^i(X,\mathscr{F})$.

A *variety* is an irreducible and reduced, separated scheme of finite type over k. A *curve* is a variety of dimension one. For every scheme X of finite type over k, we denote by $\mathrm{CDiv}(X)$ the group of Cartier divisors on X (with the operation written additively), and by $\mathrm{Pic}(X)$ the Picard group of X. We denote by $\mathscr{O}_X(D)$ the line bundle associated to the Cartier divisor D. We usually identify an effective Cartier divisor on X with the corresponding subscheme of X.

Remark 0.0.1. It is well-known that if X is an integral scheme, then for every $\mathscr{L} \in \mathrm{Pic}(X)$, there is a Cartier divisor D on X such that $\mathscr{L} \simeq \mathscr{O}_X(D)$. By a result of [Nak63], the same holds if X is a projective scheme over a field k. This is easy to see when k is infinite. Indeed, in this case one can write $\mathscr{L} \simeq \mathscr{L}_1 \otimes \mathscr{L}_2^{-1}$, with $\mathscr{L}_1$ and $\mathscr{L}_2$ very ample line bundles. If $s_i \in \Gamma(X,\mathscr{L}_i)$ are general, for $i = 1,2$, then s_i defines an effective Cartier divisor D_i with $\mathscr{O}_X(D_i) \simeq \mathscr{L}_i$, hence $\mathscr{L} \simeq \mathscr{O}_X(D_1 - D_2)$.

It follows from the above remark that in many cases it makes no difference whether we state things in terms of Cartier divisors or line bundles. However, it is sometimes more convenient to use Cartier divisors for reasons of notation.

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[1] We make this assumption in order to simplify some arguments, and to avoid having to extend too often the ground field; the key advantage is that it allows us to consider general elements in a linear system.

If A is an abelian group, then we will use the notation $A_{\mathbb{Q}}$ and $A_{\mathbb{R}}$ for $A \otimes_{\mathbb{Z}} \mathbb{Q}$ and $A \otimes_{\mathbb{Z}} \mathbb{R}$, respectively. In particular, we will consider the groups $\mathrm{CDiv}(X)_{\mathbb{Q}}$, $\mathrm{CDiv}(X)_{\mathbb{R}}$, $\mathrm{Pic}(X)_{\mathbb{Q}}$ and $\mathrm{Pic}(X)_{\mathbb{R}}$. Note that we always write the operation on $\mathrm{Pic}(X)_{\mathbb{Q}}$ and $\mathrm{Pic}(X)_{\mathbb{R}}$ additively.

An element of $\mathrm{CDiv}(X)_{\mathbb{Q}}$ is called a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor and an element of $\mathrm{CDiv}(X)_{\mathbb{R}}$ is called an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor. An *effective* $\mathbb{R}$-Cartier $\mathbb{R}$-divisor is an element of $\mathrm{CDiv}(X)_{\mathbb{R}}$ that can be written as $\sum_{i=1}^{r} t_i D_i$, where each D_i is an effective Cartier divisor and $t_i \in \mathbb{R}_{\geq 0}$.

Remark 0.0.2. Note that a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor is effective if and only if it can be written as λF, for a Cartier divisor F and $\lambda \in \mathbb{Q}_{\geq 0}$. Indeed, suppose that $D \in \mathrm{CDiv}(X)_{\mathbb{Q}}$ can be written as $D = \sum_{i=1}^{r} \alpha_i D_i$, with $a_i \in \mathbb{R}_{\geq 0}$ and D_i Cartier divisors. After possibly enlarging the set of D_i's, we may assume that we can also write $D = \sum_{i=1}^{r} b_i D_i$, with $b_i \in \mathbb{Q}$. If W is the linear span of $D_1, \ldots, D_r$ in $\mathrm{CDiv}(X)_{\mathbb{Q}}$, then a general property of convex cones (see Corollary A.5.8) implies that since D lies in the intersection of W with the convex cone generated by the D_i in $W_{\mathbb{R}}$, then we can write $D = \sum_{i=1}^{r} a'_i D_i$, with $a'_i \in \mathbb{Q}_{\geq 0}$ for all i. If m is a positive integer such that $ma'_i \in \mathbb{Z}$ for all i, then $D = \frac{1}{m} F$, where $F = \sum_{i=1}^{r} (ma'_i) D_i$ is a Cartier divisor.

For a normal variety X, we denote by $\mathrm{Div}(X)$ the abelian group of divisors on X (a *divisor* is a Weil divisor). Recall that we have an injective group homomorphism $\mathrm{CDiv}(X) \hookrightarrow \mathrm{Div}(X)$. An $\mathbb{R}$-divisor (or $\mathbb{Q}$-divisor) D on X is an element of $\mathrm{Div}(X)_{\mathbb{R}}$ (resp. $\mathrm{Div}(X)_{\mathbb{Q}}$). In this case D is called $\mathbb{R}$-Cartier ($\mathbb{Q}$-Cartier) if it lies in the image of $\mathrm{CDiv}(X)_{\mathbb{R}}$ (resp. $\mathrm{CDiv}(X)_{\mathbb{Q}}$). Note that this is compatible with the above terminology for Cartier divisors. If D is an $\mathbb{R}$-divisor, then we denote by $\mathscr{O}_X(D)$ the corresponding subsheaf of $K(X)$; its sections over $U \subseteq X$ are given by the nonzero rational functions ϕ such that $\mathrm{div}_X(\phi) + D$ is effective on U. Of course, if D is a Cartier divisor, then this is isomorphic to the line bundle associated to D.

An $\mathbb{R}$-divisor is effective if all its coefficients are non-negative. Note that if D is a Cartier divisor on a normal variety, then D is effective as a Cartier divisor if and only if it is effective as a Weil divisor. The same is true for $\mathbb{R}$-divisors, but this is less obvious.

Lemma 0.0.3. *If X is a normal variety and $D \in \mathrm{CDiv}(X)_{\mathbb{R}}$, then D is effective as an $\mathbb{R}$-divisor if and only if it is effective as an element of $\mathrm{CDiv}(X)_{\mathbb{R}}$.*

Proof. It is clear that if D is effective as an element of $\mathrm{CDiv}(X)_{\mathbb{R}}$, then it is also effective as an $\mathbb{R}$-divisor. Note also that the converse is clear if $D \in \mathrm{CDiv}(X)_{\mathbb{Q}}$. In general, let us write $D = t_1 D_1 + \ldots + t_r D_r$, with D_i Cartier divisors and $t_i \in \mathbb{R}$. Consider the prime divisors $E_1, \ldots, E_N$ that appear in $D_1, \ldots, D_r$ and let M be the free abelian group they generate. If σ is the convex cone generated by $E_1, \ldots, E_N$ in $M_{\mathbb{R}}$ and L is the linear subspace over $\mathbb{Q}$ generated by $D_1, \ldots, D_r$, then it follows from general results about rational polyhedral cones that $\sigma \cap L_{\mathbb{R}}$ is a rational polyhedral cone (see Corollary A.5.5). Therefore we can write $D = \sum_{j=1}^{s} F_j$, with each $F_j \in \sigma \cap L$. As we have mentioned, each such F_j is effective as an element of $\mathrm{CDiv}(X)_{\mathbb{R}}$, hence D has the same property. $\square$

Remark 0.0.4. If X is not normal, then it can happen that a Cartier divisor D on X is not effective, but its image in $\mathrm{CDiv}(X)_{\mathbb{Q}}$ is effective (equivalently, there is a positive integer m such that mD is effective). For example, if $X = \operatorname{Spec} k[x,y]/(x^2 - y^3)$, then the Cartier divisor D on X defined by x/y is not effective, but $2D$ is effective. However, such pathologies do not occur on normal varieties.

Chapter 1
Ample, nef, and big line bundles

1.1 The Serre criterion for ampleness

In this section, we review some basic properties of ample line bundles that follow easily from Serre's cohomological criterion for ampleness. Throughout this section we work over a Noetherian affine scheme $S = \mathrm{Spec}(R)$ (we will later be interested in the case when R is a field or a finitely generated algebra over a field).

Definition 1.1.1. A line bundle on a Noetherian scheme X is *ample* if for every coherent sheaf $\mathscr{F}$ on X, the sheaf $\mathscr{F} \otimes \mathscr{L}^m$ is globally generated for $m \gg 0$. If X is a proper scheme over S, then a line bundle $\mathscr{L}$ on X is said to be *very ample* over S if there is a closed immersion $f\colon X \hookrightarrow \mathbb{P}^N_S$ such that $f^*\mathscr{O}_{\mathbb{P}^N_S}(1) \cong \mathscr{L}$. It is a basic fact that $\mathscr{L} \in \mathrm{Pic}(X)$ is ample if and only if $\mathscr{L}^m$ is very ample over S for some positive integer m (see [Har77, Chap. II.7]).

An easy consequence of the definition is that for every line bundles $\mathscr{L}$, $\mathscr{M}$ on X as above, with $\mathscr{L}$ ample, we have $\mathscr{M} \otimes \mathscr{L}^m$ very ample (over S) for $m \gg 0$. In particular, if X has an ample line bundle, then we can write $\mathscr{M} \simeq \mathscr{L}_1 \otimes \mathscr{L}_2^{-1}$, with $\mathscr{L}_1$ and $\mathscr{L}_2$ very ample. It is also easy to see that if $\mathscr{L}$ and $\mathscr{M}$ are (very) ample line bundles on X, then so is $\mathscr{L} \otimes \mathscr{M}$.

Definition 1.1.2. A Cartier divisor D on X is *ample* (or *very ample* over S) if $\mathscr{O}_X(D)$ has this property.

Remark 1.1.3. Suppose that X is a proper scheme over a field k and K/k is a field extension. If $\mathscr{L}$ is a line bundle on X and $\mathscr{L}_K$ is the pull-back of $\mathscr{L}$ to the scheme $X_K = X \times_{\mathrm{Spec}\,k} \mathrm{Spec}\,K$, then $\mathscr{L}$ is ample if and only if $\mathscr{L}_K$ is ample. Indeed, note first that for every $m \geq 1$, we have

$$\Gamma(X_K, \mathscr{L}_K^m) \simeq \Gamma(X, \mathscr{L}^m) \otimes_k K.$$

Therefore $\mathscr{L}^m$ is globally generated if and only if $\mathscr{L}_K^m$ is globally generated, and in this case the map defined by $\mathscr{L}_K^m$ is obtained from the map defined by $\mathscr{L}^m$ by extending scalars. Therefore one map is a closed immersion if and only if the other one is.

Remark 1.1.4. Suppose that X is a proper scheme over S, $f\colon T \to S$ is a morphism, with T a Noetherian affine scheme, and $g\colon X_T = X \times_S T \to X$ is the canonical projection. It is clear from definition that if $\mathscr{L}$ is very ample over S, then $g^*(\mathscr{L})$ is very ample over T. This immediately implies that if $\mathscr{L}$ is ample, then $g^*(\mathscr{L})$ is ample.

The following ampleness criterion is well-known. We refer to [Har77, Chap. III.5] for a proof.

Theorem 1.1.5. *For a line bundle $\mathscr{L}$ on a proper scheme X over S, the following properties are equivalent:*

i) *$\mathscr{L}$ is ample.*
ii) (Asymptotic Serre vanishing). *For every coherent sheaf $\mathscr{F}$ on X, we have $H^i(X, \mathscr{F} \otimes \mathscr{L}^m) = 0$ for all $i > 0$ and all $m \gg 0$.*

We use the characterization of ampleness given in the above theorem to prove some basic properties of this notion.

Lemma 1.1.6. *If $\mathscr{L}$ is an ample line bundle on a proper scheme X over S and Y is a closed subscheme of X, then $\mathscr{L}|_Y$ is ample.*

Proof. The assertion follows easily from definition. □

Proposition 1.1.7. *Let $\mathscr{L}$ be a line bundle on a proper scheme X over S. If $X_1, \ldots, X_r$ are the irreducible components of X, considered with the reduced scheme structures, then $\mathscr{L}$ is ample if and only if $\mathscr{L}|_{X_i}$ is ample for $1 \leq i \leq r$. In particular, $\mathscr{L}$ is ample if and only if its restriction to the reduced subscheme X_{red} is ample.*

Before proving the proposition, we give a general lemma.

Lemma 1.1.8. *If $\mathscr{F}$ is a coherent sheaf on a Noetherian scheme X, then $\mathscr{F}$ has a finite filtration*

$$\mathscr{F} = \mathscr{F}_m \supseteq \mathscr{F}_{m-1} \supseteq \ldots \supseteq \mathscr{F}_1 \supseteq \mathscr{F}_0 = 0,$$

such that for every i with $1 \leq i \leq m$, the annihilator $\mathrm{Ann}_{\mathscr{O}_X}(\mathscr{F}_i/\mathscr{F}_{i-1})$ defines an integral closed subscheme Z_i of X.

Proof. Arguing by Noetherian induction, we may assume that the assertion holds whenever $\mathrm{Ann}_{\mathscr{O}_X}(\mathscr{F})$ is nonzero. Suppose that $\mathrm{Ann}_{\mathscr{O}_X}(\mathscr{F}) = 0$. If X is not reduced, let $\mathscr{I}$ be the ideal defining X_{red} in X. Let us consider the smallest integer $d \geq 2$ such that $\mathscr{I}^d = 0$. Since both $\mathscr{I}\mathscr{F}$ and $\mathscr{F}/\mathscr{I}\mathscr{F}$ are annihilated by $\mathscr{I}^{d-1} \neq 0$, it follows from the induction hypothesis that both these sheaves have filtrations as in the lemma. By concatenating these filtrations, we deduce that also $\mathscr{F}$ has a filtration with the required property.

We may thus assume that X is reduced and let $X_1, \ldots, X_r$ be the irreducible components of X, considered with the reduced scheme structures. If $r = 1$, then X is an integral scheme, and we are done since $\mathrm{Ann}_{\mathscr{O}_X}(\mathscr{F})=0$. Suppose now that $r\geq 2$. If $\mathscr{I}_j$ is the ideal defining X_j in X, then $\mathscr{I}_1\cap\ldots\cap\mathscr{I}_r=0$. Since $\mathscr{I}_1\mathscr{F}$ is annihilated by $\mathscr{I}_2\cap\ldots\cap\mathscr{I}_r\neq 0$, and $\mathscr{F}/\mathscr{I}_1\mathscr{F}$ is annihilated by $\mathscr{I}_1\neq 0$, it follows from the induction hypothesis that both $\mathscr{I}_1\mathscr{F}$ and $\mathscr{F}/\mathscr{I}_1\mathscr{F}$ admit filtrations as in the lemma. By concatenating these, we obtain such a filtration also for $\mathscr{F}$. This completes the proof of the lemma. □

Proof of Proposition 1.1.7. We only need to prove the first assertion in the proposition: the last one follows from the fact that X and X_{red} have the same irreducible components. If $\mathscr{L}$ is ample on X, then each $\mathscr{L}|_{X_i}$ is ample by Lemma 1.1.6.

Conversely, suppose that each $\mathscr{L}|_{X_i}$ is ample. We need to show that for every coherent sheaf $\mathscr{F}$ on X, we have

$$H^j(X,\mathscr{F}\otimes\mathscr{L}^m)=0\ \text{ for all } j\geq 1 \text{ and } m\gg 0. \tag{1.1}$$

Note that if

$$0\to\mathscr{F}'\to\mathscr{F}\to\mathscr{F}''\to 0$$

is an exact sequence, and $\mathscr{F}'$ and $\mathscr{F}''$ satisfy (1.1), then so does $\mathscr{F}$ (it is enough to tensor the above exact sequence with $\mathscr{L}^m$, and consider the corresponding cohomology long exact sequence). If Z is an integral closed subscheme of X, then Z is a closed subscheme of some X_i, hence $\mathscr{L}|_Z$ is ample by Lemma 1.1.6. By considering a filtration of $\mathscr{F}$ as in Lemma 1.1.8, we conclude that $\mathscr{F}$ satisfies (1.1). This completes the proof of the proposition. □

Proposition 1.1.9. *If $f\colon Y\to X$ is a finite morphism between two proper schemes over S and $\mathscr{L}$ is an ample line bundle on X, then its pull-back $f^*\mathscr{L}$ is ample. Conversely, if f is also surjective and $f^*\mathscr{L}$ is ample, then $\mathscr{L}$ is ample.*

Proof. If $\mathscr{L}$ is ample and $\mathscr{G}$ is any coherent sheaf on Y, then using the projection formula and the fact that f is finite we get

$$H^i(Y,\mathscr{G}\otimes(f^*\mathscr{L})^m)\simeq H^i(X,f_*(\mathscr{G}\otimes(f^*\mathscr{L})^m))\simeq H^i(X,(f_*\mathscr{G})\otimes\mathscr{L}^m)=0$$

for all $i>0$ and $m\gg 0$. Therefore $f^*\mathscr{L}$ is ample.

Conversely, suppose that f is surjective and $f^*\mathscr{L}$ is ample. For every irreducible component Y' of Y, there is an irreducible component X' of X such that f induces a finite, surjective morphism $X'\to Y'$. We deduce using Proposition 1.1.7 that we may assume that X and Y are irreducible and reduced.

Arguing by Noetherian induction, we may assume that the restriction of $\mathscr{L}$ to every closed subscheme of X different from X is ample. It follows from Lemma 1.1.10 below that we can find a coherent sheaf $\mathscr{G}$ on Y equipped with a morphism

$$\phi\colon f_*\mathscr{G}\to\mathscr{F}^{\oplus d},$$

where d is the degree of f, which restricts to an isomorphism over a nonempty open subset of X. Note that this suffices to prove that

$$H^i(X, \mathscr{F} \otimes \mathscr{L}^m) = 0 \text{ for } i > 0 \text{ and } m \gg 0,$$

and thus that $\mathscr{L}$ is ample. Indeed, by the inductive assumption, we have

$$H^i(X, \ker(\phi) \otimes \mathscr{L}^m) = 0 = H^i(X, \operatorname{coker}(\phi) \otimes \mathscr{L}^m)$$

for all $i > 0$ and $m \gg 0$. Therefore the vanishing of $H^i(X, \mathscr{F} \otimes \mathscr{L}^m)$ for $i > 0$ and $m \gg 0$ will follow from the vanishing of $H^i(X, (f_*\mathscr{G}) \otimes \mathscr{L}^m)$, and it is enough to note, as above, that

$$H^i(X, (f_*\mathscr{G}) \otimes \mathscr{L}^m) \simeq H^i(Y, \mathscr{G} \otimes (f^*\mathscr{L})^m) = 0 \text{ for } i \geq 1 \text{ and } m \gg 0.$$

□

Lemma 1.1.10. *If $f\colon Y \to X$ is a finite, surjective morphism of integral schemes and $\mathscr{F}$ is a coherent sheaf on X, then there is a coherent sheaf $\mathscr{G}$ on Y and a morphism*

$$\phi\colon f_*\mathscr{G} \to \mathscr{F}^{\oplus d},$$

where d is the degree of f, which restricts to an isomorphism over a nonempty open subset of X.

Proof. We fix d elements $s_1, \ldots, s_d \in K(Y)$ forming a basis for $K(Y)$ over $K(X)$. These elements generate an $\mathscr{O}_Y$-coherent sheaf $\mathscr{M}$, and there is an induced $\mathscr{O}_X$-linear map $\psi\colon \mathscr{O}_X^{\oplus d} \to f_*\mathscr{M}$ which restricts to an isomorphism over a suitable open subset of X. We consider the coherent sheaf $\mathscr{G} := \mathscr{H}om_{\mathscr{O}_Y}(\mathscr{M}, f^!\mathscr{F})$, where $f^!\mathscr{F}$ is the coherent sheaf on Y such that $f_*(f^!\mathscr{F}) \simeq \mathscr{H}om_{\mathscr{O}_X}(f_*\mathscr{O}_Y, \mathscr{F})$. Note that

$$f_*\mathscr{G} = f_*\mathscr{H}om_{\mathscr{O}_Y}(\mathscr{M}, f^!\mathscr{F}) \cong \mathscr{H}om_{\mathscr{O}_X}(f_*\mathscr{M}, \mathscr{F})$$

(see [Har77, Exercise III.6.10]). Then, composing with ψ, we obtain a map

$$\phi\colon f_*\mathscr{G} \to \mathscr{H}om_{\mathscr{O}_X}(\mathscr{O}_X^{\oplus d}, \mathscr{F}) \cong \mathscr{F}^{\oplus d}$$

which, by construction, restricts to an isomorphism over an open subset of X. □

Corollary 1.1.11. *If $f\colon X \to \mathbb{P}^n_S$ is a proper morphism over S and $\mathscr{L} = f^*(\mathscr{O}_{\mathbb{P}^n_S}(1))$, then $\mathscr{L}$ is ample if and only if f is finite.*

Proof. If f is finite, since $\mathscr{O}_{\mathbb{P}^n_S}(1)$ is ample on $\mathbb{P}^n_S(1)$, we conclude that $\mathscr{L}$ is ample by Proposition 1.1.9. Conversely, if $\mathscr{L}$ is ample, then f has finite fibers (the restriction of $\mathscr{L}$ to each fiber is both ample and trivial, hence the fiber is 0-dimensional). Since f is proper, we conclude that f is finite. □

1.2 Intersection numbers of line bundles

Our goal in this section is to define the intersection numbers of divisors and give their main properties, following [Kle66]. For similar presentations, see also [Băd01] and [Deb01]. All schemes are of finite type over a fixed infinite field k. For a coherent sheaf $\mathscr{M}$ on a complete scheme X, we denote by $\chi(\mathscr{M})$ its Euler-Poincaré characteristic

$$\chi(\mathscr{M}) := \sum_{i=0}^{\dim(X)} (-1)^i h^i(X, \mathscr{M}).$$

Both in this section and the next one, while we state the results for complete schemes, we only give the proofs in the projective case whenever this simplifies the argument. We leave the general case as an exercise for the reader.

Proposition 1.2.1. *(Snapper) Let X be a complete scheme. If $\mathscr{L}_1, \ldots, \mathscr{L}_r$ are line bundles on X and $\mathscr{F}$ is a coherent sheaf on X, then the function*

$$\mathbb{Z}^r \ni (m_1, \ldots, m_r) \to \chi(\mathscr{F} \otimes \mathscr{L}_1^{m_1} \otimes \ldots \otimes \mathscr{L}_r^{m_r}) \in \mathbb{Z}$$

is polynomial, of total degree $\leq \dim(\mathrm{Supp}(\mathscr{F}))$.

Proof. We give the proof under the assumption that X is projective. We prove the assertion by induction on $d = \dim(\mathrm{Supp}(\mathscr{F}))$. If $d = -1$, then the assertion is clear (we make the convention that $\dim(\emptyset) = -1$ and the zero polynomial has degree -1). Since X is projective, we can find very ample effective Cartier divisors A and B on X such that $\mathscr{L}_1 \simeq \mathscr{O}_X(A - B)$. Furthermore, by taking A and B to be general in their linear systems, we may assume that no associated points of $\mathscr{F}$ lie on A or B. On one hand, this gives exact sequences

$$0 \to \mathscr{F} \otimes \mathscr{O}_X(-B) \otimes \mathscr{L}_1^{m_1 - 1} \to \mathscr{F} \otimes \mathscr{L}_1^{m_1} \to \mathscr{F} \otimes \mathscr{O}_A \otimes \mathscr{L}_1^{m_1} \to 0$$

and

$$0 \to \mathscr{F} \otimes \mathscr{O}_X(-B) \otimes \mathscr{L}_1^{m_1 - 1} \to \mathscr{F} \otimes \mathscr{L}_1^{m_1 - 1} \to \mathscr{F} \otimes \mathscr{O}_B \otimes \mathscr{L}_1^{m_1 - 1} \to 0.$$

By tensoring these with $\mathscr{L}_2^{m_2} \otimes \ldots \otimes \mathscr{L}_r^{m_r}$ and taking the long exact sequences in cohomology, we obtain using the additivity of the Euler-Poincaré characteristic

$$\chi(\mathscr{F} \otimes \mathscr{L}_1^{m_1} \otimes \ldots \otimes \mathscr{L}_r^{m_r}) - \chi(\mathscr{F} \otimes \mathscr{L}_1^{m_1 - 1} \otimes \ldots \otimes \mathscr{L}_r^{m_r})$$

$$= \chi(\mathscr{F} \otimes \mathscr{O}_A \otimes \mathscr{L}_1^{m_1} \otimes \ldots \otimes \mathscr{L}_r^{m_r}) - \chi(\mathscr{F} \otimes \mathscr{O}_B \otimes \mathscr{L}_1^{m_1 - 1} \otimes \ldots \otimes \mathscr{L}_r^{m_r}).$$

On the other hand, the assumption on A and B also gives $\dim(\mathrm{Supp}(\mathscr{F} \otimes \mathscr{O}_A)) \leq d - 1$ and $\dim(\mathrm{Supp}(\mathscr{F} \otimes \mathscr{O}_B)) \leq d - 1$. It follows from these inequalities and the inductive assumption that the function

$$\mathbb{Z}^r \ni (m_1, \ldots, m_n) \to \chi(\mathscr{F} \otimes \mathscr{L}_1^{m_1} \otimes \ldots \otimes \mathscr{L}_r^{m_r}) - \chi(\mathscr{F} \otimes \mathscr{L}_1^{m_1 - 1} \otimes \ldots \otimes \mathscr{L}_r^{m_r})$$

is polynomial of total degree $\leq (d-1)$. Since the same assertion clearly also holds with respect to the other variables, it is an elementary exercise to deduce that the function

$$\mathbb{Z}^r \ni (m_1, \ldots, m_n) \to \chi(\mathscr{F} \otimes \mathscr{L}_1^{m_1} \otimes \ldots \otimes \mathscr{L}_r^{m_r})$$

is polynomial, of total degree $\leq d$. □

Definition 1.2.2. Suppose that $\mathscr{L}_1, \ldots, \mathscr{L}_r$ are line bundles on a complete scheme X and $\mathscr{F}$ is a coherent sheaf on X with $\dim(\mathrm{Supp}(\mathscr{F})) \leq r$. The intersection number $(\mathscr{L}_1 \cdot \ldots \cdot \mathscr{L}_r; \mathscr{F})$ is defined as the coefficient of $m_1 \cdots m_r$ in the polynomial $P(m_1, \ldots, m_r)$ such that $P(m_1, \ldots, m_r) = \chi(\mathscr{F} \otimes \mathscr{L}_1^{m_1} \otimes \ldots \otimes \mathscr{L}_r^{m_r})$ for all $(m_1, \ldots, m_r) \in \mathbb{Z}^r$.

If $\mathscr{F} = \mathscr{O}_Y$, for a closed subscheme Y of X, then we write $(\mathscr{L}_1 \cdot \ldots \cdot \mathscr{L}_r \cdot Y)$ instead of $(\mathscr{L}_1 \cdot \ldots \cdot \mathscr{L}_r; \mathscr{O}_Y)$ and simply $(\mathscr{L}_1 \cdot \ldots \cdot \mathscr{L}_r)$ if $Y = X$. Furthermore, if $\mathscr{L}_1 = \ldots = \mathscr{L}_r = \mathscr{L}$, then we write $(\mathscr{L}^r; \mathscr{F})$, $(\mathscr{L}^r \cdot Y)$ and $(\mathscr{L}^r)$ for the corresponding intersection numbers (similar conventions will also be used if only some of the $\mathscr{L}_i$ are equal). If $D_1, \ldots, D_r$ are Cartier divisors on X and $\mathscr{F}$ is as above, then we also write $(D_1 \cdot \ldots \cdot D_r; \mathscr{F})$ for $(\mathscr{O}_X(D_1) \cdot \ldots \cdot \mathscr{O}_X(D_r); \mathscr{F})$ and similarly for the other variants of intersection numbers.

The following lemma allows us to describe the intersection numbers as alternating sums of Euler-Poincaré characteristics.

Lemma 1.2.3. *Let P be a polynomial in r variables with coefficients in a ring R such that the total degree of P is $\leq r$. The coefficient of $x_1 \cdots x_r$ in P is equal to*

$$\sum_{J \subseteq \{1, \ldots, r\}} (-1)^{|J|} P(\delta_{J,1}, \ldots, \delta_{J,r}),$$

where the sum is over all subsets J of $\{1, \ldots, r\}$ (including the empty subset) and where $\delta_{J,j} = -1$ if $j \in I$ and $\delta_{J,j} = 0$ if $j \notin J$.

Proof. The assertion follows by induction on r, the case $r = 1$ being trivial. For the induction step, it is enough to note that the coefficient of $x_1 \cdots x_r$ in P is equal to the coefficient of $x_1 \cdots x_{r-1}$ in

$$Q(x_1, \ldots, x_{r-1}) = P(x_1, \ldots, x_{r-1}, 0) - P(x_1, \ldots, x_{r-1}, -1),$$

whose total degree is $\leq (r-1)$. This in turn follows by considering the effect of taking the difference on the right-hand side for each of the monomials in P. □

Corollary 1.2.4. *If $\mathscr{L}_1, \ldots, \mathscr{L}_r$ are line bundles on a complete scheme X and $\mathscr{F}$ is a coherent sheaf on X with* $\dim(\mathrm{Supp}(\mathscr{F})) \leq r$*, then*

$$(\mathscr{L}_1 \cdot \ldots \cdot \mathscr{L}_r; \mathscr{F}) = \sum_{J \subseteq \{1, \ldots, r\}} (-1)^{|J|} \chi(\mathscr{F} \otimes (\otimes_{j \in J} \mathscr{L}_j^{-1})).$$

We can now prove the basic properties of intersection numbers.

Proposition 1.2.5. *Let $\mathscr{L}_1,\dots,\mathscr{L}_r$ be line bundles on the complete scheme X and $\mathscr{F}$ a coherent sheaf on X, with $\dim(\mathrm{Supp}(\mathscr{F}))\le r$.*

i) If $\dim(\mathrm{Supp}(\mathscr{F}))<r$, then $(\mathscr{L}_1\cdot\ldots\cdot\mathscr{L}_r;\mathscr{F})=0$.
ii) The intersection number $(\mathscr{L}_1\cdot\ldots\cdot\mathscr{L}_r;\mathscr{F})$ is an integer. The map

$$\mathrm{Pic}(X)^r\ni(\mathscr{L}_1,\dots,\mathscr{L}_r)\to(\mathscr{L}_1\cdot\ldots\cdot\mathscr{L}_r;\mathscr{F})\in\mathbb{Z}$$

is multilinear and symmetric.
iii) If $Y_1,\dots,Y_s$ are the r-dimensional irreducible components of $\mathrm{Supp}(\mathscr{F})$ (with reduced scheme structures) and η_i is the generic point of Y_i, then

$$(\mathscr{L}_1\cdot\ldots\cdot\mathscr{L}_r;\mathscr{F})=\sum_{i=1}^{s}\ell_{\mathscr{O}_{X,\eta_i}}(\mathscr{F}_{\eta_i})\cdot(\mathscr{L}_1\cdot\ldots\cdot\mathscr{L}_r\cdot Y_i). \tag{1.2}$$

iv) (Projection formula) Suppose that $f\colon X\to Y$ is a surjective morphism of complete varieties, with $\dim(X)\le r$. If there are line bundles $\mathscr{M}_i$ on Y such that $\mathscr{L}_i\simeq f^*(\mathscr{M}_i)$ for every i, then $(\mathscr{L}_1\cdot\ldots\cdot\mathscr{L}_r)=d\cdot(\mathscr{M}_1\cdot\ldots\cdot\mathscr{M}_r)$ if f is generically finite of degree d, and $(\mathscr{L}_1\cdot\ldots\cdot\mathscr{L}_r)=0$, otherwise.
v) If $\mathscr{L}_r=\mathscr{O}_X(D)$ for some effective Cartier divisor D, then

$$(\mathscr{L}_1\cdot\ldots\cdot\mathscr{L}_r)=(\mathscr{L}_1|_D\cdot\ldots\cdot\mathscr{L}_{r-1}|_D),$$

with the convention that when $r=1$, the right-hand side is equal to $h^0(\mathscr{O}_D)$.
vi) If k' is a field extension of k, we put $X'=X\times_{\mathrm{Spec}k}\mathrm{Spec}\,k'$, and $\mathscr{L}_i'$ and $\mathscr{F}'$ are the pull-backs of $\mathscr{L}_i$ and $\mathscr{F}$, respectively, to X', then

$$(\mathscr{L}_1\cdot\ldots\cdot\mathscr{L}_r;\mathscr{F})=(\mathscr{L}_1'\cdot\ldots\cdot\mathscr{L}_r';\mathscr{F}').$$

Proof. The assertion in i) follows from definition and Proposition 1.2.1. The fact that intersection numbers are integers is clear by Corollary 1.2.4. The symmetry of the application in ii) is obvious, hence in order to prove ii) we only need to show that

$$((\mathscr{L}_1\otimes\mathscr{L}_1')\cdot\mathscr{L}_2\cdot\ldots\cdot\mathscr{L}_r;\mathscr{F})-(\mathscr{L}_1\cdot\mathscr{L}_2\cdot\ldots\cdot\mathscr{L}_r;\mathscr{F})-(\mathscr{L}_1'\cdot\mathscr{L}_2\cdot\ldots\cdot\mathscr{L}_r;\mathscr{F})=0. \tag{1.3}$$

An easy computation using the formula in Corollary 1.2.4 shows that the difference in (1.3) is equal to $-(\mathscr{L}_1\cdot\mathscr{L}_1'\cdot\mathscr{L}_2\cdot\ldots\cdot\mathscr{L}_r;\mathscr{F})$, which vanishes by i).

We note that iii) clearly holds if $\dim(\mathrm{Supp}(\mathscr{F}))<r$. It follows from definition and the additivity of the Euler-Poincaré characteristic that if

$$0\to\mathscr{F}'\to\mathscr{F}\to\mathscr{F}''\to 0$$

is an exact sequence of coherent sheaves on X, then

$$(\mathscr{L}_1\cdot\ldots\cdot\mathscr{L}_r;\mathscr{F})=(\mathscr{L}_1\cdot\ldots\cdot\mathscr{L}_r;\mathscr{F}')+(\mathscr{L}_1\cdot\ldots\cdot\mathscr{L}_r;\mathscr{F}''). \tag{1.4}$$

Since $\ell_{\mathscr{O}_{X,\eta_i}}(\mathscr{F}_{\eta_i}) = \ell_{\mathscr{O}_{X,\eta_i}}(\mathscr{F}'_{\eta_i}) + \ell_{\mathscr{O}_{X,\eta_i}}(\mathscr{F}''_{\eta_i})$ for every i, we conclude that if (1.2) holds for two of $\mathscr{F}'$, $\mathscr{F}$, and $\mathscr{F}''$, then it also holds for the third one.

Recall that by Lemma 1.1.8, $\mathscr{F}$ has a finite filtration such that the annihilator of each of the successive quotients is the ideal of an integral closed subscheme of X. We conclude that in order to prove (1.2), we may assume that X is an integral scheme. We also see that if $\mathscr{G}$ is another sheaf such that we have a morphism $\phi\colon \mathscr{F} \to \mathscr{G}$ that is an isomorphism at the generic point $\eta \in X$, then iii) holds for $\mathscr{F}$ if and only if it holds for $\mathscr{G}$ (note that in this case both $\ker(\phi)$ and $\mathrm{coker}(\phi)$ are supported in dimension $< r$). In particular, by replacing $\mathscr{F}$ by $\mathscr{F} \otimes \mathscr{O}_X(D)$, where D is a suitable effective very ample divisor, we may assume that $\mathscr{F}$ is generated by global sections. If $d = \ell_{\mathscr{O}_{X,\eta}}(\mathscr{F}_\eta)$ and $s_1, \ldots, s_d \in \Gamma(X, \mathscr{F})$ are general sections, then the induced morphism $\mathscr{O}_X^{\oplus d} \to \mathscr{F}$ is an isomorphism at η. Since (1.2) clearly holds for $\mathscr{O}_X^{\oplus d}$, this completes the proof of iii).

In order to prove iv), note first that the additivity of the Euler-Poincaré characteristic, the Leray spectral sequence, and the projection formula imply that

$$\chi(\mathscr{L}_1^{m_1} \otimes \ldots \otimes \mathscr{L}_r^{m_r}) = \sum_{i \geq 0} (-1)^i \chi(R^i f_*(\mathscr{O}_X) \otimes \mathscr{M}_1^{m_1} \otimes \ldots \otimes \mathscr{M}_r^{m_r}),$$

hence by definition of intersection numbers we have

$$(\mathscr{L}_1 \cdot \ldots \cdot \mathscr{L}_r) = \sum_{i \geq 0} (-1)^i (\mathscr{M}_1 \cdot \ldots \cdot \mathscr{M}_r; R^i f_*(\mathscr{O}_X)).$$

If f is not generically finite, then all intersection numbers on the right-hand side are zero since $\dim(Y) < r$. Suppose now that f is generically finite and $\deg(f) = d$. In this case $R^i f_*(\mathscr{O}_X)$ is supported on a proper subscheme of Y for all $i \geq 1$, while $\ell_{\mathscr{O}_{Y,\eta}}((f_*(\mathscr{O}_X)_\eta) = d$ if η is the generic point of Y. The formula in iv) now follows from iii) and i).

In order to prove v), we use Corollary 1.2.4 by considering first the subsets contained in $\{1, \ldots, r-1\}$, and then the ones contaning r. We obtain

$$(\mathscr{L}_1 \cdot \ldots \mathscr{L}_{r-1} \cdot \mathscr{O}(D)) = \sum_{J \subseteq \{1,\ldots,r-1\}} (-1)^{|J|} \chi(\otimes_{i \in J} \mathscr{L}_i^{-1})$$

$$+ \sum_{J \subseteq \{1,\ldots,r-1\}} (-1)^{|J|+1} \chi(\mathscr{O}_X(-D) \otimes (\otimes_{i \in J} \mathscr{L}_i^{-1})) = \sum_{J \subseteq \{1,\ldots,r-1\}} (-1)^{|J|} \chi(\otimes_{i \in J} \mathscr{L}_i^{-1}|_D)$$

$$= (\mathscr{L}_1|_D \cdot \ldots \cdot \mathscr{L}_{r-1}|_D),$$

where the second equality follows by tensoring the exact sequence

$$0 \to \mathscr{O}_X(-D) \to \mathscr{O}_X \to \mathscr{O}_D \to 0$$

with $\otimes_{i \in J} \mathscr{L}_i^{-1}$, and using the additivity of the Euler-Poincaré characteristic.

The equality in vi) is an immediate consequence of the definition of intersection numbers and of the fact that for every sheaf $\mathscr{M}$ on X, if $\mathscr{M}'$ is its pull-back to X', then $h^i(X, \mathscr{M}) = h^i(X', \mathscr{M}')$ for every i. $\square$

Remark 1.2.6. Suppose that X is a Cohen–Macaulay scheme of pure dimension n and $D_1,\ldots,D_n$ are effective Cartier divisors on X such that $\dim(D_1\cap\ldots\cap D_i)=n-i$ for $1\leq i\leq n$. In this case, at every point $x\in D_1\cap\ldots\cap D_i$, the local equations of $D_1,\ldots,D_i$ form a regular sequence in $\mathscr{O}_{X,x}$. Applying the assertion in v) above on $X,D_1,\ldots,D_1\cap\ldots,\cap D_{n-1}$, we obtain

$$(D_1\cdot\ldots\cdot D_n)=h^0(\mathscr{O}_{D_1\cap\ldots\cap D_n}).$$

If, in addition, the intersection points are smooth k-rational points of X and of each of the D_i and the intersection is transversal, then $(D_1\cdot\ldots\cdot D_n)$ is equal to the number of intersection points.

Remark 1.2.7. Suppose that X is a projective r-dimensional scheme and $\mathscr{L}_1,\ldots,\mathscr{L}_s$ are ample line bundles on X, for some $s\leq r$. In this case, there is a positive integer m and a closed subscheme Y of X of dimension $r-s$ such that

$$(\mathscr{L}_1\cdot\ldots\cdot\mathscr{L}_s\cdot\mathscr{L}'_1\cdot\ldots\cdot\mathscr{L}'_{r-s})=\frac{1}{m}(\mathscr{L}'_1\cdot\ldots\cdot\mathscr{L}'_{r-s}\cdot Y) \tag{1.5}$$

for every $\mathscr{L}'_1,\ldots,\mathscr{L}'_{r-s}\in\operatorname{Pic}(X)$. Indeed, if m_i is a positive integer, for $1\leq i\leq s$, such that $\mathscr{L}_i^{m_i}$ is very ample and if $D_i\in|L_i^{m_i}|$ is a general element, then the closed subscheme $Y=D_1\cap\ldots\cap D_s$ has dimension $r-s$ and a repeated application of Proposition 1.2.5 v) gives the equality in (1.5), with $m=\prod_{i=1}^s m_i$.

Remark 1.2.8. It is easy to see that properties i)–v) in Proposition 1.2.5 uniquely determine the intersection numbers $(\mathscr{L}_1\cdot\ldots\cdot\mathscr{L}_r;\mathscr{F})$. Indeed, we argue by induction on r. It follows from iii) that a general such intersection number is determined if we know the intersection numbers of the form $(\mathscr{L}_1\cdot\ldots\mathscr{L}_r)$ when X is an r-dimensional complete variety. Moreover, by Chow's lemma we can find a birational morphism $f\colon X'\to X$, with X' a projective variety, and property iv) gives $(\mathscr{L}_1\cdot\ldots\mathscr{L}_r)=(f^*\mathscr{L}_1\cdot\ldots\cdot f^*\mathscr{L}_r)$. Therefore we may assume that X is projective. By multilinearity, if we write $\mathscr{L}_1\simeq\mathscr{O}_X(A-B)$, with A and B effective very ample Cartier divisors, then

$$(\mathscr{L}_1\cdot\ldots\cdot\mathscr{L}_r)=(\mathscr{O}_X(A)\cdot\mathscr{L}_2\cdot\ldots\cdot\mathscr{L}_r)-(\mathscr{O}_X(B)\cdot\mathscr{L}_2\cdot\ldots\cdot\mathscr{L}_r).$$

On the other hand, property v) gives $(\mathscr{O}_X(A)\cdot\mathscr{L}_2\cdot\ldots\cdot\mathscr{L}_r)=(\mathscr{L}_2|_A\cdot\ldots\cdot\mathscr{L}_r|_A)$ and $(\mathscr{O}_X(B)\cdot\mathscr{L}_2\cdot\ldots\cdot\mathscr{L}_r)=(\mathscr{L}_2|_B\cdot\ldots\cdot\mathscr{L}_r|_B)$, and we are thus done by induction.

Remark 1.2.9. If $Q(x)$ is a polynomial in one variable of degree d and we consider the polynomial in r variables $P(x_1,\ldots,x_r)=Q(x_1+\ldots+x_r)$, then the total degree of P is d and the coefficient of $x_1\cdots x_r$ in P is $d!\cdot a$, where a is the coefficient of x^d in Q. It follows that if $\mathscr{L}$ is a line bundle on an n-dimensional complete scheme X, then

$$\chi(\mathscr{L}^m)=\frac{(\mathscr{L}^n)}{n!}m^n+\text{lower order terms in } m.$$

This expression is known as the *asymptotic Riemann-Roch formula*.

Remark 1.2.10. Suppose that $\mathscr{L}$ is a very ample line bundle on the n-dimensional projective scheme X. The polynomial P_X such that $P_X(m) = \chi(\mathscr{L}^m)$ is the *Hilbert polynomial* of X corresponding to the projective embedding $X \hookrightarrow \mathbb{P}^N$ given by $\mathscr{L}$ (see [Har77, Exer. III.5.2]). In particular, it follows from Remark 1.2.9 that the *degree* of X with respect to this embedding is equal to $(\mathscr{L}^n)$. Note that this is positive: if $H_1, \ldots, H_n$ are general hyperplanes in $\mathbb{P}^N$, then a repeated application of Proposition 1.2.5 v) implies that the degree of X is equal to $h^0(\mathscr{O}_{X \cap H_1 \cap \ldots \cap H_n})$ and $X \cap H_1 \cap \ldots \cap H_n$ is always non-empty.

Example 1.2.11. Suppose that X is a complete curve (recall our convention that in this case X is irreducible and reduced). If D is an effective Cartier divisor on X, then the intersection number (D) on X is equal to $h^0(\mathscr{O}_D)$ (in particular, it is nonnegative). Therefore the intersection number $(\mathscr{L})$ of a line bundle on X is equal to the usual degree $\deg(\mathscr{L})$ on X. By applying Corollary 1.2.4 for $\mathscr{L}^{-1}$, we obtain

$$\deg(\mathscr{L}) = -\deg(\mathscr{L}^{-1}) = \chi(\mathscr{L}) - \chi(\mathscr{O}_X),$$

which is the Riemann-Roch theorem for a line bundle on X.

Example 1.2.12. Let X be a smooth projective surface. If $\mathscr{L}_1$ and $\mathscr{L}_2$ are line bundles on X, then the formula in Corollary 1.2.4 applied to $\mathscr{L}_1^{-1}$ and $\mathscr{L}_2^{-1}$ gives

$$(\mathscr{L}_1 \cdot \mathscr{L}_2) = (\mathscr{L}_1^{-1} \cdot \mathscr{L}_2^{-1}) = \chi(\mathscr{O}_X) - \chi(\mathscr{L}_1) - \chi(\mathscr{L}_2) + \chi(\mathscr{L}_1 \otimes \mathscr{L}_2).$$

If we take $\mathscr{L}_2 = \omega_X \otimes \mathscr{L}_1^{-1}$, then $\mathscr{L}_1 \otimes \mathscr{L}_2 = \omega_X$ and Serre duality gives $\chi(\mathscr{L}_1) = \chi(\mathscr{L}_2)$ and $\chi(\omega_X) = \chi(\mathscr{O}_X)$. The above formula implies

$$(\mathscr{L}_1^2) - (\mathscr{L}_1 \cdot \omega_X) = 2\chi(\mathscr{L}) - 2\chi(\mathscr{O}_X),$$

the Riemann-Roch theorem for a line bundle on X.

Proposition 1.2.13. *Let $\pi \colon X \to T$ be a proper flat morphism of relative dimension n. If $\mathscr{L}_1, \ldots, \mathscr{L}_n$ are line bundles on X and for every $t \in T$ we consider the corresponding line bundles $\mathscr{L}_1|_{X_t}, \ldots, \mathscr{L}_n|_{X_t}$ on the fiber X_t, then the function*

$$T \ni t \to (\mathscr{L}_1|_{X_t} \cdot \ldots \cdot \mathscr{L}_n|_{X_t})$$

is locally constant.

Proof. The assertion follows from the definition of intersection numbers and the fact that under our assumption, every line bundle $\mathscr{L}$ on X is flat over T, hence the function $T \ni t \to \chi(\mathscr{L}|_{X_t})$ is locally constant. □

1.3 The ample and nef cones

Our goal in this section is to introduce the ample and nef cones of a projective scheme, and discuss the relation between them. This is based on the theorems of

Nakai-Moishezon and Kleiman. We keep the assumption that all schemes are of finite type over an infinite field. Our presentation follows the one in [Laz04a, Chap. 1].

1.3.1 The Nakai-Moishezon ampleness criterion

The following basic theorem describes ampleness in terms of intersection numbers with subvarieties.

Theorem 1.3.1 (Nakai-Moishezon). *A line bundle $\mathscr{L}$ on the complete scheme X is ample if and only if for every subvariety V of X with $r = \dim(V) > 0$, we have $(\mathscr{L}^r \cdot V) > 0$.*

Proof. For simplicity, we only give the argument when X is projective (see [Har70, Theorem 5.1] for a proof in the general case). If $\mathscr{L}$ is ample, then some multiple $\mathscr{M} = \mathscr{L}^d$ is very ample and $(\mathscr{L}^r \cdot V) = \frac{1}{d^r}(\mathscr{M}^r \cdot V)$. We have seen in Remark 1.2.10 that $(\mathscr{M}^r \cdot V)$ is the degree of V under the embedding given by $\mathscr{M}$, which is positive. Therefore $(\mathscr{L}^r \cdot V) > 0$.

Suppose now that $(\mathscr{L}^r \cdot V) > 0$ for every $r \geq 1$ and every r-dimensional subvariety V of X. It follows from Proposition 1.2.5 iii) that the same inequality holds for all r-dimensional closed subschemes V of X. Arguing by Noetherian induction, we may assume that $\mathscr{L}|_Y$ is ample for every closed subscheme Y of X, different from X. Using Proposition 1.1.7, we deduce that we may assume that X is an integral scheme. Let $n = \dim(X)$. If $n = 0$, then every line bundle on X is ample. Suppose now that $n > 0$.

Claim. We have $h^0(X, \mathscr{L}^m) > 0$ for $m \gg 0$. Let us suppose that this is the case. Since X is integral, it follows that we have an effective Cartier divisor D such that $\mathscr{O}_X(D) \simeq \mathscr{L}^m$. For every positive integer p, we get a short exact sequence

$$0 \to \mathscr{L}^{(p-1)m} \to \mathscr{L}^{pm} \to \mathscr{L}^{pm}|_D \to 0,$$

and a corresponding long exact sequence

$$H^0(X, \mathscr{L}^{pm}) \xrightarrow{\phi} H^0(D, \mathscr{L}^{pm}|_D) \to H^1(X, \mathscr{L}^{(p-1)m}) \xrightarrow{\psi} H^1(X, \mathscr{L}^{pm}) \to H^1(D, \mathscr{L}^{pm}|_D).$$

Since $\mathscr{L}|_D$ is ample by the inductive assumption, we have $H^1(D, \mathscr{L}^{pm}|_D) = 0$ for $p \gg 0$, hence $h^1(X, \mathscr{L}^{pm}) \leq h^1(X, \mathscr{L}^{(p-1)m})$ for $p \gg 0$. Therefore the sequence $(h^1(X, \mathscr{L}^{pm}))_{p \geq 1}$ is eventually constant, which in turn implies that for $p \gg 0$, in the above exact sequence ψ is an isomorphism, hence ϕ is surjective. Since the base-locus of $\mathscr{L}^{pm}$ is clearly contained in D, while the ampleness of $\mathscr{L}|_D$ implies that $\mathscr{L}^{pm}|_D$ is globally generated for $p \gg 0$, we conclude that $\mathscr{L}^{pm}$ is globally generated. Let $f\colon X \to \mathbb{P}^N$ be the map defined by $|\mathscr{L}^{pm}|$, so that $\mathscr{L}^{pm} \simeq f^*(\mathscr{O}_{\mathbb{P}^N}(1))$. If C is a curve contracted by f, then the projection formula gives $(\mathscr{L} \cdot C) = 0$, a contradiction. This shows that f is a finite morphism, and since $\mathscr{L}^{pm}$ is the pull-back induced by f of an ample line bundle, the ampleness of $\mathscr{L}$ follows from Proposition 1.1.9.

Therefore in order to complete the proof of the theorem it is enough to prove the above claim. Since X is projective, we can find effective Cartier divisors A and B on X such that $\mathscr{L} \simeq \mathscr{O}_X(A-B)$. For every integer m, we consider the short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{O}_X(-A)\otimes\mathscr{L}^m \arrow[r] \arrow[d, "\cong"] & \mathscr{L}^m \arrow[r] & \mathscr{L}^m|_A \arrow[r] & 0 \\ 0 \arrow[r] & \mathscr{O}_X(-B)\otimes\mathscr{L}^{m-1} \arrow[r] & \mathscr{L}^{m-1} \arrow[r] & \mathscr{L}^{m-1}|_B \arrow[r] & 0. \end{tikzcd}$$

Since both $\mathscr{L}|_A$ and $\mathscr{L}|_B$ are ample by the inductive assumption, we have

$$h^i(A,\mathscr{L}^m|_A) = 0 = h^i(B,\mathscr{L}^{m-1}|_B) \text{ for every } i \geq 1 \text{ and all } m \gg 0.$$

We deduce from the corresponding long exact sequences in cohomology that for $m \gg 0$ and $i \geq 2$ we have

$$h^i(X,\mathscr{L}^{m-1}) = h^i(X,\mathscr{O}_X(-B)\otimes\mathscr{L}^{m-1}) = h^i(X,\mathscr{O}_X(-A)\otimes\mathscr{L}^m) = h^i(X,\mathscr{L}^m).$$

On the other hand, by asymptotic Riemann-Roch we have

$$\begin{aligned}\chi(\mathscr{L}^m) &= h^0(X,\mathscr{L}^m) - h^1(X,\mathscr{L}^m) + \sum_{i\geq 2}(-1)^i h^i(X,\mathscr{L}^m) \\ &= \frac{(\mathscr{L}^n)}{n!}m^n + \text{lower order terms.}\end{aligned}$$

Since by assumption $(\mathscr{L}^n) > 0$ and each sequence $(h^i(X,\mathscr{L}^m))_{m\geq 1}$ is eventually constant, this implies that $h^0(X,\mathscr{L}^m) > 0$ for $m \gg 0$, which completes the proof of the theorem. □

Remark 1.3.2. The easy implication in the above theorem admits the following generalization: if $\mathscr{L}_1,\ldots,\mathscr{L}_r \in \mathrm{Pic}(X)$ are ample line bundles and Y is an r-dimensional closed subscheme of X, then $(\mathscr{L}_1\cdot\ldots\cdot\mathscr{L}_r\cdot Y) > 0$. Indeed, this is an immediate consequence of Remark 1.2.7.

Remark 1.3.3. It is *not* true that in order to check the ampleness of $\mathscr{L}$ in Theorem 1.3.1 one can just check that the intersection of $\mathscr{L}$ with each curve is positive. In fact, there is a smooth projective surface X and a line bundle $\mathscr{L}$ on X such that $(\mathscr{L}\cdot C) > 0$ for every curve C in X, but $(\mathscr{L}^2) = 0$, see Example 1.3.36 below. One should contrast this phenomenon with the statement of Theorem 1.3.18 below.

1.3.2 The nef cone

We now turn to a weaker notion of positivity for line bundles, which turns out to be very important.

Definition 1.3.4. A line bundle $\mathscr{L}$ on a complete scheme X is *nef*[1] if $(\mathscr{L} \cdot C) \geq 0$ for every curve C in X.

Example 1.3.5. An important example of nef line bundles is provided by semiample ones. Recall that a line bundle $\mathscr{L}$ on a complete scheme X is *semiample* if some multiple $\mathscr{L}^m$, with m a positive integer, is globally generated. If $\mathscr{L}$ is semiample, then $\mathscr{L}$ is nef: if C is a curve on X, then $\mathscr{L}^m|_C$ is globally generated for some $m > 0$; in particular, it has nonzero sections, and therefore $(\mathscr{L} \cdot C) = \deg(\mathscr{L}|_C) \geq 0$.

On the other hand, it is very easy to give examples of nef line bundles that are not semiample. Suppose that X is a smooth projective curve of genus $g \geq 1$, over an algebraically closed field k. Recall that the degree zero line bundles on X are parametrized by a g-dimensional abelian variety, the Picard variety $\mathrm{Pic}^0(X)$. If k is uncountable, all points of $\mathrm{Pic}^0(X)$ but a countable set are non-torsion[2] (note that the degree zero line bundles $\mathscr{L}$ on X with $\mathscr{L}^m \simeq \mathscr{O}_X$ correspond precisely to the m-torsion points of $\mathrm{Pic}^0(X)$, which form a finite set). It is now enough to remark that every degree zero line bundle on X is nef, and it is semiample if and only if it is torsion.

For various technical reasons that will hopefully become clear in the following chapters, in birational geometry it is very useful to work not only with divisors and line bundles, but to allow also rational, and even real coefficients. We now introduce this formalism, as well as the ambient vector space for the ample and the nef cones.

Let $\mathrm{Z}_1(X)$ denote the free abelian group generated by the curves in X. By taking the intersection number of a line bundle with a curve we obtain a $\mathbb{Z}$-bilinear map

$$\mathrm{Pic}(X) \times \mathrm{Z}_1(X) \to \mathbb{Z}, \quad (L, \alpha = \sum_{i=1}^{r} a_i C_i) \mapsto (L \cdot \alpha) := \sum_{i=1}^{r} a_i (L \cdot C_i).$$

The *numerical equivalence* of line bundles is defined by

$$\mathscr{L}_1 \equiv \mathscr{L}_2 \quad \text{if} \quad (\mathscr{L}_1 \cdot C) = (\mathscr{L}_2 \cdot C) \quad \text{for every curve} \quad C \subseteq X.$$

If $\mathscr{L} \equiv 0$, then L is *numerically trivial*. The quotient of $\mathrm{Pic}(X)$ by the subgroup of numerically trivial line bundles is the *Néron-Severi* group $\mathrm{N}^1(X) = \mathrm{Pic}(X)/\equiv$. The following is a fundamental result, known as the *theorem of the base*. For a proof, see [LN59].

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[1] This terminology stands for *numerically effective* or *numerically eventually free*.

[2] This can fail over countable fields. In fact, if $k = \overline{\mathbb{F}_p}$ is the algebraic closure of a finite field, then every degree zero line bundle on X is torsion. Indeed, note that every such line bundle is defined over some finite field. On the other hand, the set of points of an abelian variety with values in a finite field is finite, hence form a finite group.

Theorem 1.3.6. *The group* $\mathrm{N}^1(X)$ *is finitely generated.*

We note that by definition $\mathrm{N}^1(X)$ is also torsion-free. Therefore it is a finitely generated free abelian group and its rank $\rho = \rho(X)$ is the *Picard rank* of X.

By tensoring with $\mathbb{R}$, the above $\mathbb{Z}$-bilinear map gives an $\mathbb{R}$-bilinear map

$$\mathrm{Pic}(X)_{\mathbb{R}} \times \mathrm{Z}_1(X)_{\mathbb{R}} \to \mathbb{R}.$$

The Néron-Severi vector space of X is $\mathrm{N}^1(X)_{\mathbb{R}} \simeq \mathbb{R}^{\rho}$. Note that this can be identified with $\mathrm{Pic}(X)_{\mathbb{R}}/\equiv$, where for $\alpha, \beta \in \mathrm{Pic}(X)_{\mathbb{R}}$ we have $\alpha \equiv \beta$ if and only if $\alpha - \beta$ is a linear combination of numerically trivial line bundles.

Remark 1.3.7. In fact, for $\alpha \in \mathrm{Pic}(X)_{\Lambda}$, with Λ being either $\mathbb{Q}$ or $\mathbb{R}$, we have $\alpha \equiv 0$ if and only if $(\alpha \cdot C) = 0$ for every curve C in X. Indeed, if $L \subseteq \mathrm{Pic}(X)$ is the subgroup of numerically trivial line bundles, then by considering the intersection number of a line bundle with all curves on C we obtain an inclusion $\mathrm{N}^1(X) \hookrightarrow \mathbb{Z}^J$, where J is the set of all curves in X. The map $\mathrm{N}^1(X)_{\Lambda} \hookrightarrow \mathbb{Z}^J \otimes_{\mathbb{Z}} \Lambda$ obtained by tensoring with Λ is injective and our assertion follows from the fact that the canonical map $\mathrm{Z}^J \otimes_{\mathbb{Z}} \Lambda \to \Lambda^J$ is an injection. This is clearly true when $\Lambda = \mathbb{Q}$ and therefore in order to check the injectivity when $\Lambda = \mathbb{R}$ it is enough to show that for every $\mathbb{Q}$-vector space V, the canonical map $\mathbb{Q}^J \otimes_{\mathbb{Q}} V \to V^J$ is injective. Since V is the union of its finite-dimensional subspaces, it is enough to check this when V is finite-dimensional, when the assertion is straightforward.

We also have the following dual picture. We say that $\alpha, \beta \in \mathrm{Z}_1(X)_{\mathbb{R}}$ are *numerically equivalent*, and write $\alpha \equiv \beta$, if $(\mathscr{L} \cdot \alpha) = (\mathscr{L} \cdot \beta)$ for every $\mathscr{L} \in \mathrm{Pic}(X)$ (or equivalently, for every $\mathscr{L} \in \mathrm{Pic}(X)_{\mathbb{R}}$). We put $\mathrm{N}_1(X)_{\mathbb{R}} := \mathrm{Z}_1(X)_{\mathbb{R}}/\equiv$. It follows from definition that the intersection pairing induces an inclusion $j_1\colon \mathrm{N}_1(X)_{\mathbb{R}} \hookrightarrow \mathrm{Hom}(\mathrm{N}^1(X)_{\mathbb{R}}, \mathbb{R})$, hence $\mathrm{N}_1(X)_{\mathbb{R}}$ is a finite-dimensional $\mathbb{R}$-vector space. Furthermore, since by definition also $j_2\colon \mathrm{N}^1(X)_{\mathbb{R}} \to \mathrm{Hom}(\mathrm{N}_1(X)_{\mathbb{R}}, \mathbb{R})$ is injective, it follows that both j_1 and j_2 are bijective. In other words, the induced bilinear form

$$\mathrm{N}^1(X)_{\mathbb{R}} \times \mathrm{N}_1(X)_{\mathbb{R}} \to \mathbb{R}$$

is non-degenerate. In what follows, we always identify $\mathrm{N}^1(X)_{\mathbb{R}}$ with the dual of $\mathrm{N}_1(X)_{\mathbb{R}}$ via this pairing.

We denote by $\sim$, $\sim_{\mathbb{Q}}$ and $\sim_{\mathbb{R}}$ the linear equivalence relation on $\mathrm{CDiv}(X)$, $\mathrm{CDiv}(X)_{\mathbb{Q}}$, and $\mathrm{CDiv}(X)_{\mathbb{R}}$, respectively. By definition, two divisors in $\mathrm{CDiv}(X)_{\mathbb{Q}}$ or $\mathrm{CDiv}(X)_{\mathbb{R}}$ are linearly equivalent if their difference is a finite sum of principal Cartier divisors with rational, respectively real, coefficients. In particular, note that if D and E are Cartier divisors on X, then $D \sim_{\mathbb{Q}} E$ if and only if $mD \sim mE$ for some positive integer m. If D and E are elements of $\mathrm{CDiv}(X)$ or $\mathrm{CDiv}(X)_{\mathbb{R}}$, we write $D \equiv E$ if the corresponding elements of $\mathrm{Pic}(X)_{\mathbb{R}}$ are numerically equivalent.

Definition 1.3.8. If X is a complete scheme, let $\mathrm{NE}(X)$ denote the convex cone in $\mathrm{N}_1(X)_{\mathbb{R}}$ generated by the classes of curves in X. The closure $\overline{\mathrm{NE}}(X)$ of $\mathrm{NE}(X)$ is the *Mori cone* of X.

The *nef cone* $\mathrm{Nef}(X)$ of X is the dual of $\overline{\mathrm{NE}}(X)$, that is,

$$\mathrm{Nef}(X) = \{\alpha \in \mathrm{N}^1(X)_{\mathbb{R}} \mid (\alpha \cdot C) \geq 0 \text{ for every curve } C \subseteq X\}.$$

Note that $\mathrm{Nef}(X)$ is a closed convex cone in $\mathrm{N}^1(X)_{\mathbb{R}}$ whose dual is the Mori cone $\overline{\mathrm{NE}}(X)$. We refer to Appendix A for a review of duality for closed convex cones.

We say that an element of $\mathrm{CDiv}(X)_{\mathbb{R}}$ or $\mathrm{Pic}(X)_{\mathbb{R}}$ is nef if its image in $\mathrm{N}^1(X)_{\mathbb{R}}$ is nef (that is, lies in $\mathrm{Nef}(X)$). Of course, for line bundles we recover our previous definition.

Proposition 1.3.9. *Let $f\colon X \to Y$ be a morphism of complete schemes.*

i) If $\mathscr{L} \in \mathrm{Pic}(Y)$ is such that $\mathscr{L} \equiv 0$, then $f^*\mathscr{L} \equiv 0$. Therefore by pulling-back line bundles we obtain a linear map $f^*\colon \mathrm{N}^1(Y)_{\mathbb{R}} \to \mathrm{N}^1(X)_{\mathbb{R}}$ that takes $\mathrm{N}^1(Y)$ and $\mathrm{N}^1(Y)_{\mathbb{Q}}$ to $\mathrm{N}^1(X)$ and $\mathrm{N}^1(X)_{\mathbb{Q}}$, respectively.
ii) The dual of the map in i) is $f_*\colon \mathrm{N}_1(X)_{\mathbb{R}} \to \mathrm{N}_1(Y)_{\mathbb{R}}$ that takes the class of a curve C to $\deg(C/f(C))f(C)$ if $f(C)$ is a curve, and to 0, otherwise. This induces a map $\overline{\mathrm{NE}}(X) \to \overline{\mathrm{NE}}(Y)$.
iii) If f is surjective, then $f^*\colon \mathrm{N}^1(Y)_{\mathbb{R}} \to \mathrm{N}^1(X)_{\mathbb{R}}$ is injective.
iv) If $\alpha \in \mathrm{N}^1(Y)_{\mathbb{R}}$ is nef, then $f^*(\alpha)$ is nef. The converse also holds if f is surjective.

Proof. If C is a curve on X and $\mathscr{L} \in \mathrm{Pic}(Y)$, then by the projection formula we have $(f^*\mathscr{L} \cdot C) = 0$ if $f(C)$ is a point and $(f^*\mathscr{L} \cdot C) = \deg(C/f(C)) \cdot (\mathscr{L} \cdot f(C))$ if $f(C)$ is a curve. Moreover, if f is surjective, then given any curve C' in Y, there is a curve C in X with $f(C) = C'$ (see, for example, Corollary B.1.2). All the assertions in the proposition follow from these facts. □

Definition 1.3.10. If X is any scheme and Z is a closed subscheme of X that is complete, we say that $\alpha \in \mathrm{Pic}(X)_{\mathbb{R}}$ is *nef on* Z if its image in $\mathrm{Pic}(Z)_{\mathbb{R}}$ is nef. If $D \in \mathrm{CDiv}(X)_{\mathbb{R}}$, we say that D is *nef on* Z if the corresponding element in $\mathrm{Pic}(X)_{\mathbb{R}}$ is nef on Z.

Remark 1.3.11. If Y is a closed r-dimensional subscheme of the complete scheme X and $\alpha_i, \alpha_i' \in \mathrm{Pic}(X)_{\mathbb{R}}$, with $1 \leq i \leq r$ are such that $\alpha_i \equiv \alpha_i'$ for every i, then

$$(\alpha_1 \cdot \ldots \cdot \alpha_r \cdot Y) = (\alpha_1' \cdot \ldots \cdot \alpha_r' \cdot Y).$$

Indeed, it is enough to check this when $\alpha_i = \alpha_i' \in \mathrm{Pic}(X)$ for $2 \leq i \leq r$. Using the basic properties in Proposition 1.2.5 we see that we may assume that $Y = X$ is an integral scheme. Moreover, we may apply Chow's lemma to construct a proper, birational map $f\colon Y' \to Y$, with Y' projective. Since $f^*(\alpha_1) \equiv f^*(\alpha_1')$ by Proposition 1.3.9, an application of the projection formula implies that we may replace Y by Y' and thus assume that Y is projective. In this case, after writing each $\alpha_i = \mathscr{O}_X(A_i - B_i)$, for A_i and B_i very ample Cartier divisors, we reduce to the case when $\alpha_i = \mathscr{O}_X(A_i)$ for all $2 \leq i \leq r$, with A_i very ample Cartier divisors. In this case, it follows from Remark 1.2.7 that we can find a positive integer m and a one-dimensional subscheme Z in X such that

$$(\alpha_1 \cdot \ldots \cdot \alpha_r \cdot Y) = \frac{1}{m}(\alpha_1 \cdot Z) \text{ and } (\alpha_1' \cdot \alpha_2 \cdot \ldots \cdot \alpha_r \cdot Y) = \frac{1}{m}(\alpha_1' \cdot Z).$$

If the one-dimensional irreducible components of Z are $C_1, \ldots, C_d$ (considered with reduced scheme structure) and if $\ell(\mathscr{O}_{Z,C_i}) = e_i$, then it follows from Proposition 1.2.5 iii) that $(\alpha_1 \cdot Z) = \sum_{i=1}^d e_i(\alpha_1 \cdot C_i)$ and $(\alpha_1' \cdot Z) = \sum_{i=1}^d e_i(\alpha_1' \cdot C_i)$. Since $(\alpha_1 \cdot C_i) = (\alpha_1' \cdot C_i)$ for every i by assumption, we obtain the desired equality.

Remark 1.3.12. If $\mathscr{L}$ and $\mathscr{L}'$ are two numerically equivalent line bundles, then $\mathscr{L}$ is ample if and only if $\mathscr{L}'$ is ample (one says that *ampleness is a numerical property*). Indeed, it follows from the previous remark that $(\mathscr{L}^r \cdot V) = (\mathscr{L}'^r \cdot V)$ for every r-dimensional subvariety V of X, and we can use the ampleness criterion from Theorem 1.3.1.

It follows from Remark 1.3.11 that if Y is a closed r-dimensional subscheme of a complete scheme X, and $\alpha_1, \ldots, \alpha_r \in \mathrm{N}^1(X)_{\mathbb{R}}$, then the intersection product $(\alpha_1 \cdot \ldots \cdot \alpha_r \cdot Y)$ is a well-defined real number. Furthermore, the map

$$\mathrm{N}^1(X)_{\mathbb{R}}^r \to \mathbb{R},\ (\alpha_1, \ldots, \alpha_r) \to (\alpha_1 \cdot \ldots \cdot \alpha_r \cdot Y)$$

is multilinear, hence continuous.

We now introduce the other cone that we are concerned with in this section.

Definition 1.3.13. The *ample cone* $\mathrm{Amp}(X)$ of a projective scheme X is the convex cone in $\mathrm{N}^1(X)_{\mathbb{R}}$ generated by the classes of ample line bundles, that is, it is the set of classes of Cartier divisors of the form $t_1 A_1 + \ldots + t_r A_r$, where r is a positive integer, the A_i are ample Cartier divisors, and the t_i are positive real numbers. An element $\alpha \in \mathrm{N}^1(X)_{\mathbb{R}}$ is *ample* if it lies in $\mathrm{Amp}(X)$. We say that α in $\mathrm{CDiv}(X)_{\mathbb{R}}$ or in $\mathrm{Pic}(X)_{\mathbb{R}}$ is *ample* if the image of α in $\mathbb{N}^1(X)_{\mathbb{R}}$ is ample.

Remark 1.3.14. If Y is an r-dimensional closed subscheme of the projective scheme X and $\alpha_1, \ldots, \alpha_r \in \mathrm{Amp}(X)$, then $(\alpha_1 \cdot \ldots \cdot \alpha_r \cdot Y) > 0$. Indeed, it follows from definition that this intersection number is a linear combination with positive coefficients of numbers of the form $(\mathscr{L}_1 \cdot \ldots \cdot \mathscr{L}_r \cdot Y)$, where $\mathscr{L}_i \in \mathrm{Pic}(X)$ are ample, and we can apply Remark 1.3.2. This observation implies that if $\mathscr{L} \in \mathrm{Pic}(X)$ and $\lambda \in \mathbb{R}_{>0}$, then $\lambda\mathscr{L} \in \mathrm{Amp}(X)$ if and only if the line bundle $\mathscr{L}$ is ample (therefore our new definition is compatible with the definition in the case of line bundles). We also see that ampleness of Cartier $\mathbb{Q}$-divisors can be easily reduced to the case of Cartier divisors (for example, the Nakai-Moshezon ampleness criterion extends trivially to elements of $\mathrm{Pic}(X)_{\mathbb{Q}}$).

Lemma 1.3.15. *For every ample Cartier divisor D on a projective scheme X, there are ample Cartier divisors $A_1, \ldots, A_r$ on X such that the images of $D, A_1, \ldots, A_r$ in $\mathrm{N}^1(X)_{\mathbb{R}}$ give an $\mathbb{R}$-basis of this vector space.*

Proof. Since $D \not\equiv 0$, we can find Cartier divisors $A_1, \ldots, A_r$ such that the images of $D, A_1, \ldots, A_r$ give a basis of $\mathrm{N}^1(X)_{\mathbb{R}}$. Since we may replace each A_i by $A_i + mD$, for $m \gg 0$, and since these divisors are ample, we obtain the assertion in the lemma. □

Lemma 1.3.16. *For every projective scheme X, the ample cone* $\mathrm{Amp}(X)$ *is open in* $\mathrm{N}^1(X)_{\mathbb{R}}$.

Proof. It is enough to show that if $\alpha \in \mathrm{CDiv}(X)_{\mathbb{R}}$ is ample and $D_1,\ldots,D_r$ are arbitrary Cartier divisors, then $\alpha+\sum_{i=1}^r t_iD_i$ is ample if $0\leq t_i \ll 1$ for all i (for example, choose Cartier divisors $E_1,\ldots,E_n$ whose classes give a basis of $\mathrm{N}^1(X)_{\mathbb{R}}$, and let $D_1,\ldots,D_r$ be $E_1,-E_1,\ldots,E_n,-E_n$). We may replace α by a numerically equivalent divisor, hence we may assume that $\alpha=\sum_{j=1}^m s_jA_j$, for ample Cartier divisors A_j and $s_j\in\mathbb{R}_{>0}$. Clearly, it is enough to prove that $A_1+\sum_{i=1}^r t_iD_i$ is ample for $0\leq t_i\ll 1$. We choose $m\gg 0$ such that rD_i+mA_1 is ample for every i. In this case

$$A_1+\sum_{i=1}^r t_iD_i=\left(1-\sum_{j=1}^r\frac{mt_i}{r}\right)A_1+\sum_{i=1}^r\frac{t_i}{r}(rD_i+mA_1)$$

is ample if $0\leq t_i<\frac{1}{m}$ for every i. □

Corollary 1.3.17. *For every projective variety X, the Mori cone $\overline{\mathrm{NE}}(X)$ is strongly convex.*

Proof. It follows from Lemma 1.3.16 that the interior of $\mathrm{Nef}(X)$ is non-empty, since it contains $\mathrm{Amp}(X)$. Therefore $\mathrm{Nef}(X)$ is full-dimensional, which implies that its dual $\overline{\mathrm{NE}}(X)$ is strongly convex (see Appendix A). □

Theorem 1.3.18 (Kleiman). *If X is a complete scheme and $\alpha\in N^1(X)_{\mathbb{R}}$ is nef, then for every closed subscheme Y of X we have $(\alpha^n\cdot Y)\geq 0$, where $n=\dim(Y)$.*

Proof. It is clear that we may replace X by Y and thus assume that $Y=X$. We argue by induction on n. We may assume that $n\geq 2$, as otherwise there is nothing to prove. If $X_1,\ldots,X_r$ are the n-dimensional irreducible components of X and $e_i=\ell(\mathscr{O}_{X,X_i})$, then it follows from Proposition 1.2.5 that

$$(\alpha^n)=\sum_{i=1}^r e_i\cdot(\alpha^n\cdot X_i),$$

hence it is enough to consider the case when X is irreducible and reduced. By Chow's lemma, there is a proper, birational morphism $f\colon X'\to X$, with X' projective. Since $f^*(\alpha)$ is nef and $(\alpha^n)=(f^*(\alpha)^n)$ by the projection formula, after replacing X by X' we may and will assume that X is projective.

We first show that the result holds if α is the class of a nef Cartier divisor D on X. Let H be a fixed very ample effective Cartier divisor on X. For every $t\in\mathbb{R}$, we put $D_t=D+tH$ and let $P(t)=(D_t^n)$. Note that

$$P(t)=(D^n)+n(D^{n-1}\cdot H)t+\ldots+(H^n)t^n,$$

hence P is a polynomial function of t, with $\deg(P)=n$, and positive top-degree coefficient. We assume that $P(0)<0$ and aim to obtain a contradiction. Since $P(t)>0$ for $t\gg 0$, it follows that P has positive roots. Let $t_0>0$ be the largest root of P. Note that $P(t)>0$ for all $t>t_0$.

Claim. For every subscheme W of X different from X, we have $(D_t^d \cdot W) > 0$ for all $t > 0$, where $d = \dim(W)$.

Indeed, we can write

$$(D_t^d \cdot W) = t^d (H^d \cdot W) + \sum_{i=1}^{d} \binom{d}{i} t^{d-i} (H^{d-i} \cdot D^i \cdot W).$$

Since H is ample and $\dim(W) < n$, it follows from our inductive assumption and Remark 1.2.7 that $(H^{d-i} \cdot D^i \cdot W) \geq 0$ for $1 \leq i \leq d$, while $(H^d \cdot W) > 0$. This proves the claim.

Since $(D_t^n) > 0$ for every $t > t_0$, we deduce using the claim and the Nakai-Moishezon criterion that D_t is ample for every $t \in \mathbb{Q}$, with $t > t_0$. Note also that we can write

$$P(t) = (D_t^{n-1} \cdot D) + t(D_t^{n-1} \cdot H).$$

It follows using again Remark 1.2.7 that if $t > t_0$ is rational, then $(D_t^{n-1} \cdot D) \geq 0$ and $(D_t^{n-1} \cdot H) \geq 0$ (we use the fact that D_t is ample for such t, while both D and H are nef). By continuity, these inequalities must hold also for $t = t_0$. Therefore the fact that $P(t_0) = 0$ implies $(D_{t_0}^{n-1} \cdot H) = 0$. However, this contradicts the claim for $W = H$ and $t = t_0$. This completes the proof in the case $\alpha \in \mathrm{N}^1(X)$, and the case $\alpha \in \mathrm{N}^1(X)_{\mathbb{Q}}$ is an immediate consequence.

Suppose now that $\alpha \in \mathrm{N}^1(X)_{\mathbb{R}}$ is nef. We use Lemma 1.3.15 to choose $\beta_1, \ldots, \beta_\rho \in \mathrm{N}^1(X)$ ample and giving a basis of $\mathrm{N}^1(X)_{\mathbb{R}}$. For every $\varepsilon > 0$, consider the set

$$\{\alpha + t_1\beta_1 + \ldots + t_\rho\beta_\rho \mid 0 < t_i < \varepsilon \text{ for all } i\}.$$

This is an open set in $\mathrm{N}^1(X)_{\mathbb{R}}$, hence it must contain an element $\alpha_\varepsilon \in \mathrm{N}^1(X)_{\mathbb{Q}}$. It is clear that α_ε is nef, hence the case already proved gives $(\alpha_\varepsilon^n) \geq 0$. Since $\lim_{\varepsilon \to 0} \alpha_\varepsilon = \alpha$, we conclude $(\alpha^n) \geq 0$. This completes the proof of the theorem. □

Corollary 1.3.19. *If X is a projective scheme and $\alpha, \beta \in \mathrm{N}^1(X)_{\mathbb{R}}$, with α ample and β nef, then $\alpha + \beta$ is ample.*

Proof. We prove this in three steps. Suppose first that both α and β lie in $\mathrm{N}^1(X)_{\mathbb{Q}}$. After replacing α and β by $m\alpha$ and $m\beta$ for a positive integer m that is divisible enough, we see that it is enough to show that if $D, E \in \mathrm{CDiv}(X)$ are such that D is ample and E is nef, then $D + E$ is ample. By the Nakai-Moishezon criterion, it is enough to show that for every subvariety V of X of dimension $r \geq 1$, we have $((D+E)^r \cdot V) > 0$. Note that

$$((D+E)^r \cdot V) = \sum_{i=0}^{r} \binom{r}{i} (D^i \cdot E^{r-i} \cdot V).$$

Since D is ample and E is nef we have $(D^r \cdot V) > 0$, while Theorem 1.3.18 and Remark 1.2.7 imply $(D^i \cdot E^{r-i} \cdot V) \geq 0$ for $1 \leq i \leq r$. Therefore $D + E$ is ample.

Suppose now that $\beta \in \mathrm{N}^1(X)_{\mathbb{Q}}$ and α is arbitrary. We may assume that $\beta = bG$, where $b \in \mathbb{Q}_{>0}$ and G is the class of a nef line bundle, and $\alpha = a_1H_1 + \ldots + a_mH_m$, where $m \geq 1$, the H_i are classes of ample line bundles, and the a_i are positive real numbers. If a_1' is a positive rational number with $a_1' < a_1$, then $a_1'H_1 + bG$ is ample by the case we have already proved, hence

$$a_1H_1 + \ldots + a_mH_m + bG = (a_1'H_1 + bG) + (a_1 - a_1')H_1 + a_2H_2 + \ldots + a_mH_m$$

is ample. Therefore $\alpha + \beta$ is ample also in this case.

Let us prove now the general case. We apply Lemma 1.3.15 to choose a basis of $\mathrm{N}^1(X)_{\mathbb{R}}$ of the form $\alpha_1, \ldots, \alpha_\rho$, where the α_i are classes of ample line bundles. It follows from Lemma 1.3.16 that there is $\varepsilon > 0$ such that the set

$$U := \{\alpha - t_1\alpha_1 - \ldots - t_\rho\alpha_\rho \mid 0 < t_i < \varepsilon \text{ for all } i\}$$

is contained in the ample cone. Since $U + \beta$ is open in $\mathrm{N}^1(X)_{\mathbb{R}}$, it contains a class $\beta' \in \mathrm{N}^1(X)_{\mathbb{Q}}$. By assumption, $\beta' - \beta$ is ample, hence β' is clearly nef. Since

$$\alpha + \beta = (t_1\alpha_1 + \ldots + t_\rho\alpha_\rho) + \beta',$$

for $t_1, \ldots, t_\rho > 0$ and $\beta' \in \mathrm{N}^1(X)_{\mathbb{Q}}$, we conclude that $\alpha + \beta$ is ample by the case that we already proved. □

Corollary 1.3.20. *If X is a projective scheme, then*

i) $\mathrm{Amp}(X)$ *is the interior of* $\mathrm{Nef}(X)$.
ii) $\mathrm{Nef}(X)$ *is the closure of* $\mathrm{Amp}(X)$.

Proof. Since $\mathrm{Amp}(X)$ is open by Lemma 1.3.16, the ample cone of X is contained in the interior of the nef cone. Conversely, suppose that α lies in the interior of the nef cone. If $\alpha' \in \mathrm{N}^1(X)_{\mathbb{R}}$ is any ample class, then $\alpha - t\alpha' \in \mathrm{Nef}(X)$ for $0 < t \ll 1$. In this case α is ample by Corollary 1.3.19. This proves that $\mathrm{Amp}(X)$ is the interior of $\mathrm{Nef}(X)$.

The assertion in ii) is now a consequence of the general fact that every closed convex cone is the closure of its relative interior (see Corollary A.3.6). We could also argue directly: we only need to show that every $\alpha \in \mathrm{Nef}(X)$ lies in the closure of the ample cone. For every $\beta \in \mathrm{Amp}(X)$, we have $\alpha_m := \alpha + \frac{1}{m}\beta \in \mathrm{Amp}(X)$ by Corollary 1.3.19. Since $\lim_{m\to\infty}\alpha_m = \alpha$, this shows that α lies in the closure of $\mathrm{Amp}(X)$. □

Corollary 1.3.21. *If X is a projective scheme, $\alpha_1, \ldots, \alpha_n \in \mathrm{N}^1(X)_{\mathbb{R}}$, and Y is a closed n-dimensional subscheme of X, then $(\alpha_1 \cdot \ldots \cdot \alpha_n \cdot Y) \geq 0$.*

Proof. We have seen this in Remark 1.3.14 when the α_i are ample. The assertion in the corollary follows since the closure of the ample cone is the nef cone. □

Corollary 1.3.22. *If X is a projective scheme, then $\alpha \in \mathrm{N}^1(X)_{\mathbb{R}}$ is ample if and only if $(\alpha \cdot \gamma) > 0$ for every $\gamma \in \overline{\mathrm{NE}}(X) \smallsetminus \{0\}$.*

Remark 1.3.23. It is easy to see that $(\alpha \cdot \gamma) > 0$ for every $\gamma \in \overline{\mathrm{NE}}(X) \smallsetminus \{0\}$ if and only if for some (any) norm $\| - \|$ on $\mathrm{N}_1(X)_{\mathbb{R}}$, there is $\eta > 0$ such that $(\alpha \cdot C) \geq \eta \cdot \| C \|$ for every curve C in X (equivalently, $(\alpha \cdot \gamma) \geq \eta \cdot \| \gamma \|$ for every $\gamma \in \overline{\mathrm{NE}}(X)$).

Proof of Corollary 1.3.22. We refer to Appendix A for some basic facts about closed convex cones that we are going to use. It follows from Corollary 1.3.20 that α is ample if and only if α is in the interior of $\mathrm{Nef}(X)$. Note that since this interior is non-empty, it is equal to the relative interior of the cone, which is the complement of the union of the faces of $\mathrm{Nef}(X)$ different from $\mathrm{Nef}(X)$. Each such face is of the form $\mathrm{Nef}(X)\cap\gamma^{\perp}$ for some nonzero $\gamma\in\overline{\mathrm{NE}}(X)$, which gives the assertion in the corollary. □

Remark 1.3.24. By definition, $L\in\mathrm{Pic}^1(X)_{\mathbb{R}}$ is ample if and only if it is numerically equivalent to a linear combination of ample line bundles with positive real coefficients. In fact, in this case L is equal to such a combination. Indeed, suppose that $L\equiv\sum_{i=1}^r a_iA_i$, with $a_i>0$ and all A_i ample line bundles. We can thus write

$$L=\sum_{i=1}^r a_iA_i+\sum_{j=1}^s b_jB_j,$$

with $b_j\in\mathbb{R}$, and all B_j numerically trivial line bundles. If $s>0$, let us choose a positive integer $m>\frac{b_1}{a_1}$. Since we can write

$$a_1A_1+b_1B_1=\frac{b_1}{m}(A_1+mB_1)+\frac{ma_1-b_1}{m}A_1$$

and both A_1 and A_1+mB_1 are ample line bundles, we obtain our assertion by induction on s.

Remark 1.3.25. Suppose that X is a projective scheme over k and k' is a field extension of k. If $\mathscr{L}$ is a line bundle on X, then we denote by $\mathscr{L}'$ its pull-back to $X'=X\times_{\mathrm{Spec}\,k}\mathrm{Spec}\,k'$. The map $\mathscr{L}\to\mathscr{L}'$ induces a group homomorphism $\mathrm{Pic}(X)\to\mathrm{Pic}(X')$. Recall that by Remark 1.1.3, we have $\mathscr{L}$ ample if and only if $\mathscr{L}'$ is ample. We deduce that $\mathscr{L}$ is nef if and only if $\mathscr{L}'$ is nef: indeed, if $\mathscr{M}\in\mathrm{Pic}(X)$ is ample, then $\mathscr{L}$ is nef if and only if $\mathscr{L}^m\otimes\mathscr{M}$ is ample for every $m>0$, which is the case if and only if $\mathscr{L}'^m\otimes\mathscr{M}'$ is ample for every $m>0$, which is equivalent to $\mathscr{L}'$ being nef. Since $\mathscr{L}$ is numerically trivial if and only if both $\mathscr{L}$ and $\mathscr{L}^{-1}$ are nef, we deduce that $\mathscr{L}\equiv 0$ if and only if $\mathscr{L}'\equiv 0$. Therefore we have an injective group homomorphism $\phi_{k'/k}\colon \mathrm{N}^1(X)\to\mathrm{N}^1(X')$ inducing an injective linear map $\phi_{k'/k,\mathbb{R}}\colon \mathrm{N}^1(X)_{\mathbb{R}}\to\mathrm{N}^1(X')_{\mathbb{R}}$. Given $\alpha\in\mathrm{N}^1(X)$, by considering a sequence $(\alpha_m)_{m\geq 1}$ with $\alpha_m\in\mathrm{Amp}(X)$, $\alpha-\alpha_m\in\mathrm{N}^1(X)_{\mathbb{Q}}$, and $\lim_{m\to\infty}\alpha_m=\alpha$, we see that α is ample if and only if some $\alpha-\alpha_m$ is ample, which is the case if and only if $\phi_{k'/k,\mathbb{R}}(\alpha)-\phi_{k'/k,\mathbb{R}}(\alpha_m)$ is ample for some m, which in turn is equivalent to $\phi_{k'/k,\mathbb{R}}(\alpha)$ being ample. Therefore we have $\mathrm{Amp}(X)=\phi_{k'/k,\mathbb{R}}^{-1}(\mathrm{Amp}(X'))$, and arguing as before, we deduce $\mathrm{Nef}(X)=\phi_{k'/k,\mathbb{R}}^{-1}(\mathrm{Nef}(X'))$.

1.3.3 Morphisms to projective varieties and faces of the nef cone

Our next goal is to relate the faces of the nef cone to morphisms from X to projective varieties. We begin by recalling an important concept.

Definition 1.3.26. A proper morphism of schemes $f\colon X \to Y$ is a *fiber space* if $f_*(\mathscr{O}_X) = \mathscr{O}_Y$.

It is clear that every fiber space is dominant and if X is integral or normal scheme, then Y has the same property. Furthermore, it is a consequence of Zariski's Main Theorem that every fiber space has connected fibers (see [Har77, Cor. III.11.3]).

Example 1.3.27. If $f\colon X \to Y$ is a proper, birational morphism of varieties and Y is normal, then f is a fiber space. Indeed, we may assume that Y is affine. In this case $\Gamma(Y, \mathscr{O}_Y) \hookrightarrow \Gamma(X, \mathscr{O}_X)$ is a finite homomorphism between subrings of $K(X) = K(Y)$, hence it is an isomorphism since $\Gamma(Y, \mathscr{O}_Y)$ is normal.

Recall that every proper morphism $f\colon X \to Y$ admits a factorization (the *Stein factorization*)

$$X \xrightarrow{g} W = \mathscr{S}pec(f_*(\mathscr{O}_X)) \xrightarrow{h} Y,$$

in which h is finite and g is, by definition, a fiber space. In particular, if X is integral or normal, then so is W.

Remark 1.3.28. Suppose that the ground field has characteristic 0. If $f\colon X \to Y$ is a proper, dominant morphism of varieties, with Y normal and f having connected fibers, then f is a fiber space. Indeed, if $X \xrightarrow{g} W \xrightarrow{h} Y$ is the Stein factorization of f, it follows that h is bijective. We deduce from generic smoothness that h is birational. Since Y is normal, we conclude that h is an isomorphism.

In the following proposition, for a fixed scheme X, we consider equivalence classes of fiber spaces $f\colon X \to Y$, where we identify f with $f'\colon X \to Y'$ if there is an isomorphism $\phi\colon Y \to Y'$ such that $\phi \circ f = f'$. Both the statement of the proposition and its proof make use of some basic facts about closed convex cones, for which we refer to Appendix A.

Proposition 1.3.29. *For every complete scheme X, there is a natural bijection taking f to $\tau(f)$, between equivalence classes of fiber spaces $f\colon X \to Y$, with Y a projective scheme, and faces of the nef cone* $\mathrm{Nef}(X)$ *that contain in their relative interior the numerical class of a globally generated line bundle. The class of a curve C in X lies in the face of* $\overline{\mathrm{NE}}(X)$ *corresponding to $\tau(f)$ if and only if C is contracted by f.*

Proof. The key observation is that a fiber space $f\colon X \to Y$ is uniquely determined (up to equivalence) by the curves in X that are contracted by f. Indeed, note first that two (closed) points $x_1, x_2 \in X$ lie in the same fiber of f if and only if they are joined by a chain of curves that are contracted by f (since the fibers are connected, this is a consequence of Proposition B.1.4). Since f is surjective, continuous, and closed, it follows that as a topological space, Y is the quotient of X by the equivalence relation generated by $x_1 \sim x_2$ if x_1 and x_2 both lie on a curve contracted by f. Since f is a fiber space, the sheaf of rings on Y is uniquely determined by the map of topological spaces $X \to Y$, being equal to $f_*(\mathscr{O}_X)$. This proves the assertion at the beginning of the proof.

Suppose now that $f\colon X \to Y$ is a fiber space, with Y a projective scheme. We attach to f the smallest face $\tau(f)$ of $\mathrm{Nef}(X)$ that contains $f^*(\mathrm{Nef}(Y))$. Note that if $\mathscr{L}_Y$ is an ample line bundle on Y, then $\mathscr{L}_Y$ lies in the interior of $\mathrm{Nef}(Y)$ by Corollary 1.3.20, and this implies that $\tau(f)$ is the smallest face of $\mathrm{Nef}(X)$ containing $f^*(\mathscr{L}_Y)$ (therefore the globally generated line bundle $f^*(\mathscr{L}_Y)$ lies in the relative interior of $\tau(f)$). This implies that the face of $\overline{\mathrm{NE}}(X)$ corresponding to $\tau(f)$ is

$$\overline{\mathrm{NE}}(X) \cap \tau(f)^{\perp} = \overline{\mathrm{NE}}(X) \cap f^*(\mathscr{L}_Y)^{\perp}.$$

This shows that the class of a curve C in X lies in $\overline{\mathrm{NE}}(X) \cap \tau(f)^{\perp}$ if and only if $(f^*(\mathscr{L}_Y) \cdot C) = 0$, which is the case if and only if $f(C)$ is a point. We have seen that the equivalence class of f is determined by the curves contracted by f, hence the map taking f to $\tau(f)$ is injective.

In order to see that this map is also surjective, let τ be a face of $\mathrm{Nef}(X)$ containing the numerical class of a globally generated line bundle $\mathscr{L}$ in its relative interior. Therefore we have a morphism $g\colon X \to \mathbb{P}^N$ such that $\mathscr{L} \simeq g^*(\mathscr{O}_{\mathbb{P}^N}(1))$ and let $X \xrightarrow{g} Z \xrightarrow{h} \mathbb{P}^N$ be its Stein factorization. In this case g is a fiber space and there is an ample line bundle $\mathscr{L}_Z = h^*(\mathscr{O}_{\mathbb{P}^N}(1))$ on Z such that τ is the smallest face of $\mathrm{Nef}(X)$ containing $g^*(\mathscr{L}_Z)$. Therefore $\tau = \tau(g)$. This completes the proof of the proposition. □

Remark 1.3.30. Given a complete scheme X, we can put an order relation on the set of equivalence classes of fiber spaces $X \to Y$, with Y projective, as follows. If $f\colon X \to Y$ and $f'\colon X \to Y'$ are such fiber spaces, then we put $f \prec f'$ if there is a morphism $\phi\colon Y \to Y'$ such that $\phi \circ f = f'$. Arguing as in the proof of Proposition 1.3.29, we see that $f \prec f'$ if and only if every curve on X that is contracted by f is also contracted by f' (in particular, this implies that $f \prec f'$ and $f' \prec f$ if and only if f and f' lie in the same equivalence class). Note that if $f \prec f'$, then $f'^*(\mathrm{Nef}(X')) \subseteq \mathrm{Nef}(X)$, hence $\tau(f') \subseteq \tau(f)$. Conversely, if $\tau(f') \subseteq \tau(f)$, then $\overline{\mathrm{NE}}(X) \cap \tau(f)^{\perp} \subseteq \overline{\mathrm{NE}}(X) \cap \tau(f')^{\perp}$. In particular, every curve on X that is contracted by f is also contracted by f', hence $f \prec f'$.

1.3.4 Examples of Mori and nef cones

We now discuss a few examples of nef cones and Mori cones. For more examples, see [Laz04a, Chap. 1.5]. We assume that the ground field is algebraically closed.

Example 1.3.31. If X is a projective curve (recall that by assumption X is irreducible and reduced), then the map $\mathrm{Pic}(X) \to \mathbf{Z}$, $\mathscr{L} \to \deg(\mathscr{L})$ induces an isomorphism $\mathrm{N}^1(X)_{\mathbb{R}} \simeq \mathbb{R}$, with the nef cone being the half-line generated by the class of an ample line bundle on X.

Example 1.3.32. If X is a smooth projective surface, then the map that takes a curve C in X to the line bundle $\mathscr{O}_X(C)$ induces an isomorphism $\mathrm{N}_1(X)_{\mathbb{R}} \to \mathrm{N}^1(X)_{\mathbb{R}}$. We always use this isomorphism to identify these two vector spaces in the case of a surface. Note that we have $\mathrm{Nef}(X) \subseteq \overline{\mathrm{NE}}(X)$, since for every ample line bundle $\mathscr{L}$ on X, there is an irreducible curve C with $\mathscr{O}_X(C) \simeq \mathscr{L}^m$, for some $m \geq 1$.

The intersection pairing becomes a non-degenerate bilinear form on $\mathrm{N}^1(X)_{\mathbb{R}} \simeq \mathbb{R}^{\rho}$. The Hodge Index theorem says that the signature of this form is $(1, \rho - 1)$ (see [Har77, Thm. V.1.9]; we also recall the argument in Remark 1.4.22 below).

Example 1.3.33. Let $\pi\colon X \to \mathbb{P}^n$ be the blow-up of $\mathbb{P}^n$ at a point q, with $n \geq 2$. Let H be the inverse image of a hyperplane not passing through q and $E = \pi^{-1}(q)$ the exceptional divisor. The line bundles $\mathscr{O}_X(E)$ and $\mathscr{O}_X(H)$ clearly generate $\mathrm{Pic}(X)$. Note that $E \simeq \mathbb{P}^{n-1}$ and $\mathscr{O}_E(-E) \simeq \mathscr{O}_{\mathbb{P}^{n-1}}(1)$. Since $\mathscr{O}_E(H) \simeq \mathscr{O}_E$, we conclude that

$$(H^n) = (\mathscr{O}_{\mathbb{P}^n}(1)^n) = 1, (E^i \cdot H^{n-i}) = 0 \text{ for } 1 \leq i \leq n-1, \text{ and } (E^n) = (-1)^{n-1}.$$

In particular, we see that the classes of E and H give a basis for $\mathrm{N}^1(X)$.

Suppose that $D = aE + bH$ is nef. If ℓ is a line in $\mathbb{P}^n$ passing through q and $\widetilde{\ell}$ is its proper transform, then

$$(E \cdot \widetilde{\ell}) = 1 = (H \cdot \widetilde{\ell}),$$

hence $a + b \geq 0$. On the other hand, if C is a line in $E \simeq \mathbb{P}^{n-1}$, then $(D \cdot C) = -a \geq 0$. Since we can write $D = -a(H - E) + (a + b)H$, we conclude that $\mathrm{Nef}(X)$ is contained in the cone generated by the classes of H and $H - E$. In order to see that these two cones are equal, it is enough to note that both $\mathscr{O}_X(H)$ and $\mathscr{O}_X(H - E)$ are globally generated. For $\mathscr{O}_X(H) = \pi^*\mathscr{O}_{\mathbb{P}^n}(1)$ this is clear, while the fact that $\mathscr{O}_X(H - E)$ is globally generated follows from the fact that if $\mathscr{I}_q$ is the ideal defining q in $\mathbb{P}^n$, then $\mathscr{I}_q \otimes \mathscr{O}_{\mathbb{P}^n}(1)$ is globally generated. The above description of $\mathrm{Nef}(X)$ implies that the Mori cone of X is generated by $\widetilde{\ell}$ and C.

Example 1.3.34. Let X be an abelian surface. We first show that in this case $\mathrm{Nef}(X) = \overline{\mathrm{NE}}(X)$. Indeed, note first that if C is a curve on any smooth surface X, then $\mathscr{O}_X(C)$ is nef if and only if $(C^2) \geq 0$ (this is due to the fact that if C' is any curve on X different from C, then $(C \cdot C') \geq 0$). If X is an abelian surface, if C' is a translate of C different from C, then by Proposition 1.2.13 we have $(C^2) = (C \cdot C') \geq 0$. Therefore $\mathscr{O}_X(C)$ is nef, which implies that $\overline{\mathrm{NE}}(X) \subseteq \mathrm{Nef}(X)$.

Claim. [3] If $\mathscr{L}$ is a line bundle on X such that $(\mathscr{L}^2) > 0$, then there is $C > 0$ such that either $h^0(X, \mathscr{L}^m) \geq Cm^2$ for all $m \gg 0$, or $h^0(X, \mathscr{L}^{-m}) \geq Cm^2$ for all $m \gg 0$.

[3] We now give the argument when $\omega_X = \mathscr{O}_X$. However, the assertion in the claim holds on every smooth projective surface, see Example 1.4.21 below.

Since X is an abelian surface, the canonical line bundle ω_X is trivial, hence the Riemann-Roch theorem for $\mathscr{L}^m$ gives

$$\chi(\mathscr{L}^m) = \frac{1}{2}(\mathscr{L}^2)\cdot m^2 + \chi(\mathscr{O}_X).$$

Furthermore, Serre duality gives $h^2(X,\mathscr{L}^m) = h^0(X,\mathscr{L}^{-m})$, hence

$$h^0(X,\mathscr{L}^m) + h^0(X,\mathscr{L}^{-m}) \geq \frac{1}{2}(\mathscr{L}^2)\cdot m^2 + \chi(\mathscr{O}_X).$$

For every $m \geq 1$, we can not have both $h^0(X,\mathscr{L}^m) > 0$ and $h^0(X,\mathscr{L}^{-m}) > 0$ (in that case, we would get $\mathscr{L} \simeq \mathscr{O}_X$, a contradiction with $(\mathscr{L}^2) > 0$), and we obtain the assertion in the claim.

One way of distinguishing the two situations in the above claim is by choosing an ample line bundle H. We see that if $(\mathscr{L}^2) > 0$ and $(\mathscr{L}\cdot H) > 0$, then there is $C > 0$ such that $h^0(X,\mathscr{L}^m) \geq Cm^2$ for all $m \gg 0$. In fact, we have

$$\mathrm{Nef}(X) = \{\alpha \in \mathrm{N}_1(X)_{\mathbb{R}} \mid (\alpha^2) \geq 0, (\alpha\cdot H) \geq 0\}.$$

The inclusion "$\subseteq$" is trivial; the reverse inclusion follows easily using the fact that every $\alpha \in \mathrm{N}^1(X)_{\mathbb{Q}}$ with $(\alpha^2) > 0$ and $(\alpha\cdot H) > 0$ lies in $\overline{\mathrm{NE}}(X)$ by the above discussion, hence in $\mathrm{Nef}(X)$.

Suppose that $\rho := \dim_{\mathbb{R}} \mathrm{N}^1(X)_{\mathbb{R}} \geq 3$ (for example, this is the case if $X = E\times E$, where E is an elliptic curve; one can check using the intersection matrix that the curves $\{p\}\times E$, $E\times\{p\}$, and the diagonal are linearly independent in $\mathrm{N}^1(X)_{\mathbb{R}}$). If $e_1 \in \mathrm{N}^1(X)_{\mathbb{R}}$ is the class of an ample line bundle, by the Hodge Index theorem we can complete this to a basis $e_1,\ldots,e_\rho$ of $\mathrm{N}^1(X)_{\mathbb{R}}$ such that the intersection form is given by

$$\big((u_1,\ldots,u_\rho),(v_1,\ldots,v_\rho)\big) \to u_1v_1 - \sum_{i=2}^{\rho} u_iv_i.$$

It follows that in this basis, the nef cone is given by

$$\{(u_1,\ldots,u_\rho) \mid u_1 \geq 0,\, u_1^2 \geq \sum_{i=2}^{\rho} u_i^2\}.$$

In particular, we see that this is not a polyhedral cone. In fact, it has infinitely many extremal rays and most of these are not rational.

Example 1.3.35. If X is a smooth projective surface and C is a curve on X with $(C^2) < 0$, then the class of C in $\mathrm{N}_1(X)_{\mathbb{R}}$ lies on an extremal ray of $\overline{\mathrm{NE}}(X)$. Indeed, let H be an ample Cartier divisor on X and let $t_0 \in \mathbb{R}$ be such that $D = H + t_0C$ has the property that $(D\cdot C) = 0$. Note that $t_0 > 0$. By Corollary 1.3.22, for any choice of a norm $\|-\|$ on $\mathrm{N}_1(X)_{\mathbb{R}}$, we can find $\eta > 0$ such that $(H\cdot C') \geq \eta\cdot \| C' \|$ for every curve C' on X. It follows that if $C' \neq C$, then

$$(D\cdot C') \geq (H\cdot C') \geq \eta\cdot \| C' \| .$$

This easily implies that D is nef and $\overline{\mathrm{NE}}(X) \cap D^{\perp}$ is the ray containing the class of C (moreover, the only curve whose class lies on this ray is C).

Note that the face of $\overline{\mathrm{NE}}(X)$ containing the class of C corresponds to a fiber space $f\colon X \to Y$ if and only if there is a morphism $f\colon X \to Y$, with Y normal, such that $f(C)$ is a point $p \in Y$, and f is an isomorphism over $Y \smallsetminus \{p\}$. There is always such f if $C \simeq \mathbb{P}^1$ (see the proof of [Har77, Thm. V.5.7]), but in general there is no such morphism , see [Har77, Example V.5.7.3].

Example 1.3.36. We assume that $\mathrm{char}(k) = 0$ and let $\pi\colon X \to Y$ be a ruled surface over a smooth projective curve Y of genus g. Therefore $X = \mathbb{P}(\mathscr{E})$ for a rank two vector bundle $\mathscr{E}$ on Y. We assume that $\deg(\mathscr{E})$ is even, in which case we may assume that $\deg(\mathscr{E}) = 0$: if $\mathscr{L} \in \mathrm{Pic}(Y)$ is such that $\deg(\mathscr{L}) = -\frac{1}{2}\deg(\mathscr{E})$, then we have $X \simeq \mathbb{P}(\mathscr{E} \otimes \mathscr{L})$ and $\deg(\mathscr{E} \otimes \mathscr{L}) = 0$. The Picard group of X is generated by $\pi^*(\mathrm{Pic}(Y)) \simeq \mathrm{Pic}(Y)$ and $\mathscr{O}(1)$. Therefore $\mathrm{N}^1(X)_{\mathbb{R}}$ is generated by the class f of a fiber of π and the class h of $\mathscr{O}(1)$. Note that we have

$$(f^2) = 0,\ (f \cdot h) = 1, \text{ and } (h^2) = 0$$

(see [Har77, Chap. V.2] for a proof of the last formula). In particular, we see that h and f give a basis of $\mathrm{N}^1(X)_{\mathbb{R}}$.

We first note that $\mathrm{Nef}(X)$ is contained in the convex cone $\sigma \subseteq \mathrm{N}^1(X)_{\mathbb{R}}$ generated by h and f. Indeed, if $af + bh$ is ample, then $((af+bh)\cdot f) = b > 0$, and also $((af+bh)^2) = 2ab > 0$, hence $a > 0$. Note also that the morphism $\pi\colon X \to Y$ distinguishes a face of $\mathrm{Nef}(X)$ generated by f, and which is also a face of $\overline{\mathrm{NE}}(X)$. We now distinguish two cases, depending on whether $\mathscr{E}$ is semistable.

Case 1. If $\mathscr{E}$ is not semistable, then there is a surjective map $\mathscr{E} \to \mathscr{L}$, with $\mathscr{L} \in \mathrm{Pic}(Y)$ and $d = \deg(\mathscr{L}) < 0$. This corresponds to a section $s\colon Y \to X$ and if $C = s(Y)$, then $(C \cdot f) = 1$ and $s^*(\mathscr{O}(1)) \simeq \mathscr{L}$, hence $(C \cdot h) = d$. Therefore the class of C in $\mathrm{N}^1(X)_{\mathbb{R}}$ is $df + h$, hence $(C^2) = 2d < 0$. It follows from Example 1.3.35 that $df + h$ is an extremal ray of $\overline{\mathrm{NE}}(X)$, hence $\overline{\mathrm{NE}}(X)$ is the cone generated by $\{f, df+h\}$, while $\mathrm{Nef}(X)$ is generated by $\{f, -df+h\}$.

Case 2. If $\mathscr{E}$ is semistable, then we show that $\mathrm{Nef}(X) = \overline{\mathrm{NE}}(X)$ is the cone spanned by f and h. In order to prove this assertion, it is enough to show that if C is any curve in X with class $af + bh$, then $a, b \geq 0$. Given such C, we have $\mathscr{O}_X(C) \simeq \pi^*(\mathscr{M}) \otimes \mathscr{O}(m)$ for some $\mathscr{M} \in \mathrm{Pic}(Y)$ and some $m \in \mathbb{Z}$. Since f is nef, we have $(C \cdot f) \geq 0$, hence $m \geq 0$. The existence of C gives

$$0 \neq H^0(X, \pi^*(\mathscr{M}) \otimes \mathscr{O}(m)) \simeq H^0(Y, \mathscr{M} \otimes \pi_*(\mathscr{O}(m))) \simeq H^0(Y, \mathscr{M} \otimes \mathrm{Sym}^m(\mathscr{E})),$$

hence we have a nonzero map $\mathscr{M}^{-1} \xrightarrow{\phi} \mathrm{Sym}^m(\mathscr{E})$. On the other hand, since the ground field has characteristic zero, all symmetric powers $\mathrm{Sym}^j(\mathscr{E})$ are semistable, of degree 0 (see [Har70, Thm. I.10.5]). The existence of ϕ then implies that $\deg(\mathscr{M}^{-1}) \leq 0$. Since the class of C in $\mathrm{N}^1(X)_{\mathbb{R}}$ is equal to $\deg(\mathscr{M})f + mh$, this proves our assertion.

Suppose now that $\mathscr{E}$ is stable, and furthermore, that all symmetric powers $\mathrm{Sym}^m(\mathscr{E})$ are stable. With the above notation, we see that $\deg(\mathscr{M}) > 0$. In other words, there is no curve C on X whose class lies on the extremal ray generated by $\mathscr{O}(1)$. In particular, this gives an example of a nef line bundle such that no multiple is numerically equivalent to a line bundle with a nonzero section. We also see that for such X, the convex cone in $\mathrm{N}^1(X)_{\mathbb{R}}$ generated by the numerical classes of curves in X is not closed. Finally, note that in this case $\mathscr{O}(1)$ has the property that $(\mathscr{O}(1)\cdot C) > 0$ for every curve C on X, but $(\mathscr{O}(1)^2) = 0$. We mention that Hartshorne showed in [Har70, Thm. I.10.5] that when $k = \mathbb{C}$, on every curve of genus $g \geq 2$ there are rank 2, degree 0 vector bundles $\mathscr{E}$ such that $\mathrm{Sym}^m(\mathscr{E})$ is stable for every m.

1.3.5 Ample and nef vector bundles

We end this section with a brief discussion of ampleness and nefness for vector bundles. While these notions will not play an important role in what follows, we will make use of them for constructing examples. Rather than giving an in-depth treatment of ample and nef vector bundles, we just discuss the properties that we will need. For a detailed introduction, we refer the reader to [Laz04b, Chapter 6.1].

Let X be a complete scheme over k. A locally free sheaf $\mathscr{E}$ on X is *ample* or *nef* if the line bundle $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$ on $\mathbb{P}(\mathscr{E})$ has the corresponding property. Note that when $\mathscr{E}$ is a line bundle, we recover the previously defined notions. We collect in the following proposition some basic properties of ample and nef vector bundles.

Proposition 1.3.37. *Let $\mathscr{E}$ be a locally free sheaf on the complete scheme X.*

i) If there is a surjective map $\mathscr{E} \to \mathscr{E}'$, with $\mathscr{E}'$ locally free, and $\mathscr{E}$ is ample (nef), then $\mathscr{E}''$ is ample (respectively, nef) as well.
ii) If $f\colon Y \to X$ is a finite surjective morphism, where Y is a complete scheme, then $\mathscr{E}$ is ample (nef) if and only if $f^*(\mathscr{E})$ is ample (respectively, nef).
iii) If m is a positive integer such that $\mathrm{Sym}^m(\mathscr{E})$ is ample (nef), then $\mathscr{E}$ is ample (respectively, nef).
iv) If $\mathscr{E}$ is globally generated and $\mathscr{L}$ is an ample line bundle, then $\mathscr{E} \otimes \mathscr{L}$ is ample.
v) If $\mathscr{E} = \mathscr{L}_1 \oplus \ldots \oplus \mathscr{L}_r$, where the $\mathscr{L}_i$ are ample (nef) line bundles, then $\mathscr{E}$ is ample (respectively, nef).

Proof. The surjection $\mathscr{E} \to \mathscr{E}'$ induces a closed embedding $\mathbb{P}(\mathscr{E}') \hookrightarrow \mathbb{P}(\mathscr{E})$ such that $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$ restricts to $\mathscr{O}_{\mathbb{P}(\mathscr{E}')}(1)$. This gives the assertion in i). Suppose now that f is a morphism as in ii). This induces a finite surjective morphism $g\colon \mathbb{P}(f^*(\mathscr{E})) \to \mathbb{P}(\mathscr{E})$ such that $g^*(\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)) \simeq \mathscr{O}_{\mathbb{P}(f^*(\mathscr{E}))}(1)$. The assertion in ii) about ampleness then follows from Proposition 1.1.9, while the assertion concerning nefness is clear.

In order to prove iii), it is enough to note that there is a closed embedding $\mathbb{P}(\mathscr{E}) \hookrightarrow \mathbb{P}(\mathrm{Sym}^m(\mathscr{E}))$ such that the restriction of $\mathscr{O}_{\mathbb{P}(\mathrm{Sym}^m(\mathscr{E}))}$ restricts to $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(m)$. For the assertion in iv), we use the fact that a surjection $\mathscr{O}_X^{\oplus r} \to \mathscr{E}$ induces a surjection $\mathscr{L}^{\oplus r} \to \mathscr{E} \otimes \mathscr{L}$. By i), it is enough to show that $\mathscr{L}^{\oplus r}$ is ample. However, we have an isomorphism

$$\mathbb{P}(\mathscr{L}^{\oplus r}) \simeq X \times \mathbb{P}^{r-1}$$

such that the line bundle $\mathscr{O}(1)$ on the left-hand side corresponds on the right-hand side to $p^*(\mathscr{L}) \otimes q^*(\mathscr{O}_{\mathbb{P}^{r-1}}(1))$, which is clearly ample (here p and q are the two projections). This gives iv).

Suppose now that $\mathscr{L}_1, \ldots, \mathscr{L}_r$ are ample line bundles on X. Let $\mathscr{M}$ be a fixed ample line bundle. It follows from Lemma 1.3.38 below that there is a positive integer m such that $\mathscr{M}^{-1} \otimes \mathscr{L}^{i_1} \otimes \ldots \otimes \mathscr{L}_r^{i_r}$ is globally generated for all nonnegative integers $i_1, \ldots, i_r$ with $i_1 + \ldots + i_r = m$. We can write $\mathrm{Sym}^m(\mathscr{E}) \simeq \mathscr{M} \otimes \mathscr{E}'$, where

$$\mathscr{E}' = \oplus_{i_1+\ldots+i_r=m} \mathscr{M}^{-1} \otimes \mathscr{L}^{i_1} \otimes \ldots \otimes \mathscr{L}_r^{i_r}$$

is globally generated, hence $\mathrm{Sym}^m(\mathscr{E})$ is ample by iv). We thus conclude that $\mathscr{E}$ is ample using iii).

Consider now the case when $\mathscr{L}_1, \ldots, \mathscr{L}_r$ are nef and let $\pi \colon \mathbb{P}(\mathscr{E}) \to X$ be the projection. Let $\mathscr{L}$ be a fixed ample line bundle on X. For every positive integer d, consider the embedding $\mathbb{P}(\mathscr{E}) \hookrightarrow \mathbb{P}(\mathrm{Sym}^d(\mathscr{E}))$ and let $\phi \colon \mathbb{P}(\mathscr{E}) \to X$ and $\psi \colon \mathbb{P}(\mathrm{Sym}^d(\mathscr{E})) \to X$ be the canonical projections. Since $\mathrm{Sym}^d(\mathscr{E}) \otimes \mathscr{L}$ is a direct sum of ample line bundles, it is ample by what we have already proved. This implies that $\mathscr{O}_{\mathbb{P}(\mathrm{Sym}^m(\mathscr{E}))}(1) \otimes \psi^*(\mathscr{L})$ is ample, hence its restriction to $\mathscr{P}(\mathscr{E})$, equal to $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(d) \otimes \phi^*(\mathscr{L})$ is ample. Since this holds for every $d > 0$, it follows that $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$ is nef. □

Lemma 1.3.38. *If $\mathscr{L}_1, \ldots, \mathscr{L}_r$ are ample line bundles on the projective scheme X, then for every coherent sheaf $\mathscr{F}$ on X, there is a positive integer m such that $\mathscr{F} \otimes \mathscr{L}_1^{i_1} \otimes \ldots \otimes \mathscr{L}_r^{i_r}$ is globally generated for all nonnegative integers $i_1, \ldots, i_r$, with $i_1 + \ldots + i_r \geq m$.*

Proof. We prove the assertion by induction on $r \geq 1$, the case $r = 1$ being clear. If $r \geq 1$, we use the ampleness of $\mathscr{L}_1$ and the inductive hypothesis to find m_1 such that $\mathscr{L}_1^{i_1}$ is globally generated for $i_1 \geq m_1$ and $\mathscr{F} \otimes \mathscr{L}_2^{i_2} \otimes \ldots \otimes \mathscr{L}_r^{i_r}$ is globally generated if $i_2 + \ldots + i_r \geq m_1$. We now use again the ampleness of $\mathscr{L}_1$ and the inductive hypothesis to find $m_2 \geq m_1$ such that $\mathscr{F} \otimes \mathscr{L}_1^{i_1} \otimes \ldots \otimes \mathscr{L}_r^{i_r}$ is globally generated if either $i_1 \geq m_2$ and $i_2 + \ldots + i_r < m_1$ or $i_1 < m_1$ and $i_2 + \ldots + i_r \geq m_2$. It is straightforward to see that $m = m_1 + m_2$ satisfies the conclusion of the lemma. □

1.4 Big line bundles

In this section we introduce and discuss the basic properties of another class of line bundles that play a fundamental role in birational geometry, the big line bundles. Recall that we work over a fixed infinite ground field k.

1.4.1 Iitaka dimension

We begin by presenting Iitaka's classification of line bundles $\mathscr{L}$ according to the rate of growth for $h^0(\mathscr{L}^m)$. We do this in the more general context of graded linear series, following [BCL]. Let us recall this concept from [Laz04a].

Definition 1.4.1. Let X be an arbitrary variety. A *graded linear series* $V_\bullet$ on X consists of a sequence $(V_m)_{m\geq 1}$, where each V_m is a k-linear subspace of $H^0(X,\mathscr{L}^m)$ for some $\mathscr{L}\in \mathrm{Pic}(X)$, with the property that for every $p,q\geq 1$, multiplication of sections induces a linear map $V_p\otimes V_q\to V_{p+q}$. We make the convention that $V_0=k\subseteq H^0(X,\mathscr{O}_X)$.

From now on, we assume that X is a complete variety. An important example is provided by the *complete graded linear series* $V_\bullet$ with $V_m=H^0(X,\mathscr{L}^m)$ for every $m\geq 1$. Another example is given by the *restricted linear series*

$$W_m=\mathrm{Im}\left(H^0(Y,\mathscr{L}^m)\to H^0(X,\mathscr{L}^m|_X)\right),$$

where $\mathscr{L}$ is a line bundle on the complete variety Y and X is a subvariety of Y.

Let $V_\bullet$ be a graded linear series on X. Suppose first that there is $q\geq 1$ such that $V_q\neq 0$. For every such q, we consider the rational map $\phi_q\colon X\dashrightarrow \mathbb{P}(V_q)$ defined by $|V_q|$ and denote by Y_q the closure of its image. Note that for every $r\geq 1$, multiplication of sections gives a linear map $\mathrm{Sym}^r(V_q)\to V_{rq}$ and we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, dashed, "\phi_{rq}"] \arrow[d, dashed, "\phi_q"'] & \mathbb{P}(V_{rq}) \arrow[d, dashed] \\ \mathbb{P}(V_q) \arrow[r, hook, "\iota"] & \mathbb{P}(\mathrm{Sym}^r(V_q)) \end{tikzcd}$$

with ι the Veronese embedding and the right vertical map a linear projection followed by a linear embedding. We thus have a rational dominant map $\tau_{rq,q}\colon Y_{rq}\dashrightarrow Y_q$ such that $\phi_q\circ\tau_{rq,q}=\phi_{rq}$. In particular, we have $\dim(Y_{rq})\geq\dim(Y_q)$. The *Iitaka dimension* of $V_\bullet$ is

$$\kappa(V_\bullet):=\max\{\dim(Y_q)\mid V_q\neq 0\}.$$

The above discussion shows that $\dim(Y_m)=\kappa(V_\bullet)$ whenever m is divisible enough. By convention, when $V_q=0$ for all $q\geq 1$, we put $\kappa(V_\bullet)=-\infty$. Therefore we have $\kappa(V_\bullet)\in\{-\infty,0,1,\ldots,\dim(X)\}$. If $V_\bullet$ is the complete graded linear series corresponding to $\mathscr{L}$, then $\kappa(\mathscr{L})=\kappa(V_\bullet)$ is the the Iitaka dimension of $\mathscr{L}$. A line bundle $\mathscr{L}$ on X is *big* if $\kappa(\mathscr{L})=\dim(X)$.

Remark 1.4.2. It follows from definition and the above discussion that for every line bundle $\mathscr{L}$ on X, we have $\kappa(\mathscr{L})=\kappa(\mathscr{L}^m)$ for every $m\geq 1$. We may therefore define $\kappa(D)$ for every $D\in\mathrm{CDiv}(X)_{\mathbb{Q}}$ as $\kappa(\mathscr{O}_X(mD))$, where m is any positive integer such that mD is a Cartier divisor.

Example 1.4.3. It follows from definition that $\kappa(V_\bullet) = 0$ if and only if $\dim_k(V_q) \leq 1$ for every q, with equality for some $q \geq 1$.

Our goal is to give two equivalent descriptions for the Iitaka dimension. We begin with an algebraic one. Given X and $V_\bullet$ as above, the *section ring* of $V_\bullet$ is $R(X, V_\bullet) := \bigoplus_{m \geq 0} V_m$, with the product induced by multiplication of sections. Since the tensor product of two nonzero sections is nonzero, it follows from Lemma C.0.5 that $R(X, V_\bullet)$ is a domain. We denote by $K(X, V_\bullet)$ its field of fractions.

Proposition 1.4.4. *If $V_\bullet$ is a graded linear series on the complete variety X, then the following hold:*

i) There is q such that $\tau_{rq,q}\colon Y_{rq} \dashrightarrow Y_q$ is birational for every $r \geq 1$.
ii) We have $\mathrm{trdeg}_k K(X, V_\bullet) = 1 + \kappa(V_\bullet)$, with the convention that the right-hand side is 0 when $\kappa(V_\bullet) = -\infty$.

Before proving the proposition, we make some preparations. For every $q \geq 1$ such that $V_q \neq 0$, let $R^{(q)} \subseteq R(X, V_\bullet)$ denote the k-subalgebra generated by the degree q part V_q. Note that $R^{(q)}$ is again a graded domain and we consider the following subring of the fraction field of $R^{(q)}$:

$$K^{(q)} := \left\{ \frac{a}{b} \mid a, b \in R^{(q)} \text{ homogeneous of the same degree, } b \neq 0 \right\}.$$

Similarly, we put

$$K^{(0)} := \left\{ \frac{a}{b} \mid a, b \in R(X, V_\bullet) \text{ homogeneous of the same degree, } b \neq 0 \right\}.$$

It is clear that each $K^{(q)}$ is a subfield of $K(X, V_\bullet)$ and $K^{(0)} = \bigcup_{q, V_q \neq 0} K^{(q)}$. Furthermore, for every $r \geq 1$ and every $\frac{a}{b} \in K^{(q)}$, we may write $\frac{a}{b} = \frac{ab^{r-1}}{b^r}$, hence $K^{(q)} \subseteq K^{(qr)}$. Let $k(X)$ denote the function field of X.

Given $a, b \in V_q$, with $b \neq 0$, the quotient $\frac{a}{b}$ defines a rational function on X. In this way we obtain a field homomorphism $K^{(0)} \hookrightarrow k(X)$.

Lemma 1.4.5. *With the above notation, for every $q \geq 1$ such that $V_q \neq 0$, the induced homomorphism $K^{(q)} \hookrightarrow k(X)$ identifies $K^{(q)}$ with the image of $k(Y_q)$ under the homomorphism induced by the dominant rational map $\phi_q\colon X \dashrightarrow Y_q$.*

Proof. Let $s \in V_q$ be nonzero and let $U \subseteq X$ be the complement of the zero-locus of s. We have a corresponding hyperplane H_s in $\mathbb{P}(V_q)$ and if $W = Y_q \smallsetminus H_s$, then

$$\mathscr{O}(W) \simeq \left\{ \frac{a}{s^m} \mid m \geq 0, a \in R^{(q)}_{mq} \right\},$$

which identifies the function field of Y_q with $K^{(q)}$. □

Lemma 1.4.6. *With the above notation, if $V_q \neq 0$ for some $q \geq 1$, then*

$$\mathrm{trdeg}_k K(X, V_\bullet) = \mathrm{trdeg}_k K^{(0)} + 1.$$

Proof. If s is a nonzero homogeneous element of degree q in $R(X,V_\bullet)$, then it is clear that s is transcendental over $K^{(0)}$, giving the inequality "$\geq$" in the statement. On the other hand, if t is another such homogenous element of degree m, then $\frac{t^q}{s^m} \in K^{(0)}$. Since $K(X,V_\bullet)$ is generated over k by such homogeneous elements, we deduce the inequality "$\leq$" in the lemma. $\square$

Proof of Proposition 1.4.4. We have see that $K^{(0)}$ is a subfield of $k(X)$. We know that $k(X)$ is finitely generated over k, hence also $K^{(0)}$ is finitely generated over k. Since $K^{(0)} = \bigcup_q K^{(q)}$ and $K^{(q)} \subseteq K^{q'}$ whenever q divides q', it follows that $K^{(0)} = K^{(m)}$ if m is divisible enough. In particular, we get the assertion in i).

It is clear that ii) holds when $\kappa(V_\bullet) = -\infty$. On the other hand, if $\kappa(V_\bullet) \geq 0$, then it follows from what we have shown so far and Lemma 1.4.6 that if m is divisible enough, then

$$\mathrm{trdeg}_k K(X,V_\bullet) - 1 = \mathrm{trdeg}_k K^{(0)} = \mathrm{trdeg}_k K^{(m)} = \dim(Y_m) = \kappa(V_\bullet).$$

This proves ii). $\square$

Our next goal is to give a description of the Iitaka dimension of $V_\bullet$ in terms of the rate of growth of $\dim_k(V_m)$. We begin with the following general bound for the asymptotic rate of growth of the space of global sections of twists by powers of a given line bundle.

Proposition 1.4.7. *If X is an n-dimensional complete scheme, then for every coherent sheaf $\mathscr{F}$ on X and every $\mathscr{L} \in \mathrm{Pic}(X)$, there is $C > 0$ such that*

$$h^0(X,\mathscr{F} \otimes \mathscr{L}^m) \leq C \cdot m^n \textit{ for all } m \gg 0.$$

Proof. Suppose first that X is projective. Let us write $\mathscr{L} \simeq \mathscr{O}_X(A-B)$, with A and B very ample Cartier divisors. For every m, if we choose E general in the linear system $|mB|$, then a local equation of E is a nonzero divisor on $\mathscr{F}$, in which case we have an inclusion

$$H^0(X,\mathscr{F} \otimes \mathscr{L}^m) \hookrightarrow H^0(X,\mathscr{F} \otimes \mathscr{O}_X(mA)).$$

Since A is very ample, we know that there is a polynomial $P \in \mathbb{Q}[t]$ with $\deg(P) \leq n$ such that $h^0(X,\mathscr{F} \otimes \mathscr{O}_X(mA)) = P(m)$ for $m \gg 0$. Therefore $h^0(X,\mathscr{F} \otimes \mathscr{L}^m) \leq P(m) \leq C \cdot m^n$ for a suitable $C > 0$ and all $m \gg 0$.

If X is complete, we first reduce to the case when X is an integral scheme using Lemma 1.1.8. We then use Chow's lemma and Lemma 1.1.10 to reduce to the case when X is projective. We leave the details to the interested reader. $\square$

Proposition 1.4.8. *If $V_\bullet$ is a graded linear series on the complete variety X, then there are positive constants α,β such that*

$$\alpha \cdot m^{\kappa(V_\bullet)} \leq \dim_k(V_m) \leq \beta \cdot m^{\kappa(V_\bullet)}$$

for all m divisible enough, with the convention that $m^{-\infty} = 0$ for every m.

Proof. It is clear that the assertion holds when $\kappa(V_\bullet) = -\infty$, hence we assume that $d := \kappa(V_\bullet) \geq 0$. Let $\mathscr{L} \in \mathrm{Pic}(X)$ be such that $V_q \subseteq H^0(X, \mathscr{L}^q)$ for every $q \geq 1$. In order to prove the lower-bound in the proposition, note that by Proposition 1.4.4, we have $\mathrm{trdeg}_k K(X, V_\bullet) = d+1$. Let $s_1, \ldots, s_{d+1} \in R(X, V_\bullet)$ be homogeneous and algebraically independent over k. After replacing each of them by a suitable power, we may assume that $s_i \in V_q$ for all i. In this case

$$\dim(V_{qm}) \geq \binom{m+d}{m} \geq \frac{(qm)^d}{d! \cdot q^d}$$

for every $m \geq 1$.

Let us prove now the upper bound in the proposition. Note first that if $d = 0$, then $\dim_k(V_q) \leq 1$ for every q, hence the upper-bound clearly holds. Suppose now that $d \geq 1$ and let q be a positive integer such that $\dim(Y_q) = d$. Let

$$U := \{x \in X \mid s(x) \neq 0 \text{ for some } s \in V_q\}.$$

It is easy to see that there is a subvariety T of X which intersects U, with $\dim(T) = \dim(Y_q)$, and such that $\phi_q(T \cap U)$ is dense in Y_q. Indeed, if $\pi\colon Y \to X$ is a birational map such that $\phi_q \circ \pi$ is a morphism (for example, the projection onto the first component of the graph of ϕ_q), then by Corollary B.1.2, we can find a subvariety W of Y, with $\dim(W) = \dim(Y_q)$, such that $W \cap \pi^{-1}(U) \neq \emptyset$, and W surjects onto Y_q. It is then clear that we may take $T = \pi(W)$.

For every positive integer r, the rational map ϕ_{rq} is defined on U and satisfies $\tau_{rq,q} \circ \phi_{rq} = \phi_q$, hence $\phi_{rq}(T \cap U)$ is dense in Y_{rq}. Since every element of $|V_{rq}|$ is mapped by ϕ_{rq} to the intersection of Y_{rq} with a hyperplane in $\mathbb{P}(V_{rq})$, and Y_{rq} is non-degenerate in $\mathbb{P}(V_{rq})$ by construction, it follows that the composition

$$V_{rq} \hookrightarrow H^0(X, \mathscr{L}^{rq}) \to H^0(T, \mathscr{L}^{rq}|_T)$$

is injective. Proposition 1.4.7 thus implies

$$\dim_k(V_{rq}) \leq h^0(T, \mathscr{L}^{rq}|_T) \leq C \cdot (rq)^d$$

for some $C > 0$ and all $r \gg 0$. □

Remark 1.4.9. In general, it is not the case that if $\mathscr{L}_1$ and $\mathscr{L}_2$ are numerically equivalent line bundles on the complete variety X, then $\kappa(\mathscr{L}_1) = \kappa(\mathscr{L}_2)$. Consider, for example, two degree 0 line bundles $\mathscr{L}_1$ and $\mathscr{L}_2$ on a smooth, projective curve X, with $\mathscr{L}_1$ non-torsion (hence $\kappa(\mathscr{L}_1) = -\infty$) and $\mathscr{L}_2$ torsion (hence $\kappa(\mathscr{L}_2) = 0$). However, we will see in Corollary 1.4.16 that bigness only depends on the numerical equivalence class.

Definition 1.4.10. If X is a complete, smooth variety, then the *Kodaira dimension* of X is $\kappa(X) := \kappa(\omega_X)$. One says that X is of *general type* if $\kappa(X) = \dim(X)$, that is, if ω_X is big.

Remark 1.4.11. It is easy to see that the Kodaira dimension is a birational invariant. Indeed, if X and Y are birational complete varieties, then any birational map between them induces an isomorphism of k-vector spaces

$$H^0(X, \omega_X^m) \simeq H^0(Y, \omega_Y^m)$$

for every $m \geq 1$ (for example, the case $m = 1$ is proved in [Har77, Theorem II.8.19] and the same proof works for every $m \geq 1$). The assertion now follows using the description of the Iitaka dimension in Proposition 1.4.8.

1.4.2 Big line bundles: basic properties

We now study in more detail big line bundles. We begin by introducing an invariant that measures the rate of growth for the spaces of sections of the multiples of a given line bundle. If $\mathscr{L}$ is a line bundle on the n-dimensional complete variety X, the *volume* of $\mathscr{L}$ is given by

$$\mathrm{vol}_X(\mathscr{L}) := \limsup_{m \to \infty} \frac{h^0(X, \mathscr{L}^m)}{m^n/n!}.$$

Note that this is finite by Proposition 1.4.7. It is clear that if $\mathscr{L}$ is big, then $\mathrm{vol}_X(\mathscr{L}) > 0$ and the converse will follow from Theorem 1.4.13 below, at least when X is projective. The volume of a line bundle is an important invariant that only depends on its numerical class. One can extend the volume function from $\mathrm{N}^1(X)$ to a continuous function on $\mathrm{N}^1(X)_{\mathbb{R}}$. Furthermore, the limit superior in the definition is, in fact, a limit. We do not prove these facts about the volume function since we will not need them. For a thorough study of volumes of divisors, we refer to [Laz04b, Chap. 2.2.C].

Example 1.4.12. With the above notation, if $\mathscr{L}$ is ample, then it follows from asymptotic Riemann-Roch and Serre vanishing that $\mathrm{vol}_X(\mathscr{L}) = (\mathscr{L}^n)$.

In the following theorem we collect some equivalent descriptions of big line bundles on projective varieties.

Theorem 1.4.13. *If $\mathscr{L}$ is a line bundle on the n-dimensional projective variety X, then the following are equivalent:*

i) $\mathrm{vol}_X(\mathscr{L}) > 0$.
ii) There are Cartier divisors A and E, with A ample and E effective, such that $\mathscr{L}^d \simeq \mathscr{O}_X(A+E)$ for some positive integer d.
iii) There is $C > 0$ such that

$$h^0(X, \mathscr{L}^m) \geq C \cdot m^n \quad \textit{for all} \quad m \gg 0.$$

iv) For every $q > 0$ that is divisible enough, the rational map $\phi_q \colon X \dashrightarrow \mathbb{P}(H^0(X, \mathscr{L}^q))$ is birational onto its image.

v) $\mathscr{L}$ is big.

Before giving the proof of the theorem, we prove the following lemma.

Lemma 1.4.14 (Kodaira). *If $\mathscr{L}$ is a line bundle on the complete variety X such that* $\mathrm{vol}_X(\mathscr{L}) > 0$*, then for every effective Cartier divisor D, we have*

$$h^0(X, \mathscr{L}^m \otimes \mathscr{O}_X(-D)) > 0$$

for infinitely many m.

Proof. The hypothesis on $\mathscr{L}$ is equivalent to the existence of $C > 0$ such that $h^0(X, \mathscr{L}^m) \geq C \cdot m^n$ for infinitely many m, where $n = \dim(X)$. For a fixed $m \geq 1$, we have an exact sequence

$$0 \to \mathscr{L}^m \otimes \mathscr{O}_X(-D) \to \mathscr{L}^m \to \mathscr{L}^m|_D \to 0.$$

It follows that if $h^0(X, \mathscr{L}^m \otimes \mathscr{O}_X(-D)) = 0$, then $h^0(X, \mathscr{L}^m) \leq h^0(D, \mathscr{L}^m|_D)$. On the other hand, by Proposition 1.4.7, there is $C' > 0$ such that $h^0(D, \mathscr{L}^m|_D) \leq C' \cdot m^{n-1}$ for all $m \gg 0$. We conclude that $h^0(X, \mathscr{L}^m \otimes \mathscr{O}_X(-D)) > 0$ for infinitely many m. $\square$

Proof of Theorem 1.4.13. The implication i)$\Rightarrow$ii) follows from Lemma 1.4.14. Indeed, if $\mathrm{vol}_X(\mathscr{L}) > 0$ and A is an effective ample Cartier divisor, then there is a nonzero section in $H^0(X, \mathscr{L}^d \otimes \mathscr{O}_X(-A))$ for some $d > 0$, which gives the assertion in ii).

We now suppose that ii) holds and prove iii). Since A is ample, it follows from asymptotic Serre vanishing that if $0 < C < \frac{(A^n)}{n!}$, then for every i with $0 \leq i \leq d-1$, we have

$$h^0(X, \mathscr{L}^i \otimes \mathscr{O}_X(mA)) = \chi(X, \mathscr{L}^i \otimes \mathscr{O}_X(mA)) > C \cdot m^n \quad \text{for all} \quad m \gg 0.$$

We deduce that if $C' < C/d^n$ and $m \gg 0$, then by writing $m = d \cdot \lfloor m/d \rfloor + i$, with $0 \leq i \leq d-1$, we have

$$h^0(X, \mathscr{L}^m) = h^0(X, \mathscr{L}^i \otimes \mathscr{O}_X(\lfloor m/d \rfloor A + \lfloor m/d \rfloor E))$$
$$\geq h^0(X, \mathscr{L}^i \otimes \mathscr{O}_X(\lfloor m/d \rfloor A)) \geq C' \cdot m^n.$$

We thus obtain the assertion in iii).

Let us show also that ii) implies iv). After possibly replacing $\mathscr{L}$ by some power, we may assume that there is an effective Cartier divisor E such that the line bundle $\mathscr{M} := \mathscr{L} \otimes \mathscr{O}_X(-E)$ is very ample. For every $m \geq 1$, multiplication by the section defining mE gives an injective map $H^0(X, \mathscr{M}^m) \hookrightarrow H^0(X, \mathscr{L}^m)$. Let us denote by W_m its image. Let $f_1 \colon X \dashrightarrow \mathbb{P}^{n_1}$ and $f_2 \colon X \dashrightarrow \mathbb{P}^{n_2}$ be the rational maps defined by $|W_m|$ and, respectively, the complete linear series $|\mathscr{L}^m|$. Note that f_1 agrees on the complement of $\mathrm{Supp}(E)$ with the map defined by $|\mathscr{M}^m|$, which is a closed embedding. On the other hand, there is a projection $\psi\colon \mathbb{P}^{n_2} \dashrightarrow \mathbb{P}^{n_1}$ such that $\psi \circ f_2 = f_1$. This implies that f_2 is birational onto its image. Since the implications iii)⇒i) and iv)⇒v) are clear and v)⇒i) follows from Proposition 1.4.8, this completes the proof of the theorem. □

Remark 1.4.15. Once we know that a big line bundle $\mathscr{L}$ satisfies the property in Theorem 1.4.13 iii), the proof of Lemma 1.4.14 implies that for every effective Cartier divisor D, we have $h^0(X, \mathscr{L}^m \otimes \mathscr{O}_X(-D)) > 0$ for all $m \gg 0$.

Theorem 1.4.13 implies, in particular, that bigness is a numerical property.

Corollary 1.4.16. *If $\mathscr{L}_1$ and $\mathscr{L}_2$ are numerically equivalent line bundles on a projective variety X, then $\mathscr{L}_1$ is big if and only if $\mathscr{L}_2$ is big.*

Proof. Suppose that $\mathscr{L}_1$ is big. It follows from Theorem 1.4.13 that there are Cartier divisors A and E, with A ample and E effective, such that $\mathscr{L}_1^d \simeq \mathscr{O}_X(A+E)$ for some positive integer d. In this case $\mathscr{L}_2 \simeq \mathscr{O}_X((A+D)+E)$ for some Cartier divisor D, numerically equivalent to zero. Since $A+D$ is numerically equivalent to A, hence ample, we conclude that $\mathscr{L}_2$ is big applying again Theorem 1.4.13. □

The next lemma shows that when checking the bigness of $\mathscr{L}$, we may replace the powers $\mathscr{L}^m$ by $\mathscr{F} \otimes \mathscr{L}^m$ for every coherent sheaf $\mathscr{F}$ with support X.

Lemma 1.4.17. *If $\mathscr{F}$ is a coherent sheaf on the n-dimensional complete variety X, with $\mathrm{Supp}(\mathscr{F}) = X$, and $\mathscr{L}$ is a line bundle on X, then $\mathscr{L}$ is big if and only if there is $C > 0$ such that*

$$h^0(X, \mathscr{F} \otimes \mathscr{L}^m) \geq C \cdot m^n \text{ for } m \gg 0.$$

Proof. Note that by Theorem 1.4.13, $\mathscr{L}$ is big if and only if we have the lower bound in the lemma when $\mathscr{F} = \mathscr{O}_X$. Let us say that $\mathscr{F}$ satisfies property $(\star)_{\mathscr{L}}$ if it satisfies the property in the lemma. We need to show that if $\mathrm{Supp}(\mathscr{F}) = X$, then $\mathscr{F}$ satisfies $(\star)_{\mathscr{L}}$ if and only if $\mathscr{O}_X$ does. Suppose first that X is projective.

We claim that if D is an effective Cartier divisor that does not contain any associated points of $\mathscr{F}$, then $\mathscr{F}$ satisfies $(\star)_{\mathscr{L}}$ if and only if $\mathscr{F} \otimes \mathscr{O}_X(D)$ does. Indeed, it follows from the exact sequence

$$0 \to \mathscr{F} \to \mathscr{F} \otimes \mathscr{O}_X(D) \to \mathscr{F} \otimes \mathscr{O}_D(D) \to 0$$

that

$$\begin{aligned} 0 &\leq h^0(X, \mathscr{F} \otimes \mathscr{O}_X(D) \otimes \mathscr{L}^m) - h^0(X, \mathscr{F} \otimes \mathscr{L}^m) \\ &\leq h^0(D, \mathscr{F} \otimes \mathscr{O}_D(D) \otimes \mathscr{L}^m) \leq C_1 \cdot m^{n-1}, \end{aligned}$$

where the second bound holds for some $C_1 > 0$ and all $m \gg 0$ by Proposition 1.4.7. This proves our claim. In particular, after twisting $\mathscr{F}$ by a suitable effective ample Cartier divisor, we may assume that $\mathscr{F}$ is globally generated.

Let $r = \ell_{\mathscr{O}_{X,\eta}}(\mathscr{F}_\eta) > 0$, where η is the generic point of X. If $s_1, \ldots, s_r$ are general elements in $\Gamma(X, \mathscr{F})$, then the induced map $\mathscr{O}_X^{\oplus r} \xrightarrow{\phi} \mathscr{F}$ is an isomorphism at η.

Therefore the sheaves $\ker(\phi)$ and $\mathrm{coker}(\phi)$ are supported on an $(n-1)$-dimensional subscheme and since $\mathscr{O}_X^{\oplus r}$ is torsion-free, we have $\ker(\phi) = 0$. The short exact sequence

$$0 \to \mathscr{O}_X^{\oplus r} \to \mathscr{F} \to \mathrm{coker}(\phi) \to 0$$

induces after tensoring with $\mathscr{L}^m$ and passing to the long exact sequences in cohomology

$$0 \leq h^0(X, \mathscr{F} \otimes \mathscr{L}^m) - r \cdot h^0(X, \mathscr{L}^m) \leq h^0(X, \mathrm{coker}(\phi) \otimes \mathscr{L}^m) \leq C' \cdot m^{n-1}$$

for some $C' > 0$ and all $m \gg 0$ (where the last inequality follows from Proposition 1.4.7). Therefore $\mathscr{F}$ satisfies $(\star)_{\mathscr{L}}$ if and only if $\mathscr{L}$ is big.

If X is complete, then we apply Chow's lemma, Lemma 1.1.10, and Proposition 1.4.7 to reduce to the projective case. The details are left to the reader. □

Proposition 1.4.18. *If $f: X \to Y$ is a surjective, generically finite morphism of complete varieties, then $\mathscr{L} \in \mathrm{Pic}(Y)$ is big if and only if $f^*\mathscr{L}$ is big.*

Proof. Since the support of $f_*(\mathscr{O}_X)$ is Y, it follows from Lemma 1.4.17 that $\mathscr{L}$ is big if and only if there is $C > 0$ such that

$$h^0(Y, f_*(\mathscr{O}_X) \otimes \mathscr{L}^m) \geq Cm^n \text{ for all } m \gg 0, \tag{1.6}$$

where $n = \dim(X) = \dim(Y)$. The projection formula implies

$$h^0(Y, f_*(\mathscr{O}_X) \otimes \mathscr{L}^m) = h^0(X, f^*(\mathscr{L})^m),$$

hence (1.6) is equivalent to $f^*(\mathscr{L})$ being big. □

Remark 1.4.19. If $f: X \to Y$ is a surjective morphism of complete varieties such that $\dim(X) > \dim(Y)$ and $\mathscr{L}$ is a line bundle on Y, then $f^*(\mathscr{L})$ is never big. Indeed, the projection formula implies

$$h^0(X, f^*(\mathscr{L})^m) = h^0(Y, f_*(\mathscr{O}_X) \otimes \mathscr{L}^m)$$

and by Lemma 1.4.7, the right-hand side is bounded above by a polynomial of degree $\dim(Y) < \dim(X)$.

Remark 1.4.20. The assertion in Corollary 1.4.16 holds even if X is assumed to be complete, instead of projective. Indeed, by Chow's lemma we have a birational morphism $f: Y \to X$, with Y projective. If $\mathscr{L}_1$ are numerically equivalent line bundles on X, then $f^*(\mathscr{L}_1)$ and $f^*(\mathscr{L}_2)$ are numerically equivalent. Since $\mathscr{L}_i$ is big if and only if $f^*(\mathscr{L}_i)$ is big, for $i = 1, 2$, by Corollary 1.4.18, we see that $\mathscr{L}_1$ big implies $\mathscr{L}_2$ big by using Corollary 1.4.16.

Example 1.4.21. If $\mathscr{L}$ is a line bundle on a smooth projective surface X such that $(\mathscr{L}^2) > 0$, then either $\mathscr{L}$ or $\mathscr{L}^{-1}$ is big. Indeed, arguing as in Example 1.3.34, we see that Riemann-Roch together with Serre duality imply

$$h^0(X,\mathscr{L}^m)+h^0(X,\omega_X\otimes\mathscr{L}^{-m})\geq\chi(\mathscr{L}^m)=\frac{(\mathscr{L}^2)}{2}m^2-\frac{(\omega_X\cdot\mathscr{L})}{2}m+\chi(\mathscr{O}_X). \tag{1.7}$$

Note also that if for some m both $h^0(X,\mathscr{L}^m)$ and $h^0(X,\omega_X\otimes\mathscr{L}^{-m})$ are positive, then multiplication by nonzero global sections of $\mathscr{L}^m$ and $\omega_X\otimes\mathscr{L}^{-m}$ induces embeddings

$$H^0(X,\omega_X\otimes\mathscr{L}^{-m})\hookrightarrow H^0(X,\omega_X) \text{ and } H^0(X,\mathscr{L}^m)\hookrightarrow H^0(X,\omega_X).$$

In particular, $h^0(X,\mathscr{L}^m)+h^0(X,\omega_X\otimes\mathscr{L}^{-m})\leq 2h^0(X,\omega_X)$ and by (1.7) this fails for $m\gg 0$. It follows that there is m_0 such that for every $m\geq m_0$, we have $h^0(X,\mathscr{L}^m)=0$ or $h^0(X,\omega_X\otimes\mathscr{L}^{-m})=0$. If there is $d\geq m_0$ such that $h^0(X,\mathscr{L}^d)>0$, then $h^0(X,\mathscr{L}^{dm})>0$ for every $m\geq 1$ and therefore there is $C>0$ such that $h^0(X,\mathscr{L}^{dm})\geq C\cdot(dm)^n$ for $m\gg 0$. In this case $\mathscr{L}^d$ is big and therefore $\mathscr{L}$ is big. On the other hand, if $h^0(X,\mathscr{L}^m)=0$ for all $m\geq m_0$, then there is $C>0$ such that $h^0(X,\omega_X\otimes\mathscr{L}^{-m})\geq C\cdot m^n$ for all $m\gg 0$, in which case $\mathscr{L}^{-1}$ is big by Lemma 1.4.17.

Remark 1.4.22. One can use the assertion in Example 1.4.21 to give an argument for the Hodge Index theorem. Suppose that X is a smooth projective surface. In order to show that the intersection form on $\mathrm{N}^1(X)_{\mathbb{R}}\simeq\mathbb{R}^{\rho}$ has signature $(1,\rho-1)$, it is enough to show that for every ample divisor H and every divisor $D\not\equiv 0$ such that $(D\cdot H)=0$, we have $(D^2)<0$. Note first that $(D^2)\leq 0$. Indeed, if $(D^2)>0$, then we have seen in Example 1.4.21 that either $\mathscr{O}_X(-D)$ or $\mathscr{O}_X(D)$ is big. In particular, some multiple of these line bundles has sections, and therefore $(D\cdot H)=0$ implies that $\mathscr{O}_X(D)\simeq\mathscr{O}_X$, which contradicts $(D^2)>0$.

Suppose now that $(D^2)=0$. Since $D\not\equiv 0$, we can find a divisor E such that $(D\cdot E)$ is nonzero. After replacing E by $(H^2)E-(E\cdot H)H$, we may assume, in addition, that $(E\cdot H)=0$. If $D_m=mD+E$, then $(D_m\cdot H)=0$, hence by what we have already seen

$$m(D\cdot E)+(E^2)=(D_m^2)\leq 0.$$

Since $(D\cdot E)\neq 0$, this can not hold for all $m\in\mathbb{Z}$, giving a contradiction. Therefore $(D^2)<0$.

1.4.3 The big cone

It follows from Corollary 1.4.16 that bigness is well-defined for elements of $\mathrm{N}^1(X)$. Our next goal is to study this notion in $\mathrm{N}^1(X)_{\mathbb{R}}$. In this section we assume that X is a projective variety.

Definition 1.4.23. We say that $\alpha\in\mathrm{N}^1(X)_{\mathbb{Q}}$ is big if a multiple of α is the image of a big line bundle. It follows from Corollary 1.4.16 that this is independent of the inverse image in $\mathrm{Pic}(X)$ of the multiple of α. Furthermore, since a line bundle is big if and only if a multiple is big, the definition is also independent of which multiple of α is chosen. We say that an element of $\mathrm{CDiv}(X)_{\mathbb{Q}}$ or $\mathrm{Pic}(X)_{\mathbb{Q}}$ is big if its image in $\mathrm{N}^1(X)_{\mathbb{Q}}$ is big.

Definition 1.4.24. The *pseudo-effective cone* $\mathrm{PEff}(X) \subseteq \mathrm{N}^1(X)_{\mathbb{R}}$ of a projective variety X is the closure of the set of classes of effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisors in X. An element of $\mathrm{CDiv}(X)_{\mathbb{R}}$ or $\mathrm{Pic}(X)_{\mathbb{R}}$ is *pseudo-effective* if its image in $\mathrm{N}^1(X)_{\mathbb{R}}$ lies in $\mathrm{PEff}(X)$.

Remark 1.4.25. Note that since the set of classes of effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisors is a convex cone in $\mathrm{N}^1(X)_{\mathbb{R}}$, we get that $\mathrm{PEff}(X)$ is a closed convex cone in $\mathrm{N}^1(X)_{\mathbb{R}}$.

Definition 1.4.26. The *big cone* $\mathrm{Big}(X)$ of a projective variety X is the convex cone in $\mathrm{N}^1(X)_{\mathbb{R}}$ generated by classes of big line bundles.

Remark 1.4.27. Note that we have the inclusions

$$\mathrm{Amp}(X) \subseteq \mathrm{Big}(X) \subseteq \mathrm{PEff}(X)$$

(the second inclusion follows from the fact that $\mathrm{PEff}(X)$ is a convex cone, while for every big line bundle $\mathscr{L}$, we have $h^0(X, \mathscr{L}^m) > 0$ for $m \gg 0$). Furthermore, since $\mathrm{PEff}(X)$ is closed and $\mathrm{Nef}(X) = \overline{\mathrm{Amp}(X)}$, we deduce $\mathrm{Nef}(X) \subseteq \mathrm{PEff}(X)$.

Proposition 1.4.28. *Let X be a projective variety.*

i) If $D \in \mathrm{CDiv}(X)_{\mathbb{R}}$, then the class of D lies in the big cone if and only if we can write $D = A + E$, for some $A, E \in \mathrm{CDiv}(X)_{\mathbb{R}}$, with A ample and E effective. Furthermore, we may assume that $E \in \mathrm{N}^1(X)_{\mathbb{Q}}$.
ii) In particular, $D \in \mathrm{CDiv}(X)_{\mathbb{Q}}$ is big if and only if its class in $\mathrm{N}^1(X)_{\mathbb{R}}$ lies in the big cone.
iii) We have $\overline{\mathrm{Big}(X)} = \mathrm{PEff}(X)$ and $\mathrm{Big}(X)$ is the interior of $\mathrm{PEff}(X)$.

Proof. In order to prove i), suppose first that $D = A + E$, with A ample and E effective. Let us show that in this case we may assume that $E \in \mathrm{CDiv}_{\mathbb{Q}}$. Indeed, we can write $E = t_1E_1 + \ldots + t_rE_r$, with E_i effective Cartier divisors and $t_i \in \mathbb{R}_{>0}$. If $t_i' \in \mathbb{Q}_{>0}$ are such that $0 < t_i - t_i' \ll 1$ and $E' = \sum_{i=1}^r t_i'E_i$, then $E' \in \mathrm{CDiv}(X)_{\mathbb{Q}}$ is effective and $D - E'$ is ample by the openness of the ample cone.

Therefore we may assume that we have $D = A + E$ as above, such that in addition E has rational coefficients. Let A' be a fixed ample effective Cartier divisor. If $\lambda \in \mathbb{Q}_{>0}$ is such that $\lambda \ll 1$, then we write

$$D = (A - \lambda A') + (\lambda A' + E),$$

and $A - \lambda A'$ is ample by the openness of the ample cone. Since $\lambda A' + E$ is big by Theorem 1.4.13, and the class of $A - \lambda A'$ lies in $\mathrm{Amp}(X) \subseteq \mathrm{Big}(X)$, it follows that the class of D lies in $\mathrm{Big}(X)$.

Conversely, suppose that we can write $D \equiv \lambda_1 D_1 + \ldots + \lambda_s D_s$, with $s \geq 1$, $D_i \in \mathrm{CDiv}(X)$ big, and $\lambda_i \in \mathbb{R}_{>0}$. By Proposition 1.4.13, we can write $D_i = A_i + E_i$, for some $A_i, E_i \in \mathrm{CDiv}(X)_{\mathbb{Q}}$, with A_i ample and E_i effective. In this case $D = A + E$, where $A-\sum_{i=1}^{s}\lambda_i A_i$ is numerically trivial (hence A is ample) and $E=\sum_{i=1}^{s}\lambda_i E_i$ is effective. This completes the proof of i).

The assertion in ii) follows from i) and Theorem 1.4.13. Let us prove iii). It is enough to show that $\mathrm{Big}(X)$ is the interior of $\mathrm{PEff}(X)$. The fact that $\overline{\mathrm{Big}(X)}=\mathrm{PEff}(X)$ is then a consequence of the general fact that every closed convex cone is the closure of its relative interior (see Corollary A.3.6).

Recall first that $\mathrm{Big}(X)\subseteq \mathrm{PEff}(X)$. Moreover, it follows from i) that

$$\mathrm{Big}(X)=\bigcup_{D\geq 0}([D]+\mathrm{Amp}(X)),$$

where the union is over the effective Cartier $\mathbb{R}$-divisors. Since $\mathrm{Amp}(X)$ is open by Lemma 1.3.16, we conclude that $\mathrm{Big}(X)$ is open, hence it is contained in the interior of $\mathrm{PEff}(X)$.

Suppose now that the class of a Cartier $\mathbb{R}$-divisor D lies in the interior of $\mathrm{PEff}(X)$. This implies that if A is a fixed ample Cartier $\mathbb{R}$-divisor, then $D-\frac{1}{m}A$ is pseudo-effective for $m\gg 0$. Therefore in order to complete the proof of iii) it is enough to show that if $H,F\in \mathrm{CDiv}(X)_{\mathbb{R}}$ are such that H is ample and F is pseudo-effective, then the class of $H+F$ lies in the big cone. By definition, there is a sequence $(F_m)_{m\geq 1}$ of effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisors such that $\lim_{m\to\infty}F_m=F$ in $\mathrm{N}^1(X)_{\mathbb{R}}$. Since we can write $H+F=(H+F-F_m)+F_m$ and $H+F-F_m$ is ample for $m\gg 0$ by openness of the ample cone, we conclude that the class of $H+F$ lies in $\mathrm{Big}(X)$ by i). ☐

Definition 1.4.29. We say that $\alpha\in \mathrm{N}^1(X)_{\mathbb{R}}$ is *big* if it lies in $\mathrm{Big}(X)$. We also say that an element of $\mathrm{CDiv}(X)_{\mathbb{R}}$ or $\mathrm{Pic}(X)_{\mathbb{R}}$ is big if its image in $\mathrm{N}^1(X)_{\mathbb{R}}$ is big. Note that by Theorem 1.4.28 and Proposition 1.4.28, in the case of elements of $\mathrm{CDiv}(X)_{\mathbb{Q}}$, $\mathrm{Pic}(X)_{\mathbb{Q}}$, and $\mathrm{N}^1(X)_{\mathbb{Q}}$ we recover our previous definition.

Remark 1.4.30. Since $\mathrm{Big}(X)$ is the interior of $\mathrm{PEff}(X)$, we deduce that if D and E are Cartier $\mathbb{R}$-divisors, with D big and E pseudo-effective, $D+E$ is big (see Corollary A.3.5).

Example 1.4.31. If X is a smooth projective surface, then under the canonical identification $\mathrm{N}^1(X)_{\mathbb{R}}\simeq \mathrm{N}_1(X)_{\mathbb{R}}$, the cone $\mathrm{PEff}(X)$ gets identified to $\overline{\mathrm{NE}}(X)$.

Remark 1.4.32. If $f\colon X\to Y$ is a surjective morphism of projective varieties, then $f^*\colon \mathrm{N}^1(Y)_{\mathbb{R}}\hookrightarrow \mathrm{N}^1(X)_{\mathbb{R}}$ induces an injective map $\mathrm{PEff}(Y)\hookrightarrow \mathrm{PEff}(X)$. Indeed, this follows from the fact that $\mathrm{PEff}(Y)$ is generated as a closed convex cone by the classes of $\mathscr{L}\in \mathrm{Pic}(Y)$ with $h^0(Y,\mathscr{L})\geq 1$, and for such $\mathscr{L}$ we also have $h^0(X,f^*(\mathscr{L}))\geq 1$.

Note that if f is generically finite, then $\alpha\in \mathrm{N}^1(Y)_{\mathbb{R}}$ is big if and only if $f^*(\alpha)$ is big. If $\alpha\in \mathrm{N}^1(Y)_{\mathbb{Q}}$, this follows from Proposition 1.4.18. In the general case, there is a sequence $(\alpha_m)_{m\geq 1}$, with $\lim_{m\to\infty}\alpha_m=0$, and $\alpha_m\in \mathrm{Amp}(X)$ and $\alpha_m-\alpha\in \mathrm{N}^1(X)_{\mathbb{Q}}$ for all m. We have α big if and only if $\alpha-\alpha_m$ big for $m\gg 0$, which is the case if and only if $f^*(\alpha)-f^*(\alpha_m)$ is big for $m\gg 0$. This is equivalent with $f^*(\alpha)$ being big (note that each $f^*(\alpha_m)$ is clearly big, being a positive linear combination of pull-backs of ample Cartier divisor classes).

We deduce that if f is generically finite, then $\alpha \in \mathrm{N}^1(Y)_{\mathbb{R}}$ is pseudo-effective if and only if $f^*(\alpha)$ is pseudo-effective. Indeed, if $\beta \in \mathrm{N}^1(Y)_{\mathbb{R}}$ is big, then $f^*(\beta)$ is big, and we have the following equivalences:

$$\alpha \text{ is pseudo-effective} \Leftrightarrow \alpha + \frac{1}{m}\beta \text{ is big for all } m \geq 1$$

$$\Leftrightarrow f^*\left(\alpha + \frac{1}{m}\beta\right) \text{ is big for all } m \geq 1 \Leftrightarrow f^*(\alpha) \text{ is pseudo-effective.}$$

We now show that on a smooth variety, we can characterize the bigness in terms of the rate of growth of the space of sections also for $\mathbb{R}$-divisors.

Proposition 1.4.33. *If X is a smooth n-dimensional projective variety, then an $\mathbb{R}$-divisor D on X is big if and only if there is $C > 0$ such that $h^0(X, \mathscr{O}_X(mD)) \geq Cm^n$ for all $m \gg 0$.*

Proof. Recall that by definition, we have $\mathscr{O}_X(mD) = \mathscr{O}_X(\lfloor mD \rfloor)$. Suppose first that D is big, hence we can write $D = A + E$, with A ample and E effective. Since $\lfloor mD \rfloor \geq \lfloor mA \rfloor$, we see that it is enough to prove the assertion when D is ample. In this case we can write $D = \sum_{i=1}^r a_i A_i$, where the A_i are ample Cartier divisors and the a_i are positive real numbers (see Remark 1.3.24). Since

$$\lfloor mD \rfloor \geq \sum_{i=1}^r \lfloor m a_i A_i \rfloor,$$

it is clear that it is enough to prove the assertion when $D = aA$, for an ample Cartier divisor A and for $a \in \mathbb{R}_{>0}$. When $0 < t \leq 1$, there are only finitely many sheaves of the form $\mathscr{O}_X(\lfloor tA \rfloor)$; let these be $\mathscr{F}_1, \ldots, \mathscr{F}_s$. Since A is ample and $\mathrm{Supp}(\mathscr{F}_i) = X$ for all i, it follows that there is $C' > 0$ such that $h^0(X, \mathscr{F}_i \otimes \mathscr{O}_X(mA)) \geq C' \cdot m^n$ for all i and all $m \gg 0$. We conclude that if $C < C' \cdot a^n$, then

$$h^0(X, \mathscr{O}_X(\lfloor maA \rfloor)) \geq C' \cdot (\lfloor ma \rfloor)^n \geq C' \cdot (ma - 1)^n \geq C \cdot m^n$$

for all $m \gg 0$.

Conversely, if there is C as in the proposition, we prove that D is big by arguing as in the proof of Kodaira's lemma. Let us write $D = \sum_{i=1}^s \lambda_i F_i$. Let A be an effective, very ample Cartier divisor on X, that does not contain any of the F_i. It order to show that D is effective, it is enough to prove that $h^0(X, \mathscr{O}_X(mD - A)) \geq 1$ for some $m \geq 1$. Using the short exact sequence

$$0 \to H^0(X, \mathscr{O}_X(\lfloor mD \rfloor - A)) \to H^0(X, \mathscr{O}_X(\lfloor mD \rfloor)) \to H^0(X, \mathscr{O}_X(\lfloor mD \rfloor)|_A),$$

we see that it is enough to show that there is $C' > 0$ such that

$$h^0(X, \mathscr{O}_X(\lfloor mD \rfloor)|_A) \leq C' \cdot m^{n-1} \text{ for } m \gg 0. \tag{1.8}$$

Let $D' = \lceil D \rceil$. By the assumption on A, the natural inclusion $\mathscr{O}_X(\lfloor mD \rfloor) \hookrightarrow \mathscr{O}_X(mD')$ induces an inclusion

$$H^0(X, \mathscr{O}_X(\lfloor mD \rfloor)|_A) \hookrightarrow H^0(X, \mathscr{O}_X(mD')|_A),$$

and it follows from Proposition 1.4.7 that there is $C' > 0$ such that (1.8) holds. This completes the proof of the proposition. □

1.4.4 Big and nef divisors

Of particular importance are divisors that are both nef and big. Our next goal is to give two different characterizations for such divisors.

Proposition 1.4.34. *If X is a projective variety and $D \in \mathrm{CDiv}(X)_{\mathbb{R}}$, then D is big and nef if and only if there $E \in \mathrm{CDiv}(X)_{\mathbb{R}}$ effective and $A_m \in \mathrm{CDiv}(X)_{\mathbb{R}}$ ample, for $m \geq 1$, such that $D = A_m + \frac{1}{m}E_m$ for all m (or for all $m \gg 0$). Furthermore, in this case we may assume that $E \in \mathrm{CDiv}(X)_{\mathbb{Q}}$.*

Proof. If there are divisors E and A_m as in the statement, it first follows from Proposition 1.4.28 that D is big. Furthermore, since $D - \frac{1}{m}E$ is ample, hence nef for all $m \gg 0$, and $\lim_{m\to\infty}(D - \frac{1}{m}E) = D$ in $\mathrm{N}^1(X)_{\mathbb{R}}$, it follows that D is nef.

Conversely, suppose that D is big and nef. It follows from Proposition 1.4.28 that there are $A \in \mathrm{CDiv}(X)_{\mathbb{R}}$ and $E \in \mathrm{CDiv}(X)_{\mathbb{Q}}$, with A ample and E an effective, such that $D = A + E$. Since we can write

$$D = \frac{1}{m}((m-1)D + A) + \frac{1}{m}E$$

and $(m-1)D + A$ is ample, as a sum of ample and nef divisors, it follows that D satisfies the property in the proposition. □

Our next goal is to describe big divisors among the nef ones in terms of the top self-intersection. While the result also holds for real coefficients, we only prove it for $\mathbb{Q}$-divisors.

Theorem 1.4.35. *If X is an n-dimensional complete variety and $D \in \mathrm{CDiv}(X)_{\mathbb{Q}}$ is nef, then D is big if and only if $(D^n) > 0$.*

We give a proof following [Laz04a, Thm. 2.2.16], in which the subtle implication is deduced from the following more general numerical criterion, due to Siu, for a difference of two nef divisors to be big.

Theorem 1.4.36. *If X is an n-dimensional complete variety and $D, E \in \mathrm{CDiv}(X)_{\mathbb{Q}}$ are nef, such that*

$$(D^n) > n \cdot (D^{n-1} \cdot E), \tag{1.9}$$

then $D - E$ is big.

Proof. By Chow's lemma, there is a birational morphism $f\colon X' \to X$, with X' a projective variety. Note that $D' := f^*(D)$ and $E' := f^*(E)$ are nef and condition (1.9) holds with D and E replaced by D' and E', respectively. Furthermore, if $f^*(D-E)$ is big, then $D-E$ is big by Corollary 1.4.18, hence after replacing X by X', we may and will assume that X is projective.

If A is an ample Cartier divisor on X, then the condition in (1.9) still holds after replacing D and E by $D+\varepsilon A$ and $E+\varepsilon A$, respectively, where $\varepsilon \in \mathbb{Q}_{>0}$ is close to 0. Therefore we may and will assume that both D and E are ample. Furthermore, the condition in (1.9) is still satisfied if we replace D and E by multiples mD and mE, and it is enough to show that $m(D-E)$ is big. Therefore we may and will assume that both D and E are very ample Cartier divisors.

For a positive integer m, we want to give a lower bound for the dimension of $H^0(X, \mathscr{O}_X(mD-mE))$. In order to do this, we choose general Cartier divisors $E_1, \ldots, E_m$ linearly equivalent to E, put $G = E_1 + \ldots + E_m$, and use the short exact sequence

$$0 \to \mathscr{O}_X(mD-mE) \to \mathscr{O}_X(mD) \to \mathscr{O}_G(mD) \to 0.$$

This gives the lower bound

$$h^0(X, \mathscr{O}_X(mD-mE)) \geq h^0(X, \mathscr{O}_X(mD)) - h^0(G, \mathscr{O}_G(mD)). \tag{1.10}$$

Since the E_i are chosen general, it follows that at every point in X, the equations of those of the E_i passing through the point form a regular sequence. We deduce that we have an injective map

$$\mathscr{O}_G \hookrightarrow \oplus_i \mathscr{O}_{E_i},$$

and by tensoring this with $\mathscr{O}_X(mD)$ and taking global sections, we obtain

$$h^0(G, \mathscr{O}_G(mD)) \leq \sum_{i=1}^m h^0(E_i, \mathscr{O}_{E_i}(mD)). \tag{1.11}$$

On the other hand, for every i we have a short exact sequence

$$0 \to \mathscr{O}_X(mD-E) \to \mathscr{O}_X(mD) \to \mathscr{O}_{E_i}(mD) \to 0.$$

Since D is ample, we see that for $m \gg 0$ we have $H^1(X, \mathscr{O}_X(mD-E)) = 0$. Therefore we obtain

$$h^0(E_i, \mathscr{O}_{E_i}(mD)) = h^0(X, \mathscr{O}_X(mD)) - h^0(X, \mathscr{O}_X(mD-E)),$$

hence the left-hand side is independent of the choice of E_i. Using one more time the ampleness of D, we conclude that there is a polynomial $P \in \mathbb{Q}[t]$ of degree $\leq n-1$ such that $h^0(E_i, \mathscr{O}_{E_i}(mD)) = P(m)$ for $m \gg 0$. Furthermore, the coefficient of t^{n-1} in P is $\frac{(D^{n-1}\cdot E)}{(n-1)!}$. Since $h^0(X, \mathscr{O}_X(mD)) = P_1(m)$ for some $P_1 \in \mathbb{Q}[t]$ of degree n, with the coefficient of t^n equal to $\frac{(D^n)}{n!}$, we conclude from (1.10) and (1.11) that

$$h^0(X, \mathscr{O}_X(mD - mE)) \geq P_1(m) - mP(m)$$

$$= \frac{1}{n!}\big((D^n) - n\cdot(D^{n-1}\cdot E)\big)\,m^n + \text{lower order terms.}$$

Since $(D^n) - n\cdot(D^{n-1}\cdot E) > 0$, we conclude that $D-E$ is big. $\square$

Proof of Theorem 1.4.35. If D is nef and $(D^n) > 0$, then it follows from Theorem 1.4.36 that D is big (by taking $E = 0$). Conversely, suppose that D is nef and big. Since D is big, we can write $D = A + E$, for $A, E \in \mathrm{CDiv}(X)_{\mathbb{Q}}$, with A ample and E effective. In this case we have

$$(D^n) = (D^{n-1}\cdot A) + (D^{n-1}\cdot E) = (D^{n-2}\cdot A^2) + (D^{n-2}\cdot A\cdot E) + (D^{n-1}\cdot E) = \ldots$$

$$= (A^n) + \sum_{i=1}^{n}(D^{n-i}\cdot A^{i-1}\cdot E).$$

Since E is effective, D is nef, and A is ample, we have $(D^{n-i}\cdot A^{i-1}\cdot E) \geq 0$ for $1 \leq i \leq n$ and $(A^n) > 0$. Therefore $(D^n) > 0$. $\square$

Remark 1.4.37. Note that if D is big but not nef, then (D^n) can be arbitrarily negative. Indeed, suppose for example that X is the blow-up of $\mathbb{P}^2$ at a point, and let us use the notation in Example 1.3.33. We have seen that $\mathrm{PEff}(X) = \overline{\mathrm{NE}}(X)$ is generated by E and $H-E$. Therefore $D_m = H + mE$ is big for every $m \geq 1$ and $(D_m^2) = 1 - m^2$.

Example 1.4.38. If X is a complete variety and D is an effective Cartier divisor such that $\mathscr{O}_D(D)$ is ample, then D is big and nef. Indeed, using the long exact sequence in cohomology corresponding to

$$0 \to \mathscr{O}_X((m-1)D) \to \mathscr{O}_X(mD) \to \mathscr{O}_D(mD) \to 0$$

and arguing as in the proof of Theorem 1.3.1, we see that the ampleness of $\mathscr{O}_D(D)$ implies that

$$h^0(X, \mathscr{O}_X(mD)) - h^0(X, \mathscr{O}_X((m-1)D)) = h^0(D, \mathscr{O}_D(mD)) \quad \text{for} \quad m \gg 0.$$

Since the right-hand side is a polynomial function of degree $n-1$, where $n = \dim(X)$, it follows that $\mathscr{O}_X(D)$ is big. On the other hand, we have $(\mathscr{O}_X(D)^n) = (\mathscr{O}_D(D)^{n-1}) > 0$, hence $\mathscr{O}_X(D)$ is also nef.

1.5 Asymptotic base loci

In this section we introduce different flavors of asymptotic base loci that can be associated to a line bundle, following [ELM+06]. We then use these notions to describe various subcones of the pseudo-effective cone. We work over an algebraically closed ground field k.

1.5.1 The stable base locus

Recall first that if X is a complete scheme and $\mathscr{L} \in \mathrm{Pic}(X)$, then the *base-locus* of $\mathscr{L}$ is defined as the scheme-theoretic intersection

$$\mathrm{Bs}(\mathscr{L}) := \bigcap_{s \in H^0(X,\mathscr{L})} Z(s),$$

where we denote by $Z(s)$ the zero-locus of $s \in H^0(X,\mathscr{L})$.

Definition 1.5.1. The *stable base locus* of $\mathscr{L}$ is the closed subset

$$\mathrm{SB}(\mathscr{L}) := \bigcap_{m \geq 1} \mathrm{Bs}(\mathscr{L}^m)_{\mathrm{red}} \subseteq X.$$

Note that if $s \in H^0(X,\mathscr{L}^m)$, then we have a section $s^{\otimes d} \in H^0(X,\mathscr{L}^{md})$ such that $Z(s^{\otimes d})_{\mathrm{red}} = Z(s)_{\mathrm{red}}$. This implies that for every positive integers m and d, we have

$$\mathrm{Bs}(\mathscr{L}^m)_{\mathrm{red}} \supseteq \mathrm{Bs}(\mathscr{L}^{md})_{\mathrm{red}}.$$

It follows by the Noetherian property of X that the following holds:

Lemma 1.5.2. *If $\mathscr{L}$ is a line bundle on the complete scheme X, then for $m \in \mathbb{Z}_{>0}$ divisible enough, we have*

$$\mathrm{SB}(\mathscr{L}) = \mathrm{Bs}(\mathscr{L}^m)_{\mathrm{red}}.$$

In particular, this implies that the stable base locus is invariant under replacing $\mathscr{L}$ by a power.

Corollary 1.5.3. *If $\mathscr{L}$ is a line bundle on the complete scheme X, then $\mathrm{SB}(\mathscr{L}) = \mathrm{SB}(\mathscr{L}^d)$ for every positive integer d.*

We can therefore extend the definition of the stable base locus to elements of $\mathrm{Pic}(X)_{\mathbb{Q}}$.

Definition 1.5.4. If X is a projective scheme and $\lambda\mathscr{L} \in \mathrm{Pic}(X)_{\mathbb{Q}}$, for some $\lambda \in \mathbb{Q}_{>0}$ and $\mathscr{L} \in \mathrm{Pic}(X)$, then we put $\mathrm{SB}(\lambda\mathscr{L}) := \mathrm{SB}(\mathscr{L})$. It follows from Corollary 1.5.3 that this definition is independent of choices. We also define $\mathrm{SB}(D)$, for D a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor, to be the stable base locus of the corresponding element of $\mathrm{Pic}(X)_{\mathbb{Q}}$.

Example 1.5.5. Note that a line bundle $\mathscr{L}$ on X is semiample if and only if $\mathrm{SB}(\mathscr{L})$ is empty.

Example 1.5.6. It is easy to give examples of numerically equivalent line bundles whose stable base loci are different (consider, as in Example 1.3.5, two degree 0 line bundles on a smooth projective curve, one of them torsion and the other one non-torsion). We now give such an example in which both line bundles are big and nef (see [Laz04b, Example 10.3.3] for a different presentation).

Let $C \subset \mathbb{P}^n$ be a smooth, projective curve of genus $g \geq 1$ over an uncountable algebraically closed field. Consider the projective cone $Y \hookrightarrow \mathbb{P}^{n+1}$ over C and $f\colon X \to Y$ the blow-up of the vertex. We have an induced morphism $g\colon X \to C$ and if E is the exceptional divisor of f, then g induces an isomorphism $g|_E\colon E \simeq C$.

Let A be an ample line bundle on Y, B a degree 0 line bundle on C, and put $\mathscr{L} = f^*(A) \otimes g^*(B)$. Note that when we vary B, we obtain numerically equivalent line bundles on X, which are all big and nef, since f is birational and A is ample. If B is torsion, then clearly $\mathrm{SB}(\mathscr{L}) = \emptyset$. On the other hand, since $\mathscr{L}|_E$ corresponds to $B \in \mathrm{Pic}(C)$ via $g|_E$, it follows that if B is non-torsion, then $E \subseteq \mathrm{SB}(\mathscr{L})$ (in fact, this is an equality).

Lemma 1.5.7. *If* $\alpha, \beta \in \mathrm{Pic}(X)_{\mathbb{Q}}$, *then* $\mathrm{SB}(\alpha+\beta) \subseteq \mathrm{SB}(\alpha) \cup \mathrm{SB}(\beta)$. *In particular, if* β *is ample (or semiample), then* $\mathrm{SB}(\alpha+\beta) \subseteq \mathrm{SB}(\alpha)$.

Proof. The assertion is a consequence of the fact that given $\mathscr{L}_1, \mathscr{L}_2 \in \mathrm{Pic}(X)$, we have
$$\mathrm{Bs}(\mathscr{L}_1 \otimes \mathscr{L}_2)_{\mathrm{red}} \subseteq \mathrm{Bs}(\mathscr{L}_1)_{\mathrm{red}} \cup \mathrm{Bs}(\mathscr{L}_2)_{\mathrm{red}},$$
which in turn follows from the observation that if $s_1 \in H^0(X, \mathscr{L}_1)$ and $s_2 \in H^0(X, \mathscr{L}_2)$, then
$$Z(s_1 \otimes s_2)_{\mathrm{red}} = Z(s_1)_{\mathrm{red}} \cup Z(s_2)_{\mathrm{red}}.$$
$\square$

1.5.2 The augmented base locus

We will consider two variants of asymptotic base loci that are attached to small perturbations of a given divisor. They have the advantage that only depend on the numerical class of a divisor, and moreover, they can also be defined for $\mathbb{R}$-coefficients. In what follows X is a fixed projective scheme. We first introduce an upper approximation of the stable base locus.

Definition 1.5.8. If $D \in \mathrm{Pic}(X)_{\mathbb{R}}$, then the *augmented base locus* of α is
$$\mathrm{B}_+(D) := \bigcap_{\beta \in Q_D} \mathrm{SB}(D-A),$$
where
$$Q_D = \{A \in \mathrm{Pic}(X)_{\mathbb{R}} \mid A \text{ ample and } D-A \in \mathrm{Pic}(X)_{\mathbb{Q}}\}.$$
We also define the augmented base locus of an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor as the augmented base locus of the corresponding element in $\mathrm{Pic}(X)_{\mathbb{R}}$.

Remark 1.5.9. It follows from Lemma 1.5.7 that if $D \in \mathrm{Pic}(X)_{\mathbb{Q}}$, then $\mathrm{SB}(D) \subseteq \mathrm{B}_+(D)$.

In order to simplify formulations, it will be convenient to make the following convention: if $D \in \mathrm{Pic}(X)_{\mathbb{R}}$ and $\mathscr{U}$ is a subset of $\mathrm{N}^1(X)_{\mathbb{R}}$, we say that D lies in $\mathscr{U}$ if the image of D in $\mathrm{N}^1(X)_{\mathbb{R}}$ lies in $\mathscr{U}$.

Proposition 1.5.10. *The augmented base locus of D is a closed subset of X. Furthermore, there is an open neighborhood $\mathscr{U}$ of 0 in $\mathrm{N}^1(X)_{\mathbb{R}}$ such that $\mathrm{B}_+(D) = \mathrm{SB}(D-A)$ for every $A \in Q_D \cap \mathscr{U}$.*

Proof. Since $\mathrm{B}_+(D)$ is an intersection of closed subsets of X, it is clear that it is closed. Note now that if $A_1, A_2 \in Q_D$ are such that $A_1 - A_2$ is ample, then by Lemma 1.5.7 we have

$$\mathrm{SB}(D-A_2) \subseteq \mathrm{SB}(D-A_1). \tag{1.12}$$

By the Noetherian property, we may choose $A_0 \in Q_D$ with $Z = \mathrm{SB}(D-A_0)$ minimal. If $A \in Q_D$ is such that $A_0 - A$ is ample (which is the case if A lies in a suitable open neighborhood of 0 in $\mathrm{N}^1(X)_{\mathbb{R}}$), then by (1.12) we have $\mathrm{SB}(D-A) \subseteq Z$, and the minimality in the choice of A_0 implies that this is in fact an equality.

Furthermore, given any $A' \in Q_D$, we can find $A \in Q_D$ such that $A_0 - A$ and $A' - A$ are both ample, and therefore

$$Z = \mathrm{SB}(D-A) \subseteq \mathrm{SB}(D-A').$$

This implies that $Z = \mathrm{B}_+(D)$ and by what we have already proved, completes the proof of the proposition. $\square$

Remark 1.5.11. If D and A are Cartier divisors, with A ample, it follows from Proposition 1.5.10 that $\mathrm{B}_+(D) = \mathrm{SB}(mD-A)$ for any $m \in \mathbb{Z}$, with $m \gg 0$.

Remark 1.5.12. Note that if V is a subvariety of X, then $V \not\subseteq \mathrm{B}_+(D)$ if and only if we can find $A \in \mathrm{CDiv}(X)_{\mathbb{R}}$ ample and $E \in \mathrm{CDiv}(X)_{\mathbb{Q}}$ effective with $V \not\subseteq \mathrm{Supp}(E)$ and $D = A + E$ in $\mathrm{Pic}(X)_{\mathbb{R}}$. In this case, we may restrict E to V to get an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor, hence the restriction to S of the numerical class of D is big. We also mention that it is enough to find A and E as above, but with E possibly an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor: arguing as in the proof of Proposition 1.4.28 one can show that we can write $A + E = A' + E'$, where E is a Cartier $\mathbb{Q}$-divisor and $\mathrm{Supp}(E) = \mathrm{Supp}(E')$.

Proposition 1.5.13. *If $D_1, D_2 \in \mathrm{Pic}(X)_{\mathbb{R}}$ and $D_1 \equiv D_2$, then $\mathrm{B}_+(D_1) = \mathrm{B}_+(D_2)$.*

Proof. We have a bijective map $Q_{D_1} \to Q_{D_2}$ that takes A_1 to $A_2 = A_1 + (D_2 - D_1)$. Since we have $D_1 - A_1 = D_2 - A_2$, we obtain the equality $\mathrm{B}_+(D_1) = \mathrm{B}_+(D_2)$. $\square$

As a consequence of Proposition 1.5.13, we can define in the obvious way the augmented base locus $\mathrm{B}_+(\alpha)$ for $\alpha \in \mathrm{N}^1(X)_{\mathbb{R}}$. In what follows, if $\alpha \in \mathrm{N}^1(X)_{\mathbb{R}}$, we also put

$$Q_\alpha = \{\beta \in \mathrm{Amp}(X) \mid \alpha - \beta \in \mathrm{N}^1(X)_{\mathbb{Q}}\}.$$

Remark 1.5.14. If $D \in \mathrm{Pic}(X)_{\mathbb{R}}$, then

i) $\mathrm{B}_+(D) = \emptyset$ if and only if D is ample.

ii) Assuming that X is a variety, $\mathrm{B}_+(D) \neq X$ if and only if D is big.

Indeed, it follows from Proposition 1.5.10 that $\mathrm{B}_+(D)$ is empty if and only if there is $A \in Q_D$ such that $\mathrm{SB}(D-A) = \emptyset$. In this case D is a sum of an ample and a nef divisor, hence it is ample. Conversely, if D is ample, then we can find $A \in Q_D$ such that $D-A$ is ample, hence $\mathrm{SB}(D-A) = \emptyset$. The assertion in ii) is an immediate consequence of Remark 1.5.12.

Lemma 1.5.15. *If* $\alpha_1, \alpha_2 \in \mathrm{N}^1(X)_{\mathbb{R}}$, *then*

$$\mathrm{B}_+(\alpha_1 + \alpha_2) \subseteq \mathrm{B}_+(\alpha_1) \cup \mathrm{B}_+(\alpha_2).$$

In particular, if $\alpha_2 \in \mathrm{Amp}(X)$, *then* $\mathrm{B}_+(\alpha_1 + \alpha_2) \subseteq \mathrm{B}_+(\alpha_1)$.

Proof. Let us write α_1, α_2 as the images of $D_1, D_2 \in \mathrm{Pic}(X)_{\mathbb{R}}$. By Proposition 1.5.10, we can find $A_1 \in Q_{D_1}$ and $A_2 \in Q_{D_2}$ such that $\mathrm{B}_+(\alpha_1) = \mathrm{SB}(D_1 - A_1)$ and $\mathrm{B}_+(\alpha_2) = \mathrm{SB}(D_2 - A_2)$. It is clear that $(A_1 + A_2) \in Q_{D_1+D_2}$ and using Lemma 1.5.7 and the definition of the augmented base locus, we obtain

$$\begin{aligned} \mathrm{B}_+(\alpha_1+\alpha_2) &\subseteq \mathrm{SB}(D_1 + D_2 - A_1 - A_2) \\ &\subseteq \mathrm{SB}(D_1 - A_1) \cup \mathrm{SB}(D_2 - A_2) = \mathrm{B}_+(\alpha_1) \cup \mathrm{B}_+(\alpha_2). \end{aligned}$$

□

Proposition 1.5.16. *If* $\alpha \in \mathrm{N}^1(X)_{\mathbb{R}}$ *and* $\lambda > 0$, *then* $\mathrm{B}_+(\alpha) = \mathrm{B}_+(\lambda\alpha)$.

Proof. Clearly, it is enough to show that $\mathrm{B}_+(\alpha) \supseteq \mathrm{B}_+(\lambda\alpha)$, since applying this with (α, λ) replaced by $(\lambda\alpha, \lambda^{-1})$ would give the reverse inclusion. Let $D \in \mathrm{Pic}(X)_{\mathbb{R}}$ be such that its class is α, and consider $A \in Q_D$. We choose $A' \in Q_{\lambda D}$, whose class is close enough to 0, such that $\lambda A - A'$ is ample. In this case, if $\lambda' \in \mathbb{Q}_{>0}$ is close enough to λ, we have that $(\lambda D - A') - \lambda'(D-A)$ is ample, and using Lemma 1.5.7 we obtain

$$\mathrm{B}_+(\lambda D) \subseteq \mathrm{SB}(\lambda D - A') \subseteq \mathrm{SB}(\lambda'(D-A)) = \mathrm{SB}(D-A).$$

Since this holds for every $A \in Q_D$, we obtain $\mathrm{B}_+(\lambda D) \subseteq \mathrm{B}_+(D)$. □

Proposition 1.5.17. *If* $\alpha \in \mathrm{N}^1(X)_{\mathbb{R}}$, *then there is an open subset* $\mathscr{W}$ *of* α *such that* $\mathrm{B}_+(\alpha') \subseteq \mathrm{B}_+(\alpha)$ *for every* $\alpha' \in \mathscr{W}$, *with equality if* $\alpha - \alpha'$ *is ample.*

Proof. Let us write α as the image of $D \in \mathrm{Pic}(X)_{\mathbb{R}}$. We need to find an open neighborhood $\mathscr{W}$ of α such that for every $D' \in \mathrm{Pic}(X)_{\mathbb{R}}$ that lies in $\mathscr{W}$, we have $\mathrm{B}_+(D') \subseteq \mathrm{B}_+(D)$, with equality if $D - D'$ is ample.

It follows from Proposition 1.5.10 that there is an open neighborhood $\mathscr{U}$ of 0 in $\mathrm{N}^1(X)_{\mathbb{R}}$ such that $\mathrm{B}_+(D) = \mathrm{SB}(D-A)$ whenever $A \in Q_D \cap \mathscr{U}$. Fix $B \in Q_D \cap \mathscr{U}$ and let $\mathscr{V}$ be an open neighborhood of 0 such that $B - \mathscr{V} \subseteq \mathrm{Amp}(X)$ and $\mathscr{V} + \mathscr{V} \subseteq \mathscr{U}$. We show that $\mathscr{W} = \alpha - \mathscr{V}$ satisfies the conditions in the proposition.

Suppose first that $D' \in \mathscr{W}$ is such that $D - D'$ is ample. Using Lemma 1.5.15, we first obtain $\mathrm{B}_+(D) \subseteq \mathrm{B}_+(D')$. On the other hand, let $G \in Q_{D'} \cap \mathscr{V}$. This first implies $\mathrm{B}_+(D') \subseteq \mathrm{SB}(D' - G)$. We also see that $D - (D' - G) = (D - D') + G$ is ample, it lies in $\mathscr{V} + \mathscr{V} \subseteq \mathscr{U}$, and $D' - G \in \mathrm{N}^1(X)_{\mathbb{Q}}$. Therefore $D - (D' - G) \in Q_D \cap \mathscr{U}$, and by assumption we get $\mathrm{B}_+(D) = \mathrm{SB}(D' - G) \supseteq \mathrm{B}_+(D')$. We conclude that in this case $\mathrm{B}_+(D) = \mathrm{B}_+(D')$.

Suppose now that $D' \in \mathscr{W}$ is arbitrary. Note that in this case $D' - (D - B) \in B - \mathscr{V}$, hence it is ample. Therefore

$$\mathrm{B}_+(D') \subseteq \mathrm{SB}(D - B) = \mathrm{B}_+(D).$$

This completes the proof of the proposition. □

The augmented base locus was introduced in [Nak00], where this locus was described for a nef $\mathbb{Q}$-divisor, as follows.

Theorem 1.5.18. *If X is a smooth projective variety over an algebraically closed field of characteristic 0, and $\mathscr{L} \in \mathrm{Pic}(X)_{\mathbb{Q}}$ is nef, then*

$$\mathrm{B}_+(\mathscr{L}) = \bigcup_{\mathscr{L}|_V \neq \mathrm{big}} V,$$

where the union is over all subvarieties V of X such that $\mathscr{L}|_V$ is not big.

The proof in [Nak00] relies on the Kawamata-Viehweg vanishing theorem. The same result was proved for arbitrary schemes in positive characteristic in [CMM14], by making use of the Frobenius morphism. A uniform proof for schemes in arbitrary characteristic, relying on Fujita's vanishing theorem, has been recently announced in [Bir].

We note that in the context of the theorem, since $\mathscr{L}$ is nef, for a subvariety V of X the restriction $\mathscr{L}|_V$ is not big if and only if $d = \dim(V) > 0$ and $(\mathscr{L}^d \cdot V) = 0$. We also note that the inclusion "$\supseteq$" in the theorem is clear and holds without the assumption that $\mathscr{L}$ is nef: indeed, if V is not contained in $\mathrm{B}_+(\mathscr{L})$, then it follows from Remark 1.5.12 that there are $A, E \in \mathrm{Pic}(X)_{\mathbb{Q}}$, with A ample and E represented by an effective divisor whose support does not contain V, such that $D = A + E$. In this case $E|_V$ is pseudoeffective and $A|_V$ is ample, hence $\mathscr{L}|_V$ is big.

Remark 1.5.19. We also mention the following fact, due to Keel, which is particular to positive characteristic, see [Kee99] and [CMM14]. Suppose that X is a projective variety over an algebraically closed field k of characteristic $p > 0$, and $\mathscr{L} \in \mathrm{Pic}(X)_{\mathbb{Q}}$ is nef. In this case $\mathrm{SB}(\mathscr{L}) = \mathrm{SB}\left(\mathscr{L}|_{\mathrm{B}_+(\mathscr{L})}\right)$.

One can use this to recover the following result of Artin: if $k = \overline{\mathbb{F}_p}$ and $\dim(X) = 2$, then every nef and big line bundle $\mathscr{L}$ on X is semiample. Indeed, in this case every irreducible component C of $\mathrm{B}_+(\mathscr{L})$ is a curve such that $\deg(\mathscr{L}|_C) = 0$, and therefore $\mathscr{L}|_C$ is torsion. This implies that $\mathscr{L}|_{\mathrm{B}_+(\mathscr{L})}$ is torsion, hence semiample, which implies $\mathscr{L}$ semiample.

1.5.3 The non-nef locus

We now consider what happens if instead of subtracting a "small" ample divisor we add such a divisor. We keep the assumption that X is a projective scheme.

Definition 1.5.20. If $D \in \mathrm{Pic}(X)_{\mathbb{R}}$, then the *non-nef locus*[4] of D is

$$\mathrm{B}_{-}(D) := \bigcup_{A \in Q_{-D}} \mathrm{SB}(D+A).$$

It follows from Lemma 1.5.7 that if $D \in \mathrm{Pic}(X)_{\mathbb{Q}}$, then $\mathrm{B}_{-}(D) \subseteq \mathrm{SB}(D)$. Furthermore, arguing as in the proof of Proposition 1.5.13, we see that the following holds:

Proposition 1.5.21. *If $D_1, D_2 \in \mathrm{Pic}(X)_{\mathbb{R}}$ and $D_1 \equiv D_2$, then $\mathrm{B}_{-}(D_1) = \mathrm{B}_{-}(D_2)$.*

Similar arguments with those used in the proofs of Proposition 1.5.16 and Lemma 1.5.15 give the following.

Proposition 1.5.22. *If D, D_1, and D_2 are in $\mathrm{Pic}(X)_{\mathbb{R}}$ and $\lambda > 0$, then*

i) $\mathrm{B}_{-}(\alpha) = \mathrm{B}_{-}(\lambda\alpha)$.
ii) $\mathrm{B}_{-}(D_1 + D_2) \subseteq \mathrm{B}_{-}(D_1) + \mathrm{B}_{-}(D_2)$*; in particular, if D_2 is ample, then we have* $\mathrm{B}_{-}(D_1 + D_2) \subseteq \mathrm{B}_{-}(D_1)$.

It follows from Proposition 1.5.21 that we may define in the obvious way $\mathrm{B}_{-}(\alpha)$ when $\alpha \in \mathrm{N}^1(X)_{\mathbb{R}}$. The following proposition shows that $\mathrm{B}_{-}(D)$ is always a countable union of Zariski closed subsets.

Proposition 1.5.23. *If $D \in \mathrm{Pic}(X)_{\mathbb{R}}$, then for every sequence $(A_m)_{m \geq 1}$ of elements in Q_{-D}, whose classes in $\mathrm{N}^1(X)_{\mathbb{R}}$ converge to 0, we have*

$$\mathrm{B}_{-}(D) = \bigcup_{m \geq 1} \mathrm{SB}(D + A_m).$$

Proof. The inclusion "$\supseteq$" follows from definition. Suppose that A is any element in Q_{-D}. For $m \gg 0$, the difference $A - A_m$ is ample, hence $\mathrm{SB}(D+A) \subseteq \mathrm{SB}(D+A_m)$. By letting A run over Q_{-D}, we obtain the inclusion "$\subseteq$" in the proposition. $\square$

Remark 1.5.24. One can choose the sequence $(A_m)_{m \geq 1}$ in Proposition 1.5.23 such that, in addition, each $A_m - A_{m+1}$ is ample. In this case the union is non-decreasing: $\mathrm{SB}(D + A_m) \subseteq \mathrm{SB}(D + A_{m+1})$ for all m.

Remark 1.5.25. It can happen that $\mathrm{B}_{-}(D)$ is not closed in X, though such an example has only recently been obtained in [Les].

[4] This is sometimes called the *diminished base locus* or the *restricted base locus* of D.

In light of Proposition 1.5.23, it is convenient to work with $\mathrm{B}_{-}(D)$ when the ground field k is uncountable. For example, in this case it follows that a subvariety V of X is contained in $\mathrm{B}_{-}(D)$ if and only if it is contained in some $\mathrm{SB}(D+A)$, with $A \in Q_{-D}$. On the other hand, when k is countable, the non-nef locus does not provide the correct formulation for many statements. Because of this restriction, we will avoid in general working with $\mathrm{B}_{-}(D)$ and work instead with all subsets $\mathrm{SB}(D+A)$, where A varies over Q_{-D}.

Remark 1.5.26. If $D \in \mathrm{Pic}(X)_{\mathbb{R}}$, then

i) $\mathrm{B}_{-}(D) = \emptyset$ if and only if D is nef (which explains the name of $\mathrm{B}_{-}(D)$).
ii) If k is uncountable, then $\mathrm{B}_{-}(D) = X$ if and only if D is not pseudoeffective (in general, the latter condition is equivalent to $\mathrm{SB}(D+A) = X$ for some $A \in Q_{-D}$).

Indeed, for i) note that $\mathrm{B}_{-}(D) = \emptyset$ if and only if $D+A$ is semiample for every $A \in Q_{-D}$. This clearly holds if D is ample (since in this case $D+A$ is ample), and one can see that the converse holds by considering a sequence $(A_m)_{m\geq 1}$ of elements in Q_{-D} with the classes of A_m in $\mathrm{N}^1(X)_{\mathbb{Q}}$ converging to 0. Indeed, if $D+A_m$ is semiample for all such m, then $D+A_m$ is nef and by passing to limit we obtain D nef.

It is easy to see that D is pseudoeffective if and only if $D+A$ is big (or pseudoeffective) for every $A \in Q_{-D}$ (for the "if" part, consider a sequence $(A_m)_{m\geq 1}$ of elements in Q_{-D} with classes in $\mathrm{N}^1(X)_{\mathbb{R}}$ converging to 0). The assertion in ii) is an immediate consequence.

Example 1.5.27. It is easy to give examples of big line bundles $\mathscr{L}$ on projective varieties such that $\mathrm{SB}(\mathscr{L}) \neq \mathrm{B}_{+}(\mathscr{L})$: for example, it is enough to consider $\mathscr{L}$ that is globally generated, but not ample (in which case $\mathrm{SB}(\mathscr{L})$ is empty, while $\mathrm{B}_{+}(\mathscr{L})$ is nonempty). In order to find an example of a big line bundle $\mathscr{L}$ such that $\mathrm{SB}(\mathscr{L}) \neq \mathrm{B}_{-}(\mathscr{L})$, it is enough to take $\mathscr{L}$ big and nef, but not semiample (in which case $\mathrm{B}_{-}(\mathscr{L})$ is empty, but $\mathrm{SB}(\mathscr{L})$ is not). For an explicit example, see Example 1.5.6.

1.5.4 Stability in $\mathrm{N}^1(X)_{\mathbb{R}}$

We now use the asymptotic base loci introduced so far to define a notion of "stability" for line bundles (and more generally, for elements of $\mathrm{Pic}(X)_{\mathbb{R}}$), which is satisfied when the stable base loci do not change in some neighborhood.

Definition 1.5.28. We say that $D \in \mathrm{Pic}(X)_{\mathbb{R}}$ is *stable* if there is $A \in Q_{-D}$ such that $\mathrm{B}_{+}(D) = \mathrm{SB}(D+A)$.

Arguing as in the proof of Proposition 1.5.13, we see that the stability of D only depends on the numerical class of D. Therefore we can consider the stability of the elements in $\mathrm{N}^1(X)_{\mathbb{R}}$. We denote by $\mathrm{Stab}(X)$ the set of stable $\alpha \in \mathrm{N}^1(X)_{\mathbb{R}}$.

Remark 1.5.29. If k is uncountable, then $D \in \mathrm{Pic}(X)_{\mathbb{R}}$ is stable if and only if $\mathrm{B}_+(D) = \mathrm{B}_-(D)$. Indeed, this follows from the inclusion $\bigcup_{m\geq 1} \mathrm{SB}(D+A_m) \subseteq \mathrm{B}_+(D)$, where $(A_m)_{m\geq 1}$ is as in Remark 1.5.24.

Proposition 1.5.30. *If $\alpha \in \mathrm{N}^1(X)_{\mathbb{R}}$, then the following are equivalent:*

i) α is stable.
ii) There is an open neighborhood $\mathscr{U}$ of α such that $\mathrm{SB}(\mathscr{L})$ is independent of $\mathscr{L} \in \mathrm{Pic}(X)_{\mathbb{Q}}$ with image in $\mathscr{U}$.
iii) There is an open neighborhood $\mathscr{U}$ of α such that $\mathrm{B}_+(\alpha) = \mathrm{B}_+(\alpha')$ for every $\alpha' \in \mathscr{U}$.

Proof. We first show i)⇒iii). Suppose that α is the class of $D \in \mathrm{Pic}(X)_{\mathbb{R}}$ and that $A \in Q_{-D}$ is such that $\mathrm{B}_+(D) = \mathrm{SB}(D+A)$. We choose an open neighborhood $\mathscr{U}$ of α that satisfies the condition in Proposition 1.5.17 and such that $A+D-D'$ is ample whenever D' lies in $\mathscr{U}$. This implies that for every D' in $\mathscr{U}$, we have

$$\mathrm{SB}(D+A) \subseteq \mathrm{B}_+(D+A) \subseteq \mathrm{B}_+(D') \subseteq \mathrm{B}_+(D),$$

hence all these inclusions are equalities and $\mathrm{B}_+(D) = \mathrm{B}_+(D')$.

Suppose now that $\mathscr{U}$ is an open neighborhood of α such that $\mathrm{B}_+(\alpha') = Z$ for every $\alpha' \in \mathscr{U}$. If $\mathscr{L} \in \mathrm{Pic}(X)_{\mathbb{Q}}$ lies in $\mathscr{U}$, then there is $H \in \mathrm{Pic}(X)_{\mathbb{Q}}$ ample such that both $\mathscr{L}-H$ and $\mathscr{L}+H$ lie in $\mathscr{U}$. In this case we have

$$\mathrm{B}_+(\mathscr{L}+H) \subseteq \mathrm{SB}(\mathscr{L}) \subseteq \mathrm{SB}(\mathscr{L}-H) \subseteq \mathrm{B}_+(\mathscr{L}-H),$$

hence all these inclusions are equalities and $\mathrm{SB}(\mathscr{L}) = Z$.

In order to prove ii)⇒i), suppose that $\mathscr{U}$ is an open neighborhood of α that satisfies ii). If we choose $A' \in Q_D$ and $A \in Q_{-D}$ whose images in $\mathrm{N}^1(X)_{\mathbb{R}}$ are close enough to 0, the classes of $D+A$ and $D-A'$ are in $\mathscr{U}$, hence

$$\mathrm{B}_+(D) = \mathrm{SB}(D-A') = \mathrm{SB}(D+A),$$

so that D is stable. □

Since the condition in Proposition 1.5.30 iii) clearly defines an open cone in $\mathrm{N}^1(X)$, we obtain

Corollary 1.5.31. *The set* $\mathrm{Stab}(X)$ *is an open cone in* $\mathrm{N}^1(X)_{\mathbb{R}}$.

Corollary 1.5.32. *For every $\alpha \in \mathrm{N}^1(X)_{\mathbb{R}}$, there is an open neighborhood $\mathscr{U}$ of α such that α' is stable for every $\alpha' \in \mathscr{U}$, with $\alpha - \alpha'$ ample. In particular, the set $\mathscr{S}tab(X)$ is dense[5] in $\mathrm{N}^1(X)_{\mathbb{R}}$.*

Proof. The second assertion follows immediately from the first one, which in turn is a consequence of the description of stable elements of $\mathrm{N}^1(X)_{\mathbb{R}}$ in Proposition 1.5.30 iii) and of Proposition 1.5.17. □

[5] We will show in Proposition 1.5.36 below that, in fact, the complement of $\mathrm{Stab}(X)$ has Lebesgue measure zero.

Example 1.5.33. Suppose, for example, that $X = \mathrm{Bl}_{Q_1,Q_2}(\mathbb{P}^n)$ is the blow-up of $\mathbb{P}^n$ at two points Q_1 and Q_2. In this case $\mathrm{Pic}(X) = \mathrm{N}^1(X)$ is freely generated by the classes of the exceptional divisors E_1 and E_2 and of the pull-back H of a hyperplane in $\mathbb{P}^n$. An $\mathbb{R}$-divisor $D = \alpha H - \beta_1 E_1 - \beta_2 E_2$ is big if and only if

$$\alpha > \max\{\beta_1, \beta_2, 0\}.$$

We now describe a decomposition of the stable classes inside the big cone in five open cones, such that the stable base loci for the rational points are constant in each of these cones.

Consider first the region defined by $\alpha > \beta_1 > 0$ and $\beta_2 < 0$. All elements of $\mathrm{N}^1(X)_{\mathbb{Q}}$ in this region have stable base locus equal to E_2. Similarly, in the region $\alpha > \beta_2 > 0$ and $\beta_1 < 0$, all elements of $\mathrm{N}^1(X)_{\mathbb{Q}}$ have stable base locus equal to E_1. In the region $\alpha > 0$ and $\beta_1, \beta_2 < 0$ the stable base locus is equal to $E_1 \cup E_2$.

If $\beta_1, \beta_2 > 0$, then we have two regions. The first one, given by $\alpha > \beta_1 + \beta_2$ is the ample cone. In the other one, given by $0 < \alpha < \beta_1 + \beta_2$, the stable base locus of each element of $\mathrm{N}^1(X)_{\mathbb{Q}}$ is equal to the proper transform of the line joining Q_1 and Q_2. The union of these five regions is the set of stable big classes in $\mathrm{N}^1(X)_{\mathbb{R}}$.

Question 1.5.34. Is it always possible to write $\mathrm{Stab}(X)$ as a disjoint union of open convex cones such that the stable base locus is constant for the line bundles in each cone?

1.5.5 Cones defined by base loci conditions

We now use the asymptotic base loci to define some natural cones in $\mathrm{N}^1(X)_{\mathbb{R}}$. Suppose that X is a fixed projective scheme and Z is an irreducible closed subset of X. We define

$$\mathscr{C}_Z := \{\alpha \in \mathrm{N}^1(X)_{\mathbb{R}} \mid Z \not\subseteq \mathrm{B}_+(\alpha)\}$$

and $\overline{\mathscr{C}_Z}$ as the set of classes of those $D \in \mathrm{Pic}(X)_{\mathbb{R}}$ with the property that for every $A \in Q_{-D}$, we have $Z \not\subseteq \mathrm{SB}(D+A)$ (note that if $D_1 \equiv D_2$, then this condition is the same for D_1 and D_2). Note that when k is uncountable, we have

$$\overline{\mathscr{C}_Z} = \{\alpha \in \mathrm{N}^1(X)_{\mathbb{R}} \mid Z \not\subseteq \mathrm{B}_-(\alpha)\}.$$

Proposition 1.5.35. *For every irreducible closed subset Z of X, the following hold:*

i) $\mathscr{C}_Z$ is an open convex cone.
ii) $\overline{\mathscr{C}_Z}$ is a closed convex cone.
iii) $\mathscr{C}_Z$ is the interior of $\overline{\mathscr{C}_Z}$ and $\overline{\mathscr{C}_Z}$ is the closure of $\mathscr{C}_Z$.

Proof. By Proposition 1.5.17, there is an open neighborhood $\mathscr{U}$ of α such that $\mathrm{B}_+(\alpha') \subseteq \mathrm{B}_+(\alpha)$ for every $\alpha' \in \mathscr{U}$. In particular, if $Z \not\subseteq \mathrm{B}_+(\alpha)$, then $Z \not\subseteq \mathrm{B}_+(\alpha')$, hence $\mathscr{C}_Z$ is open. The fact that $\mathscr{C}_Z$ is a convex cone follows from the fact that Z is irreducible and $\mathrm{B}_+(\alpha_1+\alpha_2)\subseteq \mathrm{B}_+(\alpha_1)\cup \mathrm{B}_+(\alpha_2)$ (see Lemma 1.5.15) and $\mathrm{B}_+(\alpha)=\mathrm{B}_+(\lambda\alpha)$ for $\lambda>0$ (see Proposition 1.5.16). This completes the proof of i).

When k is uncountable, the fact that $\overline{\mathscr{C}_Z}$ is a convex cone follows as above, using the corresponding properties of the non-nef locus (see Proposition 1.5.22). We leave the general case as an exercise for the reader. Let us prove now that $\overline{\mathscr{C}_Z}$ is closed. Suppose that $\alpha_m\in\overline{\mathscr{C}_Z}$ for every $m\geq 1$ and $\lim_{m\to\infty}\alpha_m=\alpha$. We choose D and D_m for all m, whose classes are equal to α and α_m, respectively. Given $A\in Q_{-D}$, we choose $m\gg 0$ such that $A+(D-D_m)$ is ample, and then choose $A'\in Q_{-D_m}$ such that $(D+A)-(D_m+A')$ is ample. In this case $\mathrm{SB}(D+A)\subseteq \mathrm{SB}(D_m+A')$, hence Z is not contained in $\mathrm{SB}(D+A)$. We thus conclude that $\alpha\in\overline{\mathscr{C}_Z}$.

In order to prove iii), it is enough to show the first assertion (recall that every closed convex cone is the closure of its relative interior). Since we have already seen that $\mathscr{C}_Z$ is open, it is enough to show that if α lies in the interior of $\overline{\mathscr{C}_Z}$, then it lies in $\mathscr{C}_Z$. Suppose that this is not the case, hence $Z\subseteq \mathrm{B}_+(\alpha)$. Let us choose $D\in \mathrm{N}^1(X)_{\mathbb{R}}$ whose numerical class is α. By assumption, we can find A ample such that $D-A\in\mathscr{C}_Z$, hence for every $A'\in Q_{A-D}$, we have $Z\not\subseteq \mathrm{SB}(D-A+A')$. On the other hand, there is such A' with $A-A'$ is ample, in which case $Z\subseteq \mathrm{B}_+(D)\subseteq \mathrm{SB}(D-A+A')$, a contradiction. □

We use these cones to show that the set of elements of $\mathrm{N}^1(X)_{\mathbb{R}}$ that are not stable is small, in the following sense.

Proposition 1.5.36. *For every projective scheme X, the complement of* $\mathrm{Stab}(X)$ *in* $\mathrm{N}^1(X)_{\mathbb{R}}$ *has Lebesgue measure zero.*

Proof. It follows from definition that $D\in \mathrm{Pic}(X)_{\mathbb{R}}$ is unstable if and only if $\mathrm{B}_+(D)\not\subseteq \mathrm{SB}(D+A)$ for every $A\in Q_{-D}$. Note that given $A_1,A_2\in Q_{-D}$, there is $A\in Q_{-D}$ such that $\mathrm{SB}(D+A_1)\cup \mathrm{SB}(D+A_2)\subseteq \mathrm{SB}(D+A)$ (it is enough to take $A\in Q_{-D}$ such that A_1-A and A_2-A are both ample). Since $\mathrm{B}_+(D)$ has finitely many irreducible components, it follows that D is not stable if and only if there is a closed irreducible subset Z of X such that $Z\subseteq \mathrm{B}_+(D)$ but $Z\not\subseteq \mathrm{SB}(D+A)$ for every $A\in Q_{-D}$ (furthermore, Z can be taken to be an irreducible component of $\mathrm{B}_+(D)$). Therefore

$$\mathrm{N}^1(X)_{\mathbb{R}}\smallsetminus \mathrm{Stab}(X)=\bigcup_Z(\overline{\mathscr{C}_Z}\smallsetminus\mathscr{C}_Z), \tag{1.13}$$

where the union is over all irreducible closed subsets of X. Since the boundary of a closed convex cone has Lebesgue measure zero, in order to complete the proof of the proposition, it is enough to show that we may only take the union in (1.13) over countably many subsets $Z\subseteq X$.

We have seen that it is enough to consider the union in (1.13) over the irreducible components Z of $\mathrm{B}_+(\alpha)$, where $\alpha\in \mathrm{N}^1(X)_{\mathbb{R}}$. On the other hand, it follows from Proposition 1.5.17 that for every such α, there is $\alpha'\in \mathrm{N}^1(X)_{\mathbb{Q}}$ with $\mathrm{B}_+(\alpha)=\mathrm{B}_+(\alpha')$. Since there are countably many such α' and for each of these, the augmented base locus has only finitely many irreducible components, it follows that we only need to consider countably many Z. □

We now introduce some natural cones of the pseudoeffective cone of a projective variety X. For every j with $0 \leq j \leq n-1$, where $n = \dim(X)$, we put

$$\mathrm{Mov}^j(X) = \{\alpha \in \mathrm{N}^1(X)_{\mathbb{R}} \mid \mathrm{codim}(\mathrm{B}_+(\alpha)) \geq j+1\}$$

and let $\overline{\mathrm{Mov}}^j(X)$ denote the set of classes of D in $\mathrm{Pic}(X)_{\mathbb{R}}$ with $\mathrm{codim}(\mathrm{SB}(D+A)) \geq j+1$ for every $A \in Q_{-D}$. Note that if k is uncountable, then

$$\overline{\mathrm{Mov}}^j(X) = \{\alpha \in \mathrm{N}^1(X)_{\mathbb{R}} \mid \mathrm{codim}(\mathrm{B}_-(\alpha)) \geq j+1\}.$$

Proposition 1.5.37. *With the above notation, $\overline{\mathrm{Mov}}^j(X)$ is a closed convex cone and $\mathrm{Mov}^j(X)$ is its interior.*

Proof. Note that by definition, $\overline{\mathrm{Mov}}^j(X)$ is the intersection of all cones $\overline{\mathscr{C}_Z}$, where Z varies over the irreducible closed subsets of X with $\mathrm{codim}(Z) \leq j$. Therefore $\overline{\mathrm{Mov}}^j(X)$ is a closed convex cone by Proposition 1.5.35. The argument for the fact that $\mathrm{Mov}^j(X)$ is the interior of $\overline{\mathrm{Mov}}^j(X)$ is the same as in the proof of Proposition 1.5.35. $\square$

Remark 1.5.38. It is not hard to see that $\overline{\mathrm{Mov}}^j(X)$ is the closed convex cone generated by the classes of

$$\{\mathscr{L} \in \mathrm{Pic}(X) \mid \mathrm{codim}(\mathrm{Bs}(\mathscr{L})) \geq j+1\}.$$

Note that we have

$$\overline{\mathrm{Mov}}^{n-1}(X) \subseteq \overline{\mathrm{Mov}}^{n-2}(X) \subseteq \ldots \subseteq \overline{\mathrm{Mov}}^1(X) \subseteq \overline{\mathrm{Mov}}^0(X) = \mathrm{PEff}(X).$$

Note also that if $\alpha \in \mathrm{Pic}(X)_{\mathbb{Q}}$ is such that $\mathrm{SB}(\alpha)$ is zero-dimensional, then α is nef[6]. This easily implies the fact that $\overline{\mathrm{Mov}}^{n-1}(X) = \mathrm{Nef}(X)$. For $n \geq 2$, the cone $\mathrm{Mov}^1(X)$ is called the *movable cone* and plays an important role in understanding the rational maps from X to other projective varieties.

1.6 The relative setting

In this section we treat the relative versions of some of the notions that we previously encountered. We also discuss in some detail the notion of projective morphism, since the one we use is slightly different from the one in [Har77].

[6] It is a theorem of Zariski that in this case $\mathrm{SB}(\alpha)$ is empty, but we do not need this fact.

1.6.1 Relatively ample line bundles

All schemes are assumed to be Noetherian, but in the beginning we do not make any other assumptions.

Definition 1.6.1. If $f\colon X \to S$ is a proper morphism of schemes, a line bundle $\mathscr{L}$ on X is *f-ample* (or *ample over S*) if for every affine open subset $U \subseteq S$, the line bundle $\mathscr{L}|_{f^{-1}(U)}$ is ample in the sense of Definition 1.1.1. Recall also that $\mathscr{L}$ is *f-very ample* (or *very ample over S*) if there is a closed immersion $j\colon X \hookrightarrow \mathbb{P}^n_S$ of schemes over S, such that $j^*(\mathscr{O}_{\mathbb{P}^n_S}(1)) \simeq \mathscr{L}$. When $S = \operatorname{Spec} k$, we recover, of course, the definition of ample and very ample line bundles on a complete scheme over k.

Remark 1.6.2. It follows from definition that if $f\colon X \to S$ is a proper morphism, then $\mathscr{L} \in \operatorname{Pic}(X)$ is f-ample if and only if $\mathscr{L}^d$ is f-ample for some (any) $d > 0$. If $\mathscr{L}$ is f-very ample, then $\mathscr{L}^d$ is f-very ample for every positive integer d (this follows by composing a given embedding into $\mathbb{P}^n_S$ with a Veronese embedding $\mathbb{P}^n_S \hookrightarrow \mathbb{P}^N_S$, where $N = \binom{n+d}{d} - 1$).

We have the following equivalent descriptions of relative ampleness.

Proposition 1.6.3. *If $f\colon X \to S$ is a proper morphism of schemes and $\mathscr{L} \in \operatorname{Pic}(X)$, then the following are equivalent:*

i) $\mathscr{L}$ is f-ample.
ii) For every coherent sheaf $\mathscr{F}$ on X, we have $R^i f_*(\mathscr{F} \otimes \mathscr{L}^m) = 0$ for all $i \geq 1$ and all $m \gg 0$.
iii) For every coherent sheaf $\mathscr{F}$ on X, the natural map

$$f^* f_*(\mathscr{F} \otimes \mathscr{L}^m) \to \mathscr{F} \otimes \mathscr{L}^m$$

is surjective for $m \gg 0$.

Proof. Note that if $U \subseteq S$ is an open subset, then for every coherent sheaf $\mathscr{G}$ on $f^{-1}(U)$, there is a coherent sheaf $\mathscr{F}$ on X with $\mathscr{F}|_{f^{-1}(U)} \simeq \mathscr{G}$. Furthermore, if U is affine, then

$$R^i f_*(\mathscr{F} \otimes \mathscr{L}^m)|_U = 0 \text{ if and only if } H^i\left(f^{-1}(U), \mathscr{G} \otimes \mathscr{L}^m|_{f^{-1}(U)}\right) = 0.$$

Similarly, the map $f^* f_*(\mathscr{F} \otimes \mathscr{L}^m) \to \mathscr{F} \otimes \mathscr{L}^m$ is surjective on U if and only if $\mathscr{G} \otimes \mathscr{L}^m|_{f^{-1}(U)}$ is generated by global sections. Therefore the equivalences in the proposition follow from Definition 1.1.1 and Theorem 1.1.5. □

Remark 1.6.4. The description in Proposition 1.6.3 implies, in particular, that when S is affine, $\mathscr{L}$ is ample over S if and only if it is ample. More generally, given any proper morphism $f\colon X \to S$ and any affine open cover $S = \bigcup_i U_i$, a line bundle $\mathscr{L} \in \operatorname{Pic}(X)$ is f-ample if and only if $\mathscr{L}|_{f^{-1}(U_i)}$ is ample for every i . This implies that for a line bundle on X, the property of being f-ample is local on the base. We point out, however, that the *existence* of an f-ample line bundle on X is not local on the base.

Remark 1.6.5. If $f: X \to S$ is a proper morphism of schemes and $\mathscr{L} \in \mathrm{Pic}(X)$ is f-ample, then for every $\mathscr{M} \in \mathrm{Pic}(S)$, the line bundle $\mathscr{L} \otimes f^*(\mathscr{M})$ is f-ample. Indeed, it is enough to consider restrictions to subsets of the form $f^{-1}(U)$, where $U \subseteq X$ is an open subset such that $\mathscr{M}|_U$ is trivial.

Remark 1.6.6. Let $f: X \to S$ be a proper morphism and $\mathscr{L} \in \mathrm{Pic}(X)$ an f-ample line bundle. For every morphism $u: T \to S$, if $g: X \times_S T \to T$ and $v: X \times_S T \to X$ are the canonical projections, then $v^*(\mathscr{L})$ is g-ample. Indeed, when both S and T are affine, the assertion follows from Remark 1.1.4; the general case follows from this using the fact that the ampleness property is local on the base.

Remark 1.6.7. If $f: X \to S$ is proper and $\mathscr{L}, \mathscr{M} \in \mathrm{Pic}(X)$, with $\mathscr{L}$ being f-ample, then $\mathscr{M} \otimes \mathscr{L}^m$ is f-ample for all $m \gg 0$. This follows from the corresponding assertion when S is affine, due to the fact that it is enough to check ampleness over a finite affine open cover of S.

Definition 1.6.8. A morphism $f: X \to S$ is a *projective morphism* if there is a quasicoherent graded $\mathscr{O}_S$-algebra $\mathscr{A} = \bigoplus_{i \geq 0} \mathscr{A}_i$, with

i) $\mathscr{A}_0$ and $\mathscr{A}_1$ coherent $\mathscr{O}_S$-modules.
ii) $\mathscr{A}$ locally generated by $\mathscr{A}_1$ as an $\mathscr{O}_S$-algebra,

such that $X \simeq \mathscr{P}roj(\mathscr{A})$, as schemes over S. Of course, the structure morphism of X to $\mathrm{Spec}(k)$ is projective if and only if X is a projective scheme. In general, our definition is slightly weaker than the one in [Har77, Chap. II.7]. However, the two definitions agree, for example, if S has an ample line bundle, see Proposition 1.6.14 below.

Example 1.6.9. For every scheme X, if Z is a closed subscheme defined by the ideal $\mathscr{I}_Z$, then the blow-up Y of X along Z is a projective scheme over X. Indeed, we have $Y = \mathscr{P}roj\left(\bigoplus_{i \geq 0} \mathscr{I}_Z^i\right)$.

Proposition 1.6.10. *A morphism of schemes $f: X \to S$ is projective if and only if f is proper and there is $\mathscr{L} \in \mathrm{Pic}(X)$ which is f-ample.*

Proof. It is clear that if f is a projective morphism and $\mathscr{A}$ is as in the definition, then f is proper and the line bundle corresponding to $\mathscr{O}(1)$ on $\mathscr{P}roj(\mathscr{A})$ is f-ample. Conversely, suppose that f is proper and $\mathscr{L} \in \mathrm{Pic}(X)$ is f-ample. Let $X = \bigcup_i U_i$ be a finite affine open cover of S. Let us choose d such that for every i, the line bundle $\mathscr{L}^d|_{f^{-1}(U_i)}$ is very ample over U_i and the natural map

$$\mathrm{Sym}^m_{\mathscr{O}(U_i)} H^0(f^{-1}(U_i), \mathscr{L}^d) \to H^0(f^{-1}(U_i), \mathscr{L}^{dm})$$

is surjective for every $m \geq 1$. In this case the $\mathscr{O}_S$-algebra $\mathscr{A} := \bigoplus_{i \geq 0} f_*(\mathscr{L}^i)$ satisfies the conditions in Definition 1.6.8 and $X \simeq \mathscr{P}roj(\mathscr{A})$ over S. □

Example 1.6.11. If $f: X \to S$ is a finite morphism, then it follows from definition that every line bundle on X is f-ample (note that every line bundle on an affine scheme is ample). In particular, f is projective by Proposition 1.6.10.

Remark 1.6.12. Note that Remark 1.6.6 and the description of projective morphisms in Proposition 1.6.10 imply that projective morphisms are closed under base-change.

Remark 1.6.13. If $f\colon X\to Y$ and $g\colon Y\to Z$ are proper morphisms and $\mathscr{L}\in\mathrm{Pic}(X)$ is $(g\circ f)$-ample, then $\mathscr{L}$ is also f-ample. Indeed, if $U\subseteq Z$ is an affine open subset, then $\mathscr{L}|_{f^{-1}(g^{-1}(U))}$ is ample. In particular, it is ample over $g^{-1}(U)$. We deduce that if $g\circ f$ is projective, then also f is projective.

It is clear that if $f\colon X\to S$ is a proper morphism, then any f-very ample line bundle on X is f-ample.

Proposition 1.6.14. *If $f\colon X\to S$ is a proper morphism, $\mathscr{L}\in\mathrm{Pic}(X)$ is f-ample, and S has an ample line bundle $\mathscr{M}$, then there is $d>0$ such that $\mathscr{L}^d\otimes f^*(\mathscr{M})^m$ is f-very ample for every $m\gg 0$.*

Proof. We choose d and $\mathscr{A}$ as in the proof of Proposition 1.6.10. Since $\mathscr{M}$ is ample, it follows that $\mathscr{A}_1\otimes\mathscr{M}^m$ is globally generated for all $m\gg 0$. In this case, the $\mathscr{O}_S$-algebra $\mathscr{A}^{(m)}:=\bigoplus_{i\geq 0}(\mathscr{A}_i\otimes\mathscr{M}^{im})$ is a graded quotient of some $\mathscr{O}_S[x_0,\ldots,x_N]$, which induces a closed immersion

$$j\colon X\simeq \mathscr{P}roj(\mathscr{A}^{(m)})\hookrightarrow \mathscr{P}roj(\mathscr{O}_S[x_0,\ldots,x_N])=\mathbb{P}^N_S$$

such that $j^*(\mathscr{O}_{\mathbb{P}^N_S}(1))\simeq\mathscr{L}^d\otimes f^*(\mathscr{M})^m$. $\square$

Proposition 1.6.15. *Let $X\xrightarrow{f}Y\xrightarrow{g}Z$ be two proper morphisms of schemes and consider $\mathscr{L}\in\mathrm{Pic}(X)$ and $\mathscr{M}\in\mathrm{Pic}(Y)$.*

i) If $\mathscr{L}$ is f-very ample and $\mathscr{M}$ is g-very ample, then $\mathscr{L}\otimes f^*(\mathscr{M})$ is $(g\circ f)$-very ample.
ii) If $\mathscr{L}$ is f-ample and $\mathscr{M}$ is g-ample, then $\mathscr{L}\otimes f^*(\mathscr{M})^m$ is $(g\circ f)$-ample for all $m\gg 0$.

Proof. If $\mathscr{L}$ is f-very ample, then there is a closed immersion $i\colon\ \hookrightarrow\mathbb{P}^{n_1}_Y$ such that $i^*(\mathscr{O}_{\mathbb{P}^{n_1}_Y}(1))\simeq\mathscr{L}$. Similarly, if $\mathscr{M}$ is g-very ample, then we have a closed immersion $j\colon Y\hookrightarrow\mathbb{P}^{n_2}_Z$ such that $j^*(\mathscr{O}_{\mathbb{P}^{n_2}_Z}(1))\simeq\mathscr{M}$. We then obtain an embedding $\phi\colon X\hookrightarrow\mathbb{P}^N\times Z$ as the composition

$$X\overset{i}{\hookrightarrow}\mathbb{P}^{n_1}\times Y\overset{\mathrm{id}\times j}{\hookrightarrow}\mathbb{P}^{n_1}\times\mathbb{P}^{n_2}\times Z\overset{\psi\times\mathrm{id}}{\hookrightarrow}\mathbb{P}^N\times Z,$$

where $N=n_1n_2+n_1+n_2$ and ψ is the Segre embedding. Since $\phi^*(\mathscr{O}_{\mathbb{P}^N_Z}(1))\simeq\mathscr{L}\otimes f^*(\mathscr{M})$, it follows that $\mathscr{L}\otimes f^*(\mathscr{M})$ is $(g\circ f)$-very ample.

If $\mathscr{L}$ is f-ample and $\mathscr{M}$ is g-ample, then in order to check the assertion in ii) it is enough to do it over each element of a finite affine open cover of Z. Therefore we may assume that Z is affine, in which case $\mathscr{M}$ is ample. It follows from Proposition 1.6.14 that there are d and m_1 such that $\mathscr{L}^d\otimes f^*(\mathscr{M})^m$ is f-very ample for all $m\geq m_1$. Applying Proposition 1.6.14 for g, we see that there is m_2 such that

$\mathscr{M}^m$ is g-very ample for every $m \geq m_2$. We deduce from i) that $\mathscr{L}^d \otimes f^*(\mathscr{M})^m$ is $(g \circ f)$-very ample for every $m \geq m_1 + m_2$. If m_3 is such that $dm_3 \geq m_1 + m_2$, we conclude that $\mathscr{L} \otimes f^*(\mathscr{M})^m$ is f-ample for every $m \geq m_3$. □

Example 1.6.16. Suppose that X is a projective scheme over a field and $Z \hookrightarrow X$ is a closed subscheme defined by the ideal $\mathscr{I}_Z$. Let $f\colon Y \to X$ be the blow-up of X along Z, with exceptional divisor E. If $\mathscr{M} \in \operatorname{Pic}(X)$ is such that $\mathscr{I}_Z \otimes \mathscr{M}$ is globally generated, then the argument in the proof of Proposition 1.6.14 implies that $f^*(\mathscr{M}) \otimes \mathscr{O}_Y(-E)$ is f-very ample. Therefore $f^*(\mathscr{M})^m \otimes \mathscr{O}_Y(-mE)$ is f-very ample for every $m > 0$, and Proposition 1.6.15 implies that $f^*(\mathscr{M}^m \otimes \mathscr{M}') \otimes \mathscr{O}_Y(-mE)$ is a very ample line bundle on Y for every very ample line bundle $\mathscr{M}'$ on X.

Using the description of projective morphisms in Proposition 1.6.10 and part ii) in Proposition 1.6.15, we obtain the following.

Corollary 1.6.17. *A composition of two projective morphisms is again projective.*

Remark 1.6.18. If $f\colon X \to S$ is a projective morphism and S has an ample line bundle, then there is an effective Cartier divisor A on X such that $\mathscr{O}_X(A)$ is f-ample. Indeed, since S has an ample line bundle, it follows from Proposition 1.6.14 that there is a closed immersion $j\colon X \hookrightarrow \mathbb{P}^N \times S$ of schemes over S. If H is a general hyperplane in $\mathbb{P}^N$, then we may take $A = (H \times S) \cap X$.

The following proposition provides a very useful criterion for relative ampleness. For a morphism $f\colon X \to S$ and for a (not-necessarily-closed) point $s \in S$, we denote by X_s the fiber of X over s. If $\mathscr{L}$ is a line bundle on X, we denote by $\mathscr{L}|_{X_s}$ the pull-back of $\mathscr{L}$ to X_s.

Proposition 1.6.19. *If $f\colon X \to S$ is a proper morphism and $\mathscr{L} \in \operatorname{Pic}(X)$ is such that $\mathscr{L}|_{X_s}$ is ample for some $s \in S$, then there is an affine open neighborhood U of s such that $\mathscr{L}|_{f^{-1}(U)}$ is ample.*

Proof. The argument we give follows [KM98, Prop. 1.41]. Without any loss of generality, we may assume that $S = \operatorname{Spec}(A)$ and let $\mathfrak{p} \subseteq A$ be the prime ideal corresponding to s.

We first show that given any coherent sheaf $\mathscr{F}$ on X, we have $H^i(X, \mathscr{F} \otimes \mathscr{L}^m)_{\mathfrak{p}} = 0$ for all $m \gg 0$. This clearly holds if $i > \dim(X \times_{\operatorname{Spec} A} \operatorname{Spec} A_p)$, hence it is enough to show that if $i > 0$ and the property holds for $(i+1)$ and all coherent sheaves $\mathscr{F}$, then it also holds for i and all coherent sheaves $\mathscr{F}$. If $u_1, \ldots, u_N \in A$ generate $\mathfrak{p}$, then we have an exact sequence on X

$$\mathscr{F}^{\oplus N} \xrightarrow{\phi} \mathscr{F} \to \mathscr{F} \otimes_A A/\mathfrak{p} \to 0, \tag{1.14}$$

where $\phi = (u_1, \ldots, u_N)$. By assumption, for $m \gg 0$ we have

$$H^{i+1}(X, \ker(\phi) \otimes \mathscr{L}^m)_{\mathfrak{p}} = 0 \quad \text{and} \quad H^{i+1}(X, \operatorname{Im}(\phi) \otimes \mathscr{L}^m)_{\mathfrak{p}} = 0,$$

hence by tensoring (1.14) with $\mathscr{L}^m$, taking the long exact sequence in cohomology, and localizing at $\mathfrak{p}$, we obtain an exact sequence

$$H^i(X,\mathscr{F}\otimes\mathscr{L}^m)_{\mathfrak{p}}^{\oplus N}\to H^i(X,\mathscr{F}\otimes\mathscr{L}^m)_{\mathfrak{p}}\to H^i(X_s,\mathscr{F}\otimes\mathscr{L}^m|_{X_s})\to 0.$$

Since $\mathscr{L}|_{X_s}$ is ample and $i>0$, it follows that $H^i(X_s,\mathscr{F}\otimes\mathscr{L}^m|_{X_s})=0$ for $m\gg 0$. We conclude that if $m\gg 0$, then $H^i(X,\mathscr{F}\otimes\mathscr{L}^m)\otimes_A k(\mathfrak{p})=0$ and Nakayama's lemma implies $H^i(X,\mathscr{F}\otimes\mathscr{L}^m)_{\mathfrak{p}}=0$.

We apply the above property with $\mathscr{F}=\mathfrak{p}\cdot\mathscr{O}_X$ to deduce that for $m\gg 0$, the map

$$H^0(X,\mathscr{L}^m)\otimes_A k(\mathfrak{p})\to H^0(X_s,\mathscr{L}^m|_{X_s})$$

is surjective. Since $\mathscr{L}^m|_{X_s}$ is globally generated for $m\gg 0$, it follows that for some positive integer m, we have a morphism $\mathscr{O}_X^{\oplus N}\xrightarrow{\psi}\mathscr{L}^m$ that is surjective over $\operatorname{Spec} k(s)$. Therefore $\operatorname{coker}(\psi)\otimes k(\mathfrak{p})=0$, hence Nakayama's lemma implies that after possibly localizing at an element in $A\smallsetminus\mathfrak{p}$, we may assume that ψ is surjective. We thus have an induced morphism $j\colon X\to\mathbb{P}_A^{N-1}$ such that $j^*\mathscr{O}(1)\simeq\mathscr{L}^m$. Since $\mathscr{L}|_{X_s}$ is ample, the corresponding morphism over $\operatorname{Spec} k(\mathfrak{p})$ is finite. It follows that if $W\subseteq j(X)$ is the open subset over which j has zero-dimensional fibers, the image in S of $j(X)\smallsetminus W$ does not contain s. Therefore after possibly replacing S with an affine open neighborhood of s, we may assume that j is finite, hence $\mathscr{L}$ is ample by Proposition 1.1.9. □

Corollary 1.6.20. *If $f\colon X\to S$ is a proper morphism, then $\mathscr{L}\in\operatorname{Pic}(X)$ is f-ample if and only if $\mathscr{L}|_{X_s}$ is ample for every $s\in S$. Moreover, if the schemes are of finite type over a field, then it is enough to only consider the closed points $s\in S$.*

Proof. It follows from Remark 1.6.6 that if $\mathscr{L}$ is f-ample, then $\mathscr{L}|_{X_s}$ is ample on X_s for every $s\in S$. The converse follows from Proposition 1.6.19. □

From now on, we assume that all our schemes are of finite type over a field k. By combining Corollary 1.6.20 with Theorem 1.3.1, we obtain the following.

Corollary 1.6.21. *If $f\colon X\to S$ is a proper morphism, then $\mathscr{L}\in\operatorname{Pic}(X)$ is f-ample if and only if for every closed subvariety V of X with $r=\dim(V)>0$ and such that $f(V)$ is a point*[7]*, we have $(\mathscr{L}^r\cdot V)>0$.*

1.6.2 The relative ample and nef cones

We now turn to the definition of the relative Néron-Severi group. We fix a proper morphism $f\colon X\to S$ of schemes of finite type over k. We say that $\mathscr{L}\in\operatorname{Pic}(X)$ is f-*numerically trivial* if $(\mathscr{L}\cdot C)=0$ for every curve C on X such that $f(C)$ is a point, or equivalently, if $\mathscr{L}|_{X_s}$ is numerically trivial for every (closed) point $s\in S$. For two line bundles $\mathscr{L}_1,\mathscr{L}_2\in\operatorname{Pic}(X)$, we write $\mathscr{L}_1\equiv_f\mathscr{L}_2$ if $\mathscr{L}_1\otimes\mathscr{L}_2^{-1}$ is f-numerically trivial.

[7] This implies that V is a complete variety over k. Therefore the intersection number $(\mathscr{L}^r\cdot V)$ is defined as $(\mathscr{L}|_V^r)$, even though X might not be complete over k.

The quotient of $\mathrm{Pic}(X)$ by this equivalence relation is denoted by $\mathrm{N}^1(X/S)$. A relative version of Theorem 1.3.6 says that $\mathrm{N}^1(X/S)$ is a finitely generated (torsion-free) abelian group. We get the corresponding vector spaces $\mathrm{N}^1(X/S)_{\mathbb{Q}}$ and $\mathrm{N}^1(X/S)_{\mathbb{R}}$, which can also be obtained by taking the quotient of $\mathrm{Pic}(X)_{\mathbb{Q}}$ and $\mathrm{Pic}(X)_{\mathbb{R}}$, respectively, by the equivalence relation defined similarly.

We also have the dual picture: the group $Z_1(X/S)$ is the free abelian group on the set of all curves on X that are mapped by f to a point. We say that $\alpha \in Z_1(X/S)$ is f-numerically trivial if $(\mathscr{L} \cdot \alpha) = 0$ for every $\mathscr{L} \in \mathrm{Pic}(X)$. For $\alpha, \beta \in Z_1(X/S)$, we write $\alpha \equiv_f \beta$ if $\alpha - \beta$ is f-numerically trivial. The quotient of $Z_1(X/S)$ by this equivalence relation is denoted by $\mathrm{N}_1(X/S)$ and by tensoring with $\mathbb{Q}$ and $\mathbb{R}$ we obtain the vector spaces $\mathrm{N}_1(X/S)_{\mathbb{Q}}$ and $\mathrm{N}_1(X/S)_{\mathbb{R}}$. It follows from definitions that the intersection pairing induces a non-degenerate pairing

$$\mathrm{N}^1(X/S)_{\mathbb{R}} \times \mathrm{N}_1(X/S)_{\mathbb{R}} \to \mathbb{R}.$$

One defines the *relative Mori cone* $\overline{\mathrm{NE}}(X/S)$ to be the closed convex cone in $\mathrm{N}_1(X/S)_{\mathbb{R}}$ generated by the classes of curves $C \subseteq X$ that map to points. The dual of $\overline{\mathrm{NE}}(X/S)$ is the *f-nef cone* $\mathrm{Nef}(X/S)$, consisting of *f-nef* classes. Explicitly, $\alpha \in \mathrm{Pic}(X)_{\mathbb{R}}$ is f-nef if $(\alpha \cdot C) \geq 0$ for every curve $C \subseteq X$ such that $f(C)$ is a point, or equivalently, if α is nef on every fiber X_s, where $s \in S$ is a closed point.

If f is projective, then the *f-ample cone* $\mathrm{Amp}(X/S)$ of X/S is the convex cone generated by f-ample line bundle classes (note that by Corollary 1.6.21, if we have $\mathscr{L}_1 \equiv_f \mathscr{L}_2$ in $\mathrm{Pic}(X)$, then $\mathscr{L}_1$ is f-ample if and only if $\mathscr{L}_2$ is f-ample). One defines in the obvious way what it means for an element in $\mathrm{Pic}(X)_{\mathbb{R}}$ to be f-nef or f-ample, in terms of the corresponding class in $\mathrm{N}^1(X/S)_{\mathbb{R}}$. Note that by definition, $\mathscr{L} \in \mathrm{Pic}(X)$ is f-nef if and only if $\mathscr{L}|_{X_s}$ is nef for every $s \in S$.

Remark 1.6.22. If $g\colon Y \to X$ is a morphism of proper schemes over S, then it is easy to see that the pull-back of line bundles induces a linear map $g^*\colon \mathrm{N}^1(X/S) \to \mathrm{N}^1(Y/S)$ which takes $\mathrm{Nef}(X/S)$ to $\mathrm{Nef}(Y/S)$.

Example 1.6.23. If $f\colon X \to S$ is a projective morphism as above, then $\mathscr{L} \in \mathrm{Pic}(X)$ is *f-base-point free* if the canonical morphism $f^*f_*(\mathscr{L}) \to \mathscr{L}$ is surjective; equivalently, for every affine open subset $U \subseteq X$, the restriction $\mathscr{L}|_{f^{-1}(U)}$ is globally generated. We say that $\mathscr{L}$ is *f-semiample* if $\mathscr{L}^m$ is f-base-point free for some positive integer m. It is clear that if $\mathscr{L}^m$ is f-base-point free for some $m \geq 1$, then $\mathscr{L}^m|_{X_s}$ is globally generated for every $s \in S$. In particular, if $\mathscr{L}$ is f-semiample, then $\mathscr{L}$ is f-nef.

The same argument used in the absolute case gives the fact that if f is projective, then $\mathrm{N}^1(X/S)_{\mathbb{R}}$ has a basis consisting of classes of f-ample line bundles and $\mathrm{Amp}(X/S)$ is open in $\mathrm{N}^1(X/S)_{\mathbb{R}}$.

Lemma 1.6.24. *Let $f\colon X \to S$ be a projective morphism. If $\alpha \in \mathrm{N}^1(X/S)_{\mathbb{R}}$, then α is f-ample if and only if $\alpha|_{X_s}$ is ample on X_s for every closed point $s \in S$.*

Proof. The "only if" part is clear. Suppose now that $\alpha|_{X_s}$ is ample for every closed point $s \in S$. After choosing a basis of $\mathrm{N}^1(X/S)_{\mathbb{R}}$ consisting of classes of f-ample line bundles, we can find a sequence of elements $\alpha_m \in \mathrm{N}^1(X/S)_{\mathbb{Q}}$ such that $\alpha - \alpha_m$ is f-ample for every m and $\lim_{m\to\infty} \alpha_m = \alpha$. For every closed point $s \in S$, there is $m(s)$ such that $\alpha_{m(s)}|_{X_s}$ is ample. By applying Proposition 1.6.19 to a suitable multiple of $\alpha_{m(s)}$, we deduce that there is an affine open neighborhood $U = U_s$ of s such that $\alpha_{m(s)}|_{f^{-1}(U_s)}$ is ample. If $s_1, \ldots, s_r$ are such that $S = U_{s_1} \cup \ldots \cup U_{s_r}$ and if $m \gg 0$ is such that $\alpha_m - \alpha_{m(s_i)}$ is f-ample for $1 \leq i \leq r$, then we see that α_m is f-ample and therefore α is f-ample. $\square$

It is now clear that if $\alpha \in \mathrm{Amp}(X/S)$ and $\beta \in \mathrm{Nef}(X/S)$, then $\alpha + \beta \in \mathrm{Amp}(X/S)$: indeed, this follows from the fact that the same property holds in each $\mathrm{N}^1(X_s)_{\mathbb{R}}$. One then deduces as in the absolute case that $\mathrm{Nef}(X/S)$ is the closure of $\mathrm{Amp}(X/S)$ and $\mathrm{Amp}(X/S)$ is the interior of $\mathrm{Nef}(X/S)$.

Remark 1.6.25. The argument in the proof of Lemma 1.6.24 shows that if f is projective and $\alpha \in \mathrm{N}^1(X/S)_{\mathbb{R}}$ is such that $\alpha|_{X_s}$ is ample for some $s \in S$, then there is an open neighborhood U of s such that $\alpha|_{f^{-1}(U)}$ is ample over U. In particular, we see that for every $\alpha \in \mathrm{N}^1(X/S)_{\mathbb{R}}$, the set

$$\{s \in S \mid \alpha|_{X_s} \text{ is ample}\}$$

is open in S.

Remark 1.6.26. Let $f\colon X \to S$ be a projective morphism. If $\alpha \in \mathrm{N}^1(X/S)_{\mathbb{R}}$ and $\beta \in \mathrm{N}^1(X/S)_{\mathbb{R}}$ is any f-ample class, then $\alpha|_{X_s}$ is nef if and only if $(\alpha + \frac{1}{n}\beta)|_{X_s}$ is ample for every $n \geq 1$. It follows from Remark 1.6.25 that the set

$$\{s \in S \mid \alpha|_{X_s} \text{ is nef}\}$$

is the complement of a countable union of closed subsets in X. We refer to [Les] for an example over $\mathbb{C}$ in which the complement of the above set is indeed not Zariski closed.

Remark 1.6.27. The analogue of Proposition 1.6.15 fails if we replace "f-ample" by "f-nef". Suppose, for example, that k is algebraically closed, $X = C \times C$, where C is an elliptic curve, and $f\colon X \to C$ is the projection onto the first component. If $D_1 = C \times \{p\}$ for some $p \in C$ and D_2 is the diagonal, then $D_1 \equiv_f D_2$. In particular, $D = D_1 - D_2$ is f-nef. On the other hand, for every divisor M on C, the sum $D + f^*(M)$ is not nef: indeed, $((D + f^*(M))^2) = (D^2) = -2$.

Remark 1.6.28. If $f\colon X \to S$ is a morphism between two complete schemes over k, then we have a surjective linear map

$$\mathrm{N}^1(X)_{\mathbb{R}}/f^*(\mathrm{N}^1(S)_{\mathbb{R}}) \to \mathrm{N}^1(X/S)_{\mathbb{R}}. \tag{1.15}$$

In general, this is not an isomorphism. Suppose, for example, that f is the morphism in Remark 1.6.27. In this case $\mathrm{N}^1(X)_{\mathbb{R}}/f^*(\mathrm{N}^1(S)_{\mathbb{R}})$ has dimension ≥ 2, while $\mathrm{N}^1(X/S)_{\mathbb{R}}$ has dimension 1, being generated by the class of D_1. However, we will see in Example 1.6.37 below, as a consequence of the negativity lemma, that (1.15) is an isomorphism if f is birational morphism, X and S are normal, and S is $\mathbb{Q}$-factorial.

1.6.3 Relatively big line bundles

We now consider the relative version of big line bundles.

Definition 1.6.29. Let $f\colon X \to S$ be a surjective, proper morphism of varieties over k and let $\mathscr{L} \in \mathrm{Pic}(X)$. If K is the function field of S, $X_K = X \times_{\mathrm{Spec}\,k} \mathrm{Spec}\,K$, and $\mathscr{L}_K$ is the pull-back of $\mathscr{L}$ to X_K, then $\mathscr{L}$ is *f-big* if $\mathscr{L}_K$ is big on X_K.

Proposition 1.6.30. *Let $f\colon X \to S$ be a surjective, proper morphism of varieties.*

i) If $\mathscr{L} \in \mathrm{Pic}(X)$ and m is a positive integer, then $\mathscr{L}$ is f-big if and only if $\mathscr{L}^m$ is f-big.
ii) If $g\colon Y \to X$ is a proper, surjective, generically finite morphism, then $\mathscr{L}$ is f-big if and only if $g^*(\mathscr{L})$ is $(f \circ g)$-big.

Proof. Both assertions follow from definition, using the corresponding properties of line bundles on X_K. □

If $f\colon X \to S$ is a proper, surjective morphism of varieties, then the role of line bundles with nonzero sections is played by those $\mathscr{L} \in \mathrm{Pic}(X)$ such that $f_*(\mathscr{L}) \neq 0$. Note that if this is the case, then $\mathrm{rank}(f_*(\mathscr{L})) > 0$.

Remark 1.6.31. If $f\colon X \to S$ is as above and there is an ample line bundle on S, then a Cartier divisor D on X has the property that $f_*(\mathscr{O}_X(D)) \neq 0$ if and only if there is a Cartier divisor B on S and an effective Cartier divisor D' on X such that $D \sim f^*(B) + D'$. Moreover, in this case we may assume that $-B$ is ample. Indeed, it is clear that if we have such B and D', then

$$f_*(\mathscr{O}_X(D)) \simeq \mathscr{O}_S(B) \otimes f_*(\mathscr{O}_X(D')) \supseteq \mathscr{O}_S(B) \otimes f_*(\mathscr{O}_X) \neq 0.$$

Conversely, if $f_*(\mathscr{O}_X(D)) \neq 0$ and A is an ample Cartier divisor on X, then

$$H^0(X, \mathscr{O}_X(D + f^*(mA))) \simeq H^0(S, f_*(\mathscr{O}_X(D)) \otimes \mathscr{O}_S(mA)) \neq 0$$

for $m \gg 0$, hence there is an effective Cartier divisor D' on X such that $D \sim D' - mf^*(A)$.

Let $f\colon X \to S$ be a proper, surjective morphism of varieties and let $\mathscr{L} \in \mathrm{Pic}(X)$. If $f_*(\mathscr{L}^m) \neq 0$ for some $m \geq 1$, then the canonical morphism $f^*(f_*(\mathscr{L}^m)) \to \mathscr{L}^m$ induces a rational map

$$\phi_{\mathscr{L}^m,S}\colon X \dashrightarrow \mathscr{P}roj(\mathrm{Sym}_{\mathscr{O}_S}(f_*(\mathscr{L}^m)))$$

(since we are only interested in this as a rational map, it is enough to consider this over an open subset U of X such that $f_*(\mathscr{L}^m)|_U$ is locally free). Note that over K, this induces the rational map defined by $\mathscr{L}_K^m$.

Proposition 1.6.32. *Let $f\colon X \to S$ be a proper, surjective morphism of varieties, of relative dimension r (that is, $\dim(X) - \dim(S) = r$). For any $\mathscr{L} \in \mathrm{Pic}(X)$, the following are equivalent:*

i) $\mathscr{L}$ is f-big.
ii) There is $m > 0$ (equivalently, for all m divisible enough), the rational map $\phi_{\mathscr{L}^m,S}$ is defined and its image dominates S, with relative dimension r.
iii) There is $C > 0$ such that

$$\mathrm{rank}(f_*(\mathscr{L}^m)) > C \cdot m^r \quad \textit{for all} \quad m \gg 0.$$

Proof. The equivalence between i) and ii) follows from definition. On the other hand, if U is an affine open subset of S, then the function field K of S is the fraction field of $\mathscr{O}(U)$ and

$$h^0(X_K, \mathscr{L}_K^m) = \dim_K(H^0(f^{-1}(U), \mathscr{L}^m) \otimes_{\mathscr{O}(U)} K) = \mathrm{rank}(f_*(\mathscr{L}^m)).$$

Therefore the equivalence between i) and iii) follows from Proposition 1.4.8. □

In the projective case, we have the following extension of Theorem 1.4.13.

Proposition 1.6.33. *Let $f\colon X \to S$ be a surjective, projective morphism of varieties over k, of relative dimension r. For every line bundle $\mathscr{L}$ on X, the following are equivalent:*

i) $\mathscr{L}$ is f-big.
ii) For all $m \in \mathbb{Z}_{>0}$ divisible enough, the rational map $\phi_{\mathscr{L}^m,S}$ is defined and it is birational onto its image.
iii) There is $C > 0$ such that

$$\mathrm{rank}(f_*(\mathscr{L}^m)) > C \cdot m^r \quad \textit{for all} \quad m \gg 0.$$

iv) There are Cartier divisors A and E on X, with A being f-ample and $f_*\mathscr{O}_X(E) \neq 0$, such that $\mathscr{L}^d \simeq \mathscr{O}_X(A+E)$ for some $d \geq 1$.

Proof. If K is the function field of S, then the equivalence between i), ii), and iii) follows by applying Theorem 1.4.13 to $\mathscr{L}_K \in \mathrm{Pic}(X_K)$. If A and E are as in iv), then after replacing S by an affine open subset, we may assume that E is effective. Since the pull-backs of A and E to X_K are ample and effective, respectively, then we conclude that $\mathscr{L}_K$ is big by Theorem 1.4.13. For the implication iii)⇒iv), we choose an f-ample Cartier divisor A on X. After possibly replacing A by a multiple, we may assume that its pull-back to X_K can be written as a difference of two effective Cartier divisors. In this case, it follows from Lemma 1.4.14, applied to $\mathscr{L}_K$, that there is d such that $f_*(\mathscr{L}^d \otimes \mathscr{O}_X(-A)) \neq 0$, giving the assertion in iv). This completes the proof of the proposition. □

It follows from Proposition 1.6.33 that if $f\colon X\to S$ is a surjective, projective morphism of varieties, and $\mathscr{L}_1,\mathscr{L}_2\in \mathrm{Pic}(X)$ are such that $\mathscr{L}_1\equiv_f \mathscr{L}_2$, then $\mathscr{L}_1$ is f-big if and only if $\mathscr{L}_2$ is f-big. Indeed, it is enough to use the equivalent condition iv), since f-ampleness is invariant with respect to adding an f-numerically trivial line bundle.

We now introduce the relative versions of the big and pseudo-effective cones. Suppose that $f\colon X\to S$ is a projective, surjective morphism of varieties. The *f-big cone* $\mathrm{Big}(X/S)$ is the convex cone generated in $\mathrm{N}^1(X/S)_{\mathbb{R}}$ by the classes of f-big line bundles on X. The *f-pseudo-effective cone* $\mathrm{PEff}(X/S)$ is the closed convex cone generated in $\mathrm{N}^1(X/S)_{\mathbb{R}}$ by the classes of Cartier divisors D on X such that $f_*(\mathscr{O}_X(D))\neq 0$. Using the description of f-big line bundles in Proposition 1.6.33iv), we see as in the proof of Proposition 1.4.28 that a divisor $D\in \mathrm{CDiv}(X)_{\mathbb{R}}$ is f-big if and only if we can write $D=A+E$, where $A,E\in \mathrm{CDiv}(X)_{\mathbb{R}}$ are such that A is f-ample and $E=\sum_{i=1}^r\lambda_iB_i$, with $\lambda_i\geq 0$ and B_i Cartier divisors, with $f_*(\mathscr{O}_X(B_i))\neq 0$ (moreover, one can always arrange to have $\lambda_i\in\mathbb{Q}$ for all i). Using this, one sees that $\mathrm{Big}(X/S)$ is the interior of $\mathrm{PEff}(X/S)$ and $\mathrm{PEff}(X/S)$ is the closure of $\mathrm{Big}(X/S)$.

1.6.4 The negativity lemma

We end this section with two applications of the Hodge index theorem that are very useful in birational geometry. We work over an algebraically closed ground field, that in the beginning is assumed of arbitrary characteristic. If $f\colon X\to Y$ is a proper, surjective morphism of varieties and E is a prime divisor on X, we say that E is f-exceptional if $\mathrm{codim}_Y f(E)\geq 2$ (note that when f is birational and Y is normal, this agrees with the definition in Appendix B). A divisor D on X is f-exceptional if all prime divisors in D are f-exceptional.

Theorem 1.6.34 (Negativity lemma). *Let $f\colon X\to Y$ be a surjective, generically finite, projective morphism of varieties, with X normal, and let $D\in \mathrm{CDiv}(X)_{\mathbb{R}}$ be such that $-D$ is f-nef.*

i) If every prime divisor that appears in D with negative coefficient is f-exceptional, then in fact D is effective.
ii) If f has connected fibers and D is effective, then for every $y\in f(\mathrm{Supp}(D))$, we have $f^{-1}(y)\subseteq \mathrm{Supp}(D)$.

The key ingredient for the proof of Theorem 1.6.34 is the following easy application of the Hodge index theorem.

Proposition 1.6.35. *If $f\colon X\to Y$ is a projective, surjective morphism of surfaces, with X smooth, and if $E_1,\ldots,E_m$ are prime divisors on X that are contracted by f, then the intersection matrix $(E_i\cdot E_j)_{1\leq i,j\leq m}$ is negative definite.*

Proof. We need to show that if $D=\sum_{i=1}^m a_iE_i$, and some a_i is nonzero, then $(D^2)<0$. Suppose first that f is a morphism of projective varieties. If H is an ample Cartier divisor on Y, then $(f^*(H)\cdot E_i)=0$ for $1\leq i\leq m$. Since $(f^*(H)^2)=\deg(f)\cdot(H^2)>0$, it follows from the Hodge index theorem that $(D^2)\leq 0$, with equality if and only if $D\equiv 0$. Let us write $D=D_+-D_-$, where D_+ and D_- are effective divisors without common components. Suppose that $D\equiv 0$, hence $D_+\equiv D_-$. By assumption, at least one of D_+ and D_- is nonzero. Suppose, for example, that D_+ is nonzero. In this case we have

$$0\geq (D_+^2)=(D_+\cdot D_-)\geq 0,$$

hence $(D_+^2)=0$ and $D_+\equiv 0$. On the other hand, since D_+ is a nonzero effective divisor, we have $(D_+\cdot M)>0$ for every ample divisor M on X. Therefore the hypothesis that $D\equiv 0$ leads to a contradiction, and we conclude that $(D^2)<0$.

Suppose now that S is arbitrary. Since $(E_i\cdot E_j)=0$ whenever E_i and E_j lie in different fibers, it is enough to prove the proposition when all E_i lie in a fiber $f^{-1}(s)$. After replacing S by an affine open neighborhood of s, we may assume that S is affine. Let $\overline{S}$ and $\overline{X}$ be projective varieties containing S and X, respectively, as open subsets. After replacing $\overline{X}$ by its blow-up along a suitable closed subscheme whose support does not intersect X, we may assume that f extends to a morphism $g\colon \overline{X}\to\overline{S}$. Furthermore, we may consider a resolution of singularities[8] $X'\to\overline{X}$ that is an isomorphism over X, and after replacing $\overline{X}$ by X', we may assume that $\overline{X}$ is smooth. Since g is a morphism of projective surfaces and g contracts each E_i, we may apply the case we have already proved to get the assertion in the proposition. ☐

Proof of Theorem 1.6.36. The assertions are local on Y, hence we may and will assume that Y is affine. Part ii) is easy: since $f^{-1}(y)$ is connected, if $f^{-1}(y)\cap\mathrm{Supp}(D)$ is a proper, nonempty subset of $f^{-1}(y)$, then there is a curve $C\subseteq f^{-1}(y)$ such that $C\cap\mathrm{Supp}(D)$ is a nonempty, proper subset of C (see Corollary B.1.5). This implies $(D\cdot C)>0$. Since C is contained in a fiber of f, this contradicts the fact that $-D$ is f-nef.

We note that if $g\colon\widetilde{X}\to X$ is a proper, generically finite, surjective morphism, with $\widetilde{X}$ normal, then we may replace f and D by $f\circ g$ and $g^*(D)$, respectively. Indeed, if X_0 is the union of the prime divisors that appear with negative coefficient in D, then by assumption $\mathrm{codim}_Y f(X_0)\geq 2$. Since $D|_{X\smallsetminus X_0}$ is effective, it follows that the restriction of $f^*(D)$ to $g^{-1}(X\smallsetminus X_0)$ is effective, and therefore every prime divisor on $\widetilde{X}$ that appears with negative coefficient in $g^*(D)$ is supported in $g^{-1}(X_0)$, hence it is $(f\circ g)$-exceptional. Since $-g^*(D)$ is $(f\circ g)$-nef and we have the equality of Weil divisors $g_*(g^*(D))=\deg(g)\cdot D$, we conclude that it is enough to prove the theorem for the morphism $f\circ g$ and the divisor $g^*(D)$. This first implies, by applying Chow's lemma and then taking the normalization, that we may assume that X is quasi-projective. Suppose now that $g\colon\widetilde{X}\to X$ is an alteration (that is, a projective, surjective, generically finite morphism), with $\widetilde{X}$ smooth. Such an alteration exists by [dJ96], hence we may and will assume that X is smooth.

By assumption, we may write $D=A+B$, where A and B have no common components, A is effective, and all prime divisors in B are f-exceptional. We prove that D is effective by induction on $n=\dim(X)=\dim(Y)\geq 2$. We first consider the case

[8] Since we are in dimension 2, such a resolution exists in arbitrary characteristic, see [Lip78].

$n = 2$ and write $B = P - N$, such that P and N are effective, without common components. Since $-D$ is f-nef, the components of N lie in fibers of f, and A, P, and N are effective, without common components, we obtain

$$-(N^2) \leq (A \cdot N) + (P \cdot N) - (N^2) = (D \cdot N) \leq 0.$$

We then deduce from Proposition 1.6.35 that $N = 0$. This completes the proof of the case $n = 2$.

We now prove the induction step. Let $n \geq 3$. We first show that if E is a prime divisor on X with $\dim(f(E)) > 0$, then its coefficient in D is nonnegative. Let V be an open subset of Y such that f is finite over V. Suppose that we can find a closed codimension 1 subvariety $H \subset X$, with the following properties:

1) $H \cap f^{-1}(V) \neq \emptyset$.
2) H is not equal to any of the prime divisors that appear in D.
3) For every prime divisor F that appears in B, we have $\dim(f(F \cap H)) \leq n-3$.
4) $E \cap H$ is not contained in the union of the other prime divisors that appear in D.

Given such H, the restriction $u\colon H \to f(H)$ of f is generically finite by 1). If $v\colon \widetilde{H} \to H$ is the normalization of H and $w = u \circ v$, then w is generically finite. The divisor $v^*(D|_H)$ is well-defined by 2), and $-v^*(D|_H)$ is w-nef. We can write $v^*(D|_H) = v^*(A|_H) + v^*(B|_H)$, with $v^*(A|_H)$ effective, and all prime divisors in $v^*(B|_H)$ are w-exceptional by 3). Therefore the induction assumption implies that $v^*(D|_H)$ is effective and it follows from 4) that the coefficient of E in D is nonnegative.

We now show that we can find such H when $\dim(f(E)) > 0$. Since Y is affine, we can choose a closed subset Z of Y defined by a nonzero element in $\mathscr{O}(Y)$ such that the following hold:

a) Z does not contain $f(F)$ for any prime divisor F that appears in D.
b) Z does not contain $f(W)$ for any irreducible component W of $X \smallsetminus f^{-1}(V)$.
c) Z contains the image of a point $p \in E$ which does not lie on any other prime divisor that appears in D.

If H is an irreducible component of $f^{-1}(Z)$ that contains p, then H satisfies 1)-4) above.

In fact, we will only use the fact that for every E with $\dim(f(E)) = n-2$, its coefficient in D is non-negative. In order to treat the other prime divisors, consider a locally closed embedding of X in a projective space and let W be a general hyperplane section of X. Note that W satisfies the following conditions:

α) W is irreducible and smooth by Bertini's theorem. Furthermore, if $D = \sum_{i=1}^r a_i D_i$, with the D_i distinct prime divisors, then the $W \cap D_i$ are non-empty, irreducible and pairwise distinct (the irreducibility is a consequence of a version of Bertini's theorem, see [Jou83, Théorème 6.3]).

β) For every i, we have $f(D_i \cap W) = f(D_i)$ if $\dim f(D_i) \leq n-2$ and $\dim f(D_i \cap W) = n-2$ if $\dim f(D_i) = n-1$. In particular, $D_i \cap W$ is $f|_W$-exceptional if and only if either D_i is f-exceptional, or $\dim(f(D_i)) = n-2$.

γ) $W \cap f^{-1}(V) \neq \emptyset$.

Note that $D|_W = \sum_{i=1}^{r} a_i(D_i \cap W)$. The hypothesis, together with what we have already proved, implies that $a_i \geq 0$ for those i such that $D_i \cap W$ is not $f|_W$-exceptional. Since $-D|_W$ is $f|_W$-nef, we obtain applying the inductive hypothesis that $D|_W$ is effective. This implies that D is effective, and therefore completes the proof of the theorem. □

We will usually apply Theorem 1.6.34 for birational morphisms of normal varieties, when it takes the following form.

Corollary 1.6.36. *If $f\colon X \to Y$ is a proper, birational morphism of normal varieties and $D \in \mathrm{CDiv}(X)_{\mathbb{R}}$ is such that $-D$ is f-nef, then the following hold:*

i) D is effective if and only if $f_*(D)$ is effective.
ii) If D is effective, then for every $y \in f(\mathrm{Supp}(D))$, we have $f^{-1}(y) \subseteq \mathrm{Supp}(D)$.

Proof. It is clear that if D is effective, then $f_*(D)$ is effective. All other assertions follow from Theorem 1.6.34 □

Example 1.6.37. If $f\colon X \to S$ is a projective, birational morphism of normal varieties and S is $\mathbb{Q}$-factorial, then every $D \in \mathrm{CDiv}(X)_{\mathbb{R}}$ which is f-numerically trivial can be written as $f^*(E)$, for some $E \in \mathrm{CDiv}(S)_{\mathbb{R}}$. Indeed, let $E = f_*(D)$. This is $\mathbb{Q}$-Cartier since S is $\mathbb{Q}$-factorial, hence we may consider $D' = D - f^*(E)$. It is clear that also D' is f-numerically trivial, and since $f_*(D') = 0$, it follows from Corollary 1.6.36 that $D' = 0$.

We end with another useful application of Proposition 1.6.35, due to Fujita. In this case we assume that the ground field has characteristic zero.

Proposition 1.6.38. *If $f\colon X \to Y$ is a projective, surjective morphism of varieties, with X smooth, then for every effective f-exceptional divisor E on X, we have $f_*(\mathcal{O}_X(E)|_E) = 0$.*

Proof. The statement is local on Y, hence we may assume that Y is affine. If $\dim(Y) \leq 1$, then no divisor on X is f-exceptional, hence we may assume $\dim(Y) \geq 2$. We prove the proposition by induction on $\dim(X) + \dim(Y)$ and first treat the case $\dim(X) = \dim(Y) = 2$. Let us write $E = \sum_{i=1}^{m} a_i E_i$, where the E_i are pairwise distinct prime f-exceptional divisors. We argue by induction on $N := \sum_i a_i$, the case $N = 0$ being trivial. If $N \geq 1$, then it follows from Proposition 1.6.35 that $(E^2) < 0$. Therefore there is i such that $(E \cdot E_i) < 0$, in which case $H^0(X, \mathcal{O}_X(E)|_{E_i}) = 0$.

On the other hand, if $F = E - E_i$, then we have an exact sequence

$$0 \to \mathcal{O}_X(F)|_F \to \mathcal{O}_X(E)|_E \to \mathcal{O}_X(E)|_{E_i} \to 0.$$

This gives an exact sequence

$$0 \to H^0(X, \mathcal{O}_X(F)|_F) \to H^0(X, \mathcal{O}_X(E)|_E) \to H^0(X, \mathcal{O}_X(E)|_{E_i}).$$

We have seen that the third group vanishes and the first one also vanishes by induction on N. Therefore the group in the middle vanishes as well, proving the statement in dimension 2.

We now give the induction step. Note that if Z is a general member of a base-point free linear system on X, then Z is smooth by Kleiman's version of Bertini's theorem and $E|_Z$ is an effective divisor on Z. Moreover, if $H^0(X, \mathscr{O}_X(E)|_E) \neq 0$, then $H^0(Z, \mathscr{O}_Z(E|_Z)|_{E|_Z}) \neq 0$. It follows that if we can choose Z such that $E|_Z$ is $f|_Z$-exceptional, then we are done by induction. If $\dim f(\mathrm{Supp}(E)) \geq 1$, we choose $Z = f^*(H)$, where H is a general member of any base-point free linear system on Y. In this case $\dim f(\mathrm{Supp}(E|_Z)) = \dim f(\mathrm{Supp}(E)) - 1$ and $\dim(f(Z)) = \dim(Y) - 1$, hence $E|_Z$ is $f|_Z$-exceptional. Suppose now that $f(\mathrm{Supp}(E))$ is 0-dimensional. We consider a locally closed embedding of X in a projective space and let Z be a general hyperplane section. In this case $\dim(f(Z)) = \dim(Y)$ if $\dim(X) > \dim(Y)$, while $\dim(f(Z)) = \dim(Y) - 1$ if f is generically finite. Therefore $E|_Z$ is $f|_Z$-exceptional, unless $\dim(X) = \dim(Y) = 2$. This completes the proof of the theorem. $\square$

1.7 Asymptotic invariants of linear systems

In this section we define and study, following [Nak04] and [ELM+06], asymptotic invariants of linear systems $|\mathscr{L}^m|$, where $\mathscr{L}$ is a line bundle on a projective variety X. We associate such invariants, more generally, to certain sequences of ideals that we now introduce. As usual, we work over an infinite ground field k.

1.7.1 Graded sequences of ideals

Let X be an arbitrary variety.

Definition 1.7.1. A *graded sequence of ideals* on X is a sequence $\mathfrak{a}_\bullet = (\mathfrak{a}_m)_{m \geq 1}$ of coherent ideals on X such that

$$\mathfrak{a}_p \cdot \mathfrak{a}_q \subseteq \mathfrak{a}_{p+q} \text{ for all } p, q \geq 1.$$

We say that the graded sequence $\mathfrak{a}_\bullet$ is *nonzero* if some $\mathfrak{a}_p$ is nonzero. We make the convention that $\mathfrak{a}_0 = \mathscr{O}_X$.

Example 1.7.2 (Trivial graded sequences). If $\mathfrak{b}$ is an ideal on X, then by taking $\mathfrak{a}_m = \mathfrak{b}^m$ for every $m \geq 1$, we obtain a graded sequence of ideals. This is a trivial example: studying invariants for such graded sequences is equivalent to studying invariants for ideals. However, as we will see in Section 1.8, one is often interested in criteria that guarantee that a given graded sequence is of this form.

Example 1.7.3 (Graded sequence of a valuation). Suppose that $X = \mathrm{Spec} A$ is an affine variety and $v\colon k(X) \to \mathbb{R} \cup \{\infty\}$ is a real valuation of the function field of X

such that $v(A) \subseteq \mathbb{R}_{\geq 0} \cup \{\infty\}$. If we put $\mathfrak{a}_m = \{f \in A \mid v(f) \geq m\}$, then $\mathfrak{a}_\bullet$ is a graded sequence.

Example 1.7.4 (Graded sequence of a graded linear series). Suppose that $V_\bullet$ is a graded linear series on X (see Definition 1.4.1). For every $m \geq 1$, let $\mathfrak{a}_m$ be the ideal defining the base-locus of V_m, that is, if $V_m \subseteq H^0(X, \mathscr{L}^m)$, then evaluation of sections induces a surjective map

$$V_m \otimes \mathscr{O}_X \twoheadrightarrow \mathfrak{a}_m \otimes \mathscr{L}^m.$$

It follows from the definition of a graded linear series that $\mathfrak{a}_\bullet$ is a graded sequence of ideals on X. An important example is when X is complete and $V_m = H^0(X, \mathscr{L}^m)$ for all $m \geq 1$.

We will also consider the following generalization of the concept of graded sequence of ideals. Given an arbitrary monoid S and a variety X, an S-graded sequence of ideals on X is a family $\mathfrak{a}_\bullet = (\mathfrak{a}_u)_{u \in S}$ of coherent ideals on X indexed by S such that $\mathfrak{a}_0 = \mathscr{O}_X$ and $\mathfrak{a}_u \cdot \mathfrak{a}_v \subseteq \mathfrak{a}_{u+v}$ for all $u, v \in S$. Note that our previous notion of graded sequence is equivalent to that of an $\mathbb{N}$-graded sequence in the above sense.

Example 1.7.5. Let X be a complete variety and $\mathscr{L}_1, \ldots, \mathscr{L}_r$ line bundles on X. For every $u = (u_1, \ldots, u_r) \in \mathbb{N}^r$, let $\mathfrak{a}_u$ be the ideal defining the base-locus of the line bundle $\mathscr{L}_1^{u_1} \otimes \ldots \otimes \mathscr{L}_r^{u_r}$. It is clear that $\mathfrak{a}_\bullet$ is an $\mathbb{N}^r$-graded sequence of ideals.

Example 1.7.6. Suppose that S is a monoid and $\mathfrak{a}_\bullet$ is an S-graded sequence of ideals on X. For every $u \in S$, we have a graded sequence of ideals $\mathfrak{a}_{\bullet u} = (\mathfrak{a}_{mu})_{m \geq 1}$.

1.7.2 Divisors over X

In order to attach asymptotic invariants to a graded sequence of ideals on X, we use divisors over X. This notion will play an important role in Chapter 3, but for now, we only need the definition and some related terminology.

Let X be an arbitrary variety and $f\colon Y \to X$ a birational morphism, with Y normal. A prime divisor E on Y defines a discrete valuation ord_E of the function field $K(Y) = K(X)$, called a *divisorial valuation*. The corresponding DVR is the local ring $\mathscr{O}_{Y,E}$, that with a slight abuse of notation we also denote by $\mathscr{O}_{X,E}$. The *center* $c_X(E)$ of E on X is the closure of $f(E)$. Note that we have a canonical local homomorphism $\mathscr{O}_{X,c_X(E)} \hookrightarrow \mathscr{O}_{Y,E}$.

We identify two such divisors lying on varieties Y_1 and Y_2 as above if they give the same valuation. An equivalence class is a *divisor over* X. In particular, if $Y' \to Y$ is a proper morphism of normal varieties and E is a prime divisor on Y, then E and its proper transform on Y give the same divisor over X.

If E is a divisor over X and H is a Cartier divisor on X, then we put $\mathrm{ord}_E(H) := \mathrm{ord}_E(\phi)$, where ϕ is a nonzero rational function such that $H = \mathrm{div}_X(\phi)$ in a neighborhood of the generic point of $c_X(E)$. This definition extends by linearity to

$\mathrm{CDiv}(X)_{\mathbb{R}}$. We also define $\mathrm{ord}_E(\mathfrak{a})$ when $\mathfrak{a}$ is a nonzero fractional ideal on X (that is, a coherent subsheaf of the function field), as follows. If t is a uniformizer in the DVR $\mathscr{O}_{X,E}$ and the $\mathscr{O}_{X,E}$-module $\mathfrak{a}\cdot\mathscr{O}_{X,E}$ is generated by t^e, then we put $\mathrm{ord}_E(\mathfrak{a})=e$. If $\mathfrak{a}$ is an ideal on X and we present E as a prime divisor on Y such that $\mathfrak{a}\cdot\mathscr{O}_Y=\mathscr{O}_Y(-D)$ for an effective Cartier divisor D (given any Y, we may achieve this condition after replacing Y by the normalization of the blow-up along $\mathfrak{a}\cdot\mathscr{O}_Y$), then $\mathrm{ord}_E(\mathfrak{a})$ is the coefficient of E in D. If Z is the closed subscheme defined by $\mathfrak{a}$, we also write $\mathrm{ord}_E(Z)$ for $\mathrm{ord}_E(\mathfrak{a})$. Note that $\mathrm{ord}_E(Z)>0$ if and only if $c_X(E)\subseteq Z$. We make the convention that $\mathrm{ord}_E(\mathfrak{a})=\infty$ if $\mathfrak{a}$ is the zero ideal.

Remark 1.7.7. It is clear that if $\mathfrak{a}$ and $\mathfrak{b}$ are ideals on X, then $\mathrm{ord}_E(\mathfrak{a}\cdot\mathfrak{b})=\mathrm{ord}_E(\mathfrak{a})+\mathrm{ord}_E(\mathfrak{b})$. Note also that if $\mathfrak{a}\subseteq\mathfrak{b}$, then $\mathrm{ord}_E(\mathfrak{b})\leq\mathrm{ord}_E(\mathfrak{a})$.

Remark 1.7.8. For every irreducible closed subset V of X, there is a divisor E over X with $c_X(E)=V$. For example, if $f\colon Y\to X$ is the normalization of the blow-up of X along V, then we may take E to be any irreducible component of $f^{-1}(V)$ which dominates V. If V meets the smooth locus of X, then there is a unique such irreducible component of $f^{-1}(V)$. The corresponding valuation is denoted by ord_V. In this case, if $\mathfrak{a}$ is an ideal in X and I_V is the ideal defining V, then $\mathrm{ord}_V(\mathfrak{a})\geq m$ if and only if $\mathfrak{a}\subseteq I_V^m$ at the generic point of V.

1.7.3 Asymptotic invariants of graded sequences

Let $\mathfrak{a}_\bullet$ be a nonzero graded sequence of ideals on a variety X and E a divisor over X. It is clear that the set $S=\{m\in\mathbb{Z}_{>0}\mid\mathfrak{a}_m\neq 0\}$ is closed under addition.

For every $p,q\geq 1$, the inclusion $\mathfrak{a}_p\cdot\mathfrak{a}_q\subseteq\mathfrak{a}_{p+q}$ implies

$$\mathrm{ord}_E(\mathfrak{a}_{p+q})\leq\mathrm{ord}_E(\mathfrak{a}_p\cdot\mathfrak{a}_q)=\mathrm{ord}_E(\mathfrak{a}_p)+\mathrm{ord}_E(\mathfrak{a}_q). \tag{1.16}$$

Lemma 1.7.9 below implies that in this case we have

$$\inf_{m\geq 1}\frac{\mathrm{ord}_E(\mathfrak{a}_m)}{m}=\lim_{m\to\infty}\frac{\mathrm{ord}_E(\mathfrak{a}_m)}{m},$$

where the limit is over those $m\in S$. This limit is the *asymptotic order of vanishing of* $\mathfrak{a}_\bullet$ *along* E, and we denote it by $\mathrm{ord}_E(\mathfrak{a}_\bullet)$.

Lemma 1.7.9. *Let $S\subseteq\mathbb{Z}_{>0}$ be a nonempty subset closed under addition and $(\alpha_m)_{m\in S}$ a set of real numbers that satisfies $\alpha_{p+q}\leq\alpha_p+\alpha_q$ for all $p,q\in S$. In this case we have*

$$\lim_{m\to\infty, m\in S}\frac{\alpha_m}{m}=\inf_{m\in S}\frac{\alpha_m}{m}. \tag{1.17}$$

Proof. Let $T:=\inf_{m\in S}\alpha_m/m\in\mathbb{R}\cup\{-\infty\}$. We need to show that for every $\tau>T$, we have $\alpha_p/p<\tau$ if $p\gg 0$, with $p\in S$. Let $m\in S$ be such that $\alpha_m/m<\tau$. It is enough to show that for every integer q with $0 \le q < m$, if $p = m\ell + q \in S$ with $\ell \gg 0$, then $\alpha_p/p < \tau$.

If there is no ℓ such that $m\ell + q \in S$, then there is nothing to prove. Otherwise, let us choose ℓ_0 with $m\ell_0 + q \in S$. For $\ell \ge \ell_0$ we have

$$\frac{\alpha_{m\ell+q}}{m\ell+q} \le \frac{\alpha_{m\ell_0+q} + (\ell-\ell_0)\alpha_m}{m\ell+q}.$$

Since the right-hand side converges to $\alpha_m/m < \tau$ for $\ell \to \infty$, it follows that

$$\frac{\alpha_{m\ell+q}}{m\ell+q} < \tau \quad \text{for} \quad \ell \gg 0,$$

which completes the proof. $\square$

Proposition 1.7.10. *If $\mathfrak{a}_\bullet$ and $\mathfrak{b}_\bullet$ are nonzero graded sequences on the variety X such that for some nonzero ideal $\mathfrak{c}$ and for some $q \in \mathbb{Z}$ we have*

$$\mathfrak{c} \cdot \mathfrak{a}_m \subseteq \mathfrak{b}_{m+q} \quad \textit{for all} \quad m \gg 0,$$

then $\mathrm{ord}_E(\mathfrak{b}_\bullet) \le \mathrm{ord}_E(\mathfrak{a}_\bullet)$ *for all divisors E over X.*

Proof. The hypothesis implies that for $m \gg 0$, if $\mathfrak{a}_m$ is nonzero, then $\mathfrak{b}_{m+q}$ is nonzero as well. Furthermore, we have

$$\frac{\mathrm{ord}_E(\mathfrak{b}_{m+q})}{m} \le \frac{\mathrm{ord}_E(\mathfrak{a}_m)}{m} + \frac{\mathrm{ord}_E(\mathfrak{c})}{m}.$$

By considering m with $\mathfrak{a}_m \neq 0$ and letting it go to infinity, we obtain the inequality in the proposition. $\square$

Suppose now that S is a monoid and $\mathfrak{a}_\bullet$ is an S-graded sequence of ideals on X. Note that the set

$$S_+(\mathfrak{a}_\bullet) := \{u \in S \mid \mathfrak{a}_{mu} \neq 0 \text{ for some } m > 0\}$$

is a submonoid of S. Given a divisor E over X, we define a map $\mathrm{ord}_E^{\mathfrak{a}_\bullet} \colon S_+(\mathfrak{a}_\bullet) \to \mathbb{R}_{\ge 0}$ as follows. For every $u \in T$, we consider the corresponding graded sequence of ideals $\mathfrak{a}_{\bullet u}$ and put

$$\mathrm{ord}_E^{\mathfrak{a}_\bullet}(u) = \mathrm{ord}_E(\mathfrak{a}_{\bullet u}).$$

Note that if q is a positive integer, then

$$\mathrm{ord}_E^{\mathfrak{a}_\bullet}(qu) = q \cdot \mathrm{ord}_E^{\mathfrak{a}_\bullet}(u) \quad \text{for every} \quad u \in T. \tag{1.18}$$

Indeed, we have

$$\mathrm{ord}_E^{\mathfrak{a}_\bullet}(qu) = \lim_{m\to\infty} \frac{\mathrm{ord}_E(\mathfrak{a}_{qmu})}{m} = q \cdot \lim_{m\to\infty} \frac{\mathrm{ord}_E(\mathfrak{a}_{qmu})}{qm} = q \cdot \mathrm{ord}_E^{\mathfrak{a}_\bullet}(u),$$

where both limits are over those m such that $\mathfrak{a}_{qmu} \neq 0$.

We will be especially interested in the case when S is a submonoid of a finitely generated, free abelian group M. Suppose that $\mathfrak{a}_\bullet$ is an S-graded sequence of ideals and let us assume that the monoid $S_+(\mathfrak{a}_\bullet)$ considered above is finitely generated. If C is the convex cone generated by T in $M_{\mathbb{R}}$, then $C \cap M_{\mathbb{Q}}$ consists of all $\frac{1}{m}u$, with $u \in S_+(\mathfrak{a}_\bullet)$ and $m \geq 1$. We extend $\mathrm{ord}_E^{\mathfrak{a}_\bullet}$ to $C \cap M_{\mathbb{Q}}$ by putting

$$\mathrm{ord}_E^{\mathfrak{a}_\bullet}(u) = \frac{1}{m}\,\mathrm{ord}_E^{\mathfrak{a}_\bullet}(mu),$$

where m is a positive integer such that $mu \in S_+(\mathfrak{a}_\bullet)$. It follows from (1.18) that the definition is independent of m and we have

$$\mathrm{ord}_E^{\mathfrak{a}_\bullet}(\lambda u) = \lambda \cdot \mathrm{ord}_E^{\mathfrak{a}_\bullet}(u) \quad \text{for every} \quad u \in C,\ \lambda \in \mathbb{Q}_{>0}.$$

1.7.4 Asymptotic invariants of big divisors

Suppose now that X is a complete variety and $\mathscr{L} \in \mathrm{Pic}(X)$ is a line bundle such that $\kappa(\mathscr{L}) \geq 0$ (recall that by definition, this means that $h^0(X, \mathscr{L}^m) \geq 1$ for some positive integer m). We fix a divisor E over X. If $\mathfrak{a}_\bullet$ is the graded sequence of ideals such that $\mathfrak{a}_m$ is the ideal defining the base-locus of $|\mathscr{L}^m|$, then the *asymptotic order of vanishing of $\mathscr{L}$ along E* is

$$\mathrm{ord}_E(\parallel \mathscr{L} \parallel) := \mathrm{ord}_E(\mathfrak{a}_\bullet).$$

Note that if $|\mathscr{L}^m|$ is nonempty and $D \in |\mathscr{L}^m|$ is a general element, then $\mathrm{ord}_E(D) = \mathrm{ord}_E(\mathfrak{a}_m)$. Therefore we sometimes write $\mathrm{ord}_E(|\mathscr{L}^m|)$ instead of $\mathrm{ord}_E(\mathfrak{a}_m)$.

Example 1.7.11. It is clear that if $\mathscr{L}$ is semiample, then $\mathrm{ord}_E(\parallel \mathscr{L} \parallel) = 0$.

Lemma 1.7.12. *With the above notation, for every positive integer q, we have*

$$\mathrm{ord}_E(\parallel \mathscr{L}^q \parallel) = q \cdot \mathrm{ord}_E(\parallel \mathscr{L} \parallel).$$

Proof. If $\mathfrak{a}_\bullet$ is as above and $S = \{m \in \mathbb{Z}_{>0} \mid \mathfrak{a}_m \neq 0\}$, then

$$\mathrm{ord}_E(\parallel \mathscr{L}^q \parallel) = \lim_{m\to\infty, mq \in S} \frac{\mathrm{ord}_E(\mathfrak{a}_{mq})}{m} = q \cdot \lim_{m\to\infty, mq\in S} \frac{\mathrm{ord}_E(\mathfrak{a}_{mq})}{mq} = q \cdot \mathrm{ord}_E(\parallel \mathscr{L} \parallel).$$

□

We can use this homogeneity property to define $\mathrm{ord}_E(\parallel D \parallel)$ when $D \in \mathrm{CDiv}(X)_{\mathbb{Q}}$. Given $D \in \mathrm{CDiv}(X)_{\mathbb{Q}}$ such that $\kappa(D) \geq 0$, we put

$$\mathrm{ord}_E(\parallel D \parallel) = \frac{1}{m} \cdot \mathrm{ord}_E(\parallel \mathscr{O}_X(mD) \parallel),$$

where m is such that mD is a Cartier divisor. It follows from Lemma 1.7.12 that $\mathrm{ord}_E(\| D \|)$ is well-defined. Furthermore, it is a consequence of the definition that for every such D, we have $\mathrm{ord}_E(\| \lambda D \|) = \lambda \cdot \mathrm{ord}_E(\| \lambda D \|)$ for every $\lambda \in \mathbb{Q}_{>0}$.

Lemma 1.7.13. *If $D, D' \in \mathrm{CDiv}(X)_{\mathbb{Q}}$ are such that $\kappa(D), \kappa(D') \geq 0$, then*

$$\mathrm{ord}_E(\| D + D' \|) \leq \mathrm{ord}_E(\| D \|) + \mathrm{ord}_E(\| D' \|).$$

Proof. After replacing D and D' by suitable multiples, we may assume that both D and D' are Cartier divisors, such that the corresponding line bundles have nonzero sections. Let $\mathfrak{a}_m$, $\mathfrak{a}'_m$, and $\mathfrak{b}_m$ be the ideals defining the base-loci of $|mD|$, $|mD'|$, and $|m(D+D')|$, respectively. It is clear that $\mathfrak{a}_m \cdot \mathfrak{a}'_m \subseteq \mathfrak{b}_m$ for all m, hence

$$\mathrm{ord}_E(\mathfrak{b}_m) \leq \mathrm{ord}_E(\mathfrak{a}_m) + \mathrm{ord}_E(\mathfrak{a}'_m).$$

Dividing by m and letting m go to infinity gives the inequality in the lemma. □

Proposition 1.7.14. *Let $f\colon X' \to X$ be a birational morphism of complete varieties, with X normal. If E is a divisor over X, then for every $D \in \mathrm{CDiv}(X)_{\mathbb{Q}}$ with $\kappa(D) \geq 0$, we have*

$$\mathrm{ord}_E(\| D \|) = \mathrm{ord}_E(\| f^*(D) \|).$$

Proof. We first note that E can also be considered as a divisor over X' and for every nonzero ideal $\mathfrak{a}$ on X, we have $\mathrm{ord}_E(\mathfrak{a}) = \mathrm{ord}_E(\mathfrak{a} \cdot \mathscr{O}_{X'})$. After rescaling D, we may assume that D is a Cartier divisor and that $|D|$ is nonempty. Since f is birational and X is normal, we have $f_*(\mathscr{O}_{X'}) \simeq \mathscr{O}_X$. Therefore the projection formula implies that the canonical morphism $H^0(X, \mathscr{O}_X(mD)) \to H^0(X', \mathscr{O}_{X'}(mf^*(D)))$ is an isomorphism for all m. It follows that if $\mathfrak{a}_m$ is the ideal defining the base-locus of $|mD|$, then $\mathfrak{a}_m \cdot \mathscr{O}_{X'}$ defines the base-locus of $|f^*(mD)|$. Therefore we have $\mathrm{ord}_E(|mD|) = \mathrm{ord}_E(|mf^*(D)|)$ for all $m \geq 1$. Dividing by m and passing to the limit gives the assertion in the proposition. □

Our next goal is to show that for big divisors on projective varieties, the asymptotic invariants only depend on the numerical class. For this we will need the following fact.

Property 1.7.15. For every projective variety X, there is a line bundle $\mathscr{A} \in \mathrm{Pic}(X)$ such that for every nef line bundle $\mathscr{M} \in \mathrm{Pic}(X)$, the line bundle $\mathscr{A} \otimes \mathscr{M}$ is globally generated.

We will prove this in Corollary 2.4.4 below. It will be deduced from a vanishing theorem due to Fujita, for which we will give a proof in characteristic zero. We now use the above property to relate the graded sequences of base-loci ideals of numerically equivalent line bundles.

Lemma 1.7.16. *Let X be a projective variety and $\mathscr{L}$, $\mathscr{L}'$ line bundles on X, with $\mathscr{L}'$ big and such that $\mathscr{L}' \otimes \mathscr{L}^{-1}$ is nef. If $\mathfrak{a}_\bullet$ and $\mathfrak{a}'_\bullet$ are the graded sequences of* *ideals defining the base-loci of the multiples of $\mathscr{L}$ and $\mathscr{L}'$, respectively, then there is a nonzero ideal $\mathfrak{c} \subseteq \mathscr{O}_X$ and a nonnegative integer q such that*

$$\mathfrak{c} \cdot \mathfrak{a}_m \subseteq \mathfrak{a}'_{m+q} \text{ for all } m \gg 0.$$

Proof. We choose $\mathscr{A}$ as in Property 1.7.15. In particular, we have $\mathscr{L}'^m \otimes \mathscr{L}^{-m} \otimes \mathscr{A}$ globally generated for all $m \geq 1$. Since $\mathfrak{a}_m \otimes \mathscr{L}^m$ is globally generated by definition, it follows that $\mathfrak{a}_m \otimes \mathscr{L}'^m \otimes \mathscr{A}$ is globally generated for all $m \geq 1$. Therefore $\mathfrak{a}_m \subseteq \mathfrak{b}_m$ for all $m \geq 1$, where $\mathfrak{b}_m$ is the ideal defining the base locus of $\mathscr{L}'^m \otimes \mathscr{A}$.

On the other hand, since $\mathscr{L}'$ is big, it follows from Kodaira's lemma (see Lemma 1.4.14) that for $q \gg 0$, there is an effective Cartier divisor G such that $\mathscr{O}_X(G) \simeq \mathscr{L}'^q \otimes \mathscr{A}^{-1}$. In particular, we have $\mathscr{O}_X(-G) \cdot \mathfrak{b}_m \subseteq \mathfrak{a}'_{m+q}$. Therefore

$$\mathscr{O}_X(-G) \cdot \mathfrak{a}_m \subseteq \mathscr{O}_X(-G) \cdot \mathfrak{b}_m \subseteq \mathfrak{a}'_{m+q}.$$

We may thus take $\mathfrak{c} = \mathscr{O}_X(-G)$. □

Corollary 1.7.17. *If X is a projective variety and $D, D' \in \mathrm{CDiv}(X)_{\mathbb{Q}}$ are big and numerically equivalent, then $\mathrm{ord}_E(\| D \|) = \mathrm{ord}_E(\| D' \|)$ for every divisor E over X. Furthermore, the same result holds if we assume that X is complete and normal.*

Proof. After possibly rescaling D and D', we may assume that they are both Cartier. In this case the first assertion follows by combining Lemmas 1.7.10 and 1.7.16. The second assertion follows from the first one by using Chow's lemma and Proposition 1.7.14. □

In light of Corollary 1.7.17, for every projective variety X, we may consider the function $\mathrm{ord}_E(\| - \|)$ defined on $\mathrm{Big}(X) \cap \mathrm{N}^1(X)_{\mathbb{Q}}$. It follows from Lemmas 1.7.12 and 1.7.13 that this is a convex function (see Section A.8 for the definition of convex functions). Since every convex function defined on the set of rational points of an open convex subset of $\mathbb{R}^n$ is continuous (see Remark A.8.2), we obtain the continuity of asymptotic invariants on the set of numerical classes of big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors.

Proposition 1.7.18. *For every projective variety X and every divisor E over X, the function $\mathrm{ord}_E(\| - \|)$ defined on $\mathrm{Big}(X) \cap \mathrm{N}^1(X)_{\mathbb{Q}}$ is continuous.*

We now introduce a new definition of asymptotic invariants which is more formal and has the advantage of applying also to big $\mathbb{R}$-Cartier $\mathbb{R}$-divisors. We will later show that in the case of big $\mathbb{Q}$-divisors, this agrees with the above definition.

Suppose that X is a projective variety and E is a divisor over X. For every $D \in \mathrm{CDiv}(X)_{\mathbb{R}}$ which is big, we put

$$\widetilde{\mathrm{ord}}_E(\| D \|) := \inf\{\mathrm{ord}_E(B) \mid B \in \mathrm{CDiv}(X)_{\mathbb{R}},\ B \equiv D, \text{ and } B \text{ is effective}\}.$$

Note that since D is big, we can find $B \in \mathrm{CDiv}(X)_{\mathbb{R}}$ effective such that $B \equiv D$, hence this invariant is well-defined. The basic properties of this invariant follow formally from definition.

Proposition 1.7.19. *Let X be a projective variety, E a divisor over X, and $D, D' \in \mathrm{CDiv}(X)_{\mathbb{R}}$ big.*

i) If $D \equiv D'$, then $\widetilde{\mathrm{ord}}_E(\| D \|) = \widetilde{\mathrm{ord}}_E(\| D' \|)$.
ii) $\widetilde{\mathrm{ord}}_E(\| \lambda D \|) = \lambda \cdot \widetilde{\mathrm{ord}}_E(\| \lambda D \|)$ for every $\lambda \in \mathbb{R}_{>0}$.
iii) The induced function $\widetilde{\mathrm{ord}}_E(\| - \|)$ on $\mathrm{Big}(X)$ is convex, hence continuous.

Proof. The assertions in i) and ii) follow immediately from definition. For iii), note that

$$\widetilde{\mathrm{ord}}_E(\| D + D' \|) \leq \widetilde{\mathrm{ord}}_E(\| D \|) + \widetilde{\mathrm{ord}}_E(\| D' \|). \tag{1.19}$$

Indeed, if $B, B' \in \mathrm{CDiv}(X)_{\mathbb{R}}$ are effective and such that $B \equiv D$ and $B' \equiv D'$, then $B + B'$ is effective and $B + B' \equiv D + D'$. Therefore

$$\widetilde{\mathrm{ord}}_E(\| D + D' \|) \leq \mathrm{ord}_E(B + B') = \mathrm{ord}_E(B) + \mathrm{ord}_E(B').$$

This implies the inequality in (1.19). Together with the assertion in ii), this implies that $\widetilde{\mathrm{ord}}_E(\| - \|)$ is a convex function. Since every convex function on an open subset of a finite-dimensional real vector space is continuous (see Proposition A.8.1), this completes the proof of the proposition. □

Proposition 1.7.20. *If X is a projective variety and E is a divisor over X, then $\mathrm{ord}_E(\| D \|) = \widetilde{\mathrm{ord}}_E(\| D \|)$ for every big $D \in \mathrm{CDiv}(X)_{\mathbb{Q}}$.*

Proof. After replacing D by a suitable multiple, we may assume that D is a Cartier divisor and $h^0(X, \mathscr{O}_X(D)) \geq 1$. For every $m \geq 1$, if $F_m \in |mD|$ is a general element, then $\frac{1}{m} F_m$ is an effective divisor numerically equivalent to D. Therefore

$$\widetilde{\mathrm{ord}}_E(D) \leq \inf_{m \geq 1} \frac{\mathrm{ord}_E(F_m)}{m} = \inf_{m \geq 1} \frac{\mathrm{ord}_E(|mD|)}{m} = \mathrm{ord}_E(\| D \|).$$

In order to prove the reverse inequality, let us consider an arbitrary effective $B \in \mathrm{CDiv}(X)_{\mathbb{R}}$, with $B \equiv D$. We can write $B = \sum_{i=1}^r a_i G_i$, with a_i nonnegative and G_i effective Cartier divisors. If we choose sequences of nonnegative rational numbers $(a_{i,m})_{m \geq 1}$ with $\lim_{m \to \infty} a_{i,m} = a_i$ and put $B_m = \sum_{i=1}^r a_{i,m} G_i \in \mathrm{CDiv}(X)_{\mathbb{Q}}$, then

$$\mathrm{ord}_E(B) = \lim_{m \to \infty} \mathrm{ord}_E(B_m). \tag{1.20}$$

On the other hand, we have $\lim_{m \to \infty} B_m = D$ in $\mathrm{N}^1(X)_{\mathbb{Q}}$, hence Proposition 1.7.18 implies

$$\mathrm{ord}_E(\| D \|) = \lim_{m \to \infty} \mathrm{ord}_E(\| B_m \|). \tag{1.21}$$

It is an easy consequence of the definition of $\mathrm{ord}_E(\| B_m \|)$ that $\mathrm{ord}_E(\| B_m \|) \leq \mathrm{ord}_E(B_m)$. By combining this with (1.20) and (1.21), we obtain

$$\mathrm{ord}_E(B) = \lim_{m \to \infty} \mathrm{ord}_E(B_m) \geq \lim_{m \to \infty} \mathrm{ord}_E(\| B_m \|) = \mathrm{ord}_E(\| D \|).$$

Since this holds for every effective B with $B \equiv D$, we obtain

$$\widetilde{\mathrm{ord}}_E(\| D \|) \geq \mathrm{ord}_E(\| D \|).$$

This completes the proof of the proposition. □

In light of Proposition 1.7.20, from now on we write $\mathrm{ord}_E(\| D \|)$ instead of $\widetilde{\mathrm{ord}}_E(\| D \|)$ for any big $D \in \mathrm{CDiv}(X)_{\mathbb{R}}$.

Corollary 1.7.21. *If X is a projective variety and E is a divisor over X, then for every big $D \in \mathrm{CDiv}(X)_{\mathbb{R}}$ such that $c_X(E) \not\subseteq \mathrm{B}_{-}(D)$, we have $\mathrm{ord}_E(\| D \|) = 0$. In particular, $\mathrm{ord}_E(\| D \|)$ for all E and all big and nef $D \in \mathrm{CDiv}(X)_{\mathbb{R}}$.*

Proof. Let $A_m \in \mathrm{CDiv}(X)_{\mathbb{R}}$ be ample, with $D + A_m \in \mathrm{CDiv}(X)_{\mathbb{Q}}$ for all m, and such that $\lim_{m\to\infty} A_m = 0$ in $\mathrm{N}^1(X)_{\mathbb{R}}$. It follows from the continuity of the asymptotic invariants that

$$\lim_{m\to\infty} \mathrm{ord}_E(\| D + A_m \|) = \mathrm{ord}_E(\| D \|).$$

On the other hand, $\mathrm{SB}(D + A_m) \subseteq \mathrm{B}_{-}(D)$ for every m, hence by assumption $c_X(E) \not\subseteq \mathrm{SB}(D + A_m)$ and therefore $\mathrm{ord}_E(\| D + A_m \|) = 0$. We conclude that $\mathrm{ord}_E(\| D \|) = 0$. The second assertion in the corollary is an immediate consequence since $\mathrm{B}_{-}(D) = \emptyset$ whenever D is nef. □

Corollary 1.7.22. *Let $f\colon X' \to X$ be a birational morphism of projective varieties, with X normal. If E is a divisor over X, then for every $D \in \mathrm{Big}(X)$ we have*

$$\mathrm{ord}_E(\| D \|) = \mathrm{ord}_E(\| f^*(D) \|).$$

Proof. The assertion follows by continuity from Proposition 1.7.14. □

Example 1.7.23. Let $f\colon X \to \mathbb{P}^n$ be the blow-up of a point $Q \in \mathbb{P}^n$, with exceptional divisor E. If H is the pull-back of a hyperplane in $\mathbb{P}^n$, then for every $a, b \in \mathbb{Z}$ such that $aH + bE$ is pseudo-effective, we have

$$\mathrm{ord}_E(|aH + bE|) = \mathrm{ord}_E(\| aH + bE \|) = \begin{cases} b, & \text{if } a, b \geq 0; \\ 0, & \text{if } a \geq -b \geq 0. \end{cases}$$

Example 1.7.24. Let us consider again Example 1.5.6 and let $\mathscr{L} = f^*(A) \otimes g^*(B)$, where B is a non-torsion line bundle on C. We have seen that $\mathrm{ord}_E(|\mathscr{L}^m|) \geq 1$ for every $m \geq 1$. On the other hand, $\mathscr{L}^m(-E)|_E$ corresponds via the isomorphism $E \simeq C$ to the very ample line bundle $\mathscr{O}_C(1)$ induced by the embedding $C \subset \mathbb{P}^n$. Suppose that $\deg(\mathscr{O}_C(1)) \geq 2g - 1$. We leave it as an exercise for the reader to check that $H^1(X, \mathscr{L}^m(-2E)) = 0$ for $m \gg 1$. Using the exact sequence

$$0 \to \mathscr{L}^m(-2E) \to \mathscr{L}^m(-E) \to \mathscr{L}^m(-E)|_E \to 0,$$

we deduce that $\mathscr{L}^m(-E)$ is globally generated in a neighborhood of E for every $m \geq 1$. Therefore $\mathrm{ord}_E(|\mathscr{L}^m|) = 1$ for all $m \geq 1$, hence $\mathrm{ord}_E(\| \mathscr{L} \|) = 0$. Note that since $\mathscr{L}$ is big and nef, we knew by Corollary 1.7.21 that all asymptotic invariants of $\mathscr{L}$ vanish.

In what follows we study some further properties of the asymptotic invariants for big $\mathbb{R}$-divisors on smooth projective varieties.

Proposition 1.7.25. *Let X be a smooth projective variety, E a divisor over X, and $\Gamma_1,\ldots,\Gamma_r$ mutually distinct prime divisors on X. If D is a big $\mathbb{R}$-divisor on X and $0 \le s_i \le \mathrm{ord}_{\Gamma_i}(\| D \|)$ for $1 \le i \le r$, then*

i) $D' := D - \sum_{i=1}^r s_i\Gamma_i$ *is big.*
ii) $\mathrm{ord}_E(\| D' \|) = \mathrm{ord}_E(\| D \|) - \sum_{i=1}^r s_i \cdot \mathrm{ord}_E(\Gamma_i)$.
iii) The natural inclusion

$$H^0(X, \mathscr{O}_X(D')) \hookrightarrow H^0(X, \mathscr{O}_X(D))$$

induced by the effective divisor $\sum_{i=1}^r s_i\Gamma_i$ is an isomorphism.

Proof. We begin with the case when $D \in \mathrm{CDiv}(X)_{\mathbb{Q}}$ and all $s_i \in \mathbb{Q}$. For both i) and ii), we may multiply both D and the s_i by the same positive integer. Therefore we may assume that D is an integral divisor, $|D|$ is nonempty, and all $s_i \in \mathbb{Z}$. For every positive integer m, we have $ms_i \le m \cdot \mathrm{ord}_{\Gamma_i}(\| D \|) \le \mathrm{ord}_{\Gamma_i}(|mD|)$. Therefore the natural inclusion

$$H^0(X, \mathscr{O}_X(mD')) \hookrightarrow H^0(X, \mathscr{O}_X(mD)) \tag{1.22}$$

induced by multiplication with the section defining $\sum_{i=1}^r ms_i\Gamma_i$ is an isomorphism. Since D is big, we deduce that D' is big, and furthermore,

$$\mathrm{ord}_E(|mD|) = \mathrm{ord}_E(|mD'|) + \sum_{i=1}^r ms_i \cdot \mathrm{ord}_E(\Gamma_i).$$

Dividing by m and letting m go to infinity gives the formula in ii).

In order to prove iii), we need to show that for every nonzero rational function ϕ on X such that $\mathrm{div}_X(\phi) + D$ is effective, we have $\mathrm{div}_X + D' \ge 0$. Let m be such that mD is an integral divisor and all ms_i are integers. Since (1.22) is an isomorphism, it follows that $\phi^m \in H^0(X, \mathscr{O}_X(mD))$ is in the image of $H^0(X, \mathscr{O}_X(mD'))$, hence $\mathrm{div}_X(\phi^m) + mD'$ is effective. This implies that $\mathrm{div}_X(\phi) + D'$ is effective. We have thus proved the assertions in the proposition when D is a $\mathbb{Q}$-divisor and all s_i are rational.

Suppose now that D and the s_i are arbitrary, as in the proposition. We consider a sequence $(A_m)_{m\ge 1}$ of ample $\mathbb{R}$-divisors such that each $D - A_m$ is a big $\mathbb{Q}$-divisor and the classes of A_m in $\mathrm{N}^1(X)_{\mathbb{R}}$ converge to 0. Note that for every m we have

$$\mathrm{ord}_{\Gamma_i}(\| D \|) \le \mathrm{ord}_{\Gamma_i}(\| D - A_m \|).$$

We also choose sequences $(s_{i,m})_{m\ge 1}$ of rational numbers, with $s_{i,m} \le s_i$ for all i and m, and $\lim_{m\to\infty} s_{i,m} = s_i$. By choosing each $s_i - s_{i,m}$ small enough, we may assume that each

$$A_m - \sum_{i=1}^r (s_i - s_{i,m})\Gamma_i \text{ is ample.}$$

In this case $D - A_m - \sum_{i=1}^r s_{i,m}\Gamma_i$ is big by the case we already proved, hence

$$D' = \left(D - A_m - \sum_{i=1}^r s_{i,m}\Gamma_i\right) + \left(A_m - \sum_{i=1}^r (s_i - s_{i,m})\Gamma_i\right)$$

is the sum of a big and an ample divisor, hence it is big. This proves i). Furthermore, the case already proved gives

$$\mathrm{ord}_E(\| D - A_m - \sum_{i=1}^r s_{i,m}\Gamma_i \|) = \mathrm{ord}_E(\| D - A_m \|) - \sum_{i=1}^r s_{i,m}\,\mathrm{ord}_E(\Gamma_i).$$

Letting m go to infinity gives the formula in ii).

In order to prove iii), suppose that ϕ is a nonzero rational function such that $\mathrm{div}_X(\phi) + D$ is effective. Let us write $D = \sum_{j=1}^s \lambda_j D_j$, where the D_j are prime divisors. We choose sequences $(\lambda_{j,m})_{m\geq 1}$ of rational numbers such that $\lambda_{j,m} \geq \lambda_j$ for every j and m and $\lim_{m\to\infty} \lambda_{j,m} = \lambda_j$ for all j. Each $F_m = \sum_{j=1}^s \lambda_{j,m} D_j$ has the property that $F_m - D$ is effective, hence F_m is big. Since each F_m is a $\mathbb{Q}$-divisor and $\mathrm{div}_X(\phi) + F_m$ is effective, we conclude from what we have already proved that

$$\mathrm{div}_X(\phi) + F_m \geq \sum_{i=1}^r \mathrm{ord}_{\Gamma_i}(\| F_m \|) \cdot \Gamma_i.$$

Since $\lim_{m\to\infty} \mathrm{ord}_{\Gamma_i}(\| F_m \|) = \mathrm{ord}_{\Gamma_i}(\| D \|)$ for every i, by letting m go to infinity, we conclude that $\mathrm{div}_X(\phi) + D' \geq 0$. This completes the proof of the proposition. $\square$

For an $\mathbb{R}$-divisor D on a smooth variety, there is a more explicit description for $\mathrm{ord}_E(\| D \|)$, as follows.

Proposition 1.7.26. *If X is a smooth, projective variety and D is a big $\mathbb{R}$-divisor on X, then for every divisor E over X, we have*

$$\mathrm{ord}_E(\| D \|) = \lim_{m\to\infty} \frac{\mathrm{ord}_E(|\lfloor mD \rfloor|)}{m}.$$

Proof. We first show that if D is a big $\mathbb{R}$-divisor on X, then there is $t_0 \in \mathbb{R}_{>0}$ such that the linear system $|\lfloor tD \rfloor|$ is nonempty for $t \geq t_0$. Indeed, we can write $D = A + F$, with A ample and F effective, and since $\lfloor tD \rfloor \geq \lfloor tA \rfloor + \lfloor tF \rfloor$, it is enough to show that the linear system $|\lfloor tA \rfloor|$ is nonempty for $t \gg 0$. Note that we can write $A = \sum_{i=1}^r \alpha_i A_i$, with the A_i ample Cartier divisors and $\alpha_i \in \mathbb{R}_{>0}$ (see Remark 1.3.24). Since

$$\lfloor tA \rfloor \geq \sum_{i=1}^r \lfloor t\alpha_i A_i \rfloor,$$

we may assume that A is a Cartier divisor. For $0 < t \leq 1$, there are only finitely many possible sheaves $\mathscr{O}_X(\lfloor tA \rfloor)$. Since A is ample, we conclude that there is a positive integer m_0 such that $\mathscr{O}_X(tA + mA)$ is globally generated for all $0 < t \leq 1$ and all integers $m \geq m_0$. It is then clear that $|\lfloor tA \rfloor|$ is nonempty for all $t \geq m_0$.

The next step is to observe that if D_1 and D_2 are any two $\mathbb{R}$-divisors such that the linear systems $|\lfloor D_1\rfloor|$ and $|\lfloor D_2\rfloor|$ are nonempty, then the linear system $|\lfloor D_1+D_2\rfloor|$ is nonempty and

$$\mathrm{ord}_E(|\lfloor D_1+D_2\rfloor|)\leq \mathrm{ord}_E(|\lfloor D_1\rfloor|)+\mathrm{ord}_E(|\lfloor D_2\rfloor|)+\sum_{\Gamma}\mathrm{ord}_E(\Gamma), \tag{1.23}$$

where the sum is over the prime divisors Γ that appear in $\mathrm{Supp}(D_1)\cap\mathrm{Supp}(D_2)$. Indeed, the assertion follows from the fact that

$$\lfloor D_1+D_2\rfloor-(\lfloor D_1\rfloor+\lfloor D_2\rfloor)$$

is a reduced effective divisor, supported on $\mathrm{Supp}(D_1)\cup\mathrm{Supp}(D_2)$.

Given an arbitrary big $\mathbb{R}$-divisor D, we conclude that for $p,q\gg 0$ we have

$$\mathrm{ord}_E(|\lfloor (p+q)D\rfloor|)\leq \mathrm{ord}_E(|\lfloor pD\rfloor|)+\mathrm{ord}_E(|\lfloor qD\rfloor|)+\ell_D,$$

where $\ell_D=\sum_{\Gamma}\mathrm{ord}_E(\Gamma)$, with the sum being over all prime divisors in the support of D. It follows from Lemma 1.7.9 that

$$\lim_{m\to\infty}\frac{\mathrm{ord}_E(|\lfloor mD\rfloor|)}{m}=\inf_{m\geq 1}\frac{\mathrm{ord}_E(|\lfloor mD\rfloor|)+\ell_D}{m}. \tag{1.24}$$

Let us temporarily denote this limit by $\psi(D)$. It follows from the definition and (1.23) that for every two big $\mathbb{R}$-divisors D_1 and D_2, we have $\psi(D_1+D_2)\leq\psi(D_1)+\psi(D_2)$.

We now prove that for every $\mathbb{R}$-divisor D, we have

$$\lim_{\lambda\to 0}\psi(\lambda D)=0. \tag{1.25}$$

Let t_0 and ℓ_D be as above. Given $0<\lambda<1$, we take $m=\lceil t_0/\lambda\rceil$, hence $t_0\leq m\lambda<t_0+1$. When we vary λ, there are only finitely many linear systems $|\lfloor\lambda mD\rfloor|$, and by assumption, they are all nonempty. It follows from (1.24) that

$$\psi(\lambda D)\leq\frac{\mathrm{ord}_E(|\lfloor\lambda mD\rfloor|)+\ell_D}{m},$$

and since m goes to infinity when λ goes to 0, we obtain (1.25).

We can now show that $\psi(D)=\mathrm{ord}_E(\|D\|)$ for every big $\mathbb{R}$-divisor D. If D is a $\mathbb{Q}$-divisor, this is clear by taking the limit in the definition of $\psi(D)$ over divisible enough m. Suppose now that D is an arbitrary big $\mathbb{R}$-divisor. By definition, we can write $D=\sum_{i=1}^r\lambda_iF_i$, with F_i big Cartier divisors and $\lambda_i\in\mathbb{R}_{>0}$. We choose sequences $(\lambda'_{i,m})_{m\geq 1}$ and $(\lambda''_{i,m})_{m\geq 1}$ of positive rational numbers with $\lambda'_{i,m}<\lambda_i<\lambda''_{i,m}$ for all m and $\lim_{m\to\infty}\lambda'_{i,m}=\lambda_i=\lim_{m\to\infty}\lambda''_{i,m}$ for all i with $1\leq i\leq r$. If $D'_m=\sum_{i=1}^r\lambda'_{i,m}F_i$ and $D''_m=\sum_{i=1}^r\lambda''_{i,m}F_i$, then $D'_m\leq D\leq D''_m$ and D'_m, D''_m are big for all m. We have

$$\mathrm{ord}_E(\| D''_m \|) = \psi(D''_m) \leq \psi(D) + \psi(D''_m - D) \leq \psi(D) + \sum_{i=1}^{r} \psi((\lambda''_{i,m} - \lambda_i)F_i),$$

and by letting m go to infinity, we obtain $\mathrm{ord}_E(\| D \|) \leq \psi(D)$. Similarly, we have

$$\psi(D) \leq \psi(D'_m) + \psi(D - D'_m) \leq \mathrm{ord}_E(\| D'_m \|) + \sum_{i=1}^{r} \psi((\lambda_i - \lambda'_{i,m})F_i),$$

hence $\psi(D) \leq \mathrm{ord}_E(\| D \|)$. We thus conclude that $\psi(D) = \mathrm{ord}_E(\| D \|)$. □

We will return to the study of asymptotic invariants in Sections **??** and 5.1. We will then show that at least when X is a smooth variety over an uncountable field of characteristic zero, the vanishing of $\mathrm{ord}_E(\| D \|)$, for a big divisor D, is equivalent to the fact that the center of E on X is not contained in the non-nef locus of D. We will give two proofs of this result, first using the Kawamata-Viehweg vanishing theorem and then using results about asymptotic multiplier ideals.

1.7.5 Invariants of pseudo-effective divisors

We now extend the function $\mathrm{ord}_E(\| - \|)$ to pseudo-effective divisors. Let X be a projective variety and E a divisor over X. If $D \in \mathrm{CDiv}(X)_{\mathbb{R}}$ is pseudo-effective, then for every ample $A \in \mathrm{CDiv}(X)_{\mathbb{R}}$, we have $D + A$ big. We put

$$\sigma_E(D) := \sup_A \mathrm{ord}_E(\| D + A \|) \in \mathbb{R}_{\geq 0} \cup \{\infty\},$$

where the supremum is over all $A \in \mathrm{CDiv}(X)_{\mathbb{R}}$ ample. It is clear from definition that $\sigma_E(D)$ only depends on the numerical class of D, hence we may consider σ_E as a function on the pseudo-effective cone of X.

Lemma 1.7.27. *With the above notation, if $(A_m)_{m\geq 1}$ is a sequence of ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisors on X converging to 0 in $\mathrm{N}^1(X)_{\mathbb{R}}$, then*

$$\sigma_E(D) = \sup_{m\geq 1} \mathrm{ord}_E(\| D + A_m \|) = \lim_{m\to\infty} \mathrm{ord}_E(\| D + A_m \|).$$

Proof. It follows from definition that $\mathrm{ord}_E(\| D + A_m \|) \leq \sigma_E(D)$ for every m. On the other hand, for every $A \in \mathrm{CDiv}(X)_{\mathbb{R}}$ ample, we have $A - A_m$ ample for $m \gg 0$, hence

$$\mathrm{ord}_E(\| D + A \|) \leq \mathrm{ord}_E(\| D + A_m \|)$$

for $m \gg 0$. The assertion in the lemma now follows from the definition of $\sigma_E(D)$. □

Corollary 1.7.28. *If $D \in \mathrm{CDiv}(X)_{\mathbb{R}}$ is big, then $\sigma_E(D) = \mathrm{ord}_E(\| D \|)$.*

Proof. The assertion follows from Lemma 1.7.27 and the continuity of the function $\mathrm{ord}_E(\| - \|)$ on the big cone. □

Lemma 1.7.29. *Let $f\colon Y \to X$ be a projective, birational morphism of projective n-dimensional varieties, with Y normal, and E a prime divisor on Y. If H is an ample Cartier divisor on Y, then for every $D \in \mathrm{CDiv}(X)_{\mathbb{R}}$ big, we have*

$$\mathrm{ord}_E(\| D \|) \le \frac{(H^{n-1} \cdot f^*(D))}{(H^{n-1} \cdot E)}.$$

Proof. By continuity, it is enough to prove the assertion when $D \in \mathrm{CDiv}(X)_{\mathbb{Q}}$. Let m be a positive integer which is divisible enough, such that mD is Cartier and $|mD| \neq \emptyset$. Let $F \in |mD|$ be general. If $a = \mathrm{ord}_E(F)$, it follows from the ampleness of H that

$$(H^{n-1} \cdot f^*(mD)) = (H^{n-1} \cdot f^*(F)) \ge a \cdot (H^{n-1} \cdot E).$$

Since $\mathrm{ord}_E(\| D \|) \le \frac{a}{m}$, we obtain the inequality in the lemma. □

Corollary 1.7.30. *If X is a projective variety and E is a divisor over X, then for every pseudo-effective $D \in \mathrm{CDiv}(X)_{\mathbb{R}}$, we have $\sigma_E(D) < \infty$.*

Proof. Let $f\colon Y \to X$ be a projective, birational morphism, with Y normal, such that E is a prime divisor on Y, and let H be an ample Cartier divisor on Y. If $(A_m)_{m\ge 1}$ is a sequence of ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors on X whose classes converge to 0 in $\mathrm{N}^1(X)_{\mathbb{R}}$, then it follows from Lemma 1.7.29 that

$$\mathrm{ord}_E(\| D + A_m \|) \le \frac{(H^{n-1} \cdot f^*(D + A_m))}{(H^{n-1} \cdot E)}.$$

By letting m go to infinity and using Lemma 1.7.27, we obtain

$$\sigma_E(D) \le \frac{(H^{n-1} \cdot f^*(D))}{(H^{n-1} \cdot E)} < \infty.$$

□

Proposition 1.7.31. *If X is a projective variety and E is a divisor over X, then the following hold:*

i) The function $\sigma_E\colon \mathrm{PEff}(X) \to \mathbb{R}_{\ge 0}$ is lower semi-continuous[9].

ii) If $D \in \mathrm{CDiv}(X)_{\mathbb{R}}$ is pseudo-effective and $c_X(E) \not\subseteq \mathrm{B}_-(D)$, then $\sigma_E(D) = 0$. In particular, $\sigma_E(D) = 0$ for every nef D.

iii) For every $D \in \mathrm{PEff}(X)$ and every $\lambda \in \mathbb{R}_{>0}$, we have $\sigma_E(\lambda D) = \lambda \cdot \sigma_E(D)$.

iv) For every $D, D' \in \mathrm{PEff}(X)$, we have

$$\sigma_E(D + D') \le \sigma_E(D) + \sigma_E(D').$$

[9] Recall that a map $\phi\colon W \to \mathbb{R}$ is lower semi-continuous if for every $\alpha \in \mathbb{R}$, the inverse image of the interval (α, ∞) is open. Equivalently, for every $u_0 \in W$, we have $\liminf_{u \to u_0} \phi(u) \ge \phi(u_0)$.

Proof. The assertion in i) follows from definition and the fact, which is easy to check, that the supremum of every family of continuous functions is lower semi-continuous. If D is pseudo-effective and $c_X(E) \not\subseteq \mathrm{B}_-(D)$, then for every ample A we have $\mathrm{B}_-(D+A) \subseteq \mathrm{B}_-(D)$ and therefore $c_X(E) \not\subseteq \mathrm{B}_-(D+A)$. It follows from Corollary 1.7.21 that $\mathrm{ord}_E(\| D+A \|) = 0$ and since this holds for every A ample, we conclude that $\sigma_E(D) = 0$. We thus obtain the first assertion in ii) and the second one is an immediate consequence. The assertions in iii) and iv) follow from the corresponding properties of $\mathrm{ord}_E(\| - \|)$ on $\mathrm{Big}(X)$, by computing σ_E as a limit using Lemma 1.7.27. ☐

Remark 1.7.32. If $D \in \mathrm{CDiv}(X)_{\mathbb{R}}$ is effective, then $\sigma_E(D) \leq \mathrm{ord}_E(D)$. Indeed, for every ample effective Cartier divisor A, we have

$$\mathrm{ord}_E\left(\| D+\frac{1}{m}A \|\right) \leq \mathrm{ord}_E\left(D+\frac{1}{m}A\right) = \mathrm{ord}_E(D)+\frac{1}{m}\,\mathrm{ord}_E(A).$$

By letting m go to infinity and using Lemma 1.7.27, we obtain the desired inequality.

Remark 1.7.33. Nakayama gave an example in which the function σ_E is not continuous on the pseudo-effective cone (see [Nak04, Example IV.2.8]). In particular, we see that in this case the function $\mathrm{ord}_E(\| - \|)$ does not admit a continuous extension to the pseudo-effective cone.

Remark 1.7.34. If $D \in \mathrm{PEff}(X)$ and $B \in \mathrm{CDiv}(X)_{\mathbb{R}}$ is big, then for every divisor E over X, we have

$$\sigma_E(D) = \lim_{t\to 0} \mathrm{ord}_E\left(\| D+tB \|\right). \tag{1.26}$$

Indeed, note first that if $t > 0$, then $D+tB$ is big and we thus have

$$\mathrm{ord}_E\left(\| D+tB \|\right) = \sigma_E\left(D+tB\right) \leq \sigma_E(D)+t\sigma_E(B)$$

for every $t > 0$, hence

$$\limsup_{t\to 0} \mathrm{ord}_E\left(\| D+tB \|\right) \leq \sigma_E(D). \tag{1.27}$$

On the other hand, given any ample $A \in \mathrm{CDiv}(X)_{\mathbb{R}}$, for $0 < t \ll 1$ we have $A-tB$ ample, hence $\mathrm{ord}_E(\| D+A \|) \leq \mathrm{ord}_E(\| D+tB \|)$. By the definition of $\sigma_E(D)$, we obtain

$$\sigma_E(D) \leq \liminf_{t\to 0} \mathrm{ord}_E\left(\| D+tB \|\right). \tag{1.28}$$

By combining (1.27) and (1.27), we deduce (1.26).

Proposition 1.7.35. *If $f\colon Y \to X$ is a birational morphism of projective varieties, with X normal, then for every $D \in \mathrm{PEff}(X)$ and every divisor E over X we have*

$$\sigma_E(D) = \sigma_E(f^*(D)).$$

Proof. When D is big, this follows from Proposition 1.7.22. Suppose now that $A \in \mathrm{CDiv}(X)_{\mathbb{R}}$ is big, hence $B = f^*(A)$ has the same property. Using Remark 1.7.34, we obtain

$$\sigma_E(D) = \lim_{m\to\infty} \mathrm{ord}_E\left(\| D + \frac{1}{m}A \|\right) = \lim_{m\to\infty} \mathrm{ord}_E\left(\| f^*(D) + \frac{1}{m}B \|\right) = \sigma_E(f^*(D)).$$

□

We have the following version of Proposition 1.7.25 for pseudo-effective divisors.

Proposition 1.7.36. *Let X be a smooth projective variety, E a divisor over X, and $\Gamma_1, \dots, \Gamma_r$ mutually distinct prime divisors on X. If D is a pseudo-effective $\mathbb{R}$-divisor on X and $0 \le s_i \le \sigma_{\Gamma_i}(\| D \|)$ for $1 \le i \le r$, then*

i) $D' := D - \sum_{i=1}^r s_i\Gamma_i$ is pseudo-effective.
ii) $\sigma_E(D') = \sigma_E(D) - \sum_{i=1}^r s_i \cdot \mathrm{ord}_E(\Gamma_i)$.
iii) The inclusion

$$H^0(X, \mathscr{O}_X(D')) \hookrightarrow H^0(X, \mathscr{O}_X(D))$$

induced by the effective divisor $\sum_{i=1}^r s_i\Gamma_i$ is an isomorphism.

Proof. The case when D is big follows from Proposition 1.7.25. We may assume that all $s_i > 0$ by ignoring the ones that are 0. For every m such that $\frac{1}{m} < s_i$ for all i, we consider an ample $\mathbb{R}$-divisor A_m such that

$$\sigma_{\Gamma_i}(D) - \frac{1}{m} \le \sigma_{\Gamma_i}(D + A_m) \le \sigma_{\Gamma_i}(D) \quad \text{for all } i.$$

We may also assume that the classes of A_m in $\mathrm{N}^1(X)_{\mathbb{R}}$ converge to 0. For every m as above, we choose $s_{i,m}$ such that $s_i - \frac{1}{m} \le s_{i,m} \le s_i$ and $s_{i,m} \le \sigma_{\Gamma_i}(D + A_m)$ for all i and m. Since all $D + A_m$ are big, we conclude that each

$$D + A_m - \sum_{i=1}^r s_{i,m}\Gamma_i \text{ is big}$$

and by passing to limit in $\mathrm{N}^1(X)_{\mathbb{R}}$, that D' is pseudo-effective.

Furthermore, we know that

$$\sigma_E\left(D + A_m - \sum_{i=1}^r s_{i,m}\Gamma_i\right) = \sigma_E(D + A_m) - \sum_{i=1}^r s_{i,m} \cdot \mathrm{ord}_E(\Gamma_i).$$

By Lemma 1.7.27, the right-hand side converges to $\sigma_E(D) - \sum_{i=1}^r s_i \cdot \mathrm{ord}_E(\Gamma_i)$. Using the lower semi-continuity of σ_E, we deduce that

$$\sigma_E(D') \le \sigma_E(D) - \sum_{i=1}^r s_i \cdot \mathrm{ord}_E(\Gamma_i).$$

On the other hand, the opposite inequality follows from Lemma 1.7.31iv) and Remark 1.7.32. This completes the proof of ii).

We may assume that each A_m is effective and for every prime divisor Γ on X, we have $\mathrm{ord}_{\Gamma}(A_m) \leq 1/m$. Indeed, if $A_m \equiv \sum_j a_j F_j$, where all a_j are positive and all F_j are ample Cartier divisors, then we may replace A_m by $\sum_j \frac{a_j}{q} F'_j$, where $q > m \cdot \max_j a_j$ is such that all $\mathscr{O}_X(qF_j)$ are very ample, and $F'_j \in |qF_j|$ are general elements. Suppose now that ϕ is a nonzero rational function on X such that $\mathrm{div}_X(\phi) + D \geq 0$. Since each A_m is effective, it follows that $\mathrm{div}_X(\phi) + D + A_m \geq 0$, and the big case implies

$$\mathrm{div}_X(\phi) + D + A_m \geq \sum_{i=1}^{r} s_{i,m} \Gamma_i. \tag{1.29}$$

Since $\lim_{m \to \infty} \mathrm{ord}_{\Gamma}(A_m) = 0$ for every prime divisor Γ on X, we may pass to limit in (1.29) to deduce that $\mathrm{div}_X(\phi) + D' \geq 0$. This completes the proof of the proposition. $\square$

1.7.6 Divisorial Zariski decompositions

This is a notion introduced by Nakayama. In what follows, we follow the approach in [Nak04].

Lemma 1.7.37. *Let D be a pseudo-effective $\mathbb{R}$-divisor on a smooth projective variety X. If $\Gamma_1, \ldots, \Gamma_r$ are mutually distinct prime divisors on X such that $\sigma_{\Gamma_i}(D) > 0$ for all i, then*

$$\sigma_E(D + t_1\Gamma_1 + \ldots + t_r\Gamma_r) = \sigma_E(D) + \sum_{i=1}^{r} t_i \cdot \mathrm{ord}_E(\Gamma_i)$$

for every divisor E over X and every $t_1, \ldots, t_r \in \mathbb{R}_{\geq 0}$.

Proof. The inequality "$\leq$" follows from Proposition 1.7.31iv) and the fact that $\sigma_E(t_i\Gamma_i) \leq t_i \cdot \mathrm{ord}_E(\Gamma_i)$ for all i (see Remark 1.7.32). In order to prove the reverse inequality, we argue by induction on $m \in \mathbb{Z}_{\geq 0}$, where we make the assumption that $t_i \leq m \cdot \sigma_{\Gamma_i}(D)$ for all i. Note that the case $m = 0$ is trivial. Let us prove now the induction step. Suppose that $t_i \leq (m+1) \cdot \sigma_{\Gamma_i}(D)$ for all i. We may choose $0 \leq s_i \leq \sigma_{\Gamma_i}(D)$ for each i such that $t_i - s_i \leq m \cdot \sigma_{\Gamma_i}(D)$. Using the inductive hypothesis, Proposition 1.7.31, and Proposition 1.7.36, we obtain

$$2\left(\sigma_E(D) + \sum_{i=1}^{r} \frac{(t_i - s_i)}{2} \cdot \mathrm{ord}_E(\Gamma_i)\right) = 2 \cdot \sigma_E\left(D + \sum_{i=1}^{r} \frac{(t_i - s_i)}{2} \cdot \Gamma_i\right)$$

$$= \sigma_E\left(2D + \sum_{i=1}^{r} (t_i - s_i) \cdot \Gamma_i\right) \leq \sigma_E\left(D + \sum_{i=1}^{r} t_i\Gamma_i\right) + \sigma_E\left(D - \sum_{i=1}^{r} s_i\Gamma_i\right)$$

$$= \sigma_E\left(D + \sum_{i=1}^{r} t_i\Gamma_i\right) + \sigma_E(D) - \sum_{i=1}^{r} s_i \cdot \mathrm{ord}_E(\Gamma_i).$$

We conclude that

$$\sigma_E\left(D + \sum_{i=1}^{r} t_i\Gamma_i\right) \geq \sigma_E(D) + \sum_{i=1}^{r} t_i \cdot \mathrm{ord}_E(\Gamma_i),$$

which completes the proof of the induction step, and therefore that of the proposition. □

Corollary 1.7.38. *Let D be a pseudo-effective $\mathbb{R}$-divisor on a smooth projective variety X. If $\Gamma_1,\ldots,\Gamma_r$ are mutually distinct prime divisors on X such that $\sigma_{\Gamma_i}(D) > 0$ for all i, then*

$$\sigma_E(t_1\Gamma_1 + \ldots + t_r\Gamma_r) = \sum_{i=1}^{r} t_i \cdot \mathrm{ord}_E(\Gamma_i)$$

for every divisor E over X and every $t_1,\ldots,t_r \in \mathbb{R}_{\geq 0}$.

Proof. The inequality "$\leq$" follows from Proposition 1.7.31iv) and the fact that $\sigma_E(t_i\Gamma_i) \leq t_i \cdot \mathrm{ord}_E(\Gamma_i)$ for all i. Furthermore, if the inequality is strict for some $t_1,\ldots,t_r \in \mathbb{R}_{\geq 0}$, then another application of Proposition 1.7.31iv) gives

$$\sigma_E(D+t_1\Gamma_1+\ldots+t_r\Gamma_r) \leq \sigma_E(D)+\sigma_E(t_1\Gamma_1+\ldots+t_r\Gamma_r) < \sigma_E(D) + \sum_{i=1}^{r} t_i \cdot \mathrm{ord}_E(\Gamma_i).$$

This contradicts Lemma 1.7.37. □

Corollary 1.7.39. *If D is a pseudo-effective $\mathbb{R}$-divisor on a smooth projective variety X and $\Gamma_1,\ldots,\Gamma_r$ are mutually distinct prime divisors on X with $\sigma_{\Gamma_i}(D) > 0$ for all i, then the Γ_i are linearly independent in $\mathrm{N}^1(X)_{\mathbb{R}}$. In particular, we have $r \leq \rho = \dim_R \mathrm{N}^1(X)_{\mathbb{R}}$.*

Proof. Suppose that $\Gamma_1,\ldots,\Gamma_r$ are linearly dependent in $\mathrm{N}^1(X)_{\mathbb{R}}$. After reordering them, we may assume that we have a relation

$$\sum_{i=1}^{d} a_i\Gamma_i \equiv \sum_{i=d+1}^{r} a_i\Gamma_i, \tag{1.30}$$

where all $a_i \in \mathbb{R}_{\geq 0}$ and $a_1 > 0$. On one hand, Corollary 1.7.38 implies

$$\sigma_{\Gamma_1}\left(\sum_{i=1}^{d} a_i\Gamma_i\right) = a_1 > 0.$$

On the other hand, since σ_{Γ_1} only depends on the numerical class of a divisor, using Remark 1.7.32 we deduce from (1.30)

$$\sigma_{\Gamma_1}\left(\sum_{i=1}^{d} a_i\Gamma_i\right) = \sigma_{\Gamma_1}\left(\sum_{i=d+1}^{r} a_i\Gamma_i\right) = 0.$$

This gives a contradiction and thus proves the assertion in the corollary. $\square$

Definition 1.7.40. Let D be a pseudo-effective $\mathbb{R}$-divisor on the smooth projective variety X. By Corollary 1.7.39,

$$N_\sigma(D) := \sum_\Gamma \sigma_\Gamma(D)\Gamma,$$

where Γ varies over the prime divisors on X, is an $\mathbb{R}$-divisor on X. Note that by definition, this only depends on the numerical class of D. One puts $P_\sigma(D) := D - N_\sigma(D)$ and the decomposition

$$D = N_\sigma(D) + P_\sigma(D)$$

is the *divisorial Zariski decomposition of* D, while $N_\sigma(D)$ and $P_\sigma(D)$ are the *negative* and, respectively, the *positive* part of this decomposition. Note that Propositions 1.7.36 and 1.7.25 imply that $P_\sigma(D)$ is pseudo-effective and it is big if D is big.

Definition 1.7.41. If D is a pseudo-effective $\mathbb{R}$-divisor on the smooth projective variety X, one says that D has a *Zariski decomposition* if the divisor $P_\sigma(D)$ is nef, and in this case the decomposition $D = N_\sigma(D) + P_\sigma(D)$ is the Zariski decomposition of D.

Proposition 1.7.42. *Let D be a pseudo-effective $\mathbb{R}$-divisor on the smooth projective variety X. If D has a Zariski decomposition, then for every projective, birational morphism $f\colon Y \to X$, with Y smooth, $f^*(D)$ has a Zariski decomposition and*

$$N_\sigma(f^*(D)) = f^*(N_\sigma(D)) \text{ and } P_\sigma(f^*(D)) = f^*(P_\sigma(D)).$$

Proof. If E is a prime divisor on Y, then $\sigma_E(f^*(D)) = \sigma_E(D)$ by Proposition 1.7.35. On the other hand, it follows from Proposition 1.7.36 that

$$\sigma_E(D) = \sigma_E(P_\sigma(D)) + \mathrm{ord}_E(N_\sigma(D)) = \mathrm{ord}_E(N_\sigma(D)),$$

where the second equality follows from the fact that by assumption $P_\sigma(D)$ is nef. This implies that $N_\sigma(f^*(D)) = f^*(N_\sigma(D))$, and therefore $P_\sigma(f^*(D)) = f^*(P_\sigma(D))$. In particular, $P_\sigma(f^*(D))$ is nef, and therefore $f^*(D)$ has a Zariski decomposition. $\square$

Remark 1.7.43. Let D be a big $\mathbb{Q}$-divisor on the smooth, projective variety X. It follows from definition that if E is a prime divisor on X such that $\sigma_E(D) > 0$, then $E \subseteq \mathrm{SB}(D)$. In particular, if $\mathrm{codim}_X(\mathrm{SB}(D)) \geq 2$, then $N_\sigma(D) = 0$. This implies that if D is such a divisor which is not nef, then D does not have a Zariski decomposition. Starting with dimension 3, it is easy to construct such examples (as we will see in

Section 5.1.3, Zariski decompositions always exist in dimension 2). On the other hand, on can ask the following: given a pseudo-effective (or big) $\mathbb{R}$-divisor D on the smooth, projective variety X, is there a projective, birational morphism $f\colon Y \to X$, with Y smooth, such that $f^*(D)$ has a Zariski decomposition? Nakayama [Nak04, Chap. IV.2] gave a 3-dimensional example for which there is no such morphism.

We will return to the discussion of the concept of (divisorial) Zariski decomposition in Section 5.1.3.

1.7.7 Asymptotic invariants in the relative setting

We discuss briefly how the definitions and the basic results about asymptotic invariants extend to the relative setting. Since most proofs follow as in the absolute case, we omit them and only point out the differences from that setting. Let $f\colon X \to S$ be a proper morphism of varieties. Given a line bundle $\mathscr{L}$ on X, for every $m \geq 1$, we consider the canonical morphism $f^*f_*(\mathscr{L}^m) \to \mathscr{L}^m$. Its image can be written as $\mathfrak{a}_m \otimes \mathscr{L}^m$ for a unique coherent ideal $\mathfrak{a}_m$. Note that $\mathfrak{a}_m = 0$ if and only if $f_*(\mathscr{L}^m) = 0$. If U is an affine, open subset of S, then the restriction of $\mathfrak{a}_m$ to $f^{-1}(U)$ is the ideal defining the base locus of $\mathscr{L}^m|_{f^{-1}(U)}$. It is clear that $\mathfrak{a}_\bullet$ is a graded sequence of ideals on X.

With the above notation, suppose that $f_*(\mathscr{L}^m)$ is nonzero for some $m \geq 1$. For every divisor E over X, we put

$$\mathrm{ord}_E(\parallel \mathscr{L}/S \parallel) := \mathrm{ord}_E(\mathfrak{a}_\bullet).$$

Note that if U is an affine open subset of S that intersects the image of E, then E also gives a divisor over $f^{-1}(U)$ and if $\mathscr{L}_U$ is the restriction of $\mathscr{L}$ to $f^{-1}(U)$, then $\mathrm{ord}_E(\parallel \mathscr{L}/S \parallel) = \mathrm{ord}_E(\parallel \mathscr{L}_U/U \parallel)$. In this way, we can always reduce the study of asymptotic invariants in the relative setting to the case when S is affine, when almost everything follows as in the absolute case.

In particular, we have $\mathrm{ord}_E(\parallel \mathscr{L}/S \parallel) = \frac{1}{m} \cdot \mathrm{ord}_E(\parallel \mathscr{L}^m/S \parallel)$ for every positive integer m. Using this, we can define $\mathrm{ord}_E(\parallel D/S \parallel)$ for every $D \in \mathrm{CDiv}(X)_{\mathbb{Q}}$ such that $f_*(\mathscr{O}_X(mD)) \neq 0$ for some m such that mD is Cartier. This satisfies the following properties:

i) $\mathrm{ord}_E(\parallel \lambda D/S \parallel) = \lambda \cdot \mathrm{ord}_E(\parallel D/S \parallel)$ for every $\lambda \in \mathbb{Q}_{>0}$.
ii) $\mathrm{ord}_E(\parallel D+D'/S \parallel) \leq \mathrm{ord}_E(\parallel D/S \parallel) + \mathrm{ord}_E(\parallel D'/S \parallel)$ if both $\mathrm{ord}_E(\parallel D/S \parallel)$ and $\mathrm{ord}_E(\parallel D'/S \parallel)$ are defined.
iii) $\mathrm{ord}_E(\parallel D/S \parallel) = 0$ if $\mathscr{O}_X(mD)$ is f-base-point free for m divisible enough.

Suppose now that $f\colon X \to S$ is a projective, surjective morphism of varieties. Note that if D is f-big, then it follows from Proposition 1.6.32 that $\mathrm{ord}_E(\parallel D/S \parallel)$ is defined. Moreover, if D and D' are f-big and $D \equiv_f D'$, then

$$\mathrm{ord}_E(\parallel D/S \parallel) = \mathrm{ord}_E(\parallel D'/S \parallel).$$

In order to see this, we may assume that S is affine. In this case, one can prove a variant of Lemma 1.7.16, using the fact that there is a line bundle $\mathscr{A}$ on X such that for every f-nef $\mathscr{M} \in \mathrm{Pic}(X)$, the line bundle $\mathscr{A} \otimes \mathscr{M}$ is globally generated (see Corollary 2.6.7).

We thus obtain a continuous function $\mathrm{ord}_E(\| -/S \|)$ defined on the rational points of the f-big cone $\mathrm{Big}(X/S)$. In fact, this can be extended to a continuous function on the whole f-big cone. In order to describe this, we may assume that S is affine. In this case, for every $D \in \mathrm{CDiv}(X)_{\mathbb{R}}$ which is f-big we put

$$\mathrm{ord}_E(\| D/S \|) := \min\left\{\mathrm{ord}_E(B) \mid B \in \mathrm{CDiv}(X)_{\mathbb{R}}, B \equiv_f D, \text{ and } B \text{is effective}\right\}.$$

This is compatible with the previous definition and gives a convex, hence continuous function on $\mathrm{Big}(X/S)$ (cf. Proposition 1.7.20 and 1.7.20). In particular, we see that if $D \in \mathrm{CDiv}(X)_{\mathbb{R}}$ is f-big and f-nef, then $\mathrm{ord}_E(\| D/S \|) = 0$ for every divisor E over X. As in the absolute case, we see that if $\mu\colon Y \to X$ is a projective, birational morphism of normal varieties, then $\mathrm{ord}_E(\| D/S \|) = \mathrm{ord}_E(\| \mu^*(D)/S \|)$ for every $D \in \mathrm{CDiv}(X/S)_{\mathbb{R}}$ which is f-big.

Proposition 1.7.25 has an analogue in the relative setting as follows. Let $f\colon X \to S$ be a projective, surjective morphism of varieties, with X smooth. If $D \in \mathrm{CDiv}(X)_{\mathbb{R}}$ is f-big, $\Gamma_1, \ldots, \Gamma_r$ are prime divisors on X, and $0 \leq s_i \leq \mathrm{ord}_{\Gamma_i}(\| D/S \|)$, then $D' := D - \sum_{i=1}^r s_i \Gamma_i$ is f-big and for every divisor E over X, we have

$$\mathrm{ord}_E(\| D'/S \|) = \mathrm{ord}_E(\| D/S \|) - \sum_{i=1}^r s_i \cdot \mathrm{ord}_E(\Gamma_i).$$

Furthermore, the natural inclusion $\pi_*\mathscr{O}_X(D') \hookrightarrow \pi_*\mathscr{O}_X(D)$ induced by the effective divisor $\sum_{i=1}^r s_i\Gamma_i$ is an isomorphism.

If $D \in \mathrm{CDiv}(X)_{\mathbb{R}}$ is pseudo-effective and E is a divisor over X, then we put

$$\sigma_E(D/S) := \sup\left\{\mathrm{ord}_E(\| D + A/S \|) \mid A \text{ is } f-\text{ample}\right\}.$$

If $(A_m)_{m\geq 1}$ is a sequence of f-ample divisors, then in fact

$$\sigma_E(D/S) = \sum_{m\geq 1} \mathrm{ord}_E(\| D + A_m/S \|) = \lim_{m\to\infty} \mathrm{ord}_E(\| D + A_m/S \|).$$

If D is f-big, then $\mathrm{ord}_E(\| D/S \|) = \sigma_E(D)$. It is clear that we thus obtain a function on $\mathrm{PEff}(X/S)$ with values in $\mathbb{R}_{\geq 0} \cup \{\infty\}$. The main difference from the absolute case is that it can happen that $\sigma_E(D/S)$ is infinite. Otherwise, the general properties of this function given in Propositions 1.7.31, 1.7.35, 1.7.36 and Remark 1.7.34 also hold in the relative setting.

Lemma 1.7.37 and Corollaries 1.7.38 and 1.7.39 also hold in the relative setting. We can still define the divisorial Zariski decomposition $D = N_\sigma(D/S) + P_\sigma(X/S)$ when D is f-big. When D is only f-pseudo-effective, this does not make sense since some of the invariants $\sigma_\Gamma(D/S)$ might be infinite, hence $N_\sigma(D/S)$ cannot be defined.

1.8 Finitely generated section rings

In this section we discuss the section ring associated to a finite set of line bundles, with a focus on the good properties that hold when such a ring is finitely generated. Let us fix some notation for what follows. Given line bundles $\mathscr{L}_1,\ldots,\mathscr{L}_r$ on a variety X, for every $u=(u_1,\ldots,u_r)\in\mathbb{N}^r$, we put $\mathscr{L}^u=\mathscr{L}_1^{u_1}\otimes\ldots\otimes\mathscr{L}_r^{u_r}$. Similarly, if $D_1,\ldots,D_r$ are Cartier divisors on X and $u\in\mathbb{R}^r_{\geq 0}$, we put $D_u=\sum_{i=1}^r u_iD_i$.

1.8.1 The ring of sections of a line bundle

Let X be a fixed complete variety over a field k. Given line bundles $\mathscr{L}_1,\ldots,\mathscr{L}_r$ on X, the *section ring* of $\mathscr{L}_1,\ldots,\mathscr{L}_r$ is

$$R(X;\mathscr{L}_1,\ldots,\mathscr{L}_r):=\bigoplus_{u\in\mathbb{N}^r}\Gamma(X,\mathscr{L}^u).$$

Multiplication of sections makes this an $\mathbb{N}^r$-graded k-algebra whose degree 0 part is $H^0(X,\mathscr{O}_X)$. Note that since $H^0(X,\mathscr{O}_X)$ is a finite k-algebra, it follows that $R(X;\mathscr{L}_1,\ldots,\mathscr{L}_r)$ is a finitely generated k-algebra if and only if it is finitely generated as an algebra over its degree 0 part. When $\mathscr{L}_i=\mathscr{O}_X(D_i)$ for Cartier divisors $D_1,\ldots,D_r$, we also write $R(X;D_1,\ldots,D_r)$ for the corresponding section ring. As we will see, such rings are not, in general, finitely generated k-algebras. However, this property holds in important special cases and has such nice consequences, that it is worth studying it.

We begin by noting that $R(X;\mathscr{L}_1,\ldots\mathscr{L}_r)$ is a domain. This is an immediate consequence of Lemma C.0.5 and of the fact that since X is a variety, if s_1 and s_2 are nonzero sections of the line bundles $\mathscr{M}_1$ and $\mathscr{M}_2$, respectively, then $s_1\otimes s_2\in\Gamma(X,\mathscr{M}_1\otimes\mathscr{M}_2)$ is nonzero. Since $R(X;\mathscr{L}_1,\ldots,\mathscr{L}_r)$ is a domain, a general property of graded rings (see Proposition C.0.6) implies the following often useful fact.

Proposition 1.8.1. *If X is a complete variety and $\mathscr{L}_1,\ldots,\mathscr{L}_r\in\mathrm{Pic}(X)$, then for every positive integers $d_1,\ldots,d_r$, the k-algebra $R(X;\mathscr{L}_1,\ldots,\mathscr{L}_r)$ is finitely generated if and only if $R(X;\mathscr{L}_1^{d_1},\ldots,\mathscr{L}_r^{d_r})$ is finitely generated.*

Remark 1.8.2. One can sometimes reduce the study of the section ring of several line bundles to that associated to one line bundle, as follows. If $\mathscr{L}_1,\ldots,\mathscr{L}_r$ are line bundles on the complete variety X, let $W=\mathbb{P}(\mathscr{E})$, where $\mathscr{E}=\mathscr{L}_1\oplus\ldots\oplus\mathscr{L}_r$. If we consider the line bundle $\mathscr{L}=\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$ on W, then for every positive integer m we have a canonical isomorphism

$$\Gamma(W,\mathscr{L}^m)\simeq\bigoplus_{i_1+\ldots+i_r=m}\Gamma(X,\mathscr{L}_1^{i_1}\otimes\ldots\otimes\mathscr{L}_r^{i_r}).$$

We thus obtain an isomorphism of k-algebras $R(X;\mathscr{L}_1,\ldots,\mathscr{L}_r)\simeq R(W;\mathscr{L})$.

Remark 1.8.3. If X is a complete variety and $\mathscr{L}_1,\ldots,\mathscr{L}_r \in \mathrm{Pic}(X)$ are such that $R(X;\mathscr{L}_1,\ldots,\mathscr{L}_r)$ is a finitely generated k-algebra, then for every line bundles $\mathscr{M}_1,\ldots,\mathscr{M}_s$ that lie in the submonoid of $\mathrm{Pic}(X)$ generated by $\mathscr{L}_1,\ldots,\mathscr{L}_r$, the k-algebra $R(X;\mathscr{M}_1,\ldots,\mathscr{M}_s)$ is finitely generated, too. This follows from Proposition C.0.10.

We have the following general criterion for finite generation.

Proposition 1.8.4. *If $\mathscr{L}_1,\ldots,\mathscr{L}_r$ are semiample line bundles on the complete variety X, then $R(X;\mathscr{L}_1,\ldots,\mathscr{L}_r)$ is finitely generated.*

Proof. Proposition 1.8.1 implies that we may replace each $\mathscr{L}_i$ by a suitable power, hence we may and will assume that each $\mathscr{L}_i$ is globally generated. We first consider the case of one line bundle $\mathscr{L}$ which is globally generated.

Let $f\colon X \to \mathbb{P}^N$ be the map defined by $\mathscr{L}$ and consider the Stein factorization $X \xrightarrow{g} Y \xrightarrow{h} \mathbb{P}^N$ of f. Since h is finite, the line bundle $\mathscr{M} := h^*(\mathscr{O}_{\mathbb{P}^N}(1))$ is ample on Y and by definition we have $g^*(\mathscr{M}) \simeq \mathscr{L}$. Since $g_*(\mathscr{O}_X) \simeq \mathscr{O}_Y$, we have an isomorphism of k-algebras $R(X;\mathscr{L}) \simeq R(Y;\mathscr{M})$. Furthermore, if m is a positive integer such that $\mathscr{M}^m$ is very ample and gives a projectively normal embedding, then $R(Y;\mathscr{M}^m)$ is a quotient of the homogeneous coordinate ring of Y in the embedding given by $\mathscr{M}^m$. Therefore $R(Y;\mathscr{M}^m)$ is finitely generated, hence $R(Y;\mathscr{L}) \simeq R(Y;\mathscr{M})$ is finitely generated by Proposition 1.8.1.

Suppose now that $\mathscr{L}_1,\ldots,\mathscr{L}_r$ are globally generated line bundles on X. We use the trick described in Remark 1.8.2 to reduce to the case of one line bundle. Note that $\mathscr{E} = \mathscr{L}_1 \oplus \ldots \oplus \mathscr{L}_r$ is a globally generated vector bundle. If $\pi\colon W = \mathbb{P}(\mathscr{E}) \to X$ is the corresponding projectivized bundle, then on W we have a surjection $\pi^*(\mathscr{E}) \to \mathscr{L} = \mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$. Since $\pi^*(\mathscr{E})$ is globally generated, it follows that $\mathscr{L}$ is globally generated. The k-algebra $R(W;\mathscr{L})$ is finitely generated by what we have already proved and we have an isomorphism $R(X;\mathscr{L}_1,\ldots,\mathscr{L}_r) \simeq R(W;\mathscr{L})$. This completes the proof of the proposition. □

Remark 1.8.5. Let X be a complete variety and $\mathscr{L}$ a semiample line bundle on X. It follows from the proof of Proposition 1.8.4 that we have a fiber space $f_{\mathscr{L}}\colon X \to \mathrm{Proj}(R(X;\mathscr{L}))$. This is constructed as the Stein factorization of the morphism defined by some globally generated $\mathscr{L}^m$. Since $\mathscr{L}^m$ is the pull-back of an ample line bundle via $f_{\mathscr{L}}$, it follows that a curve C on X is contracted if and only if $(\mathscr{L}\cdot C)$=0. This uniquely determines the fiber space $f_{\mathscr{L}}$ up to equivalence (see the proof of Proposition 1.3.29). In particular, $f_{\mathscr{L}}$ is independent of the integer m used in the construction. Note that we can interpret $f_{\mathscr{L}}$ via Proposition 1.3.29 as the fiber space corresponding to the face of $\mathrm{Nef}(X)$ containing $\mathscr{L}$ in its relative interior. When $\mathscr{L} = \mathscr{O}_X(D)$, we also write f_D instead of $f_{\mathscr{L}}$.

Suppose now that we have, in addition, a fiber space $\pi\colon Z \to X$. Since the canonical morphism $\mathscr{O}_X \to \pi_*(\mathscr{O}_Z)$ is an isomorphism, it follows from the projection formula that we have a canonical isomorphism $R(X,\mathscr{L}) \simeq R(Z,\pi^*(\mathscr{L}))$ and a commutative diagram

$$\begin{tikzcd} Z \arrow[r, "f_{\pi^*(\mathscr{L})}"] \arrow[d, "\pi"] & \mathrm{Proj}(R(Z;\pi^*(\mathscr{L}))) \arrow[d, equal] \\ X \arrow[r, "f_{\mathscr{L}}"] & \mathrm{Proj}(R(X;\mathscr{L})). \end{tikzcd}$$

We now discuss some of the consequences of the finite generation of the section ring in the case of one line bundle. For simplicity, we assume that we work on a normal variety. Note that $R(X;\mathscr{L})$ is trivially finitely generated if $h^0(X,\mathscr{L}^m)=0$ for all $m\geq 1$.

Proposition 1.8.6. *Let D be a Cartier divisor on the complete, normal variety X such that $R(X;D)$ is finitely generated. We denote by $\mathfrak{a}_m$ the ideal defining the base-locus of $\mathscr{O}_X(mD)$ and assume that some $\mathfrak{a}_m$ is nonzero.*

i) There is a positive integer ℓ such that $\mathfrak{a}_{\ell m}=\mathfrak{a}_\ell^m$ for all $m\geq 1$.
ii) Let $\pi\colon W\to X$ be a projective, birational morphism, with W normal, which factors through the blowing-up of X along $\mathfrak{a}_\ell$. If we write $\mathfrak{a}_\ell\cdot\mathscr{O}_W=\mathscr{O}_W(-N)$ and $P:=\pi^*(\ell D)-N$, then $\mathscr{O}_W(P)$ is globally generated and for every positive integer m, we have an isomorphism $H^0(W,\mathscr{O}_W(mP))\to H^0(W,\mathscr{O}_W(\ell m\pi^*(D)))$ induced by multiplication with a section defining mN.
iii) If $f\colon W\to \mathrm{Proj}(R(X;D))$ is the canonical morphism defined by the globally generated line bundle $\mathscr{O}_X(P)$, then the rational map $f_D:=f\circ\pi^{-1}$ is independent of the choice of ℓ and π.
iv) For every divisor E over X, we have $\mathrm{ord}_E(\parallel D\parallel)=\frac{1}{\ell}\,\mathrm{ord}_E(\mathfrak{a}_\ell)$.
v) If D is big, then D is nef if and only if $\mathscr{O}_X(D)$ is semiample. Moreover, if we also assume that X is projective, then $\mathrm{B}_-(D)=\mathrm{SB}(D)$.

Proof. It follows from Proposition C.0.9 that there is a positive integer ℓ such that $R(X;\ell D)$ is generated in degree 1. This implies that $\mathfrak{a}_{\ell m}=\mathfrak{a}_\ell^m$ for all $m\geq 1$, proving i). In particular, by the assumption on $\mathscr{O}_X(D)$, we see that $\mathfrak{a}_\ell$ is nonzero.

Suppose now that $\pi\colon W\to X$ is a birational morphism with W normal, such that $\mathfrak{a}_\ell\cdot\mathscr{O}_W=\mathscr{O}_W(-N)$ for some effective Cartier divisor N on W. It follows from the definition of the base-locus that since $P=\pi^*(\ell D)-N$, then P is effective and $\mathscr{O}_W(P)$ is globally generated. Furthermore, the canonical map induced by the effective Cartier divisor N

$$H^0(W,\mathscr{O}_W(P))\to H^0(W,\mathscr{O}_W(\pi^*(\ell D)))$$

is an isomorphism.

Since X is normal, the canonical morphism $\mathscr{O}_X\to\pi_*(\mathscr{O}_W)$ is an isomorphism, hence the projection formula gives a canonical isomorphism

$$H^0(X,\mathscr{O}_X(mD))\simeq H^0(W,\mathscr{O}_W(\pi^*(mD)))$$

for every $m\geq 1$. Since $\mathfrak{a}_{\ell m}=\mathfrak{a}_\ell^m$ for every $m\geq 1$, we can run the above argument with ℓD replaced by ℓmD, to deduce that we have canonical isomorphisms

$$H^0(X, \mathscr{O}_X(m\ell D)) \simeq H^0(W, \mathscr{O}_W(\pi^*(m\ell D))) \simeq H^0(W, \mathscr{O}_W(mP)).$$

In particular, we obtain the assertion in ii).

Since $\mathscr{O}_W(P)$ is globally generated, it defines a fiber space

$$f = f_P \colon W \to \mathrm{Proj}(R(W;P)) \simeq \mathrm{Proj}(R(X;\ell D)) \simeq \mathrm{Proj}(R(X;D))$$

(see Remark 1.8.5). Let us show that $f \circ \pi^{-1}$ does not depend on ℓ and π. Regarding ℓ, it is enough to consider what happens when we replace ℓ by ℓm for some $m \geq 1$. Since $\mathfrak{a}_{\ell m} = \mathfrak{a}_\ell^m$, the morphism π still satisfies the condition in ii). We replace P my mP, but the resulting fiber space in unchanged (see Remark 1.8.5). In order to check independence of π, since any two such π can be dominated by a third one, it is enough to consider a birational morphism $g \colon Z \to W$, with Z normal, and compare the rational maps corresponding to π and $\pi \circ g$. Note that in this case we have the corresponding decomposition of $(\pi \circ g)^*(\ell D)$ as $g^*(P) + g^*(N)$ and since g is a fiber space, we have $f_{g^*(D)} = f_D \circ g$ (see Remark 1.8.5). This implies the equality of the rational maps corresponding to π and $\pi \circ g$.

Since for every positive integer m, we have

$$\mathrm{ord}_E(\mathfrak{a}_{\ell m}) = \mathrm{ord}_E(\mathfrak{a}_\ell^m) = m \cdot \mathrm{ord}_E(\mathfrak{a}_\ell),$$

the assertion in iv) follows from the fact that we may compute $\lim_{m\to\infty} \frac{1}{m}\,\mathrm{ord}_E(\mathfrak{a}_m)$ by restricting to those m that are multiple of ℓ.

Finally, in order to prove v), suppose first that X is projective and let us prove the second assertion. It is enough to show that $\mathrm{SB}(D) \subseteq \mathrm{B}_-(D)$, since the reverse inclusion always holds. With ℓ as above, it is clear that $\mathrm{SB}(D) = V(\mathfrak{a}_\ell)$. If V is an irreducible component of $V(\mathfrak{a}_\ell)$, then there is a divisor E over X with center V (see Remark 1.7.8). By iv), we have $\mathrm{ord}_E(\parallel D \parallel) = \frac{1}{\ell}\,\mathrm{ord}_E(\mathfrak{a}_\ell) > 0$. On the other hand, if $V \not\subseteq \mathrm{B}_-(D)$, then $V \not\subseteq \mathrm{SB}\left(D + \frac{1}{m}A\right)$ for every ample Cartier divisor A, hence $\mathrm{ord}_E(\parallel D + \frac{1}{m}A \parallel) = 0$. Since $\mathrm{ord}_E(\parallel - \parallel)$ is continuous on the big cone by Proposition 1.7.18, we obtain $\mathrm{ord}_E(\parallel D \parallel) = 0$, a contradiction. This holds for every irreducible component of $\mathrm{SB}(D)$ and therefore $\mathrm{SB}(D) \subseteq \mathrm{B}_-(D)$.

Since $\mathrm{SB}(D)$ is empty if and only if $\mathscr{O}_X(D)$ is semiample and $\mathrm{B}_-(D)$ is empty if and only if D is nef, we see that D is nef if and only if it is semiample. Moreover, this holds even if X is not projective. Indeed, by Chow's lemma we have a proper, birational morphism $h \colon X' \to X$ such that X' is projective. After possibly replacing X' by its normalization, we may assume that it is normal. Note that $h^*(D)$ is big and, since h is a fiber space, we have an isomorphism $R(X;D) \simeq R(X', h^*(D))$. Finally, D is nef or semiample if and only if $h^*(D)$ has the same property. This completes the proof of the proposition. $\square$

Remark 1.8.7. With the notation in Proposition 1.8.6, suppose that W is smooth and D is big. Assertion iv) in the proposition implies that $N = N_\sigma(\ell D)$, hence the decomposition $\pi^*(D) = \frac{1}{\ell}N + \frac{1}{\ell}P$ is the divisorial Zariski decomposition of $\pi^*(D)$. Furthermore, since P is nef, this is a Zarsiki decomposition.

Example 1.8.8. If D is not big, it can happen that D is nef, the ring $R(X;\mathscr{L})$ is finitely generated, but $\mathscr{O}_X(D)$ is not semiample. A trivial example is given by a non-torsion line bundle of degree 0 on a curve. A more interesting example, in which the section ring is different from k, is obtained as follows (see [Laz04a, Example 2.3.16]). Let C be a smooth, projective curve of genus $g \geq 2$ over an uncountable, algebraically closed field k and let $\mathscr{M} \in \mathrm{Pic}^0(X)$ be a non-torsion element. We take $X = \mathbb{P}(\mathscr{O}_C \oplus \mathscr{M})$ and consider the line bundle $\mathscr{O}(1)$ on X. Since both $\mathscr{O}_C$ and $\mathscr{M}$ are nef on C, it follows that $\mathscr{O}(1)$ is nef on X (see Example 1.3.37). Note that since $H^0(C,\mathscr{M}^i) = 0$ for every $i \neq 0$, we have

$$H^0(X,\mathscr{O}(m)) \simeq H^0(C,\mathrm{Sym}^m(\mathscr{O}_C \oplus \mathscr{M})) \simeq H^0(C,\mathscr{O}_C).$$

Therefore we have a nonzero section $s \in H^0(X,\mathscr{O}(1))$ such that $H^0(X,\mathscr{O}(m)) = k \cdot s^{\otimes m}$ for every m. On one hand, this shows that $Z(s) \subseteq \mathrm{SB}(\mathscr{O}(1))$, hence $\mathscr{O}(1)$ is not semiample, but on the other hand, it implies that $R(X,\mathscr{O}(1)) \simeq k[x]$, hence it is a finitely generated k-algebra.

Example 1.8.9. We have described in Example 1.7.24 a big and nef line bundle $\mathscr{L}$ on a surface X such that for a curve E on X we have $\mathrm{ord}_E(|\mathscr{L}^m|) = 1$ for all $m \geq 1$. It follows from Proposition 1.8.6 that in this case the k-algebra $R(X;\mathscr{L})$ is not finitely generated.

Example 1.8.10. We now give an example due to Zariski [Zar61] of a big and nef divisor D on a surface whose section ring is not finitely generated. We start with an elliptic curve C over an uncountable, algebraically closed ground field, embedded in $\mathbb{P}^2$ by a divisor ℓ of degree 3. We choose points $P_1,\ldots,P_{12}$ very general such that the line bundle $\xi = \mathscr{O}_C(4\ell - P_1 - \ldots - P_{12}) \in \mathrm{Pic}^0(C)$ is non-torsion. We consider the blow-up $\pi\colon X \to \mathbb{P}^2$ along $P_1,\ldots,P_{12}$, with exceptional divisor E. Let H be the pull-back of the hyperplane class of $\mathbb{P}^2$ and $\widetilde{C}$ the proper transform of C on X. Therefore π induces an isomorphism $\widetilde{C} \to C$ and we have $\widetilde{C} = \pi^*(C) - E \sim 3H - E$. Let $D = H + \widetilde{C}$. Note that $\mathscr{O}_X(H)$ is globally generated and big, being the pull-back of an ample, globally generated line bundle by a birational morphism. Since $\widetilde{C}$ is effective, it follows that D is big. Furthermore, $\mathscr{O}_X(D)|_{\widetilde{C}}$ corresponds via $\widetilde{C} \simeq C$ to ξ. First, since $D = H + \widetilde{C}$, with H being nef and $\widetilde{C}$ a prime divisor, and $(D \cdot \widetilde{C}) = 0$, we conclude that D is nef. On the other hand, if m is a positive integer and $s \in H^0(X,\mathscr{O}_X(mD))$ is such that $\widetilde{C} \not\subseteq Z(s)$, then ξ^m has a nonzero section, hence it is trivial, a contradiction. This implies that $\widetilde{C} \subseteq \mathrm{SB}(D)$. In particular, $\mathscr{O}_X(D)$ is not semiample and using assertion v) in Proposition 1.8.6, we see that $R(X;D)$ is not finitely generated.

Example 1.8.11. If $\mathscr{L}_1,\ldots,\mathscr{L}_r$ are line bundles on a complete toric variety X, then the k-algebra $R(X;\mathscr{L}_1,\ldots,\mathscr{L}_r)$ is finitely generated. Indeed, note first that since $\mathbb{P}(\mathscr{L}_1 \oplus \ldots \oplus \mathscr{L}_r)$ admits a structure of toric variety (see [Oda88, pp. 58–59]), the argument in Remark 1.8.2 implies that it is enough to prove the assertion when we have only one line bundle $\mathscr{L}$. In this case, the assertion follows easily from the basic properties of line bundles on toric varieties (see [Ful93, Chapter 3.3]). Indeed, if D is a torus-invariant Cartier divisor on X such that $\mathscr{L} \simeq \mathscr{O}_X(D)$, then D defines a polytope P_D in $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$, where M is a lattice (the lattice dual to that containing the fan defining X). If one considers $P_D \times \{1\} \subseteq M_{\mathbb{R}} \times \mathbb{R}$ and σ is the cone over $P_D \times \{1\}$, then $R(X, \mathscr{L})$ is isomorphic to the monoid ring $k[\sigma \cap (M \times \mathbb{Z})]$. Since σ is a rational polyhedral cone, it follows from Gordan's lemma (see Lemma A.6.1) that $\sigma \cap (M \times \mathbb{Z})$ is a finitely generated monoid, hence $R(X; \mathscr{L})$ is a finitely generated k-algebra.

Example 1.8.12. It has been a long-standing conjecture that for every smooth projective variety X, the canonical ring $R(X; \omega_X)$ is finitely generated. This has been recently proved in [BCHM10]. We discuss in Chapter 4 the proof of this result, following [CL12].

1.8.2 Finite generation and asymptotic invariants

In this section we study the consequences of the finite generation of the section ring associated to several line bundles. It is convenient to state the main technical result more generally, in terms of finitely generated Rees algebras associated to S-graded sequences of ideals.

Let S be a submonoid of a finitely generated, free abelian group M. Given an S-graded sequence of ideals $\mathfrak{a}_\bullet$ on the variety X, we obtain an S-graded $\mathscr{O}_X$-algebra

$$\mathscr{R}(\mathfrak{a}_\bullet) := \bigoplus_{u \in S} \mathfrak{a}_u,$$

where the multiplication is induced by the multiplication in $\mathscr{O}_X$. We say that an $\mathscr{O}_X$-algebra $\mathscr{R}$ is finitely generated if for every affine open subset U in X, the $\mathscr{O}_X(U)$-algebra $\mathscr{R}(U)$ is finitely generated (as usual, it is enough to test this for a family of affine open subsets covering X).

Example 1.8.13. Let X be a complete variety and $\mathscr{L}_1, \ldots, \mathscr{L}_r$ line bundles on X such that the k-algebra $R(X; \mathscr{L}_1, \ldots, \mathscr{L}_r)$ is finitely generated. If $\mathfrak{a}_\bullet$ is the $\mathbb{N}^r$-graded sequence of base loci defined in Example 1.7.5, then the $\mathscr{O}_X$-algebra $\mathscr{R}(\mathfrak{a}_\bullet)$ is finitely generated.

Recall that if $\mathfrak{a}_\bullet$ is an S-graded sequence of ideals as above, then for every divisor E over X, we defined in Section 1.7.3 a function

$$\mathrm{ord}_E^{\mathfrak{a}_\bullet} \colon S_+(\mathfrak{a}_\bullet) = \{u \in S \mid \mathfrak{a}_{mu} \neq 0 \text{ for some } m > 0\} \to \mathbb{R}_{\geq 0}.$$

Moreover, when $S_+(\mathfrak{a}_\bullet)$ is finitely generated and C is the convex cone generated by $S_+(\mathfrak{a}_\bullet)$ in $M_{\mathbb{R}}$, then $\mathrm{ord}_E^{\mathfrak{a}_\bullet}$ naturally extends as a degree one homogeneous function to $C \cap M_{\mathbb{Q}}$. The following is the key technical result of this section.

Proposition 1.8.14. *Let X be a variety, M a finitely generated, free abelian group, and S a submonoid of M. If $\mathfrak{a}_\bullet$ is an S-graded sequence of ideals on X such that $\mathscr{R}(\mathfrak{a}_\bullet)$ is a finitely generated $\mathscr{O}_X$-algebra, then the following hold:*

i) The monoid $S_+(\mathfrak{a}_\bullet)$ is finitely generated.
ii) For every divisor E over X, the map $\mathrm{ord}_E^{\mathfrak{a}_\bullet}$ is piecewise linear and convex on $C\cap M_{\mathbb{Q}}$, where C is the convex cone generated by $S_+(\mathfrak{a}_\bullet)$ in $M_{\mathbb{R}}$.
iii) Moreover, there is a rational fan Δ with support equal to C such that for every cone $\sigma\in\Delta$ and every divisor E over X, the function $\mathrm{ord}_E^{\mathfrak{a}_\bullet}$ is linear on $\sigma\cap M_{\mathbb{Q}}$.
iv) There is $d\in\mathbb{Z}_{>0}$ such that $\mathrm{ord}_E^{\mathfrak{a}_\bullet}(du)=\mathrm{ord}_E(\mathfrak{a}_{du})$ for every divisor E over X and every $u\in S_+(\mathfrak{a}_\bullet)$.

Proof. Note first that we may assume that X is affine. Indeed, if $X=U_1\cup\ldots\cup U_r$ is an affine open cover and $\mathfrak{a}_\bullet|_{U_i}$ is the restriction of $\mathfrak{a}_\bullet$ to U_i, then it is clear that $S_+(\mathfrak{a}_\bullet)=S_+(\mathfrak{a}_\bullet|_{U_i})$ for every i and $\mathrm{ord}_E^{\mathfrak{a}_\bullet}=\mathrm{ord}_E^{\mathfrak{a}_\bullet|_{U_i}}$ for every divisor E over X whose center meets U_i. It is clear that if Δ_i is a fan that satisfies iii) for $\mathfrak{a}_\bullet|_{U_i}$, then any common refinement of $\Delta_1,\ldots,\Delta_r$ satisfies the condition for $\mathfrak{a}_\bullet$ (note that such a refinement exists by Lemma A.7.6). Similarly, if d_i satisfies the condition iv) for $\mathfrak{a}_\bullet|_{U_i}$, then the least common multiple of the d_i satisfies the condition for $\mathfrak{a}_\bullet$.

Suppose from now on that $X=\mathrm{Spec}(A)$. The monoid $T:=\{u\in S\mid \mathfrak{a}_u\neq 0\}$ is finitely generated by Lemma C.0.3. Since $S_+(\mathfrak{a}_\bullet)$ is equal to the saturation T^{sat} of T, we deduce that $S_+(\mathfrak{a}_\bullet)$ is finitely generated by Proposition A.6.2.

It is easy to see that each of the functions $\mathrm{ord}_E^{\mathfrak{a}_\bullet}$ is convex: this follows from definition as in the proof of Lemma 1.7.13. We also note that for every $u\in S_+(\mathfrak{a}_\bullet)$, the ring $\bigoplus_{m\geq 0}\mathfrak{a}_{mu}$ is finitely generated by Proposition C.0.8. Applying Proposition C.0.9, we see that there is a positive integer d_u such that $\mathfrak{a}_{md_uu}=\mathfrak{a}_{d_uu}^m$ for all positive integers m. In particular, we have $\mathrm{ord}_E(\mathfrak{a}_{md_uu})=m\cdot\mathrm{ord}_E(\mathfrak{a}_{d_uu})$ for every $m\geq 1$ and every divisor E over X.

In order to prove the assertion in iii), hence also that in ii), consider generators $y_1,\ldots,y_n$ for the A-algebra $\mathscr{R}(\mathfrak{a}_\bullet)$. We may assume that each y_i is nonzero and homogeneous, with $\deg(y_i)=u_i$. Every element of degree u of $\mathscr{R}(\mathfrak{a}_\bullet)$ can be written as a linear combination, with coefficients in A, of monomials $y_1^{m_1}\cdots y_n^{m_n}$, with $\sum_{i=1}^n m_iu_i=u$. This implies that for every $u\in S$, we have $\mathfrak{a}_u=\sum_{m_1,\ldots,m_n}\mathfrak{a}_{u_1}^{m_1}\cdots\mathfrak{a}_{u_n}^{m_n}$, where the sum is over the nonnegative integers $m_1,\ldots,m_n$ such that $\sum_{i=1}^n m_iu_i=u$. We thus conclude that for every divisor E over X and every $u\in T$, we have

$$\mathrm{ord}_E(\mathfrak{a}_u)=\min\left\{\sum_{i=1}^n m_i\cdot\mathrm{ord}_E(\mathfrak{a}_{u_i})\mid m_1\ldots,m_n\in\mathbb{Z}_{\geq 0},u=\sum_{i=1}^n m_iu_i\right\}. \tag{1.31}$$

We claim that for every such E and every $u\in C\cap M_{\mathbb{Q}}$, we have

$$\mathrm{ord}_E^{\mathfrak{a}_\bullet}(u)=\min\left\{\sum_{i=1}^n \lambda_i\cdot\mathrm{ord}_E(\mathfrak{a}_{u_i})\mid \lambda_1,\ldots,\lambda_n\in\mathbb{Q}_{\geq 0},u=\sum_{i=1}^n \lambda_iu_i\right\}. \tag{1.32}$$

Indeed, the inequality "$\leq$" follows from the convexity of $\mathrm{ord}_E^{\mathfrak{a}_\bullet}$ and the fact that $\mathrm{ord}_E^{\mathfrak{a}_\bullet}(u_i)\leq\mathrm{ord}_E(\mathfrak{a}_{u_i})$. In order to prove the reverse inequality, we apply (1.31) for

$d_u u$ to find nonnegative integers $m_1,\ldots,m_n$ such that $\sum_{i=1}^n m_i u_i = d_u u$ and

$$\sum_{i=1}^n m_i \cdot \mathrm{ord}_E(\mathfrak{a}_{u_i}) = \mathrm{ord}_E(\mathfrak{a}_{d_u u}) = \mathrm{ord}_E^{\mathfrak{a}_\bullet}(d_u u) = d_u \cdot \mathrm{ord}_E^{\mathfrak{a}_\bullet}(u).$$

It follows that if we take $\lambda_i = m_i/d_u$, then $\sum_{i=1}^n \lambda_i u_i = u$ and $\mathrm{ord}_E^{\mathfrak{a}_\bullet}(u) = \sum_i \lambda_i \cdot \mathrm{ord}_E(\mathfrak{a}_{u_i})$. We thus have (1.32) and the assertion in iii) is now a consequence of Proposition A.9.6.

Suppose now that Δ is a rational fan as in iii) and let us prove iv). Fix a cone $\sigma \in \Delta$ and let $w_1,\ldots,w_N$ be generators of $\sigma \cap M$. Let d_σ be the least common multiple of the d_{w_i}. Given $u \in \sigma \cap M$, we write $u = \sum_{i=1}^N m_i w_i$ for non-negative integers $m_1,\ldots,m_N$ and we have

$$\mathrm{ord}_E(\mathfrak{a}_{d_\sigma u}) \leq \sum_{i=1}^N m_i \cdot \mathrm{ord}_E(\mathfrak{a}_{d_\sigma w_i}) = \sum_{i=1}^N m_i \cdot \mathrm{ord}_E^{\mathfrak{a}_\bullet}(d_\sigma w_i) = \mathrm{ord}_E^{\mathfrak{a}_\bullet}(d_\sigma u) \leq \mathrm{ord}_E(\mathfrak{a}_{d_\sigma u}).$$

Therefore all these inequalities are equalities. In particular, we have $\mathrm{ord}_E^{\mathfrak{a}_\bullet}(d_\sigma u) = \mathrm{ord}_E(\mathfrak{a}_{d_\sigma u})$. Since $S_+(\mathfrak{a}_\bullet) = \bigcup_{\sigma\in\Delta}(\sigma \cap M)$, it follows that $d := \mathrm{lcm}(d_\sigma \mid \sigma \in \Delta)$ satisfies the condition in iv). $\square$

Remark 1.8.15. Under the assumptions of Proposition 1.8.14, when X is normal, one can reformulate the conclusion as saying that there is a fan Δ with support C and a positive integer d such that for every cone $\sigma \in \Delta$ and every $u, v \in \sigma \cap M$, we have $\overline{\mathfrak{a}_{du}} \cdot \overline{\mathfrak{a}_{dv}} = \overline{\mathfrak{a}_{du+dv}}$. Indeed, this is a consequence of assertions iii) and iv) in the proposition and of the fact that two ideals $\mathfrak{a}$ and $\mathfrak{b}$ have the same integral closure if and only if $\mathrm{ord}_E(\mathfrak{a}) = \mathrm{ord}_E(\mathfrak{b})$ for all divisors E over X (we refer to Appendix D for the basic facts about integral closure of ideals).

We now apply Proposition 1.8.14 in the setting of finitely generated rings of sections. Let X be a a complete variety and $D_1,\ldots,D_r$ Cartier divisors on X.

Corollary 1.8.16. *With the above notation, if the section ring $R(X;D_1,\ldots,D_r)$ is finitely generated, then the following hold:*

i) The monoid $T := \{u \in \mathbb{N}^r \mid h^0(X,\mathscr{O}_X(D_u)) \geq 1\}$ is finitely generated.
ii) If C is the convex cone generated by T in $\mathbb{R}^r$, then there is a rational fan Δ with support C such that for every divisor E over X and every $\sigma \in \Delta$, the function $u \to \mathrm{ord}_E(\| D_u \|)$ is linear on σ.
iii) There is a positive integer d such that $\mathrm{ord}_E(\| D_{du} \|) = \mathrm{ord}_E(|D_{du}|)$ for every $u \in T^{\mathrm{sat}}$.

Proof. If $\mathfrak{a}_u$ is the ideal defining the base locus of $\mathscr{O}_X(D_u)$, then $\mathfrak{a}_\bullet$ is an $\mathbb{N}^r$-graded sequence such that $\mathscr{R}(\mathfrak{a}_\bullet)$ is a finitely generated $\mathscr{O}_X$-algebra. Since $\mathrm{ord}_E^{\mathfrak{a}_\bullet}(u) = \mathrm{ord}_E(\| D_u \|)$ for every $u \in \mathbb{Q}^r_{\geq 0}$, the assertions i)-iii) follow from Proposition 1.8.14. $\square$

Corollary 1.8.17. *With the notation in Corollary 1.8.16, suppose in addition that X is normal. If σ is a cone in Δ and $u \in \mathrm{Relint}(\sigma) \cap \mathbb{Z}^r$, then the stable base locus*

$\mathrm{SB}(D_u)$ *and the rational map* f_{D_u} *are independent of* u. *Moreover, if* $v \in \sigma \cap \mathbb{Z}^r$ *is arbitrary, then* $\mathrm{SB}(D_v) \subseteq \mathrm{SB}(D_u)$ *and there is a morphism* $\phi\colon \mathrm{Proj}(R(X;D_u)) \to \mathrm{Proj}(R(X;D_v))$ *such that* $\phi \circ f_{D_u} = f_{D_v}$.

Proof. Since X is normal and the section ring $R(X;D_u)$ is finitely generated for every $u \in \mathbb{N}^r$, we may apply Proposition 1.8.6 for D_u. Suppose first that $u,v \in \sigma \cap \mathbb{Z}^r$ and $w = u - v \in \sigma$. Let W be a normal variety such that we have a projective, birational morphism $\pi\colon W \to X$ and effective Cartier divisors N_u, N_v, N_w on W such that $\mathfrak{a}_{du} \cdot \mathscr{O}_W = \mathscr{O}_W(-N_u)$, $\mathfrak{a}_{dv} \cdot \mathscr{O}_W = \mathscr{O}_W(-N_v)$, and $\mathfrak{a}_{dw} \cdot \mathscr{O}_W = \mathscr{O}_W(-N_w)$. By Corollary 1.8.16, we have $\mathrm{ord}_E(\mathfrak{a}_{du}) = \mathrm{ord}_E(\mathfrak{a}_{dv}) + \mathrm{ord}_E(\mathfrak{a}_{dw})$ for every divisor E on W, hence $N_u = N_v + N_w$. If we write $P_u = \pi^*(dD_u) - N_u$, $P_v = \pi^*(dD_v) - N_v$, and $P_w = \pi^*(dD_w) - N_w$, then $P_u = P_v + P_w$ and the line bundles $\mathscr{O}_W(P_u)$, $\mathscr{O}_W(P_v)$, and $\mathscr{O}_W(P_w)$ are globally generated, hence nef.

We first deduce that

$$\mathrm{SB}(D_v) = \pi(\mathrm{Supp}(N_v)) \subseteq \pi(\mathrm{Supp}(N_u)) = \mathrm{SB}(D_u).$$

Moreover, if C is a curve in W such that $(P_u \cdot C) = 0$, then $(P_v \cdot C) = 0$. This implies that $f_{P_u} \prec f_{P_v}$, that is, there is a morphism $\phi\colon \mathrm{Proj}(R(X;D_u)) \to \mathrm{Proj}(R(X;D_v))$ such that $f_{P_v} = \phi \circ f_{P_u}$. Therefore we have the equality of rational functions

$$f_{D_v} = f_{P_v} \circ \pi^{-1} = \phi \circ f_{P_u} \circ \pi^{-1} = f_{D_u}.$$

The second assertion in the corollary now follows from the fact that if $u \in \mathrm{Relint}(\sigma)$ and $v \in \sigma$, then $mu - v \in \sigma$ for all integers $m \gg 0$ and we have $\mathrm{SB}(D_u) = \mathrm{SB}(D_{mu})$ and $f_{D_u} = f_{D_{mu}}$. By symmetry, we also obtain the first assertion. $\square$

Remark 1.8.18. Suppose that we are in the setting of Corollary 1.8.16, with X a normal variety. In this case there is a projective, birational morphism $\pi\colon W \to X$, with W normal, such that for every $u \in C \cap \mathbb{N}^r$, we have a decomposition $\pi^*(D_u) = N_u + P_u$, with N_u effective, $\mathscr{O}(dP_u)$ globally generated, and for all positive integers m, we have an isomorphism

$$H^0(W, \mathscr{O}_W(dmP_u)) \simeq H^0(W, \mathscr{O}_W(dm\pi^*(D_u)))$$

induced by a section defining dmN_u. Furthermore, the maps $u \to N_u, P_u$ are linear. Indeed, it is enough to consider for each maximal cone $\sigma \in \Delta$ a system of generators for $\sigma \cap \mathbb{Z}^r$. If $\{u_1, \ldots, u_d\}$ is the union of these systems of generators and $\pi\colon W \to X$, with W normal, is a projective, birational morphism that factors through the blow-up along each $\mathfrak{a}_{du_i}$, then π satisfies the required properties. This follows easily from Proposition 1.8.6 and Remark 1.8.15.

We keep the assumptions and notation in Corollary 1.8.16, with X a normal, projective variety. Let $\Phi\colon \mathbb{R}^r \to \mathrm{N}^1(X)_{\mathbb{R}}$ be the linear map given by $\Phi(u) = D_u$. Note that for every divisor E over X, Corollary 1.8.16 implies that the function $C \cap \mathbb{Q}^r \ni w \to \mathrm{ord}_E(\| D_u \|)$ admits a (unique) piecewise linear extension ψ_E to C. It is clear that ψ_E is continuous.

Corollary 1.8.19. *With the above notation, if there is $u \in \mathbb{R}^r$ such that D_u is big, then the following hold:*
i) For every $v \in \mathbb{Z}^r_{\geq 0}$, the divisor D_v is pseudo-effective if and only if $|dD_v| \neq \emptyset$. Moreover, we have $\mathbb{R}^r_{\geq 0} \cap \Phi^{-1}(\mathrm{PEff}(X)) = C$, hence this is a rational polyhedral cone.
ii) For every $v \in C$ and every divisor E over X, we have $\sigma_E(D_v) = \psi_E(v)$.
iii) For every $v \in \mathbb{N}^r$, we have $\mathrm{B}_-(D_v) = \mathrm{SB}(D_v) = \mathrm{Bs}(|dD_v|)_{\mathrm{red}}$. In particular, D_v is nef if and only if $\mathscr{O}_X(dD_v)$ is globally generated. Moreover, we have

$$\mathbb{R}^r_{\geq 0} \cap \Phi^{-1}(\mathrm{Nef}(X)) = \{v \in C \mid \sigma_E(D_v) = 0 \text{ for all divisors } E \text{ over } X\} \tag{1.33}$$

and this is a rational polyhedral cone.

Proof. If $H^0(X, \mathscr{O}_X(dD_v)) \neq 0$, then it is clear that D_v is pseudo-effective. Conversely, if D_v is pseudo-effective, then for every rational number $t > 0$, we have $D_v + t \cdot D_u$ big. Therefore $v + tu$ lies in C and since C is a closed cone, we have $v \in C$. In this case, Corollary 1.8.16 implies that $H^0(X, \mathscr{O}_X(dD_v)) \neq 0$. The second assertion in i) is also clear.

Given any $v \in C$ and any divisor E over X, it follows from Remark 1.7.34 that $\sigma_E(D_v) = \lim_{t \to 0} \mathrm{ord}_E(\| D_{v+tu} \|)$. On the other hand, by Proposition 1.7.19, the map $w \to \mathrm{ord}_E(\| D_w \|)$ is continuous on $\Phi^{-1}(\mathrm{Big}(X))$ and since the rational points are dense in $\Phi^{-1}(\mathrm{Big}(X))$, it follows that $\psi_E(w) = \mathrm{ord}_E(\| D_w \|)$ whenever D_w is big. In particular, we have

$$\sigma_E(D_v) = \lim_{t \to 0} \mathrm{ord}_E(\| D_{v+tu} \|) = \lim_{t \to 0} \psi_E(v + tu) = \psi_E(v),$$

giving the assertion in ii).

We now show that if $v \in \mathbb{N}^r \cap C$ and E is a divisor over X such that $\sigma_E(D_v) = 0$, then $c_X(E)$ is not contained in $\mathrm{Bs}(|dD_v|)$. Since D_v is pseudo-effective, part i) gives $v \in C$ and using part ii) we get

$$\mathrm{ord}_E(\| D_v \|) = \psi_E(v) = \sigma_E(D_v) = 0.$$

We thus conclude using Corollary 1.8.16 that $\mathrm{ord}_E(|dD_v|) = 0$, that is, $c_X(E) \not\subseteq \mathrm{Bs}(|dD_v|)$.

For every $v \in \mathbb{N}^r$, we clearly have the inclusions

$$\mathrm{B}_-(D_v) \subseteq \mathrm{SB}(D_v) \subseteq \mathrm{Bs}(|dD_v|)_{\mathrm{red}}. \tag{1.34}$$

If $v \notin C$, then D_v is not pseudo-effective, hence $\mathrm{B}_-(D_v) = X$ and the above inclusions are all equalities. Suppose now that $v \in C$ and let V be an irreducible component of $\mathrm{Bs}(|dD_v|)_{\mathrm{red}}$. Consider a divisor E over X with $c_X(E) = V$ (see Remark 1.7.8). As we have seen, in this case $\sigma_E(D_v) > 0$ and Proposition 1.7.31 implies $V \subseteq \mathrm{B}_-(D_v)$. Therefore the inclusions in (1.34) are equalities for every $v \in \mathbb{N}^r$. This proves the first assertion in iii) and the second one is a special case.

The inclusion "$\subseteq$" in (1.33) is a general fact (see Proposition 1.7.31), hence in order to prove the equality we only need to show the reverse inclusion. Given $v \in C$

such that $\sigma_E(D_v) = 0$ for every divisor E over X, let τ be the cone in Δ such that $v \in \mathrm{Relint}(\tau)$. For every E and every $w \in \tau$, we have $\psi_E(w) = 0$. Indeed, since ψ_E is linear and non-negative on τ and $mv - w \in \tau$ for $m \gg 0$, we deduce

$$\sigma_E(w) = \psi_E(w) = m \cdot \psi_E(v) - \psi_E(mv - w) = -\psi_E(mv - w) \leq 0,$$

and since $\sigma_E(w) \geq 0$, we conclude that $\sigma_E(w) = 0$. If in addition $w \in \mathbb{N}^r$, it follows from what we have already shown that D_w is nef. By applying this to integer points on each of the rays of τ, we conclude that $\tau \subseteq \Phi^{-1}(\mathrm{Nef}(X))$, giving (1.33). Moreover, we see that $\mathbb{R}^r_{\geq 0} \cap \Phi^{-1}(\mathrm{Nef}(X))$ is generated as a convex cone by those rays in Δ that are contained in it, hence it is a rational, polyhedral cone. $\square$

1.8.3 Relative section rings

In this section we consider the relative version of the finite generation of section rings. Suppose that $g \colon X \to S$ is a proper morphism of varieties over k and $\mathscr{L}_1, \ldots, \mathscr{L}_r$ are line bundles on X. The *relative section ring* of $\mathscr{L}_1, \ldots, \mathscr{L}_r$ is the $\mathbb{N}^r$-graded $\mathscr{O}_S$-algebra

$$R(X/S; \mathscr{L}_1, \ldots, \mathscr{L}_r) := \bigoplus_{u \in \mathbb{N}^r} g_*(\mathscr{L}^u).$$

Note that this is a finitely generated $\mathscr{O}_S$-algebra if and only if for every affine open subset U of S, the $\mathbb{N}^r$-graded $\mathscr{O}(U)$-algebra

$$R(g^{-1}(U); \mathscr{L}_1, \ldots, \mathscr{L}_r) = \bigoplus_{u \in \mathbb{N}^r} \Gamma(g^{-1}(U), \mathscr{L}^u)$$

is finitely generated (in fact, it is enough to only consider a family of such U that cover X). Therefore for most questions it is enough to consider the case when S is affine. When $\mathscr{L}_i = \mathscr{O}_X(D_i)$ for Cartier divisors $D_1, \ldots, D_r$, we also write $R(X/S; D_1, \ldots, D_r)$ instead of $R(X/S; \mathscr{L}_1, \ldots, \mathscr{L}_r)$.

Since X is a variety, it follows again from Lemma C.0.5 that for every affine open subset U of X, the ring $R(g^{-1}(U); \mathscr{L}_1, \ldots, \mathscr{L}_r)$ is a domain. In particular, it follows from Proposition C.0.6 that for every positive integers $d_1, \ldots, d_r$, the $\mathscr{O}_S$-algebra $R(X/S; \mathscr{L}_1, \ldots, \mathscr{L}_r)$ is finitely generated if and only if $R(X/S; \mathscr{L}_1^{d_1}, \ldots, \mathscr{L}_r^{d_r})$ has this property.

Proposition C.0.10 implies that if $R(X/S; \mathscr{L}_1, \ldots, \mathscr{L}_r)$ is finitely generated, then for every $\mathscr{M}_1, \ldots, \mathscr{M}_s$ that lie in the submonoid of $\mathrm{Pic}(X)$ generated by $\mathscr{L}_1, \ldots, \mathscr{L}_r$, we have $R(X/S; \mathscr{M}_1, \ldots, \mathscr{M}_s)$ finitely generated. We also deduce from Proposition C.0.9 that if $\mathscr{L} \in \mathrm{Pic}(X)$ is such that $R(X/S; \mathscr{L})$ is finitely generated, then there is a positive integer d such that $R(X/S; \mathscr{L}^d)$ is generated in degree 1 (if we have a finite cover $S = U_1 \cup \ldots \cup U_\ell$ and if d_i is such that $\Gamma(g^{-1}(U_i); \mathscr{L}^{d_i})$ is generated in degree 1, then we may take d to be the least common multiple of the d_i). The following is the relative version of Proposition 1.8.4 and the proof follows as in the absolute case, hence we omit it.

Proposition 1.8.20. *If $g\colon X \to S$ is a proper morphism of varieties and $\mathscr{L}_1,\ldots,\mathscr{L}_r$ are g-semiample line bundles on X, then $R(X/S;\mathscr{L}_1,\ldots,\mathscr{L}_r)$ is a finitely generated $\mathscr{O}_S$-algebra.*

If $g\colon X \to S$ is as above and $\mathscr{M} \in \mathrm{Pic}(X)$ is semiample, then there is a canonical fiber space $f_{\mathscr{M}}\colon X \to \mathscr{P}roj(R(X/S;\mathscr{M}))$, which is a morphism over S. This is characterized by the fact that a curve C contracted by π is contracted also by $f_{\mathscr{M}}$ if and only if $(\mathscr{M}\cdot C)=0$. Suppose now that $\mathscr{L}$ is a line bundle on X such that $R(X/S;\mathscr{L})$ is a finitely generated $\mathscr{O}_S$-algebra and some $g_*(\mathscr{L}^m)$ is nonzero. Let d be a positive integer such that $R(X/S;\mathscr{L}^d)$ is generated in degree 1. The image of the canonical morphism $g^*g_*(\mathscr{L}^d) \to \mathscr{L}^d$ is equal to $\mathfrak{a}\otimes\mathscr{L}^d$ for some nonzero ideal $\mathfrak{a}$. Suppose now that X is normal and $\pi\colon W \to X$ is a proper, birational morphism, with W normal, such that $\mathfrak{a}\cdot\mathscr{O}_W = \mathscr{O}_W(-N)$, for an effective Cartier divisor N. If $\mathscr{L}_W := \pi^*(\mathscr{L}^d)\otimes\mathscr{O}_W(-N)$, then $\mathscr{L}_W$ is globally generated, we have $R(W/S;\mathscr{L}_W)\simeq R(X/S;\mathscr{L}^d)$, and the rational map $f_{\mathscr{L}} := f_{\mathscr{L}_W}\circ\pi^{-1}\colon X \dashrightarrow \mathscr{P}roj(R(X/S;\mathscr{L}))$ is independent of m and π. When $\mathscr{M}=\mathscr{O}_X(D)$, for a Cartier divisor D, we also write f_D for $f_{\mathscr{M}}$. Like in the absolute case, we see that if g is projective and $\mathscr{L}$ is g-big, then $\mathscr{L}$ is g-nef if and only if $\mathscr{L}$ is g-semiample.

Consider now Cartier divisors $D_1,\ldots,D_r$ on X such that $R(X/S;D_1,\ldots,D_r)$ is a finitely generated $\mathscr{O}_S$-algebra. For every $u=(u_1,\ldots,u_r)\in\mathbb{R}^r_{\geq 0}$, we put $D_u = \sum_{i=1}^r u_iD_i$. For $u\in\mathbb{N}^r$, let $\mathfrak{a}_u$ be the ideal in $\mathscr{O}_X$ such that the image of $g^*g_*(\mathscr{O}_X(D_u)) \to \mathscr{O}_X(D_u)$ is equal to $\mathfrak{a}_u\otimes\mathscr{O}_X(D_u)$. It is clear that $\mathfrak{a}_\bullet = (\mathfrak{a}_u)_{u\in\mathbb{N}^r}$ is an $\mathbb{N}^r$-graded sequence of ideals and the $\mathscr{O}_X$-algebra $\bigoplus_{u\in\mathbb{N}^r}\mathfrak{a}_u$ is finitely generated. Therefore we may apply Proposition 1.8.14. We first deduce that the monoid

$$T := \{u\in\mathbb{N}^r \mid g_*(\mathscr{O}_X(mD_u))\neq 0\} = \{u\in\mathbb{N}^r \mid \mathfrak{a}_u\neq 0\}$$

is finitely generated. Moreover, if C is the convex cone generated by T, then there is a rational fan Δ with support C such that for every $\sigma\in\Delta$ and every divisor E over X, the function $\mathbb{Q}^r_{\geq 0}\ni u \to \mathrm{ord}_E(\| D_u/S \|)$ is linear on σ. There is also a positive integer d such that $\mathrm{ord}_E(\| D_{du}/S \|) = \mathrm{ord}_E(\mathfrak{a}_{du})$ for every divisor E over X and every $u\in T^{\mathrm{sat}} = C\cap\mathbb{N}^r$. Arguing as in the proof of Corollary 1.8.18, we see that for every $u\in T^{\mathrm{sat}}$, the rational map f_{D_u} only depends on the cone in Δ that contains u in its relative interior.

Suppose, in addition, that g is projective and there is $u\in\mathbb{N}^r$ such that D_u is g-big. In this case, for every $v\in\mathbb{N}^r_{\geq 0}$, the divisor D_v is pseudo-effective if and only if $\mathfrak{a}_{dv}$ is nonzero. If $\Phi\colon\mathbb{R}^r\to\mathrm{N}^1(X/S)_{\mathbb{R}}$ takes v to the class of D_v, then $\mathbb{R}^r_{\geq 0}\cap\Phi^{-1}(\mathrm{PEff}(X/S)) = C$, hence the left-hand side is a rational polyhedral cone. Moreover, for every divisor E over X, the map

$$\mathbb{R}^r_{\geq 0}\cap\Phi^{-1}(\mathrm{PEff}(X/S)) \ni v \to \sigma_E(D_v/S)$$

coincides on each cone $\tau\in\Delta$ with the unique linear extension of the map $\tau\cap\mathbb{Q}^r\ni v\to\mathrm{ord}_E(\| D_v/S \|)$. Finally, for $v\in\mathbb{R}^r_{\geq 0}$, the $\mathbb{R}$-divisor D_v is g-nef if and only if

$v \in C$ and $\sigma_E(D_v) = 0$ for every divisor E over X. The set of all $v \in \mathbb{R}^r_{\geq 0}$ such that D_v is nef is a rational polyhedral cone. All these assertions follow as in the proof of Corollary 1.8.19. Moreover, if $v \in \mathbb{N}^r \cap C$ is such that D_v is nef, then $\mathscr{O}_X(dD_v)$ is g-base-point free. Indeed, note first that $v \in C$, consider the cone $\tau \in \Delta$ that contains v in its relative interior, and let $v_1, \ldots, v_r$ be a system of generators of $\tau \cap \mathbb{N}^r$. For every $w \in \tau$, if $m \in \mathbb{Z}$ is large enough, then $mv - w \in \tau$. It follows that if E is a divisor over X, then

$$0 = m \cdot \sigma_E(D_v/S) = \sigma_E(D_{mv-w}/S) + \sigma_E(D_w/S) \geq \sigma_E(D_w/S),$$

hence $\sigma_E(D_w/S) = 0$. Applying this for $w = v_i$, we conclude that $\mathrm{ord}_E(\| D_{v_i}/S \|) = 0$ for every divisor E over X, hence $\mathfrak{a}_{dv_i} = \mathscr{O}_X$ for all i. Since each $\mathscr{O}_X(D_{dv_i})$ is g-base-point free and we can write $D = \sum_{i=1}^r a_i D_{v_i}$, with $a_i \in \mathbb{N}$, it follows that $\mathscr{O}_X(dD_v)$ is g-base point free.

Chapter 2
Vanishing theorems

2.1 Kodaira-Akizuki-Nakano vanishing

Let X be a smooth projective variety of dimension n over an algebraically closed field k. Recall that the canonical line bundle on X is the sheaf of top-differential forms $\omega_X = \Omega_X^n$ on X. One reason for the important role played by this line bundle comes from Serre duality (see [Har77, Cor. III.7.7]: if $\mathscr{E}$ is a locally free sheaf on X, then there are canonical isomorphisms

$$H^i(X,\mathscr{E}) \simeq H^{n-i}(X,\omega_X \otimes \mathscr{E}^\vee)^*$$

for every i, where $\mathscr{E}^\vee$ is the dual of $\mathscr{E}$, and W^* denotes the dual of a k-vector space W.

The other important feature of ω_X is its presence in vanishing theorems. As these only hold in characteristic zero, from now on, unless explicitly mentioned otherwise, we assume that the ground field has characteristic 0. Our main goal in this section is to prove the following vanishing theorem.

Theorem 2.1.1 (Kodaira). *If $\mathscr{L}$ is an ample line bundle on the smooth projective variety X, then*

$$H^i(X,\omega_X \otimes \mathscr{L}) = 0$$

for every $i \geq 1$.

Remark 2.1.2. By Serre duality, the assertion in the theorem is equivalent to the fact that $H^i(X,\mathscr{L}^{-1}) = 0$ for all $i < n = \dim(X)$.

In fact, we will prove the following more general version of the above theorem, that also treats the sheaves of lower differential forms.

Theorem 2.1.3 (Akizuki-Nakano). *If $\mathscr{L}$ is an ample line bundle on the smooth n-dimensional projective variety X, then*

$$H^q(X,\Omega_X^p \otimes \mathscr{L}) = 0$$

for all p and q such that $p+q>n$.

Remark 2.1.4. Using the bilinear map $\Omega_X^p \otimes \Omega_X^{n-p} \to \omega_X$, one checks that $(\Omega_X^p)^\vee \simeq \omega_X^{-1} \otimes \Omega_X^{n-p}$. It thus follows from Serre duality that the vanishing in Theorem 2.1.3 is equivalent with $H^q(X, \Omega_X^p \otimes \mathscr{L}^{-1}) = 0$ for all p and q with $p+q<n$.

There is an algebraic proof of Theorem 2.1.3 due to Deligne and Illusie [DI87]. This proceeds by reduction to positive characteristic, using the properties of the de Rham complex of a smooth projective algebraic variety over a field k of positive characteristic, when the variety admits a flat lifting to the ring of Witt vectors $W_2(k)$. On the other hand, we stress that in positive characteristic the above vanishing theorems can fail (see [Ray78] for examples of surfaces on which Theorem 2.1.1 does not hold).

The proof that we give for Theorem 2.1.3 uses transcendental methods. Note that standard arguments allow us to reduce to the case when the ground field k is the field $\mathbb{C}$ of complex numbers. Indeed, suppose that K/k is an extension of algebraically closed fields, $X_K = X \times_{\operatorname{Spec} k} \operatorname{Spec} K$, and $\mathscr{L}_K$ is the pull-back to X_K of the line bundle $\mathscr{L}$ on X. It follows from Remark 1.1.3 that $\mathscr{L}$ is ample if and only if $\mathscr{L}_K$ is ample, while

$$H^q(X_K, \Omega_{X_K}^p \otimes \mathscr{L}_K) \simeq H^q(X, \Omega_X^p \otimes \mathscr{L}) \otimes_k K.$$

Therefore Theorem 2.1.3 holds for $(X, \mathscr{L})$ if and only if it holds for $(X_K, \mathscr{L}_K)$. Given X over k as in Theorem 2.1.3, we can find $k_0 \subseteq k$ algebraically closed and of finite type over $\overline{\mathbb{Q}}$ such that the pair $(X, \mathscr{L})$ is obtained by extending the scalars from a similar pair defined over k_0. Since k_0 admits an embedding in $\mathbb{C}$, it follows that it is enough to prove the theorem when $k = \mathbb{C}$. In this case, we can make use of singular cohomology and Hodge theory. Before giving the proof of Theorem 2.1.3, we need to make some preparations.

2.1.1 Cyclic covers

Let X be any scheme of finite type over k (where k is algebraically closed, of arbitrary characteristic). Suppose that m is a positive integer not divisible by $\operatorname{char}(k)$, $\mathscr{L}$ is a line bundle on X, and $s \in H^0(X, \mathscr{L}^m)$ is a section whose zero-locus $Z(s) = D$ is an effective Cartier divisor on X.

The section s induces a moprhism $\phi_s \colon \mathscr{L}^{-m} \to \mathscr{O}_X$, and we consider the $\mathscr{O}_X$-algebra $\mathscr{A}$, given as a quotient of $\bigoplus_{i \geq 0} \mathscr{L}^{-i} t^i$ by the ideal generated by $ut^m - \phi_s(u)$, where u is a local section of $\mathscr{L}^{-m}$ (here t is a variable which keeps track of the grading). It is clear that as an $\mathscr{O}_X$-module, $\mathscr{A}$ is isomorphic to $\bigoplus_{i=0}^{m-1} \mathscr{L}^{-i}$; in particular, it is coherent. The *m-cyclic cover* corresponding to s is the finite map $\pi \colon Y = \mathscr{S}pec(\mathscr{A}) \to X$ defined by $\mathscr{A}$. Note that by construction, we have $\pi_*(\mathscr{O}_Y) \simeq \mathscr{A}$.

It is easy to see that if X is complete, then up to isomorphism, the construction only depends on D, and not on the section s. Indeed, if s' is another section defining the same divisor, then we can write $s' = \lambda s$, for some $\lambda \in k^*$. Let us choose α

such that $\alpha^m = \lambda$. If $\mathscr{A}'$ is the algebra corresponding to s', then we have an isomorphism of $\mathscr{O}_X$-algebras $\mathscr{A} \to \mathscr{A}'$ that in degree j is given by multiplication by α^{-j}. Therefore in this case we also refer to Y as the m-cyclic cover corresponding to D.

It is useful to keep in mind the local description of a cyclic cover. Suppose that $U \subseteq X$ is an affine open subset of X on which we have a trivialization $\mathscr{L}|_U \simeq \mathscr{O}_U$. Using the induced trivialization $\mathscr{L}^m|_U \simeq \mathscr{O}_U$, we see that $s|_U$ corresponds to $f \in \mathscr{O}(U)$ and $\pi^{-1}(U) \simeq \mathrm{Spec}(\mathscr{O}(U)[y]/(y^m - f))$.

We collect in the following lemmas some basic properties of this construction. We keep the above notation.

Lemma 2.1.5. *There is an effective Cartier divisor R on Y such that $\pi^*(D) = mR$ and π induces an isomorphism of schemes $R \simeq D$.*

Proof. We describe R locally. Suppose that $U \subseteq X$ is an affine open subset on which we have a trivialization $\mathscr{L}|_U \simeq \mathscr{O}_U$. Let $f \in \mathscr{O}(U)$ denote the regular function corresponding to s via the induced trivialization of $\mathscr{L}^m|_U$. Consider the subscheme defined in $\pi^{-1}(U) \simeq \mathrm{Spec}(\mathscr{O}(U)[y]/(y^m - f))$ by (y). Since $y^m = f$ in $\mathscr{O}(\pi^{-1}(U))$, which is a free $\mathscr{O}(U)$-module, and f is a non-zero divisor on $\mathscr{O}(U)$, it follows that y is a non-zero divisor in $\mathscr{O}(\pi^{-1}(U))$, hence it defines an effective Cartier divisor. It is easy to see that the definition is independent of the choice of trivialization, hence we obtain an effective Cartier divisor R on Y. By looking at the local description, it is clear that $\pi^*(D) = mR$ and the induced morphism of schemes $R \to D$ is an isomorphism. $\square$

Lemma 2.1.6. *If R is as in Lemma 2.1.5, then $\pi^*(\mathscr{L}) \simeq \mathscr{O}_Y(R)$. In particular, for every $j \in \mathbb{Z}$ we have*

$$\pi_*(\mathscr{O}_Y(-jR)) = \bigoplus_{i=j}^{j+m-1} \mathscr{L}^{-j}.$$

Proof. With the notation in the proof of Lemma 2.1.5, note that the trivialization $\mathscr{L}|_U \simeq \mathscr{O}_U$ induces a trivialization $\pi^*(\mathscr{L})|_{\pi^{-1}(U)} \simeq \mathscr{O}_{\pi^{-1}(U)}$. By composing this with the isomorphism $\mathscr{O}_{\pi^{-1}(U)} \simeq \mathscr{O}_Y(R)|_{\pi^{-1}(U)}$ given by $g \to g/y$, we obtain the desired isomorphism over $\pi^{-1}(U)$. It is straightforward to check that the definition is independent of the trivialization of $\mathscr{L}|_U$ and therefore these isomorphisms glue to give $\pi^*(\mathscr{L}) \simeq \mathscr{O}_Y(R)$. The last assertion follows from the fact that since $\pi_*(\mathscr{O}_Y) \simeq \bigoplus_{i=0}^{m-1} \mathscr{L}^{-i}$, the projection formula gives

$$\pi_*(\mathscr{O}_Y(-jR)) \simeq \pi_*(\pi^*(\mathscr{L}^{-j})) \simeq \mathscr{L}^{-j} \otimes \left(\bigoplus_{i=0}^{m-1} \mathscr{L}^{-i} \right) \simeq \bigoplus_{i=j}^{j+m-1} \mathscr{L}^{-j}.$$

$\square$

Lemma 2.1.7. *The morphism $\pi \colon Y \to X$ is étale over $X \smallsetminus D$.*

Proof. It is enough to show that π is étale over any affine open subset $U \subseteq X \smallsetminus D$, and therefore we may assume that $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(S)$, where

$S = A[y]/(y^m - f)$ and f is invertible in A. A standard computation gives $\Omega_{S/A} \simeq Sdy/my^{m-1}dy$. Since m is invertible in k and y^{m-1} is invertible in S (since $y^m = f$ is invertible), it follows that $\Omega_{S/A} = 0$. □

Lemma 2.1.8. *If X and D are both smooth, then Y and R are smooth, too.*

Proof. Since X is smooth and $Y \smallsetminus R \to X \smallsetminus D$ is étale by Lemma 2.1.7, it follows that $Y \smallsetminus R$ is smooth. On the other hand, R is smooth being isomorphic to D, and since R is a Cartier divisor in Y, it follows that Y is smooth along R as well. □

Remark 2.1.9. Under the assumptions of Lemma 2.1.8, if $D = 0$, then it can happen that Y is reducible, even if X is irreducible (for example, if $\mathscr{L} = \mathscr{O}_X$ and $s = 1$, then Y is a disjoint union of m copies of X). On the other hand, if X is irreducible and D is nonzero, then Y is irreducible as well. Indeed, since we know that Y is smooth, it is enough to show that it is connected. If Y_1 and Y_2 are non-empty open subsets of Y such that $Y = Y_1 \sqcup Y_2$, since π is finite and flat, both $\pi(Y_1)$ and $\pi(Y_2)$ are open and closed in X, hence $\pi(Y_1) = X = \pi(Y_2)$. It follows that if D_0 is an irreducible component of D and R_0 is the corresponding irreducible component of R, then Y_1 and Y_2 intersect R_0. The decomposition $R_0 = (R_0 \sqcup Y_1) \sqcup (R_0 \sqcup Y_2)$ contradicts the fact that R_0 is connected.

2.1.2 The de Rham complex with log poles

Suppose that X is a smooth n-dimensional variety (to begin with, we make no assumption on the ground field k). Recall that an effective divisor D on X has *simple normal crossings* (SNC, for short) if for every $p \in X$, there are (algebraic) coordinates $x_1, \ldots, x_n$ in an affine neighborhood U of p[1] such that D is defined in U by an equation of the form $x_1^{a_1} \cdots x_n^{a_n}$, with $a_1, \ldots, a_n$ nonnegative integers. Note that in this case the irreducible components of D are smooth and they intersect transversely.

Suppose that D is a reduced divisor on D, having simple normal crossings. We now define the *sheaf of* 1*-forms on X with log poles along D*, denoted by $\Omega_X(\log D)$. This is the subsheaf of $\Omega_X \otimes K(X)$ described locally as follows. Suppose that U is an affine open subset of X and $x_1, \ldots, x_n$ are coordinates on U such that D is defined in U by $x_1 \cdots x_r$. In this case $\Omega_X(\log D)|_U$ is generated by

$$\frac{dx_1}{x_1}, \ldots, \frac{dx_r}{x_r}, dx_{r+1}, \ldots, dx_n.$$

Note that this is independent of the choice of coordinates: if $h \in \mathscr{O}(U)^*$, then for $1 \leq i \leq r$ we have

[1] This means that $dx_1, \ldots, dx_n$ give a trivialization of $\Omega_X|_U$, or equivalently, the map $U \to \mathbb{A}^n$ defined by $x_1, \ldots, x_n$ is étale; this is also equivalent with saying that for every closed point $q \in U$, $x_1 - x_1(q), \ldots, x_n - x_n(q)$ generate the maximal ideal in $\mathscr{O}_{X,q}$.

$$\frac{d(hx_i)}{hx_i} = \frac{dh}{h} + \frac{dx_i}{x_i} \in \mathscr{O}(U) \cdot \frac{dx_i}{x_i} + \sum_{j=1}^{n} \mathscr{O}(U) \cdot dx_j.$$

It is clear from definition that $\Omega_X(\log D)$ is a locally free sheaf of rank n containing Ω_X. For every nonnegative integer p, we put

$$\Omega_X^p(\log D) := \wedge^p \Omega_X(\log D).$$

In particular, $\Omega_X^0(\log D) = \mathscr{O}_X$ and it follows easily from definition that $\Omega_X^n(\log D) \simeq \omega_X \otimes \mathscr{O}_X(D)$.

Recall that we have the de Rham complex $\Omega_X^\bullet$ on X:

$$0 \to \mathscr{O}_X \xrightarrow{d} \Omega_X^1 \xrightarrow{d} \ldots \xrightarrow{d} \Omega_X^n \to 0.$$

This induces the de Rham complex $\Omega_X^\bullet \otimes K(X)$ of meromorphic forms on X, and it is easy to see that the de Rham differential preserves the forms with log poles along D (the key fact is that $d(dx_i/x_i) = 0$). We thus obtain the de Rham complex with log poles $\Omega_X^\bullet(\log D)$.

In the following two propositions we collect two facts that we will need about forms with log poles, in the case of a smooth divisor.

Proposition 2.1.10. *Let X be a smooth variety, $\mathscr{L}$ a line bundle on X, m a positive integer not divisible by* $\mathrm{char}(k)$*, and $s \in \Gamma(X, \mathscr{L}^m)$ a section whose zero-locus is a smooth effective divisor D. If $\pi\colon Y \to X$ is the m-cyclic cover corresponding to s, and R is the effective divisor on Y such that $\pi^*(D) = mR$, then for every non-negative integer p we have a canonical isomorphism*

$$\pi^*(\Omega_X^p(\log D)) \simeq \Omega_Y^p(\log R).$$

Proof. Since both sheaves are canonically isomorphic to subsheaves of $\Omega^p_{K(Y)}$, it is enough to check that we have equality locally. Furthermore, it is enough to check this equality for $p = 1$, since the general case follows by taking exterior powers. The assertion is clear on $Y \smallsetminus R$, since π is étale on this open subset, hence $\pi^*(\Omega_X) = \Omega_Y$ on $Y \smallsetminus R$. Suppose now that $U \simeq \mathrm{Spec}(A)$ is an affine open subset in X and we have $x_1, \ldots, x_n$ coordinates on U such that $\mathscr{L}|_U \simeq \mathscr{O}_U$ and $s|_U$ corresponds to tx_1, with $t \in \mathscr{O}_X(U)$ invertible. Since on $\pi^{-1}(U) \simeq \mathrm{Spec}(A[y]/(y^m - tx_1))$ we have algebraic coordinates $y, x_2, \ldots, x_n$ and

$$\pi^*(dx_i) = dx_i \text{ for } i \geq 2 \text{ and } \pi^*\left(\frac{dx_1}{x_1}\right) = \frac{d(y^m)}{y^m} = m \cdot \frac{dy}{y},$$

we obtain the identification on U for the two sheaves in the proposition, when $p = 1$. $\square$

Proposition 2.1.11. *If X is a smooth variety and D is a smooth divisor on X, then for every non-negative integer p, we have an exact sequence*

$$0 \to \Omega_X^p(\log D) \otimes \mathscr{O}_X(-D) \xrightarrow{i} \Omega_X^p \xrightarrow{\tau} \Omega_D^p \to 0,$$

where i is the natural inclusion and τ is given by restriction of forms.

Proof. Since the restriction map τ is surjective, it is enough to check locally that its kernel is equal to $\Omega_X^p(\log D) \otimes \mathscr{O}_X(-D)$. Let U be an affine open subset of X on which we have coordinates $x_1, \ldots, x_n$ such that D is defined by (x_1). In this case, the kernel of $\tau|_U$ is

$$dx_1 \wedge \Omega_U^{p-1} + x_1 \cdot \Omega_U^p = \Gamma(U, \Omega_X^p(\log D) \otimes \mathscr{O}_X(-D)).$$

This gives the assertion in the proposition. □

Suppose now that $k = \mathbb{C}$. In this case, every smooth n-dimensional algebraic variety over k has a canonical structure of complex n-dimensional manifold X^{an}. In particular, we may consider the singular cohomology of X^{an}. The following is a fundamental theorem that shows that the hypercohomology of the de Rham complex with log poles computes the singular cohomology with complex coefficients for the complement of an SNC divisor.

Theorem 2.1.12. *(Grothendieck-Deligne) If X is a smooth complex algebraic variety and D is a simple normal crossing divisor on X, then there is a canonical isomorphism*

$$H^i(X, \Omega_X^\bullet(\log D)) \simeq H^i(X^{\mathrm{an}} \smallsetminus D^{\mathrm{an}}; \mathbb{C}).$$

Remark 2.1.13. We will use the above theorem in the case when X is projective (and D is a smooth divisor). Note that if $D = 0$, then the statement follows by combining the following consequence of GAGA

$$H^i(X, \Omega_X^\bullet) \simeq H^i(X^{\mathrm{an}}, \Omega_{X^{\mathrm{an}}}^\bullet)$$

with the fact that $\Omega_{X^{\mathrm{an}}}^\bullet$ gives a resolution of the constant sheaf $\underline{\mathbb{C}}_X$ (in the analytic topology), which in turn is a consequence of the complex-analytic Poincaré Lemma. The case when we also have a divisor D can be deduced without much effort by induction on the number of irreducible components of D.

Suppose now that X is a smooth complex projective variety and D is a divisor with simple normal crossings on X. The "stupid" filtration on the de Rham complex induces a Hodge-to-de Rham spectral sequence

$$E_1^{p,q} = H^q(X, \Omega_X^p(\log D)) \underset{p}{\Rightarrow} H^{p+q}(X, \Omega_X^\bullet(\log D)). \tag{2.1}$$

The following is a fundamental consequence of Hodge theory[2].

[2] Deligne and Illusie gave an algebraic proof of this result in [DI87], using reduction to positive characteristic.

Theorem 2.1.14. *If X is a smooth complex projective variety and D is a divisor with simple normal crossings on X, then the Hodge-to-de Rham spectral sequence (2.1) degenerates at E_1.*

By combining Theorems 2.1.14 and 2.1.12, we obtain the following corollary, which is the result that we will need.

Corollary 2.1.15. *If X is a smooth complex projective variety and D is a divisor with simple normal crossings on X, then*

$$\dim_{\mathbb{C}} H^i(X^{\mathrm{an}} \smallsetminus D^{\mathrm{an}}; \mathbb{C}) = \sum_{p+q=i} h^q(X, \Omega_X^p(\log D))$$

for every $i \geq 0$.

2.1.3 Cohomology of smooth complex affine algebraic varieties

Our goal in this subsection is to prove the following theorem concerning the topology of smooth affine complex varieties. In doing this, we follow the presentation in [Laz04a, Chap. 3.1.A].

Theorem 2.1.16 (Andreotti-Frankel). *If $M \hookrightarrow \mathbb{C}^N$ is a closed n-dimensional complex submanifold, then M has the homotopy type of a CW-complex of (real) dimension $\leq n$. In particular, we have $H^i(M, \mathbb{Z}) = 0$ and $H_i(M, \mathbb{Z}) = 0$ for all $i > n$.*

The proof of Theorem 2.1.16 uses some basic results from Morse theory, that we briefly review. We refer the reader to [Mil63] for proofs and details. Suppose that M is a $\mathscr{C}^\infty$ (real) manifold and $\phi \colon M \to \mathbb{R}$ is a $\mathscr{C}^\infty$ map. If $p \in M$ is a critical point of ϕ (that is, $d_p\phi = 0$), then there is a symmetric bilinear form on T_pM, the *Hessian* $\mathrm{Hess}_p(\phi)$. If $x_1, \ldots, x_d$ are local coordinates around p, then with respect to the basis of T_pM given by $\frac{\partial}{\partial x_i}(p)$, this form is given by the matrix $\left(\frac{\partial^2\phi}{\partial x_i \partial x_j}(p)\right)_{i,j}$. One says that the critical point $p \in M$ is *non-degenerate* if $\mathrm{Hess}_p(\phi)$ is non-degenerate. A lemma due to Morse asserts that if $p \in M$ is non-degenerate, then one can choose coordinates $x_1, \ldots, x_n$ around p such that

$$\phi = \phi(p) - x_1^2 - \ldots - x_r^2 + x_{r+1}^2 + \ldots + x_n^2$$

in a neighborhood of p. Of course, in this case $(n-r, r)$ is the signature of $\mathrm{Hess}_p(\phi)$ and one defines the *index* of ϕ at p to be r.

The function ϕ is a *Morse function* if all critical points of ϕ are non-degenerate. One way to obtain Morse functions is the following.

Proposition 2.1.17. *If $M \subsetneq \mathbb{R}^N$ is a closed real submanifold, then for almost all $c \in \mathbb{R}^N$, the function*

$$M \ni p \to \phi_c(p) = d(p, c)^2$$

is a Morse function, where $d(x, y)$ is the standard product metric on $\mathbb{R}^N$.

The following fundamental result of Morse theory relates the topology of a manifold to the critical points of a Morse function.

Theorem 2.1.18. *Let $\phi: M \to \mathbb{R}$ be a Morse function on a $\mathscr{C}^\infty$ manifold M such that for every $a \in \mathbb{R}$, the subset $\phi^{-1}((-\infty, a]) \subseteq M$ is compact. In this case M has the homotopy type of a CW complex, with a cell of dimension r for every non-degenerate point of ϕ of index r.*

Remark 2.1.19. Note that the Morse functions described in Proposition 2.1.17 satisfy the condition in Theorem 2.1.18. Indeed, each subset $\phi_c^{-1}((-\infty, a])$ is by definition bounded, and it is closed in M, which in turn is closed in $\mathbb{R}^N$. Therefore $\phi_c^{-1}((-\infty, a])$ is compact.

By combining Theorem 2.1.18 and Proposition 2.1.17, we see that the assertion in Theorem 2.1.16 follows from the following proposition.

Proposition 2.1.20. *Let $M \subseteq \mathbb{C}^N = \mathbb{R}^{2N}$ be a closed complex submanifold of (complex) dimension n. For every $c \in \mathbb{C}^N$, if $p \in M$ is a critical point of the function $\phi: M \to \mathbb{R}$ given by $\phi(p) = d(p,c)^2$, and the signature of $\mathrm{Hess}_p(\phi)$ is (s,r), then $r \leq n$.*

Proof. We may clearly assume that $p = 0$. Furthermore, since M is an n-dimensional complex submanifold of $\mathbb{C}^N$, after possibly relabeling the coordinates we may assume that the projection onto the first n components induces a map $M \to \mathbb{C}^n$ which is biholomorphic in a neighborhood of 0. Therefore there are holomorphic maps $f_1, \ldots, f_N$ defined in a neighborhood U of 0 in $\mathbb{C}^n$, with $f_i = z_i$ for $1 \leq i \leq n$, such that around p we have

$$M = \{(f_1(z), \ldots, f_N(z)) \mid z \in U\}.$$

Therefore it is enough to consider the Hessian at $0 \in \mathbb{C}^n$ for the function

$$g: U \to \mathbb{R},\ g(z_1, \ldots, z_n) = \sum_{i=1}^N |f_i(z) - c_i|^2 = \sum_{i=1}^N (f_i(z) - c_i)(\overline{f_i(z)} - \overline{c_i}),$$

where $c = (c_1, \ldots, c_N)$. For every i with $1 \leq i \leq N$, let us consider the Taylor expansion of f_i around 0, namely $f_i = \sum_{\ell \geq 1} f_{i,\ell}$, with each $f_{i,\ell}$ a homogeneous polynomial function of degree ℓ. An easy computation gives $\mathrm{Hess}_0(g) = \mathrm{Hess}_0(h)$, where

$$h(z) = \sum_{i=1}^N |f_{i,1}(z)|^2 - 2\sum_{i=1}^N \mathrm{Re}(\overline{c_i} \cdot f_{i,2}(z)).$$

Since $\sum_{i=1}^N |f_{i,1}(z)|^2$ is a positive definite real quadratic form (recall that $f_i(z) = z_i$ for $1 \leq i \leq n$), it follows that if V is a real subspace of $\mathbb{C}^n = \mathbb{R}^{2n}$ such that $\mathrm{Hess}_0(h)$ is negative definite on V, then the real quadratic form $\sum_{i=1}^N \mathrm{Re}(\overline{c_i} \cdot f_{i,2}(z))$ is positive definite on V. Therefore, in order to complete the proof of the proposition, it is enough to show that if Q is a complex, symmetric, bilinear form on $\mathbb{C}^n = \mathbb{R}^{2n}$, and (a,b) is the signature of the real quadratic form $z \to \mathrm{Re}(Q(z,z))$, then $a \leq n$. Note that we may find a basis of $\mathbb{C}^n$ such that $Q(z,z) = \sum_{i=1}^r z_i^2$. By writing $z_i = u_i + \sqrt{-1}v_i$, we see that

$$\mathrm{Re}(Q(z,z)) = \sum_{i=1}^r u_i^2 - \sum_{i=1}^r v_i^2,$$

hence $a = b = r \leq n$. □

2.1.4 The proof of the Akizuki-Nakano vanishing theorem

By putting together the ingredients discussed in the previous sections, we can give a proof of the Akizuki-Nakano vanishing theorem.

Proof of Theorem 2.1.3. As we have already mentioned, we may assume that the ground field is $\mathbb{C}$. We prove the theorem by induction on n, the case $n = 0$ being trivial.

Since $\mathscr{L}$ is ample, there is $m \geq 1$ such that $\mathscr{L}^m$ is very ample. By Bertini's theorem, we can find a smooth divisor $D \in |\mathscr{L}^m|$. Let $\pi\colon Y \to X$ be the m-cyclic cover corresponding to D, and let R be the effective divisor on Y such that $\pi^*(D) = mR$.

Since D is ample and π is finite, it follows from Proposition 1.1.9 that $\pi^*(D)$ is ample, hence R is ample. Therefore $Y \smallsetminus R$ is affine, and Theorem 2.1.16 implies

$$H^i(Y^{\mathrm{an}} \smallsetminus R^{\mathrm{an}}; \mathbb{C}) = 0 \quad \text{for all} \quad i > n.$$

By combining this with Corollary 2.1.15, we obtain

$$H^q(Y, \Omega_Y^p(\log R)) = 0 \quad \text{for} \quad p+q > n.$$

On the other hand, Proposition 2.1.10 gives $\Omega_Y^p(\log R) \simeq \pi^*(\Omega_X^p(\log D))$ and we deduce using the projection formula (recall that π is finite)

$$H^q(X, \Omega_X^p(\log D) \otimes \pi_*(\mathscr{O}_Y)) = 0 \quad \text{for} \quad p+q > n.$$

Since $\pi_*(\mathscr{O}_Y) = \bigoplus_{j=0}^{m-1} \mathscr{L}^{-j}$, we have

$$H^q(X, \Omega_X^p(\log D) \otimes \mathscr{L}^{-j}) = 0 \quad \text{for} \quad p+q > n \quad \text{and} \quad 0 \leq j \leq m-1. \tag{2.2}$$

Recall now that by Proposition 2.1.11, for every $p \geq 0$ we have an exact sequence

$$0 \to \Omega_X^p(\log D) \otimes \mathscr{O}_X(-D) \to \Omega_X^p \to \Omega_D^p \to 0$$

and by tensoring with $\mathscr{L}$, the long exact sequence in cohomology gives

$$H^q(X,\Omega_X^p(\log D)\otimes\mathscr{L}^{-m+1})\to H^q(X,\Omega_X^p\otimes\mathscr{L})\to H^q(D,\Omega_D^p\otimes\mathscr{L}|_D). \quad (2.3)$$

For $p+q>n$ the first term in (2.3) vanishies by (2.2) and the third term vanishes by the induction hypothesis, hence $H^q(X,\Omega_X^p\otimes\mathscr{L})=0$. □

2.2 The Kawamata–Viehweg vanishing theorem

Our goal in this section is to prove an important extension of Kodaira's vanishing theorem, due to Kawamata and Viehweg. This extension goes in two directions. First, one replaces the "ample" condition by "big and nef". Second, one allows small perturbations supported on a simple normal crossing divisor. We keep the assumption that the ground field is algebraically closed, of characteristic zero.

For a real number u, we denote by $\lfloor u\rfloor$ the largest integer that is $\leq u$, and by $\lceil u\rceil$ the smallest integer $\geq u$. If X is a normal variety and $D=\sum_{i=1}^r a_iD_i$ is an $\mathbb{R}$-divisor on X, with the D_i pairwise distinct prime divisors, then we put

$$\lfloor D\rfloor := \sum_{i=1}^r \lfloor a_i\rfloor D_i \text{ and } \lceil D\rceil := \sum_{i=1}^r \lceil a_i\rceil D_i.$$

By definition, both $\lceil D\rceil$ and $\lfloor D\rfloor$ are integral divisors on X.

If X is a smooth variety, we say that an $\mathbb{R}$-divisor $\sum_{i=1}^r a_iD_i$ has simple normal crossings if $\sum_{i=1}^r D_i$ has simple normal crossings. We can now state the main result of this section.

Theorem 2.2.1 (Kawamata–Viehweg). *If X is a smooth projective variety and D is a big and nef $\mathbb{Q}$-divisor such that $\lceil D\rceil - D$ has simple normal crossings, then*

$$H^i(X,\omega_X\otimes\mathscr{O}_X(\lceil D\rceil))=0 \quad \textit{for all} \quad i\geq 1.$$

We give the proof of the theorem following [KM98, Chap. 2.5]. We begin with some preparations.

Lemma 2.2.2. *Let X be a projective variety and $\mathscr{M}\in\operatorname{Pic}(X)$. For every positive integer m, there is a finite, surjective morphism $f\colon Y\to X$ from a projective variety Y with $\mathscr{L}\in\operatorname{Pic}(Y)$ such that $f^*(\mathscr{M})\simeq\mathscr{L}^m$. Furthermore, if X is smooth and Δ is an effective, reduced, simple normal crossing divisor on X, we may find f such that Y is smooth and $f^*(\Delta)$ is reduced and has simple normal crossings.*

Proof. If $\mathscr{M}$ is very ample, then it defines an embedding $j\colon X\hookrightarrow\mathbb{P}^N$. Consider the finite morphism $g\colon\mathbb{P}^N\to\mathbb{P}^N$ given by $g(x_0,\ldots,x_N)=(x_0^m,\ldots,x_N^m)$, which has the property that $g^*\mathscr{O}_{\mathbb{P}^N}(1)\simeq\mathscr{O}_{\mathbb{P}^N}(m)$. Let Y be the reduced scheme structure on an irreducible component of the fiber product of j and g that dominates X. We have a commutative diagram

$$\begin{tikzcd} Y \arrow[r, hook, "h"] \arrow[d, "f"'] & \mathbb{P}^N \arrow[d, "g"] \\ X \arrow[r, hook, "j"] & \mathbb{P}^N, \end{tikzcd}$$

with f finite and surjective and it is clear that if $\mathscr{L} = h^*\mathscr{O}_{\mathbb{P}^N}(1)$, then $f^*(\mathscr{M}) \simeq \mathscr{L}^m$.

If X is smooth and Δ is an effective, reduced divisor on X with simple normal crossings, then we replace g by $\sigma \circ g$, where $\sigma \in \mathrm{Aut}(\mathbb{P}^N)$ is a general element. Since $\mathrm{char}(k) = 0$, Kleiman's version of Bertini's theorem (see [Har77, Thm. III.10.8]) implies that the fiber product of $\mathbb{P}^N$ with X and with each intersection of irreducible components of Δ is again smooth, of the expected dimension (though possibly disconnected). After taking Y to be a connected component of the fiber product of P^N with X, we also satisfy the second condition in the lemma.

For an arbitrary line bundle $\mathscr{M}$, let us write $\mathscr{M} \simeq \mathscr{M}_1 \otimes \mathscr{M}_2^{-1}$, for suitable very ample line bundles $\mathscr{M}_1$ and $\mathscr{M}_2$. We first construct as above $f_1 \colon Y_1 \to X$ such that $f_1^*(\mathscr{M}_1) \simeq \mathscr{L}_1^m$ for some $\mathscr{L}_1 \in \mathrm{Pic}(Y_1)$. The pull-back $f_1^*(\mathscr{M}_2)$ is ample by Proposition 1.1.9, hence we may choose a positive integer r, relatively prime to m, such that $f_1^*(\mathscr{M}_2)^r$ is very ample. We then construct $f_2 \colon Y \to Y_1$ as above such that $f_2^*(f_1^*(\mathscr{M}_2))^r \simeq \mathscr{L}_2^m$ for some $\mathscr{L}_2 \in \mathrm{Pic}(Y)$. If a and b are integers such that $ar + bm = 1$ and $\mathscr{L} = f_2^*(\mathscr{L}_1) \otimes (\mathscr{L}_2^a \otimes f_2^*(\mathscr{M}_2)^b)^{-1}$, then $f^*(\mathscr{M}) \simeq \mathscr{L}^m$. $\square$

Definition 2.2.3. A normal variety X is $\mathbb{Q}$-factorial if for every Weil divisor D on X, there is a positive integer m such that mD is Cartier. Equivalently, the $\mathbb{Q}$-linear map $\mathrm{CDiv}(X)_{\mathbb{Q}} \to \mathrm{Div}(X)_{\mathbb{Q}}$ is an isomorphism.

The following lemma is a general result that is useful also in other situations.

Lemma 2.2.4. *If $f \colon Y \to X$ is a birational projective morphism of normal varieties, with X being $\mathbb{Q}$-factorial and carrying an ample line bundle, then there is an effective exceptional Cartier divisor[3] F on Y such that $-F$ is f-ample.*

Proof. Since f is projective and X has an ample line bundle, it follows from Remark 1.6.18 that there is an f-ample effective Cartier divisor H on Y. Let m be a positive integer such that $mf_*(H)$ is Cartier. If $F = f^*(mf_*(H)) - mH$, then F is effective and $-F$ is f-ample. $\square$

Remark 2.2.5. With the notation in Lemma 2.2.4, the exceptional locus $\mathrm{Exc}(f)$ is equal to $\mathrm{Supp}(F)$. Indeed, if $y \in \mathrm{Exc}(f)$, then there is a curve $C \subseteq f^{-1}(f(y))$ containing y (see Lemma B.2.2). Since $-F$ is f-ample, we have $(F \cdot C) < 0$, hence $C \subseteq \mathrm{Supp}(F)$. In particular, $y \in \mathrm{Supp}(F)$. Since by construction $\mathrm{Supp}(F) \subseteq \mathrm{Exc}(f)$, we have in fact equality. In particular, we conclude that for every projective, birational morphism $f \colon Y \to X$ between normal varieties, with X being $\mathbb{Q}$-factorial, all irreducible components of $\mathrm{Exc}(f)$ have codimension 1 (note that this property is local on X, hence we may assume that X is affine).

[3] We refer to Appendix B for a review of some basic facts concerning exceptional divisors.

Lemma 2.2.6. *Let X be a smooth variety, $\mathscr{L} \in \mathrm{Pic}(X)$, m a positive integer, and $s_0 \in H^0(X, \mathscr{L}^m)$ defining a smooth effective divisor D. Suppose that $D_1, \ldots, D_r$ are smooth divisors on X such that $D, D_1, \ldots, D_r$ have no common components and $D + \sum_{i=1}^r D_i$ has simple normal crossings. If $f \colon Y \to X$ is the m-cyclic cover corresponding to s_0 and R is the divisor on Y such that $f^*(D) = mR$, then the divisors $R, f^*(D_1), \ldots, f^*(D_r)$ have no common components, are all smooth, and $R + \sum_{i=1}^r f^*(D_i)$ has simple normal crossings.*

Proof. The assertion is clear over $X \smallsetminus D$ since f is étale over this open subset by Lemma 2.1.7. Given a point $p \in D$, we choose a local trivialization of $\mathscr{L}$ in an affine open neighborhood U of p and a system of coordinates $x_1, \ldots, x_n$ in U such that s is described in U by x_1 and every D_i intersecting U is defined in U by some (x_{ℓ_i}), with $\ell_i \geq 2$. Note that on $f^{-1}(U)$ we have coordinates $y, x_2, \ldots, y_n$ such that $x_1 = y^m$ and R is defined by (y). The assertions in the lemma are now clear. $\square$

The next lemma is a useful fact, in characteristic zero, about the behavior of cohomology of vector bundles under pull-back by finite morphisms.

Lemma 2.2.7. *Let $f \colon Y \to X$ be a finite morphism of varieties, with X normal. If $\mathscr{E}$ is a locally free sheaf on X, then the canonical map of $\mathscr{O}(X)$-modules*

$$H^i(X, \mathscr{E}) \to H^i(Y, f^*(\mathscr{E}))$$

is a split injection.

Proof. Consider the induced field extension $K(X) \hookrightarrow K(Y)$ between the function fields of X and Y and let $\mathrm{Tr} \colon K(Y) \to K(X)$ be the corresponding trace map. Since X is normal, Tr induces a morphism of $\mathscr{O}_X$-modules $\alpha \colon g_*(\mathscr{O}_Y) \to \mathscr{O}_X$ such that $\frac{1}{\deg(f)} \alpha$ gives a splitting of the natural inclusion $j \colon \mathscr{O}_X \hookrightarrow g_*(\mathscr{O}_Y)$.

If $\mathscr{E}$ is a locally free sheaf on X, we deduce that also the map

$$\mathscr{E} \overset{1 \otimes j}{\hookrightarrow} \mathscr{E} \otimes g_*(\mathscr{O}_Y) \simeq g_*(g^*(\mathscr{E}))$$

is a split injection. Therefore the map induced on cohomology

$$H^i(X, \mathscr{E}) \to H^i(X, g_*(g^*(\mathscr{E}))) \simeq H^i(Y, f^*(\mathscr{E}))$$

is a split injection for every $i \geq 0$. $\square$

The following proposition is the tool that will allow us in the proof of Theorem 2.2.1 to replace $\mathbb{Q}$-divisors by integral divisors.

Proposition 2.2.8. *Let X be a smooth, projective variety, F a divisor on X, and D, E two $\mathbb{Q}$-divisors on X such that $F \sim_{\mathbb{Q}} D + E$. If E has simple normal crossings and $\lfloor E \rfloor = 0$, then there is a finite, surjective morphism $p \colon W \to X$, with W smooth, and a divisor D_W on W such that the following conditions hold:*

i) $D_W \sim_{\mathbb{Q}} p^*(D)$.

ii) $H^i(X, \mathscr{O}_X(-F))$ *is a summand of* $H^i(W, \mathscr{O}_W(-D_W))$ *for every* $i \geq 0$.

Proof. Let us write $E = \sum_{i=1}^r a_i E_i$. It is convenient to not require the E_i to be irreducible, but require the E_i to have no common components. We prove the assertion in the proposition by induction on r, the case $r = 0$ being trivial. For the induction step, we choose a positive integer m such that $ma_1 \in \mathbb{Z}$. We apply Lemma 2.2.2 to construct a finite surjective morphism $g\colon Z \to X$, with Z smooth and a divisor E_1' on Z such that $\sum_{j=1}^r g^*(E_j)$ is reduced, has simple normal crossings and $g^*(E_1) \sim mE_1'$. The divisor $g^*(E_1)$ corresponds to a section of $\mathscr{O}_Z(E_1')^m$, hence we may construct a corresponding m-cyclic cover $h\colon Y \to Z$ and put $f = g \circ h$. Since $g^*(E_1), \ldots, g^*(E_r)$ are smooth, have no common components, and $\sum_{j=1}^r g^*(E_j)$ has simple normal crossings, it follows from Lemma 2.2.6 that Y is smooth, the divisors $f^*(E_j)$, for $j \geq 2$, are smooth, without common components, and $\sum_{j=2}^r f^*(E_j)$ has simple normal crossings.

Let $ma_1 = b$, hence b is an integer with $0 \leq b \leq m-1$. We can write

$$g^*(F) - bE_1' \sim_{\mathbb{Q}} g^*(D) + \sum_{j=2}^r a_j g^*(E_j) \tag{2.4}$$

so that if $F_Y = h^*(g^*(F) - bE_1')$, then

$$F_Y \sim_{\mathbb{Q}} f^*(D) + \sum_{j=2}^r a_j f^*(E_j).$$

Therefore we may apply the inductive assumption to F_Y to construct a finite surjective morphism $q\colon W \to Y$, with W smooth, such that there is a divisor D_W on W with $q^*(f^*(D)) \sim_{\mathbb{Q}} D_W$ and $H^i(Y, \mathscr{O}_Y(-F_Y))$ a direct summand of $H^i(W, \mathscr{O}_W(-D_W))$.

By taking $p = f \circ q$, we see that it is enough to show that $H^i(X, \mathscr{O}_X(-F))$ is a direct summand of $H^i(Y, \mathscr{O}_Y(-E_Y))$. On one hand, Lemma 2.2.7 implies that $H^i(X, \mathscr{O}_X(-F))$ is a direct summand of $H^i(Z, \mathscr{O}_Z(-g^*(F)))$. On the other hand, by the definition of the m-cyclic cover, we have the decomposition

$$h_*(\mathscr{O}_Y) \simeq \bigoplus_{\ell=0}^{m-1} \mathscr{O}_Z(-\ell E_1'),$$

and via the projection formula this induces the decomposition

$$\begin{aligned} H^i(Y, \mathscr{O}_Y(-F_Y)) &\simeq H^i(Z, \mathscr{O}_Z(-g^*(F) + bE_1') \otimes h_*(\mathscr{O}_Y)) \\ &\simeq \bigoplus_{\ell=0}^{m-1} H^i(Z, \mathscr{O}_Z(-g^*(F) + (b-\ell)E_1')). \end{aligned}$$

By taking $\ell = b$, we deduce that $H^i(Z, \mathscr{O}_Z(-g^*(F)))$ is a direct summand of $H^i(Y, \mathscr{O}_Y(-E_Y))$. We thus conclude that $H^i(X, \mathscr{O}_X(-F))$ is a direct summand of $H^i(Y, \mathscr{O}_(-E_Y))$, which completes the proof of the induction step. $\square$

The next lemma gives a variant for the characterization of nef and big divisors in Proposition 1.4.34.

Lemma 2.2.9. *If D is a an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on the projective variety X, then D is big and nef if and only if there is a birational morphism $f\colon Y \to X$, with Y smooth and projective, and an effective simple normal crossing $\mathbb{R}$-divisor E on Y such that $f^*(D) - \frac{1}{m}E$ is ample for all integers $m \geq 1$. Furthermore, in this case E can be chosen a $\mathbb{Q}$-divisor.*

Proof. Recall first that the pull-back of an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor is big and nef if and only if the original divisor is big and nef (see Remark 1.4.32 and Proposition 1.3.9). If we can find f and E as in the lemma, it follows from Proposition 1.4.34 that $f^*(D)$ is big and nef, hence D is big and nef.

Conversely, suppose that D is big and nef. We first choose a resolution of singularities $g\colon X_1 \to X$ of X. Since $g^*(D)$ is big, it follows from Proposition 1.4.28 that we can find $F \in \mathrm{CDiv}(X_1)_{\mathbb{Q}}$ effective such that $g^*(D) - F$ is ample. We now consider a log resolution $h\colon Y \to X_1$ of the pair (X_1, F) and let $f = g \circ h$. Since X_1 is smooth, hence $\mathbb{Q}$-factorial, it follows from Lemma 2.2.4 that there is an h-exceptional effective divisor G on Y such that $-G$ is h-ample. Proposition 1.6.15 implies that $h^*(g^*(D) - F) - \frac{1}{q}G$ is ample for some positive integer q. Note also that since G is supported on the exceptional locus of h, the divisor $E = h^*(F) + \frac{1}{q}G$ has simple normal crossings. The divisor

$$mf^*(D) - E = h^*(g^*(D) - F) - \frac{1}{q}G + (m-1)f^*(D)$$

is ample, being a sum of an ample divisor and a nef divisor, hence E satisfies the conditions in the lemma. □

Finally, we will need the following proposition which is useful also in other situations. Given a morphism $f\colon Y \to X$, the proposition gives the vanishing of the higher direct images of a sheaf $\mathscr{F}$ on Y when one knows the vanishing of the higher cohomology groups of suitable twists of $\mathscr{F}$.

Proposition 2.2.10. *Let $f\colon Y \to X$ be a morphism of projective schemes, $\mathscr{F}$ a coherent sheaf on Y, and $\mathscr{L}$ an ample line bundle on X. If $j_0 \in \mathbb{Z}_{\geq 0}$ is such that we have $H^i(Y, \mathscr{F} \otimes f^*(\mathscr{L})^j) = 0$ for all $i \geq 1$ and $j \geq j_0$, then*

i) $R^i f_*(\mathscr{F}) = 0$ for every $i \geq 1$, and
ii) $H^i(X, f_*(\mathscr{F}) \otimes \mathscr{L}^j) = 0$ for every $i \geq 1$ and $j \geq j_0$.

Proof. Using the projection formula, we can write the Leray spectral sequence for f and the sheaf $\mathscr{F} \otimes f^*(\mathscr{L})^j$ as

$$E_2^{p,q} = H^p(X, R^q f_*(\mathscr{F}) \otimes \mathscr{L}^j) \Rightarrow H^{p+q}(Y, \mathscr{F} \otimes f^*(\mathscr{L})^j).$$

Note that for $j \gg 0$, since $\mathscr{L}$ is ample, we have $H^p(X, R^q f_*(\mathscr{F}) \otimes \mathscr{L}^j) = 0$ for all $p \geq 1$. The above spectral sequence implies that for such j, we have

$$H^0(X, R^q f_*(\mathscr{F}) \otimes \mathscr{L}^j) \simeq H^q(Y, \mathscr{F} \otimes f^*(\mathscr{L})^j) = 0 \tag{2.5}$$

for all $q \geq 1$, where the vanishing follows by hypothesis. Using one more time the ampleness of $\mathscr{L}$, we see that $R^q f_*(\mathscr{F}) \otimes \mathscr{L}^j$ is generated by global sections for $j \gg 0$, and therefore (2.5) implies $R^q f_*(\mathscr{F}) = 0$ for $q \geq 1$, giving the assertion in i). The above spectral sequence for $j \geq j_0$ gives

$$H^p(X, \pi_*(\mathscr{F}) \otimes \mathscr{L}^j) \simeq H^p(f, \mathscr{F} \otimes f^*(\mathscr{L})^j) = 0$$

for every $p \geq 1$, hence ii). □

We can now give the proof of the Kawamata-Viehweg vanishing theorem.

Proof of Theorem 2.2.1. Note that the vanishing in the theorem is equivalent via Serre duality with

$$H^i(X, \mathscr{O}_X(-\lceil D \rceil)) = 0 \quad \text{for} \quad i < n = \dim(X). \tag{2.6}$$

It will be convenient to use both formulations. We divide the proof in two steps.
Step 1. We apply Proposition 2.2.8 with $F = \lceil D \rceil$ to construct a finite surjective morphism $p \colon W \to X$, with W smooth, and a divisor D_W on W such that $D_W \sim_{\mathbb{Q}} p^*(D)$ and $H^i(X, \mathscr{O}_X(-\lceil D \rceil))$ is a direct summand of $H^i(W, \mathscr{O}_W(-D_W))$. The last condition implies that it is enough to show that $H^i(W, \mathscr{O}_W(-D_W)) = 0$ for $i < n$.

First, note that we are done if D is ample. Indeed, since p is finite, we have D_W ample and $H^i(W, \mathscr{O}_W(-D_W)) = 0$ for $i < n$ by Kodaira's vanishing theorem.

Second, in the general case when D is big and nef, we have $p^*(D)$ big and nef, and therefore D_W has the same property. This shows that in order to prove the theorem, we may assume that D is an (integral) divisor.
Step 2. Let H be a fixed ample divisor on X. We apply Lemma 2.2.9 to construct a projective, birational morphism $f \colon Y \to X$, with Y smooth, and an effective, simple normal crossing divisor E, such that $f^*(D) - \frac{1}{m}E$ is ample for every $m \geq 1$. For $m \gg 0$, the coefficients of $\frac{1}{m}E$ are rational numbers in $[0,1)$, Therefore we may apply the case already proved for the ample $\mathbb{Q}$-divisor $f^*(D + jH) - \frac{1}{m}E$, with $j \geq 0$, to get

$$H^i(Y, \omega_Y \otimes f^*(\mathscr{O}_X(D + jH))) = 0 \quad \text{for all} \quad i \geq 1 \quad \text{and} \quad j \geq 0. \tag{2.7}$$

We can now apply Lemma 2.2.10 with $\mathscr{F} = \omega_Y \otimes f^*(\mathscr{O}_X(D))$ to conclude that

$$H^i(X, f_*(\omega_Y) \otimes \mathscr{O}_X(D)) = 0 \quad \text{for all} \quad i \geq 1.$$

Since $f_*(\omega_Y) \simeq \omega_X$ by Corollary B.2.6, we obtain the vanishing in the theorem. □

2.3 Grauert–Riemenschneider and Fujita vanishing theorems

In this section we give some easy, but important consequences of the Kawamata–Viehweg vanishing theorem. We begin with a result concerning the vanishing of the higher direct images of the canonical line bundle via a birational morphism.

Corollary 2.3.1 (Grauert–Riemenschneider). *If $f\colon Y \to X$ is a birational morphism between projective varieties, with Y smooth, and $D \in \mathrm{CDiv}(X)_{\mathbb{Q}}$ is nef and such that $\lceil D \rceil - D$ is a simple normal crossing divisor, then*

$$R^i f_*(\omega_Y \otimes \mathscr{O}_Y(\lceil D \rceil)) = 0 \quad \textit{for all} \quad i \geq 1.$$

In particular, $R^i f_*(\omega_Y) = 0$ for all $i \geq 1$.

Proof. Let H be an ample Cartier divisor on X. If j is a positive integer, then $E = D + f^*(jH)$ is a nef and big $\mathbb{Q}$-divisor on Y and $\lceil E \rceil - E$ has simple normal crossings. Therefore Theorem 2.2.1 implies

$$H^i(Y, \omega_Y \otimes \mathscr{O}_Y(\lceil D \rceil + jf^*(H))) = 0 \quad \text{for all} \quad i, j \geq 1.$$

Proposition 2.2.10 then implies $R^i f_*(\omega_Y \otimes \mathscr{O}_Y(\lceil D \rceil)) = 0$ for all $i \geq 1$. □

Remark 2.3.2. If $f\colon Y \to X$ is any projective, birational morphism of varieties, with Y smooth, then $R^i f_*(\omega_Y) = 0$ for all $i \geq 1$. Indeed, since the assertion is local on X, we may assume that X is affine. Consider an open immersion $j\colon X \hookrightarrow \overline{X}$, with $\overline{X}$ a projective variety. In this case we can find a Cartezian diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "f"] & \overline{Y} \arrow[d, "g"] \\ X \arrow[r, "j"] & \overline{X} \end{tikzcd}$$

such that $\overline{Y}$ is a smooth projective variety. By Corollary 2.3.1, we have $R^i g_*(\omega_{\overline{Y}}) = 0$, hence $R^i f_*(\omega_Y) = 0$ for all $i \geq 1$. Getting the full relative version of Corollary 2.3.1 is more subtle. However, this is a consequence of the relative version of the Kawamata–Viehweg vanishing theorem that we discuss in Section 2.6.

For an arbitrary variety X, one can define an analogue of the canonical line bundle that on projective varieties satisfies a Kodaira-type vanishing theorem. This is the *Grauert–Riemenschneider sheaf* ω_X^{GR}, defined as follows. If X is an arbitrary variety and $f\colon Y \to X$ is a resolution of singularities, then

$$\omega_X^{\mathrm{GR}} := f_*(\omega_Y).$$

Remark 2.3.3. Note that this is independent of the chosen resolution. Indeed, using Proposition B.3.3, we see that it is enough to check that if $g\colon Z \to Y$ is a projective birational morphism, with Z smooth, then $(f \circ g)_*(\omega_Z) \simeq f_*(\omega_Y)$. This is clear, since

$g_*(\omega_Z) \simeq \omega_Y$ by Proposition B.2.6. In particular, this shows that if X is smooth, then $\omega_X^{\mathrm{GR}} \simeq \omega_X$.

Corollary 2.3.4. *If X is a projective variety and $\mathscr{L} \in \mathrm{Pic}(X)$ is big and nef, then*

$$H^i(X, \omega_X^{\mathrm{GR}} \otimes \mathscr{L}) = 0 \quad \textit{for all} \quad i \geq 1.$$

Proof. Let $f\colon Y \to X$ be a resolution of singularities. It follows from Corollary 2.3.1 that

$$R^i f_*(\omega_Y \otimes f^*(\mathscr{L})) \simeq R^i f_*(\omega_Y) \otimes \mathscr{L} = 0 \quad \text{for all} \quad i \geq 1.$$

Therefore the Leray spectral sequence for f and $\omega_Y \otimes f^*(\mathscr{L})$ gives isomorphisms

$$H^p(Y, \omega_Y \otimes f^*(\mathscr{L})) \simeq H^p(X, \omega_X^{\mathrm{GR}} \otimes \mathscr{L}) \quad \text{for all} \quad p \geq 0. \tag{2.8}$$

On the other hand, $f^*(\mathscr{L})$ is big and nef, hence the left-hand side of (2.8) vanishes by Theorem 2.2.1. This completes the proof of the corollary. □

We now turn to a theorem due to Fujita [Fuj83], which gives a version of asymptotic Serre vanishing in which one is able to twist by arbitrary nef line bundles.

Theorem 2.3.5 (Fujita). *If X is a projective scheme, $\mathscr{L} \in \mathrm{Pic}(X)$ is ample, and $\mathscr{F}$ is a coherent sheaf on X, then there is a positive integer m such that*

$$H^i(X, \mathscr{F} \otimes \mathscr{L}^m \otimes \mathscr{L}') = 0 \quad \textit{for all} \quad i \geq 1 \quad \textit{and} \quad \mathscr{L}' \in \mathrm{Pic}(X) \textit{ nef.} \tag{2.9}$$

Proof. We prove the theorem by induction on $n = \dim(X)$, the case $n = 0$ being trivial. We say that the theorem holds for $\mathscr{F}$ if we can find m such that (2.9) is satisfied (note that in this case all integers $m' \geq m$ have the same property). Suppose now that we have an exact sequence of coherent sheaves on X

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0.$$

After tensoring this with $\mathscr{L}^m \otimes \mathscr{L}'$, with m large enough, we see using the long exact sequence in cohomology that if the theorem holds for both $\mathscr{F}'$ and $\mathscr{F}''$, then it also holds for $\mathscr{F}$. By Lemma 1.1.8, every $\mathscr{F}$ has a finite filtration with each successive quotient having support on an integral closed subscheme of X. Moreover, given a coherent sheaf $\mathscr{F}$ supported on a closed subscheme Y of X, if the theorem holds for $\mathscr{F}$ as a sheaf on Y, then it also does when considering $\mathscr{F}$ as a sheaf on X. Therefore we may assume that X is an integral scheme.

It is clear that for every integer ℓ, the theorem holds for $\mathscr{F}$ if and only if it holds for $\mathscr{F} \otimes \mathscr{L}^\ell$. Let $j \gg 0$ be such that $\mathscr{F} \otimes \mathscr{L}^j$ is globally generated. By considering r general sections in $H^0(X, \mathscr{F} \otimes \mathscr{L}^j)$, where $r = \mathrm{rank}(\mathscr{F})$, we obtain a morphism $\phi\colon \mathscr{O}_X^{\oplus r} \to \mathscr{F} \otimes \mathscr{L}^j$, which is an isomorphism at the generic point of X. In particular, ϕ has to be injective. Since the theorem holds for $\mathrm{coker}(\phi)$, which is supported in dimension $< n$, we see that it is enough show that the theorem holds for $\mathscr{F} = \mathscr{L}^d$, for some integer d.

The key fact is that the theorem holds for the sheaf ω_X^{GH}. Indeed, Corollary 2.3.4 implies that $H^i(X, \omega_X^{GH} \otimes \mathscr{L} \otimes \mathscr{L}') = 0$ for every $i \geq 1$ and every nef line bundle $\mathscr{L}'$, since $\mathscr{L} \otimes \mathscr{L}'$ is ample.

It follows from definition that ω_X^{GH} is a torsion-free rank one sheaf on X and therefore its dual $(\omega_X^{\mathrm{GH}})^\vee$ has the same properties. Furthermore, the canonical map to the double dual $\omega_X^{\mathrm{GH}} \to (\omega_X^{\mathrm{GH}})^{\vee\vee}$ is injective. We claim that there is an integer q and an injective morphism $\omega_X^{\mathrm{GH}} \hookrightarrow \mathscr{L}^q$. Indeed, if q is such that $(\omega_X^{\mathrm{GH}})^\vee \otimes \mathscr{L}^q$ is globally generated, then any nonzero section of this sheaf induces a short exact sequence

$$0 \to \mathscr{O}_X \to (\omega_X^{\mathrm{GH}})^\vee \otimes \mathscr{L}^q \to \mathscr{G} \to 0,$$

where $\mathscr{G}$ is a torsion sheaf. Applying $\mathscr{H}om_{\mathscr{O}_X}(-, \mathscr{O}_X)$ to this exact sequence gives an injective map $((\omega_X^{\mathrm{GH}})^\vee \otimes \mathscr{L}^q)^\vee \hookrightarrow \mathscr{O}_X$, hence an inclusion $\psi \colon \omega_X^{\mathrm{GH}} \hookrightarrow \mathscr{L}^q$. Since the theorem holds for $\mathrm{coker}(\psi)$, which is supported in dimension $< n$, and also for ω_X^{GH}, it follows that it holds for $\mathscr{L}^q$. As we have seen, this implies that the theorem holds for all coherent sheaves on X. □

Remark 2.3.6. The above proof of Theorem 2.3.5 made use of vanishing theorems, and is thus restricted to characteristic zero. However, the result also holds in positive characteristic, in which case the proof makes explicit use of the Frobenius morphism, see [Fuj83].

2.4 Castelnuovo-Mumford regularity

In this section we review the definition and basic results concerning Castelnuovo-Mumford regularity. In the presence of vanishing results, this notion can be applied to obtain global generation of sheaves. On the other hand, it is a topic of independent interest, that has attracted a lot of attention in connection with a diverse set of topics, from the construction of Hilbert schemes to complexity of graded free resolutions. Unless stated otherwise, in this section we work over a field k of arbitrary characteristic.

Definition 2.4.1. Let X be a projective scheme and $\mathscr{L}$ an ample and globally generated line bundle on X. Given an integer m, a coherent sheaf $\mathscr{F}$ on X is *m-regular* with respect to $\mathscr{L}$ if

$$H^i(X, \mathscr{F} \otimes \mathscr{L}^{m-i}) = 0 \quad \text{for all} \quad i \geq 1.$$

If $X = \mathbb{P}^n$ and $\mathscr{L} = \mathscr{O}_{\mathbb{P}^n}(1)$, we simply say that $\mathscr{F}$ is *m-regular*.

Remark 2.4.2. If X and $\mathscr{L}$ are as in the above definition, then $\mathscr{L}$ defines a morphism $f \colon X \to \mathbb{P}(H^0(X, \mathscr{L})) \simeq \mathbb{P}^n$ such that $f^*(\mathscr{O}_{\mathbb{P}^n}(1)) \simeq \mathscr{L}$. The morphism is finite since $\mathscr{L}$ is ample (see Corollary 1.1.11). Using this and the projection formula, we obtain

$$H^i(X, \mathscr{F} \otimes \mathscr{L}^j) \simeq H^i(\mathbb{P}^n, f_*(\mathscr{F}) \otimes \mathscr{O}_{\mathbb{P}^n}(j)) \text{ for every } i \text{ and } j. \tag{2.10}$$

Therefore $\mathscr{F}$ is m-regular with respect to $\mathscr{L}$ if and only if $f_*(\mathscr{F})$ is m-regular as a sheaf on $\mathbb{P}^n$. This can be used to reduce the study of the general notion of regularity to that of sheaves on the projective space.

The following is the basic result concerning Castelnuovo-Mumford regularity.

Theorem 2.4.3 (Mumford). *Let X be a projective scheme and $\mathscr{L}$ a line bundle on X which is ample and globally generated. If $\mathscr{F}$ is a coherent sheaf on X that is m-regular with respect to $\mathscr{L}$, then*

i) $\mathscr{F}$ is m'-regular with respect to $\mathscr{L}$ for every $m' \geq m$, that is,

$$H^i(X, \mathscr{F} \otimes \mathscr{L}^j) = 0 \quad \textit{for all} \quad i \geq 1 \quad \textit{and} \quad j \geq m - i.$$

ii) The natural map induced by multiplication of sections

$$H^0(X, \mathscr{L}) \otimes H^0(X, \mathscr{F} \otimes \mathscr{L}^m) \to H^0(X, \mathscr{F} \otimes \mathscr{L}^{m+1})$$

is surjective.

iii) The sheaf $\mathscr{F} \otimes \mathscr{L}^m$ is globally generated.

Proof. Note first that the assertion in iii) follows from i) and ii). Indeed, ii) implies that if $\mathscr{F} \otimes \mathscr{L}^{m+1}$ is globally generated, then $\mathscr{F} \otimes \mathscr{L}^m$ is globally generated. Furthermore, by i) the same holds if we replace $\mathscr{F}$ by $\mathscr{F} \otimes \mathscr{L}^j$ for every $j \geq 0$. Since $\mathscr{L}$ is ample, we have $\mathscr{F} \otimes \mathscr{L}^{m+j}$ globally generated for $j \gg 0$, and a repeated application of ii) implies that $\mathscr{F} \otimes \mathscr{L}^m$ is globally generated.

Let us consider the finite morphism $f : X \to \mathbb{P}(H^0(X, \mathscr{L})) \simeq \mathbb{P}^n$ defined by $\mathscr{L}$. It follows from (2.10) that it is enough to prove the assertions in i) and ii) for the sheaf $f_*(\mathscr{F})$ on $\mathbb{P}^n$. Therefore we may and will assume that $X = \mathbb{P}^n$ and $\mathscr{L} = \mathscr{O}_{\mathbb{P}^n}(1)$. After replacing $\mathscr{F}$ by $\mathscr{F} \otimes \mathscr{L}^m$, we may assume that $m = 0$.

If $V = H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(1))$, then the natural surjective map $V \otimes \mathscr{O}_{\mathbb{P}^n}(-1) \to \mathscr{O}_{\mathbb{P}^n}$ induced by evaluating the sections of $\mathscr{O}_{\mathbb{P}^n}(1)$ gives an exact Koszul complex

$$0 \to \wedge^{n+1} V \otimes \mathscr{O}_{\mathbb{P}^n}(-n-1) \xrightarrow{d_{n+1}} \ldots \to \wedge^i V \otimes \mathscr{O}_{\mathbb{P}^n}(-i) \xrightarrow{d_i} \ldots \to V \otimes \mathscr{O}_{\mathbb{P}^n}(-1) \xrightarrow{d_1} \mathscr{O}_{\mathbb{P}^n} \to 0.$$

Let $\mathscr{E}_i = \ker(d_i)$, for $1 \leq i \leq n+1$, hence $\mathscr{E}_{n+1} = 0$, and we also put $\mathscr{E}_0 = \mathscr{O}_{\mathbb{P}^n}$. Note that each $\mathscr{E}_i$ is locally free and the above complex breaks into short exact sequences

$$(C_i) \quad 0 \to \mathscr{E}_i \to \wedge^i V \otimes \mathscr{O}_{\mathbb{P}^n}(-i) \to \mathscr{E}_{i-1} \to 0,$$

with $1 \leq i \leq n+1$.

Let us prove i). Recall that we assume $m = 0$, and it is enough to show that $\mathscr{F}$ is 1-regular, that is, $H^j(\mathbb{P}^n, \mathscr{F}(1-j)) = 0$ for every j, with $1 \leq j \leq n$. For $0 \leq i \leq n-1$, the long exact sequence in cohomology for $(C_{i+1}) \otimes \mathscr{F}(1-j)$ gives

$$\wedge^{i+1} V \otimes H^{i+j}(\mathscr{F}(-i-j)) \to H^{i+j}(\mathscr{E}_i \otimes \mathscr{F}(1-j)) \to H^{i+j+1}(\mathscr{E}_{i+1} \otimes \mathscr{F}(1-j)).$$

Since the first term vanishes by hypothesis, we obtain by letting i vary

$$h^j(\mathscr{F}(1-j)) \leq \ldots \leq h^{i+j}(\mathscr{E}_i \otimes \mathscr{F}(1-j)) \leq h^{i+j+1}(\mathscr{E}_{i+1} \otimes \mathscr{F}(1-j))$$
$$\leq \ldots \leq h^{n+j}(\mathscr{E}_n \otimes \mathscr{F}(1-j)) = 0.$$

Therefore $h^j(\mathscr{F}(1-j)) = 0$ for every $j \geq 1$, which completes the proof of i).

The long exact sequence in cohomology for $(C_1) \otimes \mathscr{F}(1)$ gives

$$V \otimes H^0(\mathbb{P}^n, \mathscr{F}) \to H^0(\mathbb{P}^n, \mathscr{F}(1)) \to H^1(\mathbb{P}^n, \mathscr{E}_1 \otimes \mathscr{F}(1)),$$

hence in order to prove ii), it is enough to show that $H^1(\mathbb{P}^n, \mathscr{E}_1 \otimes \mathscr{F}(1)) = 0$. For $1 \leq i \leq n$, the long exact sequence in cohomology for $(C_{i+1}) \otimes \mathscr{F}(1)$ gives

$$\wedge^{i+1} V \otimes H^i(\mathbb{P}^n, \mathscr{F}(-i)) \to H^i(\mathbb{P}^n, \mathscr{E}_i \otimes \mathscr{F}(1)) \to H^{i+1}(\mathbb{P}^n, \mathscr{E}_{i+1} \otimes \mathscr{F}(1)).$$

Since the first term vanishes by assumption, we obtain

$$h^1(\mathscr{E}_1 \otimes \mathscr{F}(1)) \leq \ldots \leq h^i(\mathscr{E}_i \otimes \mathscr{F}(1)) \leq h^{i+1}(\mathscr{E}_{i+1} \otimes \mathscr{F}(1))$$
$$\leq \ldots \leq h^{n+1}(\mathscr{E}_{n+1} \otimes \mathscr{F}(1)) = 0.$$

This completes the proof of ii), hence that of the theorem. □

If $\mathscr{L}$ is ample and globally generated on X, then for every coherent sheaf $\mathscr{F}$ there is m such that $\mathscr{F}$ is m-regular with respect to $\mathscr{L}$ (this simply follows from Serre's asymptotic vanishing). The *(Castelnuovo-Mumford) regularity* of $\mathscr{F}$ is the smallest m with this property.

One can combine Fujita's vanishing theorem with Castelnuovo-Mumford regularity to obtain the following uniform global generation result for twists by nef line bundles.

Corollary 2.4.4. *If X is a projective scheme over an algebraically closed field k, then there is a line bundle $\mathscr{A}$ on X such that for every nef $\mathscr{L} \in \mathrm{Pic}(X)$, we have $\mathscr{L} \otimes \mathscr{A}$ globally generated.*

Proof. Let $\mathscr{M}$ be a very ample line bundle on X and let $n = \dim(X)$. It follows from Theorem 2.3.5 that there is q such that $H^i(X, \mathscr{M}^q \otimes \mathscr{L}') = 0$ for all $i \geq 1$ and all nef line bundles $\mathscr{L}'$ on X. In particular, if $\mathscr{L}$ is a nef line bundle, then $H^i(X, \mathscr{M}^{q+n-i} \otimes \mathscr{L})$ for all positive integers i. We put $\mathscr{A} = \mathscr{M}^{q+n}$. We see that if $\mathscr{L}$ is nef, then $\mathscr{L} \otimes \mathscr{A}$ is 0-regular with respect to $\mathscr{M}$ and therefore Theorem 2.4.3 implies that $\mathscr{L} \otimes \mathscr{A}$ is globally generated. □

Remark 2.4.5. Everything in this section works if instead of working over a ground field, we work over a Noetherian ring, and by further globalizing, over a Noetherian scheme. We thus obtain analogous notions and results in the relative case. More precisely, suppose that $f \colon X \to S$ is a projective morphism of Noetherian schemes and $\mathscr{L}$ is an f-ample and f-base-point free line bundle on X. We say that a coherent sheaf $\mathscr{F}$ on X is m-regular (over S) with respect to $\mathscr{L}$ if $R^i f_*(\mathscr{F} \otimes \mathscr{L}^{m-i}) = 0$ for all $i \geq 1$. In this case, $\mathscr{F}$ is also m'-regular, for all $m' \geq m$, and furthermore, $\mathscr{F} \otimes \mathscr{L}^m$

is f-base-point free. In order to show this, we may assume that S is affine, and in this case the proof is the same as that of Theorem 2.4.3.

By combining Theorem 2.4.3 with Kawamata–Viehweg vanishing, we obtain the following more explicit variant of Corollary 2.4.4 when working on a variety.

Corollary 2.4.6. *If X is an n-dimensional projective variety over an algebraically closed field k of characteristic 0, then for every line bundles $\mathscr{L}$ and $\mathscr{L}'$ on X, with $\mathscr{L}$ ample and globally generated, and $\mathscr{L}'$ big and nef, the sheaf $\omega_X^{\mathrm{GR}} \otimes \mathscr{L}^n \otimes \mathscr{L}'$ is globally generated.*

Proof. Let $\mathscr{F} := \omega_X^{\mathrm{GR}} \otimes \mathscr{L}^n \otimes \mathscr{L}'$. For every i with $1 \leq i \leq n$, the line bundle $\mathscr{L}^{n-i} \otimes \mathscr{L}'$ is big and nef, hence Corollary 2.3.4 implies $H^i(X, \mathscr{F} \otimes \mathscr{L}^{-i}) = 0$. Therefore $\mathscr{F}$ is 0-regular with respect to $\mathscr{L}$, hence globally generated by Theorem 2.4.3. $\square$

Theorem 2.4.3 shows that having explicit regularity bounds gives global generation results for the twists of $\mathscr{F}$ by powers of $\mathscr{L}$. Effective bounds for Castelnuovo-Mumford regularity are important in many contexts. For example, Mumford showed that for ideal sheaves in $\mathbb{P}^n$ there are regularity bounds only depending on the Hilbert polynomial of the ideal, and he used these bounds to simplify Grothendieck's proof of the existence of the Hilbert scheme, see [Mum66].

In commutative algebra bounds for regularity are important because of the connection with the Betti numbers in a graded free resolution. Suppose that M is a finitely generated graded module over the polynomial ring $S = k[x_0, \ldots, x_n]$ and $\widetilde{M}$ is the corresponding coherent sheaf on $\mathbb{P}^n$. Assume for simplicity that $\operatorname{depth}(M) \geq 2$ (equivalently, the canonical morphism $M \to \bigoplus_{i \in \mathbb{Z}} H^0(\mathbb{P}^n, \widetilde{M}(i))$ is an isomorphism). In this case, if the minimal free resolution of M over S is given by

$$0 \to F_{n+1} \to \ldots \to F_1 \to F_0 \to M \to 0,$$

and $F_i = \bigoplus_j S(-i-j)^{\beta_{i,j}}$ for every i, then

$$\min\{m \mid \widetilde{M} \text{ is } m-\text{regular}\} = \max\{j \mid \beta_{i,j} \neq 0 \text{ for some } i\}$$

(see [Eis95, Chap. 20.5] for a proof and a more general statement).

Partly motivated by the above connections, there has been a lot of work devoted to finding upper-bounds for the regularity of ideal sheaves in projective space. An example of Mayr and Meyer [MM82] shows that in general, the regularity can grow doubly exponentially in the number of variables. On the other hand, much better bounds are expected (and known, in small dimensions) for ideals of smooth varieties; see [GLP83], [Laz87], and [Kwa98] for the case of curves, surfaces, and respectively, 3-folds and 4-folds.

2.5 Seshadri constants

The Seshadri constant of a line bundle is an invariant introduced by Demailly [Dem92]. It measures the local positivity of the line bundle at a given point. The definition and the general properties of the invariant work on arbitrary projective schemes, though the more interesting properties require restricting to smooth points. In the beginning we assume that the ground field is algebraically closed, of arbitrary characteristic.

Definition 2.5.1. Let X be a projective scheme and $x \in X$ a (closed) point. Consider the blow-up $f\colon X' = \mathrm{Bl}_x(X) \to X$ of X at x, with exceptional divisor E, so that $\mathscr{O}_{X'}(-E) = \mathfrak{m}_x \cdot \mathscr{O}_{X'}$, where $\mathfrak{m}_x$ is the ideal defining x. If $D \in \mathrm{CDiv}(X)_{\mathbb{R}}$ is nef, then the *Seshadri constant* of D at x is

$$\varepsilon_x(D) := \sup\{t \geq 0 \mid f^*(D) - tE \text{ is nef}\}.$$

The *Seshadri constant* of D on X is

$$\varepsilon(X,D) := \inf\{\varepsilon_x(D) \mid x \in X\}.$$

Note that the set in the definition of $\varepsilon_x(D)$ is non-empty, since it contains 0. We will see in Proposition 2.5.2 below that if $x \in X$ is not an isolated point, then $\varepsilon_x(D)$ is finite. Note that when $x \in X$ is an isolated point, then X' is empty, and we make the convention $\varepsilon_x(D) = \infty$.

Since the nef cone is closed, if the supremum in Definition 2.5.1 is finite, it is in fact a maximum. Furthermore, if $D_1 \equiv D_2$, then $f^*(D_1) \equiv f^*(D_2)$, and since nefness only depends on the numerical equivalence class, we conclude that $\varepsilon_x(D_1) = \varepsilon_x(D_2)$ for every $x \in X$. In particular, we may consider $\varepsilon_x(\mathscr{L})$ for $\mathscr{L} \in \mathrm{Pic}(X)$ or $\varepsilon_x(\alpha)$ for $\alpha \in \mathrm{N}^1(X)_{\mathbb{R}}$.

For a scheme X and a point $x \in X$, we denote by $\mathrm{mult}_x(X)$ the Samuel multiplicity of the local ring $\mathscr{O}_{X,x}$. With the notation in Definition 2.5.1, this can be described as $(\mathscr{O}_E(-E)^{n-1})$, where $n = \dim(\mathscr{O}_{X,x})$ (note that this intersection number is defined for an arbitrary scheme X, since E is always a projective scheme).

Proposition 2.5.2. *For every projective scheme X and every $D \in \mathrm{CDiv}(X)_{\mathbb{R}}$, we have*

$$\varepsilon_x(D) = \inf_{V \ni x} \left(\frac{(D^{\dim(V)} \cdot V)}{\mathrm{mult}_x(V)} \right)^{1/\dim(V)},$$

where the infimum is over all positive-dimensional subvarieties V of X containing x. Furthermore, it is enough to let V vary over the curves containing x.

Proof. Let $f\colon X' \to X$ be as in Definition 2.5.1. By definition, we have $D_t := f^*(D) - tE$ nef if and only if $(D_t \cdot C') \geq 0$ for every curve C' in X. Note first that since $\mathscr{O}_E(-E)$ is an ample line bundle on E and $f^*(D)$ maps to 0 in $\mathrm{Pic}(E)_{\mathbb{R}}$, if $C' \subseteq E$, then $(D_t \cdot C') > 0$ for every $t > 0$. On the other hand, if $C' \not\subseteq E$ and C is the image of C' in X, then either $x \notin C$, in which case $(D_t \cdot C') = (D \cdot C) \geq 0$, or $x \in C$, in which case $f|_{C'}: C' \to C$ is the blow-up of C at x, hence $(D_t \cdot C) = (D \cdot C) - t \cdot \mathrm{mult}_x(C)$. This implies the formula in the proposition, with V varying over the curves on X containing x.

If V is a subvariety of X containing x, with $\dim(V) = r > 0$, and D_t is nef on X', then Theorem 1.3.18 implies $(D_t^r \cdot V') \geq 0$, where V' is the proper transform of V. Using the fact that $(f^*(D)^i \cdot E^{r-i} \cdot V') = 0$ for $1 \leq i \leq r-1$, we deduce

$$(D_t^r \cdot V') = (D^r \cdot V) - t^r \cdot \mathrm{mult}_x(V).$$

We thus obtain the formula in the proposition in terms of arbitrary positive-dimensional subvarieties containing x. □

Remark 2.5.3. The argument in the proof of Proposition 2.5.2 shows, using the notation in that proof, that $f^*(D) - tE$ is nef if and only if $0 \leq t \leq \varepsilon_x(D)$.

Proposition 2.5.4. *Let X be a projective scheme, $x \in X$ a point, and $D, D' \in \mathrm{CDiv}(X)_{\mathbb{R}}$.*

i) $\varepsilon_x(\lambda D) = \lambda \cdot \varepsilon_x(D)$ *for every positive real number* λ.
ii) $\varepsilon_x(D + D') \geq \varepsilon_x(D) + \varepsilon_x(D')$.
iii) If $D' - D$ *is nef, then* $\varepsilon_x(D') \geq \varepsilon_x(D)$.

Proof. All assertions follow easily from the definition of Seshadri constants. The first one is a consequence of the fact that if $\lambda > 0$, then a divisor M is nef if and only if λM is nef. The second and the third assertions follow from the fact that a sum of two nef divisors is nef. □

Proposition 2.5.5. *If $f: Y \to X$ is a birational morphism of projective varieties and $x \in X$ lies in the domain of f^{-1}, then for every $D \in \mathrm{CDiv}(X)_{\mathbb{R}}$ we have*

$$\varepsilon_x(D) = \varepsilon_{f^{-1}(x)}(f^*(D)).$$

Proof. Let $\pi_X: \mathrm{Bl}_x(X) \to X$ and $\pi_Y: \mathrm{Bl}_{f^{-1}(x)}(Y) \to Y$ be the blow-ups of X and Y at x, and respectively $f^{-1}(x)$, with exceptional divisors E_X and E_Y. We have an induced birational morphism $g: \mathrm{Bl}_{f^{-1}(x)}(Y) \to \mathrm{Bl}_x(X)$, such that $g^*(E_X) = E_Y$. Therefore $\pi_X^*(D) - tE_X$ is nef if and only if

$$g^*(\pi_X^*(D) - tE_X) = \pi_Y^*(f^*(D)) - tE_Y$$

is nef, which implies the assertion in the proposition. □

Example 2.5.6. If $X = \mathbb{P}^n$, then $\varepsilon_q(\mathscr{O}_{\mathbb{P}^n}(1)) = 1$ for every $q \in X$. Indeed, let D be a hyperplane in $\mathbb{P}^n$ and $f: X' \to X$ the blow-up of $\mathbb{P}^n$ at q, with exceptional divisor E. It follows from Example 1.3.33 that $f^*(D) - tE$ is nef if and only if $0 \leq t \leq 1$, which gives our assertion.

Example 2.5.7. If $\mathscr{L}$ is an ample and globally generated line bundle on the projective scheme X, then $\varepsilon_x(\mathscr{L}) \geq 1$ for every $x \in X$. Indeed, by Proposition 2.5.2, it is enough to show that for every curve C on X containing x, we have $(\mathscr{L}\cdot C)\geq \mathrm{mult}_x(C)$. Note first that we can find $D\in|\mathscr{L}|$ such that $x\in D$, but $C\not\subseteq D$. Indeed, since $\mathscr{L}$ is globally generated and ample, it defines a finite morphism $\phi\colon X\to\mathbb{P}^N$, and it is enough to take $D=\phi^*(H)$, where H is a general hyperplane containing $\phi(x)$. In this case, we have

$$(\mathscr{L}\cdot C)=\deg(D|_C)\geq \ell(\mathscr{O}_{D,x})\geq \mathrm{mult}_x(C),$$

where the last inequality is a well-known (and easy) estimate for the Samuel multiplicity of a one-dimensional local domain.

Example 2.5.8. On the other hand, the following example due to Miranda, shows that the Seshadri constant of an ample line bundle at a point can be arbitrarily small, even on smooth projective surfaces. Let C be a fixed irreducible curve in $\mathbb{P}^2$ of degree $d\geq 3$, having a point $y\in C$ of multiplicity m. Suppose that $C'\subset\mathbb{P}^2$ is a general curve of degree d. In particular, C and C' intersect in d^2 reduced points. Since the codimension of the space of reducible curves in $|\mathscr{O}_{\mathbb{P}^2}(d)|$ is

$$\begin{aligned}&\binom{d+2}{2}-\max_{1\leq i\leq d-1}\left(\binom{i+2}{2}+\binom{d-i+2}{2}\right)+1\\ &\geq \frac{(d+1)(d+2)}{2}-\left(\frac{d}{2}+1\right)\left(\frac{d}{2}+2\right)+1=\frac{d^2}{4}\geq 2,\end{aligned}$$

and C' is general, we may assume that every curve in the linear system $|W|$ spanned by C and C' is irreducible.

Let $\pi\colon X\to\mathbb{P}^2$ be the blow-up along $C\cap C'$, hence there are d^2 exceptional curves $E_1,\ldots,E_{d^2}$ on X. Since we have blown-up the base locus of $|W|$, it follows that W induces a morphism $g\colon X\to\mathbb{P}^1$. If T is a curve in $|W|$, then $\pi^*(T)=\widetilde{T}+\sum_{i=1}^{d^2}E_i$, and $\widetilde{T}$ is a fiber of g; furthermore, every fiber is of this form. We claim that if $\ell\geq 2$, then $\mathscr{M}_\ell=\mathscr{O}_X(E_1)\otimes g^*(\mathscr{O}_{\mathbb{P}^1}(\ell))$ is ample on X. Indeed, note first that since $\mathscr{O}_X(\widetilde{C})\simeq g^*(\mathscr{O}_{\mathbb{P}^1}(1))$ and $(\widetilde{C}\cdot E_i)=1$ for every i, we have $(\mathscr{M}_\ell^2)=2\ell-1$ and $(\mathscr{M}_\ell\cdot E_1)=\ell-1$. If Z is a curve on X different from E_1, then

$$(\mathscr{M}_\ell\cdot Z)=(\mathscr{O}_X(E_1)\cdot Z)+(g^*(\mathscr{O}_{\mathbb{P}^1}(\ell))\cdot Z)\geq 0, \tag{2.11}$$

and equality implies that both terms in (2.11) are zero. In particular, $g(Z)$ is a point. In this case, our assumption on $|W|$ implies that $Z\sim\widetilde{C}$, and therefore $(\mathscr{O}_X(E_1)\cdot Z)=1$, a contradiction. We thus conclude by the Nakai-Moishezon criterion that $\mathscr{M}_\ell$ is ample for every $\ell\geq 2$.

On the other hand, $(\mathscr{M}_\ell\cdot\widetilde{C})=(E_1\cdot\widetilde{C})=1$, and since $\widetilde{C}$ has a point $x=\pi^{-1}(y)$ of multiplicity m, it follows from Proposition 2.5.2 that $\varepsilon_x(\mathscr{M}_\ell)\leq\frac{1}{m}$. We also note that $\lim_{\ell\to\infty}(\mathscr{M}_\ell^2)=\infty$.

The name of the Seshadri constant comes from the following ampleness criterion, due to Seshadri. We note that while we work, as usual, on a projective scheme, the criterion is valid on arbitrary complete schemes. For a curve C, we put

$$\mu_{\max}(C) := \max_{x \in C} \operatorname{mult}_x(C).$$

Proposition 2.5.9. *Let X be a projective scheme and $D \in \operatorname{CDiv}(X)_{\mathbb{Q}}$. The following are equivalent:*

i) D is ample.
ii) D is nef and $\varepsilon(X,D) > 0$.
iii) There is $\delta > 0$ such that $(D \cdot C) \geq \delta \cdot \mu_{\max}(C)$ for every curve C in X.
iv) D is nef and $\varepsilon_x(D) > 0$ for every $x \in X$.

Proof. If D is ample and r is a positive integer such that rD is an integral divisor and $\mathscr{O}_X(rD)$ is globally generated, then it follows from Example 2.5.7 that

$$\varepsilon_x(D) = \frac{1}{r} \cdot \varepsilon_x(rD) \geq \frac{1}{r}$$

for every $x \in X$, hence $\varepsilon(X,D) \geq \frac{1}{r}$. This gives the implication i)$\Rightarrow$ii). Since the equivalence of ii) and iii) follows from Proposition 2.5.2, and the implication ii)$\Rightarrow$iv) is trivial, in order to complete the proof it is enough to show the implication iv)$\Rightarrow$i).

Suppose that $\varepsilon_x(D) > 0$ for every $x \in X$. If V is a subvariety of X of dimension $r > 0$, let us choose any $x \in V$. It follows from Proposition 2.5.2 that

$$(D^r \cdot V) \geq \operatorname{mult}_x(V) \cdot \varepsilon_x(D)^r > 0.$$

Since this holds for every V, we conclude that D is ample by Theorem 1.3.1. $\square$

Remark 2.5.10. One can make the criterion in Proposition 2.5.9 more precise, as follows: if X is a smooth projective variety and D is a nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X, then

$$\mathrm{B}_+(D) = \{x \in X \mid \varepsilon_x(D) > 0\}.$$

Indeed, note first that if $x \not\in \mathrm{B}_+(D)$, then we can write $D = A + E$ for $A, E \in \operatorname{CDiv}(X)_{\mathbb{Q}}$, with A ample, E effective, and such that $x \not\in \operatorname{Supp}(E)$. If C is a curve containing x, then $(D \cdot C) \geq (A \cdot C)$, hence $\varepsilon_x(D) \geq \varepsilon_x(A) > 0$. Conversely, if $x \in \mathrm{B}_+(D)$, it follows from Theorem 1.5.18 that there is a subvariety V of X of dimension $r > 0$ such that $x \in V$ and $(D^r \cdot V) = 0$. It then follows from Proposition 2.5.2 that $\varepsilon_x(D) = 0$.

Proposition 2.5.11. *Let X be a projective scheme and $f \colon X' \to X$ the blow-up of X at a point x, with exceptional divisor E. If $D \in \operatorname{CDiv}(X)_{\mathbb{R}}$ is ample, then $\varepsilon_x(D) > 0$ and $f^*(D) - tE$ is ample if and only if $0 < t < \varepsilon_x(D)$.*

Proof. Since D is ample, we can find $D' \in \operatorname{CDiv}(X)_{\mathbb{Q}}$ ample such that $D - D'$ is ample. Using Proposition 2.5.9, we obtain $\varepsilon_x(D) \geq \varepsilon_x(D') > 0$. If $f^*(D) - tE$ is ample, then the restriction to E is ample, which implies $t > 0$. We also have $t < t_0$: otherwise, the ampleness of $\operatorname{Amp}(X')$ would imply the existence of $t' > \varepsilon_x(D)$ such that $f^*(D) - t'E$ is ample, hence nef.

Conversely, suppose that $0<t<\varepsilon_x(D)$. In this case the restriction of (the class of) $f^*(D)-tE$ to E is ample, and the computation in the proof of Proposition 2.5.2 shows that for every positive-dimensional subvariety V' of X' not contained in E, we have $((\pi^*(D)-tE)^{\dim(V')}\cdot V')>0$. We conclude that $\pi^*(D)-tE$ is ample by Theorem 1.3.1. $\square$

Proposition 2.5.12. *Let X be a projective scheme and $X_{\rm sm}$ the smooth locus of X. If $D\in\mathrm{CDiv}(X)_{\mathbb{R}}$ is ample, then for every $\alpha\geq 0$, the set*

$$U_\alpha := \{x\in X_{\rm sm}\mid \varepsilon_x(D)>\alpha\}$$

is open in $X_{\rm sm}$, while the set

$$V_\alpha := \{x\in X_{\rm sm}\mid \varepsilon_x(D)\geq\alpha\}$$

is the complement in $X_{\rm sm}$ of a countable union of closed subsets.

Proof. We put $U=X_{\rm sm}$. Let $p\colon U\times X\to U$ and $q\colon U\times X\to X$ be the canonical projections, and let $\Delta\hookrightarrow U\times X$ be the graph of the inclusion $U\hookrightarrow X$. We consider the blow-up $f\colon Y\to U\times X$ along Δ, with exceptional divisor E, and for every $x\in U$, we denote by $f_x\colon Y_x\to X$ the fiber of f over x. If I_Δ is the ideal of Δ in $U\times X$, then for every $m\geq 1$, $I_\Delta^m/I_\Delta^{m+1}$ is locally free over $\mathscr{O}_\Delta$, which is flat over U (being isomorphic to $\mathscr{O}_U$). We deduce by induction on m that I_Δ^m is flat over U for every $m\geq 1$. This in turn implies that for every $x\in U$, the morphism f_x is the blow-up of X at x, the exceptional divisor being given by the fiber E_x of E over x. It follows from Proposition 2.5.11 that

$$U_\alpha=\{x\in U\mid (f^*(q^*(D))-\alpha E)_x \text{ is ample}\},$$

and this is open in U by Remark 1.6.25. Similarly, we have

$$V_\alpha=\{x\in U\mid (f^*(q^*(D))-\alpha E)_x \text{ is nef}\},$$

hence this is the complement of a countable union of Zariski closed subsets by Remark 1.6.26. $\square$

We now turn to some more subtle properties of Seshadri constants, which require only considering smooth points of X. Our first goal is to give the description of the Seshadri constants in terms of separation of jets. Recall that if X is a projective scheme and $\mathscr{L}$ is a line bundle on X, then $\mathscr{L}$ *separates i-jets at a point* $x\in X$ if the canonical restriction map

$$H^0(X,\mathscr{L})\to H^0(X,\mathscr{L}\otimes\mathscr{O}_X/\mathfrak{m}_x^{i+1})$$

is surjective, where $\mathfrak{m}_x$ is the ideal defining x. It follows from the long exact sequence in cohomology corresponding to

$$0\to\mathfrak{m}_x^{i+1}\otimes\mathscr{L}\to\mathscr{L}\to\mathscr{L}\otimes\mathscr{O}_X/\mathfrak{m}_x^{i+1}\to 0$$

that $\mathscr{L}$ separates i jets at x if and only if $H^1(X, \mathfrak{m}_x^{i+1} \otimes \mathscr{L}) = 0$ (and the converse holds if $H^1(X, \mathscr{L}) = 0$). We denote by $s(\mathscr{L}; x)$ the largest $i \geq 0$ such that $\mathscr{L}$ separates i-jets at x (if there is no such i, we put by convention $s(\mathscr{L}; x) = 0$). The following result, due to Demailly, relates Seshadri constants to separation of jets.

Theorem 2.5.13. *If X is a projective variety and $x \in X$ is a smooth point, then for every ample Cartier divisor D on X, we have*

$$\varepsilon_x(D) = \sup_{m \geq 1} \frac{s(\mathscr{O}_X(mD); x)}{m} = \lim_{m \to \infty} \frac{s(\mathscr{O}_X(mD); x)}{m}.$$

We first prove a lemma describing the higher direct images of the ideals that define the multiples of the exceptional divisor on a smooth blow-up.

Lemma 2.5.14. *Let Z be a smooth closed subvariety of a variety X, of codimension r, defined by the ideal $\mathscr{I}_Z$, and such that Z is contained in the smooth locus of X. If $f \colon Y \to X$ is the blow-up of X along Z, with exceptional divisor E, then for every $m \in \mathbb{Z}$, with $m \geq -r+1$, we have*

$$R^i f_* \mathscr{O}_Y(-mE) = \begin{cases} 0, & \text{if} \quad i \geq 1; \\ \mathscr{I}_Z^m, & \text{if} \quad i = 0, \end{cases}$$

with the convention that $\mathscr{I}_Z^m = \mathscr{O}_X$ if $m \leq 0$.

Proof. Recall that by definition $Y = \mathscr{P}roj\left(\bigoplus_{m \geq 0} \mathscr{I}_Z^m\right)$ and $\mathscr{O}_Y(1) \simeq \mathscr{O}_Y(-E)$ is f-ample. Furthermore, since both X and Z are smooth in a neighborhood of Z, $E \simeq \mathscr{P}roj(\mathscr{S}ym(\mathscr{I}_Z/\mathscr{I}_Z^2))$ is a projective bundle over Z, of relative dimension $r-1$. In particular, we have $R^i f_*(\mathscr{O}_E(m)) = 0$ for $m \geq -r+1$ and $i \geq 1$, and $f_*(\mathscr{O}_E(m)) \simeq \mathscr{I}_Z^m/\mathscr{I}_Z^{m+1}$ for $m \geq 0$. On the other hand, by a general property of the $\mathscr{P}roj$ construction, we know that the formula in the lemma holds for all i and all $m \gg 0$. Therefore it is enough to show that if $m \geq -r+1$ and the formula holds for $(m+1)$, then it also holds for m. Consider the exact sequence

$$0 \to \mathscr{O}_Y(-(m+1)E) \to \mathscr{O}_Y(-mE) \to \mathscr{O}_E(m) \to 0. \tag{2.12}$$

If $i \geq 1$, then $R^i f_*(\mathscr{O}_Y(-(m+1)E)) = 0$ and $R^i f_*(\mathscr{O}_E(m)) = 0$, hence the long exact sequence in cohomology of (2.12) gives $R^i f_*(\mathscr{O}_Y(-mE)) = 0$. If $m \leq 0$, then $f_*(\mathscr{O}_Y(-mE)) = \mathscr{O}_X$ by (B.2.5). Let us show now that if $m > 0$, then $f_*(\mathscr{O}_Y(-mE)) = \mathscr{I}_Z^m$. Since $R^1 f_* \mathscr{O}_Y(-(m+1)E) = 0$, we have a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{I}_Z^{m+1} \arrow[r] \arrow[d, "\alpha"] & \mathscr{I}_Z^m \arrow[r] \arrow[d, "\beta"] & \mathscr{I}_Z^m/\mathscr{I}_Z^{m+1} \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & f_*\mathscr{O}_Y(-(m+1)E) \arrow[r] & f_*\mathscr{O}_Y(-mE) \arrow[r] & f_*\mathscr{O}_E(m) \arrow[r] & 0
\end{tikzcd}$$

in which α and γ are isomorphisms, hence β is an isomorphism as well. This completes the proof of the lemma. $\square$

Proof of Theorem 2.5.13. We may assume that $n = \dim(X) \geq 1$, since otherwise the assertion is trivial. Let $f\colon X' \to X$ be the blow-up of X at x, with exceptional divisor E.

We first show that $\varepsilon_x(D) \geq s(\mathscr{O}_X(D);x)$. Suppose that $s := s(\mathscr{O}_X(D);x) > 0$, and let C be a curve on X with $x \in C$. Let $\mathfrak{a}$ and $\mathfrak{b}$ denote the ideals defining x in X and C, respectively. By definition, the restriction map

$$H^0(X, \mathscr{O}_X(D)) \to H^0(X, \mathscr{O}_X(D) \otimes \mathscr{O}_X/\mathfrak{a}^{i+1}) \tag{2.13}$$

is surjective. By choosing a nonzero element in $\mathfrak{b}^i/\mathfrak{b}^{i+1}$ and lifting it to $\mathfrak{a}^i/\mathfrak{a}^{i+1}$, we deduce from the surjectivity of (2.13) that there is an effective Cartier divisor $D' \sim D$ with $\mathrm{mult}_x(D') = i$ and such that C is not contained in D'. We may write $f^*(D') = \widetilde{D}' + iE$, for en effective Cartier divisor $\widetilde{D}'$ whose support does not contain E. If $\widetilde{C}$ is the proper transform of C, then $\widetilde{C}$ is not contained in $\widetilde{D}'$, hence

$$(D' \cdot C) = (f^*(D') \cdot \widetilde{C}) = (\widetilde{D}' \cdot \widetilde{C}) + i(E \cdot \widetilde{C}) \geq i \cdot \mathrm{mult}_x(C).$$

Proposition 2.5.2 implies $\varepsilon_x(D) \geq i$, and applying this to mD, we obtain $\varepsilon_x(D) = \frac{1}{m}\varepsilon_x(mD) \geq \frac{s(\mathscr{O}_X(mD);x)}{m}$, hence

$$\varepsilon_x(D) \geq \sup_{m\geq 1} \frac{s(\mathscr{O}_X(mD);x)}{m}.$$

In order to complete the proof of the theorem, it is enough to show that for every $\alpha < \varepsilon_x(D)$, we have $s(\mathscr{O}_X(mD);x) > \alpha m$ for all $m \gg 0$. Let us fix $\beta \in \mathbb{Q}$, with $\alpha < \beta < \varepsilon_x(D)$. Note that by Proposition 2.5.11, the $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $f^*(D) - \beta E$ is ample. It follows from Theorem 2.3.5 (see also Remark 2.3.6) that we can find a positive integer d such that $d(f^*(D) - \beta E)$ is an integral divisor, and for every nef Cartier divisor A on X', we have $H^1(X', \mathscr{O}_{X'}(d(f^*(D) - \beta E) + A)) = 0$.

Given a positive integer $m \geq d$, we put $i = \lfloor m/d \rfloor$. Note that

$$mf^*(D) - di\beta E = (m - di)f^*(D) + di(f^*(D) - \beta E),$$

and since both $f^*(D)$ and $d(f^*(D) - \beta E)$ are nef, we conclude that

$$H^1(X, I_x^{di\beta} \otimes \mathscr{O}_X(mD)) \simeq H^1(X', \mathscr{O}_{X'}(mf^*(D) - di\beta E)) = 0,$$

where I_x denotes the ideal defining x (the isomorphism follows from Lemma **??**). This implies that $s(\mathscr{O}_X(mD);x) \geq di\beta - 1$. Moreover, for $m \gg 0$ we have

$$di\beta - 1 = d\lfloor m/d \rfloor \beta - 1 \geq m\beta - d\beta - 1 > m\alpha,$$

and this completes the proof of the theorem. $\square$

An important feature of Seshadri constants is that they control the positivity properties of the corresponding adjoint line bundles. In particular, the next theorem shows that lower bounds for Seshadri constants at *all* points imply the global generation or very ampleness of the adjoint bundles. In this result, we assume that the ground field has characteristic zero.

Theorem 2.5.15 (Demailly). *Let X be a smooth n-dimensional projective variety and $\mathscr{L}$ a big and nef line bundle on X.*

i) If $\varepsilon_x(L) > n$, then x is not in the base-locus of $\omega_X \otimes \mathscr{L}$. More generally, if $\varepsilon_x(L) > n+i$, then $\omega_X \otimes \mathscr{L}$ separates i-jets at x.
ii) If $\varepsilon_x(L) > 2n$, then $\omega_X \otimes \mathscr{L}$ defines a rational map that in a neighborhood of x is a locally closed immersion.
iii) If $\varepsilon_x(L) > 2n$ for every $x \in X$, then $\omega_X \otimes \mathscr{L}$ is very ample.

Proof. Let $\mathfrak{m}_x$ denote the ideal defining x and let $f\colon Y \to X$ be the blow-up at x, with exceptional divisor E. We fix a Cartier divisor D with $\mathscr{O}_X(D) \simeq \mathscr{L}$. In order to prove that the restriction map

$$H^0(X, \omega_X \otimes \mathscr{L}) \to H^0(X, \omega_X \otimes \mathscr{L} \otimes \mathscr{O}_X/\mathfrak{m}_x^{i+1})$$

is surjective, it is enough to show that $H^1(X, \mathfrak{m}_x^{i+1} \otimes \omega_X \otimes \mathscr{L}) = 0$. Furthermore, it follows from Lemma 2.5.14 that it is enough to show that

$$H^1(Y, f^*(\omega_X \otimes L) \otimes \mathscr{O}_Y(-(i+1)E)) = 0.$$

On the other hand, Example B.2.4 gives $\omega_Y \simeq f^*(\omega_X) \otimes \mathscr{O}_Y((n-1)E)$. Since we can write

$$f^*(\omega_X \otimes \mathscr{L}) \otimes \mathscr{O}_Y(-(i+1)E)) \simeq \omega_Y \otimes f^*(\mathscr{L}) \otimes \mathscr{O}_Y(-(i+n)E),$$

it follows from Theorem 2.2.1 that the desired vanishing follows if the line bundle $f^*(\mathscr{L}) \otimes \mathscr{O}_Y(-(i+n)E)$ is big and nef. This holds since

$$f^*(D) - (i+n)E = \left(1 - \frac{i+n}{\varepsilon_x(D)}\right) f^*(D) + \frac{i+n}{\varepsilon_x(D)}(f^*(D) - \varepsilon_x(D)E)$$

is the sum of a big and nef divisor with a nef one, hence it is big and nef. We thus obtain the assertion in i).

If $\varepsilon_x(\mathscr{L}) > 2n$, then it follows from Proposition 2.5.12 that $\varepsilon_{x'}(\mathscr{L}) > 2n$ for all x' in a neighborhood U of x. In order to prove both ii) and iii), it is enough to show that for every such U, the map $\phi\colon X \dashrightarrow \mathbb{P}^N$ defined by $\omega_X \otimes \mathscr{L}$ is a locally closed immersion on U (we get iii) by taking $U = X$). Note first that by i), ϕ defines a morphism on U that separates tangent vectors. In order to prove that it is a locally closed immersion on U, it is enough to check that it also separates points.

Suppose that x_1 and x_2 are distinct point in U. Let $g\colon W \to X$ be the blow-up along $Z = \{x_1, x_2\}$, with exceptional divisor F, and denote by $\mathscr{I}_Z$ the ideal defining Z. If $f_1\colon Y_1 \to X$ and $f_2\colon Y_2 \to X$ are the blow-ups along x_1 and x_2, respectively, then we have morphisms $g_1\colon W\to Y_1$ and $g_2\colon W\to Y_2$ such that $g=f_1\circ g_1=f_2\circ g_2$. Furthermore, if F_i is the exceptional divisor of f_i, then $F=g_1^*(E_1)+g_2^*(E_2)$. If $\alpha\in\mathbb{Q}$ is such that $\varepsilon_{x_i}(D)>\alpha>2n$ for $i=1,2$, then

$$g^*(D)-\frac{\alpha}{2}F=\frac{1}{2}g_1^*(f_1^*(D)-\alpha F_1)+\frac{1}{2}g_2^*(f_2^*(D)-\alpha F_2)$$

is nef. Arguing as in the proof of i), we see that $g^*(D)-nF$ is big and nef. Furthermore, applying twice the formula for the relative canonical divisor in Example B.2.4 (note that g_1 is the blow-up of X_1 at $f_1^{-1}(x_2)$), we get $\omega_W=g^*(\omega_X)\otimes\mathscr{O}_W((n-1)F)$, hence Theorem 2.2.1 gives

$$H^1(W,g^*(\omega_X\otimes\mathscr{L})\otimes\mathscr{O}_W(-F))=0.$$

Using Lemma **??**, we obtain $H^1(X,\omega_X\otimes\mathscr{L}\otimes\mathscr{I}_Z)=0$, and therefore the restriction map

$$H^0(X,\omega_X\otimes\mathscr{L})\to H^0(X\,\omega_X\otimes\mathscr{L}\otimes\mathscr{O}_X/\mathscr{I}_Z)$$

is surjective. This implies that $\omega_X\otimes\mathscr{L}$ separates x_1 and x_2, and therefore ϕ is a locally closed immersion on U. $\square$

In particular, we obtain the following global generation statement.

Corollary 2.5.16. *If X is an n-dimensional smooth projective variety and $\mathscr{L}\in\operatorname{Pic}(X)$ is ample and globally generated, then $\omega_X\otimes\mathscr{L}^m$ is globally generated for every $m\geq n+1$ and very ample for every $m\geq 2n+1$.*

Proof. Both assertions follow from Theorem 2.5.15, since $\varepsilon_x(\mathscr{L}^m)=m\cdot\varepsilon_x(\mathscr{L})\geq m$ for every m, where the inequality follows from Example 2.5.7. $\square$

Remark 2.5.17. While the proof of Theorem 2.5.15 made use of characteristic zero via vanishing theorems, most of the assertions still hold in positive characteristic. More precisely, if $\mathscr{L}$ is assumed to be ample, then the global generation statement in i), as well as ii) and iii) still hold in positive characteristic, see [MS14].

In light of Theorem 2.5.15, it is very useful to have lower bounds for the Seshadri constants of ample (or big and nef) line bundles. Note, however, that as Example 2.5.8 illustrates, one can not hope to have universal lower bounds at *all* points on a variety. The most one can hope is the following:

Conjecture 2.5.18 (Ein-Lazarsfeld). If $\mathscr{L}$ is an ample line bundle on a smooth projective variety X over a field k of characteristic 0, then for every $\alpha<1$, we have $\varepsilon_x(L)>\alpha$ for $x\in X$ general. In particular, if k is uncountable, then for a very general point $x\in X$, we have $\varepsilon_x(L)\geq 1$.

It is known that in characteristic 0, the assertion in the conjecture holds if we replace 1 by $\frac{1}{\dim(X)}$, see [EKL95]. We end with the following result from [EL93b], giving a proof of the conjecture for surfaces (the case of curves being, of course, trivial).

Theorem 2.5.19. *If X is a smooth projective surface over a field k of characteristic 0, and $\mathscr{L}$ is an ample line bundle on X, then for every $\alpha < 1$ we have $\varepsilon_x(\mathscr{L}) > \alpha$ for all but a finite set of points $x \in X$. In particular, if k is uncountable, then $\varepsilon_x(L) \geq 1$ for all but a countable set of points $x \in X$.*

Proof. We may assume that we work over $\mathbb{C}$. Indeed, suppose first that $k \subset K$ is a field extension, with K algebraically closed, and let $X_K = X \times_{\operatorname{Spec} k} \operatorname{Spec} K$ and $\mathscr{L}_K$ the pull-back of $\mathscr{L}$ to X_K. If

$$U_\alpha = \{x \in X \mid \varepsilon_x(\mathscr{L}) > \alpha\},$$

it follows from Proposition 2.5.12 that U_α is open in X, and the description of U_α in the proof of that proposition, together with Remark 1.1.3 implies that

$$U_\alpha \times_{\operatorname{Spec} k} \operatorname{Spec} K = \{x \in X_K \mid \varepsilon_x(\mathscr{L}_K) > \alpha\}.$$

Therefore the theorem holds for the pair $(X, \mathscr{L})$ if and only if it holds for $(X_K, \mathscr{L}_K)$. After first choosing a finitely generated extension k_0 of $\overline{\mathbb{Q}}$ such that both X and $\mathscr{L}$ are defined over k_0, and then embedding k_0 in $\mathbb{C}$, we see that it is enough to prove the theorem when $k = \mathbb{C}$. Furthermore, since each U_α is open, the finiteness of $X \smallsetminus U_\alpha$ for all $\alpha < 1$ is equivalent to the fact that $\varepsilon_x(\mathscr{L}) \geq 1$ for all but a countable set of points $x \in X$.

It follows from Proposition 2.5.2 that if $\varepsilon_x(\mathscr{L}) < 1$, then there is a curve C containing x, and such that $(\mathscr{L} \cdot C) < \operatorname{mult}_x(C)$. Note that this implies $\operatorname{mult}_x(C) > 1$, hence there are only finitely many such points on each curve C. On the other hand, for every m and d, the pairs (Z, x), with Z a one-dimensional subscheme and $x \in Z$ with $(\mathscr{L} \cdot Z) = d$ and $\operatorname{mult}_x(Z) \geq m$ are parametrized by countably many varieties; this follows from the fact that the Hilbert schemes of subschemes of X are parametrized by the countable set of possible Hilbert polynomials, see [Mum66]. Such a parameter space for which the corresponding curve is fixed only contributes finitely many points $x \in X$ with $\varepsilon_x(\mathscr{L}) < 1$. Therefore it is enough to prove the following: if S is variety, $\mathscr{C} \hookrightarrow X \times S$ is a relative Cartier divisor (over S), and $\sigma \colon S \to \mathscr{C}$ is a section of $X \times S \to S$ such that

i) The map $g \colon S \to \operatorname{Hilb}_P(X)$ given by $s \to \mathscr{C}_s$ is not constant, where P is the corresponding Hilbert polynomial.
ii) $\operatorname{mult}_{\sigma(s)}(\mathscr{C}_s) \geq m$ for every $s \in S$.
iii) The set $\{s \in S \mid \mathscr{C}_s \text{ is integral}\}$ is dense in S,

then $(\mathscr{L} \cdot \mathscr{C}_s) \geq m$ for some (every) $s \in S$. In fact, we will show that under the above conditions, we have $(\mathscr{C}_s^2) \geq m(m-1)$. The Hodge index theorem then gives (see [Har77, Exer. V.1.9])

$$(\mathscr{L} \cdot \mathscr{C}_s)^2 \geq (\mathscr{C}_s^2) \cdot (\mathscr{L}^2) \geq m(m-1).$$

Since $(\mathscr{L} \cdot \mathscr{C}_s)$ is a positive integer, it follows that $(\mathscr{L} \cdot \mathscr{C}_s) \geq m$, as required.

After possibly replacing S by an open subset, we may assume that S is smooth, and by generic smoothness, that g gives a smooth morphism onto a locally closed subset of $\mathrm{Hilb}_P(X)$. Let us choose $s_0 \in S$ such that $C = \mathscr{C}_{s_0}$ is integral. After replacing S by a suitable locally closed subset, we may assume that S is a smooth curve and that the tangent map $dg_{s_0} \colon T_{s_0}(S) \to T_{g(s_0)}\mathrm{Hilb}_P(X)$ is injective. Recall that we have an isomorphism $T_{g(s_0)}\mathrm{Hilb}_P(X) \simeq H^0(C, \mathscr{O}_C(C))$, (see for example [Mum66, Chap. 22]).

We now come to the crux of the argument: we claim that if t is a local coordinate at s_0 and $\mathfrak{a}$ denotes the ideal defining $\sigma(s_0)$ in C, then

$$\alpha := dg_{s_0}\left(\frac{\partial}{\partial t}(s_0)\right) \in \mathfrak{a}^{m-1}.$$

Note that this gives[4] $(C^2) = \deg(\mathscr{O}_C(C)) \geq m(m-1)$, which is precisely what we wanted to show.

In order to prove our claim, we choose coordinates $u = (u_1, u_2)$ at $\sigma(s_0)$. If Φ defines $\mathscr{C}$ in a neighborhood of $(s_0, \sigma(s_0))$, then in this neighborhood we have an isomorphism of $\mathscr{O}_C(C)$ and $\mathscr{O}_C$ such that α corresponds to the restriction of $\frac{\partial \Phi}{\partial t}|_{t=0}$ to C. Therefore it is enough to show that $\frac{\partial \Phi}{\partial t}|_{t=0} \in (u_1, u_2)^{m-1}$.

Let $u_i \circ \sigma = a_i$ for $i = 1, 2$. By treating Φ and a_1, a_2 in terms of the corresponding power series expansions at $(s_0, \sigma(s_0))$ and s_0, respectively, the assumption ii) on our family $\mathscr{C}$ implies that

$$\Phi(t, u_1 - a_1(t), u_2 - a_2(t)) \in (u_1, u_2)^m \subseteq \mathbb{C}[\![t, u_1, u_2]\!].$$

By differentiating with respect to u_1 and u_2, we obtain

$$\frac{\partial \Phi}{\partial u_i}(t, u_1 - a_2(t), u_2 - a_2(t)) \in (u_1, u_2)^{m-1} \text{ for } i = 1, 2,$$

and by differentiating with respect to t, we obtain

$$\frac{\partial \Phi}{\partial t}(t, u_1 - a_2(t), u_2 - a_2(t)) - \sum_{i=1}^{2} \frac{\partial \Phi}{\partial u_i}(t, u_1 - a_1(t), u_2 - a_2(t)) \cdot \frac{\partial a_i}{\partial t} \in (x, y)^m.$$

Therefore $\frac{\partial \Phi}{\partial t}(t, u_1 - a_2(t), u_2 - a_2(t)) \in (u_1, u_2)^{m-1}$, and by making $t = 0$ we obtain $\frac{\partial \Phi}{\partial t}(t, u_1, u_2)|_{t=0} \in (u_1, u_2)^{m-1}$, as claimed. This completes the proof of the theorem. $\square$

In the twenty years since they have been introduced, Seshadri constants found connections with many different points of view in the study of linear series. We refer to [Laz04a] and [BDRH$^+$09] for more in-depth introductions to these interesting invariants.

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[4] We are using the fact that if $(R, \mathfrak{a})$ is a local ring of dimension 1 and $h \in \mathfrak{a}^i$ is not a zero divisor, then $\ell(R/(h)) \geq i \cdot e(\mathfrak{a}; R)$; recall that for a zero-dimensional ideal $\mathfrak{b}$ in a local ring R, one denotes by $e(\mathfrak{b}; R)$ the Samuel multiplicity of R with respect to $\mathfrak{b}$. The inequality follows from $\ell(R/(h)) = e((h); R) \geq e(\mathfrak{a}^i; R) = i \cdot e(\mathfrak{a}; R)$.

2.6 Relative vanishing

In this section we prove the relative version of the Kawamata–Viehweg vanishing theorem, following [KMM87]. As in the absolute case, we assume that the ground field has characteristic zero.

Theorem 2.6.1. *Let $f\colon X \to S$ be a projective, surjective morphism of varieties, with X smooth. If D is a $\mathbb{Q}$-divisor on X such that*

i) D is f-big,
ii) D is nef on $X_{s_0} = f^{-1}(s_0)$ for some $s_0 \in S$, and
iii) $\lceil D \rceil - D$ is a divisor with simple normal crossings,

then $R^i f_*(\omega_X \otimes \mathscr{O}_X(\lceil D \rceil))_{s_0} = 0$ for every $i \geq 1$. In particular, if instead of condition ii)*, we assume that D is f-nef, then $R^i f_*(\omega_X \otimes \mathscr{O}_X(\lceil D \rceil)) = 0$ for every $i \geq 1$.*

Remark 2.6.2. Note that when S is a point, the above theorem is the Kawamata–Viehweg vanishing theorem. Another important special case is when f is birational, when condition i) is automatically satisfied.

Remark 2.6.3. Theorem 2.6.1 is usually stated under the assumption that D is f-nef. Note that this version does not directly imply the first assertion in the theorem, since the set of points $s \in S$ for which D is nef on X_s is not necessarily open in S (see Remark 1.6.26).

For the proof of Theorem 2.6.1 we will need the following lemma. Note that if $g\colon Z \to X$ is a projective morphism that is an isomorphism over an open subset U of X, in general there might be no divisor supported on $g^{-1}(X \smallsetminus U)$ which is g-ample, even if Z is smooth (for example, it might happen that $g^{-1}(X \smallsetminus U)$ has codimension ≥ 2 in Z). The lemma gives a way to fix this problem.

Lemma 2.6.4. *If $g\colon Z \to X$ is a projective morphism that is an isomorphism over an open subset U of X, then there is a morphism $h\colon W \to Z$ that is an isomorphism over $g^{-1}(U)$, with W smooth, and a Cartier divisor F on W such that*

i) F is effective and supported on $(g \circ h)^{-1}(X \smallsetminus U)$.
ii) $-F$ is $(g \circ h)$-ample.

In fact, note that if h has this property and $h'\colon W' \to W$ is any projective morphism that is an isomorphism over $(g \circ h)^{-1}(U)$, with W' normal, we can find a Cartier divisor F' on W' that satisfies i) and ii) above

Proof. By Remark B.3.13, we can find a resolution of singularities $f\colon Y \to X$ that is an isomorphism over U and such that there is an effective Cartier divisor E supported on $f^{-1}(X \smallsetminus U)$ such that $-E$ is f-ample. In this case, for any resolution of singularities $h\colon W \to Z$ that is an isomorphism over $g^{-1}(U)$, and such that $g \circ h$ factors through f, we can find F as required by using Lemma 2.2.4 and Proposition 1.6.15). The last assertion in the lemma also follows by combining Lemma 2.2.4 and Proposition 1.6.15). □

Proof of Theorem 2.6.1. Let $\Delta = \lceil D \rceil - D$. We argue in several steps.

Step 1. Suppose first that X and S are projective and D is f-ample. Let H be an ample Cartier divisor on S. It follows from Proposition 1.6.15 that there is m_0 such that $D + mf^*(H)$ is ample for all $m \geq m_0$. In this case, Theorem 2.2.1 implies

$$H^i(X, \omega_X \otimes \mathscr{O}_X(\lceil D \rceil + mf^*(H)) = 0 \text{ for all } i \geq 1 \text{ and } m \geq m_0.$$

We deduce the fact that $R^i f_*(\omega_X \otimes \mathscr{O}_X(\lceil D \rceil)) = 0$ for $i \geq 1$ by Lemma 2.2.10.

Step 2. We now consider that case when D is f-ample, but X and S are not necessarily projective, and show that $R^i f_*(\omega_X \otimes \mathscr{O}_X(\lceil D \rceil)) = 0$ for all $i \geq 1$. By taking an affine open cover of S, we reduce to the case when S is affine. In this case, if m is a divisible enough positive integer, there is a closed immersion $j \colon X \hookrightarrow \mathbb{P}^N_S$ over S, such that $\mathscr{O}_X(mD) \simeq j^*(\mathscr{O}_{\mathbb{P}^N_S}(1))$. Let S' be the closure of S in some projective space, $\overline{X}$ the closure of X in $\mathbb{P}^N_{S'}$, and $\overline{f} \colon \overline{X} \to S'$ the induced morphism. Using Remark B.3.14 and Lemma 2.6.4, we construct a projective morphism $g \colon X' \to \overline{X}$, with the following properties:

1) g is an isomorphism over X.
2) X' is smooth.
3) $X' \smallsetminus X$ is an effective Cartier divisor F on X'.
4) There is a divisor Δ' on X' without common components with F such that $\Delta'|_X = \Delta$ and $\Delta' + F$ has simple normal crossings.
5) There is an effective divisor G supported on $X' \smallsetminus X$ such that $-G$ is g-ample.

We put $f' = \overline{f} \circ g$. Note that by construction, we have a Cartezian diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ X \arrow[r] & S'. \end{tikzcd}$$

Furthermore, there is $H \in \mathrm{CDiv}(\overline{X})$ which is $\overline{f}$-ample and such that $H|_X \sim_{\mathbb{Q}} D$. If $H' = -\frac{1}{m}G + g^*(H)$, with $m \gg 0$, then H' is f'-ample by Proposition 1.6.15 and $H'|_X \sim_{\mathbb{Q}} D$. It follows that there is an f'-ample $\mathbb{Q}$-divisor D' on X' such that $D'|_X = D$ and $\lceil D' \rceil - D'$ is supported on $\mathrm{Supp}(\Delta' + F)$, and thus has simple normal crossings. Since $R^i f'_*(\omega_{X'} \otimes \mathscr{O}_{X'}(\lceil D' \rceil)) = 0$ for all $i \geq 1$ by Step 1, we conclude that $R^i f_*(\omega_X \otimes \mathscr{O}_X(\lceil D \rceil)) = 0$ for all $i \geq 1$.

Step 3. We consider the general case. Note that we may replace S by an open neighborhood of s_0. In particular, we may assume that S is affine. Since D is f-big, it follows from Proposition 1.6.33 that D can be written as a sum of two $\mathbb{Q}$-divisors, the first one f-ample, and the second one effective. Arguing as in the proof of Lemma 2.2.9, we find a projective birational morphism $g \colon Y \to X$, with Y smooth, and a decomposition $g^*(D) = A + E$ for $\mathbb{Q}$-divisors A and E, with A being $(f \circ g)$-ample and E effective, such that $g^*(\Delta) + \mathrm{Exc}(f) + E$ has simple normal crossings.

Note that for every $\varepsilon \in \mathbb{Q}$ with $0 < \varepsilon < 1$, the $\mathbb{Q}$-divisor $g^*(D) - \varepsilon E$ is nef on $Y_{s_0} = (f \circ g)^{-1}(s_0)$. Indeed, since D is nef on X_{s_0}, it follows that $g^*(D)$ is nef on Y_{s_0},

and we can write
$$g^*(D) - \varepsilon E = (1-\varepsilon)g^*(D) + \varepsilon A.$$
We fix $\varepsilon \in \mathbb{Q}$, with $0 < \varepsilon \ll 1$, such that $\lceil g^*(D) - \varepsilon E \rceil = \lceil g^*(D) \rceil$. By Remark 1.6.25, after possibly replacing S by an open neighborhood of s_0, we may assume that $g^*(D) - \varepsilon E$ is $(f \circ g)$-ample (hence also g-ample). Furthermore, since

$$\mathrm{Supp}(\lceil g^*(D) \rceil - (g^*(D) - \varepsilon E)) \subseteq \mathrm{Supp}(g^*(\Delta)) \cup \mathrm{Supp}(E) \cup \mathrm{Exc}(g),$$

which has simple normal crossings, it follows that we may apply the case in Step 2 for $g^*(D) - \varepsilon E$ and the morphisms $f \circ g$ and g to conclude that

$$R^i(f \circ g)_*(\omega_Y \otimes \mathscr{O}_Y(\lceil g^*(D) \rceil)) = 0 \text{ and } R^i g_*(\omega_Y \otimes \mathscr{O}_Y(\lceil g^*(D) \rceil)) = 0$$

for all $i \geq 1$. The Leray spectral sequence implies

$$R^i f_*(g_*(\omega_Y \otimes \mathscr{O}_Y(\lceil g^*(D) \rceil))) = 0 \text{ for all } i \geq 1.$$

Therefore in order to complete the proof of the theorem, it is enough to show that

$$g_*(\omega_Y \otimes \mathscr{O}_Y(\lceil g^*(D) \rceil)) \simeq \omega_X \otimes \mathscr{O}_X(\lceil D \rceil). \tag{2.14}$$

Recall that by Lemma B.2.3, we have an effective g-exceptional divisor $K_{Y/X}$ on Y such that $\omega_Y \simeq g^*(\omega_X) \otimes \mathscr{O}_Y(K_{Y/X})$. Since $\lceil g^*(D) \rceil = g^*(\lceil D \rceil) - \lfloor g^*(\Delta) \rfloor$, the isomorphism in (2.14) follows from the following equality of subsheaves of the function field of X:
$$g_*\mathscr{O}_Y(K_{Y/X} - \lfloor g^*(\Delta) \rfloor) = \mathscr{O}_X. \tag{2.15}$$
Since $\lfloor g^*(\Delta) \rfloor$ is effective, we obtain $g_*\mathscr{O}_Y(K_{Y/X} - \lfloor g^*(\Delta) \rfloor) \subseteq g_*\mathscr{O}_Y(K_{Y/X}) = \mathscr{O}_X$, where the equality follows from Lemma B.2.5. On the other hand, we will see in Chapter 3 that since Δ is a simple normal crossing divisor with $\lfloor \Delta \rfloor = 0$, the divisor $K_{Y/X} - \lfloor g^*(\Delta) \rfloor$ is effective. Therefore $g_*\mathscr{O}_Y(K_{Y/X} - \lfloor g^*(\Delta) \rfloor) \supseteq g_*(\mathscr{O}_Y) = \mathscr{O}_X$. This completes the proof of the theorem. ☐

Using the relative Kawamata-Viehweg theorem, we obtain relative versions of some of the results that we discussed in the previous sections. Since the proofs follow closely the ones in the absolute case, we omit them.

Corollary 2.6.5 (cf. Corollary 2.3.4). *If $f\colon X \to S$ is a projective morphism and $\mathscr{L} \in \mathrm{Pic}(X)$ is f-big and f-nef, then*

$$R^i f_*(\omega_X^{\mathrm{GR}} \otimes \mathscr{L}) = 0 \textit{ for all } i \geq 1.$$

Theorem 2.6.6 (cf. Theorem 2.3.5). *Let $f\colon X \to S$ be a projective morphism and $\mathscr{L}$ a line bundle on X which is f-ample. For every coherent sheaf $\mathscr{F}$ on X, there is m such that $R^i f_*(\mathscr{F} \otimes \mathscr{L}^m \otimes \mathscr{L}') = 0$ for every $i \geq 1$ and every f-nef line bundle $\mathscr{L}'$ on X.*

Corollary 2.6.7 (cf. Corollary 2.4.4). *If $f: X \to Y$ is a projective morphism, then there is $\mathscr{A} \in \mathrm{Pic}(X)$ such that for every $\mathscr{L} \in \mathrm{Pic}(X)$ which is f-nef, the line bundle $\mathscr{A} \otimes \mathscr{L}$ is f-base-point free.*

2.7 The injectivity theorem

We now turn to a theorem which applies under fairly general conditions, without any positivity assumptions. While this does not directly give the vanishing of cohomology groups, it provides the injectivity of suitable maps in cohomology. As we will see, this is strong enough to imply Kodaira's vanishing theorem (we thus obtain a second proof of this theorem), but it also has other important applications. We keep the assumption that the ground field has characteristic zero.

Theorem 2.7.1. *Let X be a smooth projective variety and $\Delta = \sum_{i=1}^r \Delta_i$ a simple normal crossing divisor on X, with the Δ_i distinct prime divisors. If B is a Cartier divisor on X such that $B \sim_{\mathbb{Q}} \sum_{i=1}^r b_i\Delta_i$, with $0 < b_i \leq 1$ for all i, then for every effective divisor D, with $\mathrm{Supp}(D) \subseteq \mathrm{Supp}(\Delta)$, the map*

$$H^q(X, \omega_X \otimes \mathscr{O}_X(B)) \to H^q(X, \omega_X \otimes \mathscr{O}_X(B+D))$$

induced in cohomology by multiplication with an equation defining D is injective for all $q \geq 0$.

The original injectivity theorem is due to Kollár [Kol86]. Esnault and Viehveg generalized the result and gave a new proof in [EV92]. This was further strengthened to the above form by Ambro [Amb]. We follow [Amb] for the first part of the argument (the case $B = \Delta$). In order to deduce the general case of the theorem, we imitate the argument in the proof of the Kawamata–Viehweg vanishing theorem (this allows us to only consider cyclic coverings with respect to smooth divisors, and thus makes the proof more elementary). We start with the following proposition, which is where Hodge theory comes into play.

Proposition 2.7.2. *If X is a smooth projective variety and $\Delta = \sum_{i=1}^r \Delta_i$ a simple normal crossing divisor on X, then the map*

$$H^q(X, \omega_X \otimes \mathscr{O}_X(\Delta)) \to H^q(U, \omega_U),$$

induced by restriction to $U = X \smallsetminus \mathrm{Supp}(\Delta)$, is injective for every $q \geq 0$.

Proof. Let $j: U \hookrightarrow X$ be the inclusion. Note that we have an injective map of complexes

$$\iota: \Omega_X^\bullet(\log \Delta) \hookrightarrow j_*\Omega_U^\bullet$$

and it is a basic result that this is a quasi-isomorphism (see [Gro66]). We consider the two spectral sequences corresponding to the "stupid" filtrations on these two complexes, namely

$$E_1^{p,q} = H^q(X, \Omega_X^p(\log \Delta)) \underset{p}{\Rightarrow} H^{p+q}(X, \Omega_X^\bullet(\log \Delta)) \text{ and} \tag{2.16}$$

$$\widetilde{E}_1^{p,q} = H^q(X, j_*\Omega_U^p) \underset{p}{\Rightarrow} H^{p+q}(X, j_*\Omega_U^\bullet). \tag{2.17}$$

Since both $E_1^{p,q}$ and $\widetilde{E}_1^{p,q}$ vanish for $p > n$, it follows that we get canonical morphisms

$$E_1^{n,q} \to H^{n+q}(X, \Omega_X^\bullet(\log \Delta)) \quad \text{and} \quad \widetilde{E}_1^{n,q} \to H^{n+q}(X, j_*\Omega_U^\bullet).$$

We thus obtain a commutative diagram

$$\begin{tikzcd} H^q(X, \omega_X \otimes \mathscr{O}_X(\Delta)) \arrow[r, "\gamma"] \arrow[d, "\alpha"'] & H^{n+q}(X, \Omega_X^\bullet(\log \Delta)) \arrow[d, "\beta"] \\ H^q(X, j_*\omega_U) \arrow[r, "\delta"] & H^{n+q}(X, j_*\Omega_U^\bullet) \end{tikzcd}$$

in which α and β are induced by the inclusion ι. Since ι is a quasi-isomorphism, it follows that β is an isomorphism. On the other hand, since the spectral sequence in (2.16) degenerates at the E_1 term (see Theorem 2.1.14), it follows that γ is injective. We conclude from the commutative diagram that α is injective. It is now enough to note that since j is an affine morphism[5], we have an isomorphism $H^q(X, j_*\omega_U) \simeq H^q(U, \omega_U)$ such that α gets identified with the map in the proposition. □

Corollary 2.7.3. *If X and Δ are as in Proposition 2.7.2, then for every effective divisor D with* $\mathrm{Supp}(D) \subseteq \mathrm{Supp}(\Delta)$*, the natural map*

$$H^q(X, \omega_X \otimes \mathscr{O}_X(\Delta)) \to H^q(X, \omega_X \otimes \mathscr{O}_X(\Delta + D)),$$

induced by multiplication with an equation of D is injective for every $q \geq 0$.

Proof. With the notation in the proof of Proposition 2.7.2, we have

$$\mathscr{O}_X \hookrightarrow \mathscr{O}_X(D) \hookrightarrow j_*\mathscr{O}_U,$$

where the first map is given by multiplication with the section defining D. By tensoring this with $\omega_X \otimes \mathscr{O}_X(\Delta)$ and taking the q^{th} cohomology, we obtain

$$H^q(X, \omega_X \otimes \mathscr{O}_X(\Delta)) \to H^q(X, \omega_X \otimes \mathscr{O}_X(\Delta + D)) \to H^q(X, j_*(\omega_U)) \simeq H^q(U, \omega_U),$$

where the isomorphism follows from the fact that j is affine. Since the composition map is injective by Proposition 2.7.2, it follows that the first map is injective. □

We can now prove the injectivity theorem.

[5] In general, if R is an effective Cartier divisor on a scheme Y, the inclusion $Y \smallsetminus \mathrm{Supp}(R) \hookrightarrow Y$ is affine. Indeed, this property is local on Y, hence we may assume that R is defined by an equation in $\mathscr{O}(Y)$. In this case, the assertion is clear.

Proof of Theorem 2.7.1. Our goal is to reduce the assertion to the case when $b_i = 1$ for every i, which is a consequence of Corollary 2.7.3. Note that by Serre duality, the injectivity of the maps in the theorem is equivalent to the surjectivity of the maps

$$H^i(X, \mathscr{O}_X(-B-D)) \to H^i(X, \mathscr{O}_X(-B)),$$

induced by multiplication with an equation of D, for all $i \geq 0$.

For the purpose of doing induction, it is convenient to allow the Δ_i to be reducible, but require that they have no common components (of course, we keep the assumption that Δ has simple normal crossings). We argue by induction on the cardinality of $\{i \mid b_i < 1\}$. If this is 0, then $B \sim_{\mathbb{Q}} \Delta$. Let n be a positive integer such that $nB \sim n\Delta$. If $n = 1$, then the assertion we need follows from Corollary 2.7.3. If $n \geq 2$, then we consider $\mathscr{M} = \mathscr{O}_X(B - \Delta)$ and construct the n-cyclic cover $\mu\colon W \to X$ corresponding to a section of $\mathscr{M}^n$ that does not vanish anywhere. Therefore μ is étale and we have $\mu^*(B) \sim \mu^*(\Delta)$ (see Lemma 2.1.6). We may thus apply Corollary 2.7.3 for $B_W := \mu^*(B)$ and $D_W := \mu^*(D)$ to deduce that all maps

$$H^i(W, \mathscr{O}_W(-B_W - D_W)) \to H^i(W, \mathscr{O}_W(-B_W)) \tag{2.18}$$

induced by multiplication with a section defining D_W are surjective. Since $\mu_*(\mathscr{O}_W) \simeq \oplus_{j=0}^{n-1} \mathscr{M}^{-j}$, using the projection formula and the fact that μ is finite, we obtain

$$H^i(W, \mathscr{O}_W(-B_W - D_W)) \simeq \bigoplus_{j=0}^{n-1} H^i(X, \mathscr{O}(-B-D) \otimes \mathscr{M}^{-j}) \quad \text{and}$$

$$H^i(W, \mathscr{O}_W(-B_W)) \simeq \bigoplus_{j=0}^{n-1} H^i(X, \mathscr{O}(-B) \otimes \mathscr{M}^{-j}).$$

By taking the component of the map in (2.18) corresponding to $j = 0$, we conclude that all morphisms

$$H^i(X, \mathscr{O}_X(-B-D)) \to H^i(X, \mathscr{O}_X(-B))$$

are surjective. This completes the proof in this case.

Suppose now that $b_1 < 1$ and let m be a positive integer such that $mb_1 = a_1 \in \mathbb{Z}$. By Lemma 2.2.2, we can find a finite surjective morphism $f\colon Y \to X$ such that $\mathscr{O}_Y(f^*\Delta_1) \simeq \mathscr{L}^m$ for a line bundle $\mathscr{L}$ on Y. Furthermore, we may assume that Y is smooth and $f^*(\Delta)$ is reduced and has simple normal crossings. In this case, it is enough to prove the theorem for $B_Y := f^*(B) \sim_{\mathbb{Q}} \sum_{i=1}^r b_i f^*(\Delta_i)$ and $D_Y := f^*(D)$. Indeed, note that these divisors satisfy the assumptions in the theorem and we have a commutative diagram

$$\begin{tikzcd}
H^i(Y, \mathscr{O}_Y(-B_Y - D_Y)) \arrow[r] \arrow[d] & H^i(Y, \mathscr{O}_Y(-B_Y)) \arrow[d] \\
H^i(X, \mathscr{O}_X(-B - D)) \arrow[r, "\delta"] & H^i(X, \mathscr{O}_X(-B)
\end{tikzcd}$$

in which the vertical maps are the surjective maps induced by the trace map $\mathrm{Tr}\colon K(Y) \to K(X)$ (see Lemma 2.2.7). The commutativity of the diagram follows from the fact that Tr is $K(X)$-linear. It is clear now that the surjectivity of the top horizontal map in the diagram implies the subjectivity of the bottom one.

After replacing X by Y, we may thus assume that there is a line bundle $\mathscr{L}$ on X such that $\mathscr{L}^m \simeq \mathscr{O}_X(\Delta_1)$. Let $g\colon Z \to X$ be the m-cyclic cover corresponding to a section of $\mathscr{L}^m$ defining Δ_1. Since Δ_1 is smooth and Δ has simple normal crossings, it follows from Lemma 2.2.6 that Z is smooth and if $f^*(\Delta_1) = m\Delta_1'$ and $\Delta_i' = f^*(\Delta_i)$ for $i \geq 2$, then each Δ_i' is smooth, the Δ_i' have no common components, and $\Delta_Z := \sum_{i=1}^r \Delta_i'$ has simple normal crossings. Let

$$D_Z := g^*(D) \quad \text{and} \quad B_Z := g^*(B) + (1 - a_1)\Delta_1' \sim_{\mathbb{Q}} \Delta_1' + \sum_{i=2}^r b_i \Delta_i'.$$

Note that we may apply the inductive hypothesis to B_Z and D_Z to conclude that all maps

$$H^q(Z, \mathscr{O}_Z(-D_Z - B_Z)) \to H^q(Z, \mathscr{O}_Z(-B_Z)) \tag{2.19}$$

induced by multiplication with an equation of D_Z are surjective. Since g is finite, we deduce using the projection formula and Lemma 2.1.6 that

$$\begin{aligned}
H^q(Z, \mathscr{O}_Z(-B_Z)) &\simeq H^q(X, \mathscr{O}_X(-B) \otimes g_*\mathscr{O}_Z((a_1 - 1)\Delta_1')) \\
&\simeq \bigoplus_{j=0}^{m-1} H^q(X, \mathscr{O}_X(-B) \otimes \mathscr{L}^{a_1 - 1 - j}).
\end{aligned}$$

We similarly have an isomorphism

$$H^q(Z, \mathscr{O}_Z(-B_Z - D_Z)) \simeq \bigoplus_{j=0}^{m-1} H^q(X, \mathscr{O}_X(-B) \otimes \mathscr{L}^{a_1 - 1 - j}).$$

By assumption, we have $1 \leq a_1 \leq m - 1$, and by taking the component of the map (2.19) corresponding to $j = a_1 - 1$, we obtain the surjectivity of

$$H^q(X, \mathscr{O}_X(-B - D)) \to H^q(X, \mathscr{O}_X(-B))$$

for every q. This completes the proof of the theorem. □

Remark 2.7.4. Note that Theorem 2.7.1 implies Kodaira's vanishing theorem, hence we obtain a second proof of this result. Indeed, suppose that $\mathscr{L}$ is an ample line bundle on the smooth, projective variety X. Let $m \gg 0$ be such that $\mathscr{L}^m$ is very ample and $H^i(X, \omega_X \otimes \mathscr{L}^{m+1}) = 0$ for all $i > 0$. By Bertini's theorem, there is $\Delta \in$

$|\mathscr{L}^m|$ smooth. If B is a Cartier divisor with $\mathscr{O}_X(B) \simeq \mathscr{L}$, then $B \sim_{\mathbb{Q}} \frac{1}{m}\Delta$, hence Theorem 2.7.1 implies that multiplication by an equation defining Δ induces an injective map

$$H^i(X, \omega_X \otimes \mathscr{O}_X(B)) \to H^i(X, \omega_X \otimes \mathscr{O}_X(B+\Delta)) = H^i(X, \omega_X \otimes \mathscr{L}^{m+1})$$

for every $i \geq 0$. We conclude that $H^i(X, \omega_X \otimes \mathscr{L}) = 0$ for $i > 0$.

The injectivity theorem is often applied via the following corollary.

Corollary 2.7.5. *Let E be a semiample divisor on a smooth projective variety X. If F is an effective divisor on X such that $H^0(X, \mathscr{O}_X(mE - F)) \neq 0$ for some $m \geq 1$, then multiplication by an equation of F induces an injective map*

$$H^q(X, \omega_X \otimes \mathscr{O}_X(dE)) \to H^q(X, \omega_X \otimes \mathscr{O}_X(dE+F))$$

for every $d \geq 1$ and $q \geq 0$.

Proof. By hypothesis, we can pick $D \in |mE|$ such that $D - F$ is effective. Let $f\colon Y \to X$ be a log resolution of (X, D) and write $f^*(D) = \sum_j a_j \Delta_j$, where the Δ_j are distinct prime divisors. Let ℓ be a sufficiently divisible integer, such that $\mathscr{O}_X(\ell E)$ is globally generated, $\ell > m$, and $\ell > da_j$ for every j. Since $f^*\mathscr{O}_X(\ell E)$ is globally generated, it follows from Kleiman's version of Bertini's theorem (see [Har77, Thm. III.10.8]) that there is a smooth effective divisor $G \in |f^*(\ell E)|$ without common components with $f^*(D)$ and such that $f^*(D) + G$ has simple normal crossings. Note that the divisor

$$H := \frac{d}{\ell}\left(f^*(D) + \left(1 - \frac{m}{\ell}\right)G\right)$$

is linearly equivalent to $f^*(dE)$, it has simple normal crossing support, and $\lfloor H \rfloor = 0$. Since

$$\mathrm{Supp}(f^*(F)) \subseteq \mathrm{Supp}(f^*(D)) \subseteq \mathrm{Supp}(H),$$

it follows that we may apply Theorem 2.7.1 to conclude that all maps

$$H^q(Y, \omega_Y \otimes f^*\mathscr{O}_Y(dE)) \to H^q(Y, \omega_Y \otimes f^*\mathscr{O}_Y(dE+F)), \tag{2.20}$$

induced by multiplication with an equation of F, are injective. On the other hand, we have $f_*(\omega_Y) \simeq \omega_X$ (see Corollary B.2.6) and the Grauert–Riemenschneider vanishing theorem implies $R^i f_*(\omega_Y) = 0$ for all $i \geq 0$. It follows from the Leray spectral sequence for f and the projection formula that the map in (2.20) is identified with

$$H^q(X, \omega_X \otimes \mathscr{O}_X(dE)) \to H^q(X, \omega_X \otimes \mathscr{O}_X(dE+F)).$$

This gives the assertion in the corollary. $\square$

2.8 Higher direct images of canonical line bundles

We now explain how the injectivity theorem implies Kollár's results on higher direct images of canonical line bundles. As in the previous section, we assume that the ground field has characteristic zero. The following is the main result of this section.

Theorem 2.8.1 (Kollár). *If $g\colon X\to Z$ is a surjective morphism of projective varieties, with X smooth, then for every $j\geq 0$, the following hold:*

i) $R^jg_*(\omega_X)$ is a torsion-free sheaf, and
ii) $H^i(Z,R^jg_*(\omega_X)\otimes\mathscr{O}_Z(A))=0$ for every $i>0$ and every ample Cartier divisor A on Z.

Proof. We fix $j\geq 0$, an arbitrary ample Cartier divisor A on Z, and let $A'=g^*(A)$. By asymptotic Serre vanishing, we can choose a positive integer m_0 such that

$$H^q(Z,R^pg_*(\omega_X)\otimes\mathscr{O}_Z(mA))=0\quad\text{for all}\quad p\geq 0,q>0,\text{ and }m\geq m_0.\tag{2.21}$$

Using the Leray spectral sequence and the projection formula, we deduce

$$H^0(Z,R^jg_*(\omega_X)\otimes\mathscr{O}_Z(mA))\cong H^j(X,\omega_X\otimes\mathscr{O}_X(mA'))\quad\text{for all}\quad m\geq m_0.\tag{2.22}$$

Let $\mathscr{F}$ be the torsion subsheaf of $R^jg_*(\omega_X)$, which consists of all local sections of $R^jg_*(\omega_X)$ that are zero at the generic point of Z. In order to prove i), we need to show that $\mathscr{F}=0$. By assumption, the coherent ideal $\mathrm{Ann}_{\mathscr{O}_Z}(\mathscr{F})$ is nonzero. We pick an integer $\ell\geq m_0$ large enough such that the following conditions are satisfied:

- $\mathscr{F}\otimes\mathscr{O}_Z(\ell A)$ is generated by its global sections;
- The sheaf $\mathrm{Ann}_{\mathscr{O}_Z}(\mathscr{F})\otimes\mathscr{O}_Z(\ell A)$ is globally generated. In particular, there is a nonzero global section s of $\mathscr{O}_Z(\ell A)$ that annihilates $\mathscr{F}$.

These conditions imply that multiplication by the section s induces a map

$$H^0(Z,R^jg_*(\omega_X)\otimes\mathscr{O}_Z(\ell A))\to H^0(Z,R^jg_*(\omega_X)\otimes\mathscr{O}_Z(2\ell A)),$$

that cannot be injective, unless $\mathscr{F}=0$. Note that since $\ell\geq m_0$, the above map gets identified via the isomorphisms (2.22) to the map

$$H^j(X,\omega_X\otimes\mathscr{O}_X(\ell A'))\to H^j(X,\omega_X\otimes\mathscr{O}_X(2\ell A'))\tag{2.23}$$

induced by multiplication with the section $g^*(s)$ of $\mathscr{O}_X(\ell A')$. On the other hand, A' is semiample, hence the map in (2.23) is injective by Corollary 2.7.5. We thus conclude that $\mathscr{F}$ is trivial, hence $R^jg_*(\omega_X)$ is torsion-free. This proves i).

We prove ii) by induction on $n=\dim(Z)$, the case $n=0$ being trivial. Let $m\geq m_0$ be a fixed large enough integer, such that $\mathscr{O}_X(mA)$ is very ample, and let $H'\in|mA'|$ be the pullback of a general divisor $H\in|mA|$. It follows from Kleiman's version of Bertini's theorem that we may assume that H and H' are smooth (though possibly disconnected). We have an exact sequence

$$0 \to \omega_X \otimes \mathscr{O}_X(A') \to \omega_X \otimes \mathscr{O}_X((m+1)A') \to \omega_{H'} \otimes \mathscr{O}_X(A')|_{H'} \to 0$$

induced by multiplication with a section defining H'. Since all higher direct images of $\omega_X \otimes \mathscr{O}_X(A')$ are torsion-free by i), and since the sheaves $R^j g_*(\omega_{H'} \otimes \mathscr{O}_X(A')|_{H'})$ are clearly torsion on Z, we obtain short exact sequences

$$0 \to R^j g_*(\omega_X \otimes \mathscr{O}_X(A')) \to R^j g_*(\omega_X \otimes \mathscr{O}_X((m+1)A')) \to R^j g_*(\omega_{H'} \otimes \mathscr{O}_X(A')|_{H'}) \to 0$$

for every $j \geq 0$. On the other hand, applying the projection formula and the inductive hypothesis to each connected component of H', we conclude that

$$H^i(Z, R^j g_*(\omega_{H'} \otimes \mathscr{O}_X(A')|_{H'})) = 0 \quad \text{for all} \quad i \geq 1.$$

Furthermore, we have by (2.21)

$$H^i(Z, R^j g_*(\omega_X \otimes \mathscr{O}_X((m+1)A'))) = 0 \quad \text{for all} \quad i \geq 1.$$

By taking the cohomology long exact sequence corresponding to the above short exact sequence of sheaves on Z, we conclude that $H^i(Z, R^j g_*(\omega_X \otimes \mathscr{O}_X(A'))) = 0$ for every $i > 1$.

We still need to prove the vanishing for $i = 1$. Note that if we have a first-quadrant spectral sequence $E_2^{p,q} \Rightarrow H^{p+q}$ such that $E_2^{p,q} = 0$ unless $p \in \{0, 1\}$, then $E_\infty^{1,q}$ is a subspace of H^{q+1} for every q (and the quotient is isomorphic to $E_\infty^{0,q+1}$). In general, we have an injective map $E_2^{1,q} \hookrightarrow E_\infty^{1,q}$ and we thus get an injective map $E_2^{1,q} \hookrightarrow H^{q+1}$.

We deduce that in our setting we have a commutative diagram

$$\begin{tikzcd}
H^1(Z, R^j g_*(\omega_X \otimes \mathscr{O}_X(A'))) \arrow[r, "\phi"] \arrow[d] & H^{j+1}(X, \omega_X \otimes \mathscr{O}_X(A')) \arrow[d, "\psi"] \\
H^1(Z, R^j g_*(\omega_X \otimes \mathscr{O}_X((m+1)A'))) \arrow[r] & H^{j+1}(X, \omega_X \otimes \mathscr{O}_X((m+1)A')),
\end{tikzcd}$$

where the horizontal maps are the canonical injective maps coming, as described above, out of the Leray spectral sequences, and the vertical maps are induced by multiplication with sections defining H' and H.

The map ψ is injective by Corollary 2.7.5, hence the composition $\psi \circ \phi$ is injective. On the other hand, we have $H^1(Z, R^j g_*(\omega_X \otimes \mathscr{O}_X((m+1)A'))) = 0$, and we thus conclude that $H^1(Z, R^j g_*(\omega_X \otimes \mathscr{O}_X(A'))) = 0$. This completes the proof of the theorem. □

Corollary 2.8.2. *Under the same assumptions as in Theorem 2.8.1, if $\mathscr{L}$ and $\mathscr{L}'$ are line bundles on Z, with $\mathscr{L}$ ample and globally generated, and $\mathscr{L}'$ nef, then the sheaf $R^j g_*(\omega_X) \otimes \mathscr{L}^m \otimes \mathscr{L}'$ is globally generated for every $m \geq \dim(Z) + 1$ and every $j \geq 0$.*

Proof. It follows from Theorem 2.8.1 that $H^i(Z, R^j g_*(\omega_X) \otimes \mathscr{L}^{m-i} \otimes \mathscr{L}') = 0$ for every $i \geq 1$. Therefore the sheaf $R^j g_*(\omega_X) \otimes \mathscr{L}^m \otimes \mathscr{L}'$ is 0-regular with respect to $\mathscr{L}$, hence globally generated by Theorem 2.4.3. □

Kawamata observed that the following stronger version of Theorem 2.8.1 holds.

Theorem 2.8.3 (Kawamata). *With the same assumptions as in Theorem 2.8.1, suppose that M is a divisor on X that is numerically equivalent to a $\mathbb{Q}$-divisor D having simple normal crossings, and such that $\lfloor D \rfloor = 0$. In this case, the following hold for every $j \geq 0$:*

i) $R^j g_*(\omega_X \otimes \mathscr{O}_X(M))$ is a torsion-free sheaf, and
ii) $H^i(Z, R^j g_*(\omega_X \otimes \mathscr{O}_X(M)) \otimes \mathscr{O}_Z(A)) = 0$ for every $i > 0$ and every ample divisor A on Z.

The proof of this variant is similar to that of Theorem 2.8.1, using a more general version of Corollary 2.7.5, which in turn can be deduced from Theorem 2.7.1. We close this section by applying Theorem 2.8.1 to give a proof of the following result of Fujita and Kawamata. The proof given here is due to Kollár.

Theorem 2.8.4 (Fujita-Kawamata). *If $g\colon X \to Z$ is a smooth, projective morphism between smooth projective varieties, then the locally free sheaf $g_*(\omega_{X/Z})$ is nef.*

Remark 2.8.5. Recall that if g is a smooth, projective morphism, of relative dimension d, then one defines $\omega_{X/Z} := \wedge^d \Omega_{X/Z}$. It is a general fact that all sheaves $R^q g_*(\omega_{X/Z})$ are locally free. Indeed, for every $z \in Z$, the restriction of $\omega_{X/Z}$ to $X_z = g^{-1}(z)$ is isomorphic to ω_{X_z}. Note that since $\omega_{X/Z}$ is flat over Z, it follows from the base-change theorems (see [Har77, Cor. III.12.9]) that in order to show that $R^q g_*(\omega_{X/Z})$ is locally free, it is enough to show that the function $Z \ni z \to h^q(X_z, \omega_{X_z})$ is constant.

This is an easy consequence of Hodge theory. First, we may assume that the ground field is $\mathbb{C}$. Since g is a smooth projective morphism, it follows from a theorem of Ehresman that in the $\mathscr{C}^\infty$-category, g is a locally trivial fibration. In particular, all fibers X_z are diffeomorphic, and therefore the map $Z \ni z \to \dim_{\mathbb{C}} H^i(X_z^{\mathrm{an}}; \mathbb{C})$ is constant. On the other hand, the Hodge decomposition gives

$$\dim_{\mathbb{C}} H^i(X_z^{\mathrm{an}}; \mathbb{C}) = \sum_{p+q=i} h^q(X_z, \Omega^p_{X_z})$$

(see Corollary 2.1.15). By the semicontinuity theorem (see [Har77, Thm. III.12.8]), each function $Z \ni z \to h^q(X_z, \Omega^p_{X_z})$ is upper-semicontinuous on Z and since the sum of these functions is constant, we conclude that each of these functions is constant on Z. In particular, by taking $p = d$, we obtain our assertion.

Note that if, in addition, Z is smooth, then X is smooth too and $\omega_{X/Z} \simeq \omega_X \otimes g^*(\omega_Z)^{-1}$. We thus see that in this case $R^q g_*(\omega_X)$ is locally free for every q.

Proof of Theorem 2.8.4. For any positive integer m, consider the m-fold fiber product

$$X_m = \overbrace{X \times_Z \cdots \times_Z X}^{m \text{ times}}$$

of X over Z, and denote by $g_m \colon X_m \to Z$ the natural projection. Let $\mathscr{E} = g_* \omega_{X/Z}$. We have seen in Remark 2.8.5 that this is a locally free sheaf on Z.

Claim. $(g_m)_* \omega_{X_m/Z} = \mathscr{E}^{\otimes m}$.

We check the claim by induction on m, the case $m = 1$ being trivial. Applying flat base-change to the Cartezian diagram with smooth maps

$$\begin{tikzcd} X_m \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, "g_m"] & X_{m-1} \arrow[d, "g_{m-1}"] \\ X \arrow[r, "g"] & Z \end{tikzcd}$$

we see that $g_{m-1}^* g_* (\omega_{X/Z}) \simeq q_* p^* (\omega_{X/Z})$. By combining this with the inductive assumption, and using the fact that $\omega_{X_m/Z} \simeq q^*(\omega_{X_{m-1}/Z}) \otimes p^*(\omega_{X/Z})$, we obtain

$$\begin{aligned} (g_m)_* \omega_{X_m/Z} &\simeq (g_{m-1})_* q_* (q^* \omega_{X_{m-1}/Z} \otimes p^* \omega_{X/Z}) \\ &\simeq (g_{m-1})_* (\omega_{X_{m-1}/Z} \otimes q_* p^* \omega_{X/Z}) \\ &\simeq (g_{m-1})_* (\omega_{X_{m-1}/Z} \otimes g_{m-1}^* g_* \omega_{X/Z}) \\ &\simeq (g_{m-1})_* \omega_{X_{m-1}/Z} \otimes g_* \omega_{X/Z} \simeq \mathscr{E}^{\otimes (m-1)} \otimes \mathscr{E} \simeq \mathscr{E}^{\otimes m}. \end{aligned}$$

Fix a very ample divisor H on Z and let A be a divisor such that $\mathscr{O}_Z(A) \simeq \omega_Z \otimes \mathscr{O}_Z((n+1)H)$, where $n = \dim(Z)$. By applying Corollary 2.8.2 to g_m, we deduce that the sheaf

$$(g_m)_*(\omega_{X_m}) \otimes \mathscr{O}_Z((n+1)H) \simeq (g_m)_*(\omega_{X_m/Z}) \otimes \mathscr{O}_Z(A) \simeq \mathscr{E}^{\otimes m} \otimes \mathscr{O}_Z(A)$$

is generated by its global sections. Therefore the sheaf $\mathrm{Sym}^m(\mathscr{E}) \otimes \mathscr{O}_Z(A)$, being a quotient of $\mathscr{E}^{\otimes m} \otimes \mathscr{O}_Z(A)$, is globally generated, too. Consider $\pi \colon \mathbb{P}(\mathscr{E}) \to Z$. Note that we have a surjective morphism

$$\pi^* \pi_* \mathscr{O}_{\mathbb{P}(\mathscr{E})}(m) \simeq \pi^*(\mathrm{Sym}^m(\mathscr{E})) \to \mathscr{O}_{\mathbb{P}(\mathscr{E})}(m),$$

and we thus deduce that $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(m) \otimes \pi^* \mathscr{O}_Z(A)$ is globally generated, hence nef, for every $m \geq 1$. This implies that $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$ is nef, that is, $\mathscr{E}$ is nef. $\square$

Chapter 3
Singularities of pairs

3.1 Pairs and log discrepancies

In this section we set up the framework for measuring the singularities of higher-dimensional algebraic varieties, and more generally, of pairs and triples. While the definitions can be given without any assumptions on the ground field, the main tool for understanding singularities in this context is provided by resolution of singularities. Furthermore, some of the deeper results rely on vanishing theorems. As a consequence, in this chapter we assume that we work over a field of characteristic zero.

3.1.1 The canonical divisor

Let X be an n-dimensional normal variety. Note that if U is an open subset of X such that $\mathrm{codim}_X(X \smallsetminus U) \geq 2$, then we have a group isomorphism

$$\mathrm{Div}(X) \to \mathrm{Div}(U),\ D \to D|_U,$$

which induces an isomorphism of class groups $\mathrm{Cl}(X) \simeq \mathrm{Cl}(U)$. If $i \colon U \hookrightarrow X$ is the inclusion, then for every divisor D on X we have an equality $i_* \mathscr{O}_U(D|_U) = \mathscr{O}_X(D)$ of subsheaves of the function field of X.

Suppose now that U is smooth (for example, U can be the smooth locus of X). It follows that there is a divisor K_X on X, called *canonical divisor*, such that $\mathscr{O}_U(K_X|_U) \simeq \omega_U = \Omega_U^n$. It is clear that K_X is uniquely defined up to linear equivalence and the definition is independent of the choice of U. Furthermore, if V is an arbitrary open subset of X, then $(K_X)|_V$ is a canonical divisor on V.

Lemma 3.1.1. *If $f \colon Y \to X$ is a proper, birational morphism of normal varieties, and K_Y is a canonical divisor on Y, then $f_*(K_Y)$ is a canonical divisor on X.*

Proof. If $U = X_{\mathrm{sm}} \smallsetminus f(Y \smallsetminus Y_{\mathrm{sm}})$, then $\mathrm{codim}_X(X \smallsetminus U) \geq 2$, hence it is enough to check the assertion on U. Therefore we may assume that both X and Y are smooth. In this case the assertion follows from the fact that if K_X is a canonical divisor on X, then there is an exceptional divisor E on Y such that $f^*(K_X) + E$ is a canonical divisor on Y (see Lemma B.2.3). If ϕ is a nonzero rational function such that $K_Y = \mathrm{div}_Y(\phi) + f^*(K_X) + E$, then $f_*(K_Y) = \mathrm{div}_X(\phi) + K_X$. □

In what follows, when considering a variety X, we will fix a canonical divisor K_X on X. This particular choice will not play any role. The important fact is that whenever considering another normal variety having a proper birational morphism $f\colon Y \to X$, we choose as canonical divisor K_Y on Y the unique one with the property $f_*(K_Y) = K_X$. Existence and uniqueness of such K_Y follows from Lemma 3.1.1 and the fact that for every nonzero rational function ϕ we have $f_*(\mathrm{div}_Y(\phi)) = \mathrm{div}_X(\phi)$ and $\mathrm{div}_X(\phi) = 0$ if and only if $\mathrm{div}_Y(\phi) = 0$ (both conditions being equivalent to $\phi \in \mathscr{O}_X(X)^* = \mathscr{O}_Y(Y)^*$).

Remark 3.1.2. If $f\colon Y \to X$ is a proper, birational morphism between normal varieties, then for every integer m we have an inclusion

$$f_*\mathscr{O}_Y(mK_Y) \hookrightarrow \mathscr{O}_X(mK_X)$$

of subsheaves of the function field. Indeed, if V is an open subset of X and ϕ is a nonzero rational function such that $\mathrm{div}_Y(\phi) + mK_Y$ is effective on $f^{-1}(V)$, then its push-forward $f_*(\mathrm{div}_Y(\phi) + mK_Y) = \mathrm{div}_X(\phi) + mK_X$ is effective on V.

In particular, by taking f a resolution of singularities of X and $m = 1$, we obtain an inclusion $\omega_X^{\mathrm{GR}} \hookrightarrow \mathscr{O}_X(K_X)$.

Remark 3.1.3. Let $f\colon Y \to X$ and $g\colon Z \to Y$ be proper birational morphisms between normal varieties. If K_X is a canonical divisor on X and K_Y and K_Z are canonical divisors on Y and Z, respectively, such that $f_*(K_Y) = K_X$ and $(f \circ g)_*(K_Z) = K_X$, then $g_*(K_Z) = K_Y$. This is clear from the uniqueness of K_Y and K_Z with these properties: if D is the unique canonical divisor on Z such that $g_*(D) = K_Y$, then $(f \circ g)_*(D) = f_*(K_Y) = K_X$, hence $D = K_Z$.

Remark 3.1.4. Suppose that X is a normal variety and H is a normal, irreducible, effective Cartier divisor on X. If D is a $\mathbb{Q}$-divisor on X that does not contain H in its support, then we can define the restriction $D|_H$ as a $\mathbb{Q}$-divisor, as follows. The smooth locus H_{sm} of H can be written as $U \cap H$ for some open subset U of X, and since H is a Cartier divisor, after possibly replacing U by a smaller subset, we may assume that $U \subseteq X_{\mathrm{sm}}$. Therefore $D|_U$ is $\mathbb{Q}$-Cartier, and we define $D|_H$ to be the unique $\mathbb{Q}$-divisor on H whose restriction to H_{sm} is equal to the restriction of $D|_U$ to H_{sm}.

If m is a positive integer such that $m(K_X + D)$ is Cartier, then

$$\mathscr{O}_H(mK_H + mD|_H) \simeq \mathscr{O}_X(mK_X + mD + mH)|_H. \tag{3.1}$$

In particular, $m(K_H + D|_H)$ is Cartier. In order to check (3.1), note that if $j\colon H_{\mathrm{sm}} \hookrightarrow H$ is the inclusion of the smooth locus, then

$$\mathscr{O}_H(mK_H + mD|_H) \simeq j_* j^* \mathscr{O}_H(mK_H + mD|_H), \text{ and}$$

$$\mathscr{O}_X(mK_X + mD + mH)|_H \simeq j_* j^* (\mathscr{O}_X(mK_X + mD + mH)|_H)$$

(the second equality follows since by assumption, $\mathscr{O}_X(m(K_X + D) + mH)|_H$ is a line bundle on H). Therefore, in order to check (3.1), we may assume that H is smooth, and after replacing X by an open neighborhood of H, also that X is smooth. In this case, the assertion follows from the adjunction isomorphism $\omega_H \simeq \omega_X \otimes \mathscr{O}_X(H)|_H$.

In particular, we see that if mK_X is Cartier, then

$$\mathscr{O}_H(mK_H) \simeq \mathscr{O}_X(mK_X + mH)|_H,$$

hence mK_H is Cartier.

Remark 3.1.5. Recall that every separated scheme X of finite type over k carries a *dualizing sheaf* ω_X°. We refer to [Har77, Chap. III.7] for the construction in the case when X is projective and to [Har66] for the general case. The construction and main properties in the case of an algebraic variety can also be found in [Kun08, Chap. 9]. If X can be embedded in a smooth variety Y and $\mathrm{codim}_Y(X) = c$, then $\omega_X^\circ \simeq \mathscr{E}xt^c_{\mathscr{O}_Y}(\mathscr{O}_X, \omega_Y)$. In particular, when X is smooth, ω_X° is the sheaf of top differential forms on X. If X is normal, ω_X° is a reflexive sheaf (see [Kun08, Cor. 9.8]), hence it is isomorphic to the push-forward of its restriction to X_{sm}. Therefore we have an isomorphism $\omega_X^\circ \simeq \mathscr{O}_X(K_X)$.

3.1.2 Divisors over X, revisited

The notion of divisor over X, introduced in Section 1.7.2, will play an important role in what follows. Recall that given an arbitrary variety X, a divisor E over X is given by a prime divisor E on a normal variety Y that has a birational morphism to X. The corresponding valuation on the function field $K(X)$ is ord_E, and we identify two such divisors if they give the same valuation.

Whenever considering Y and E as above, it is convenient to assume that Y is proper over X. This is no restriction, since we can always embed Y as an open subset of a normal variety Y' which is proper over X (this is a theorem due to Nagata and Deligne, see [Con07]), and we may replace E by its closure in Y'. Furthermore, given a proper birational morphism $Y' \to Y$, we may replace Y by Y' and E by its proper transform on Y'. It follows that by Chow's lemma, we may assume that Y is projective over X, and using a log resolution of (Y,E), that both Y and E are smooth. In the presence of some ideals or divisors on X, we may further assume that Y gives a log resolution of these ideals and divisors in the sense of Section B.3.

Suppose that $f_i\colon Y_i \to X$ are proper birational morphisms, for $i = 1,2$, with Y_1 and Y_2 normal. Note that there is a normal variety Y with proper birational morphisms $g_i\colon Y \to Y_i$ such that $f_1 \circ g_1 = f_2 \circ g_2$ (for example, we may take Y to be the normalization of the unique irreducible component of $Y_1 \times_X Y_2$ that dominates X). If E_1 and E_2 are prime divisors on Y_1 and Y_2, respectively, then they define the same divisor over X if and only if their proper transforms on Y are the same. Note that if this is the case, then the centers of E_1 and E_2 on X are the same.

If E is a divisor over X and $g\colon W \to X$ is a proper birational morphism, with W normal, then we may also consider E as a divisor over W. Indeed, if E is given as a prime divisor on some Y as above and we choose a normal variety W' over X, with proper birational morphisms (over X) to Y and W, then the valuation ord_E of $K(X) = K(W)$ corresponds to the proper transform of E on W'. In particular, we may consider the center of E on any such variety W.

3.1.3 Log discrepancy for pairs

In what follows we will consider two types of pairs, of which we now introduce the first one. A *log pair* (or simply *pair*, when there is no risk of confusion) (X,D) consists of a normal variety X and an $\mathbb{R}$-divisor D on X such that the divisor $K_X + D$ is an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor. Note that if D is a $\mathbb{Q}$-divisor, then $K_X + D$ is $\mathbb{R}$-Cartier if and only if it is $\mathbb{Q}$-Cartier. A pair (X,D) as above is *effective* if D is an effective $\mathbb{R}$-divisor and it is *rational* if D is a $\mathbb{Q}$-divisor. Here and in what follows K_X is a fixed canonical divisor on X. For every proper birational morphism $f\colon Y \to X$, with Y normal, we fix the canonical divisor K_Y on Y such that $f_*(K_Y) = K_X$, and define a divisor D_Y on Y by

$$K_Y + D_Y = f^*(K_X + D) \tag{3.2}$$

(note that the pull-back of $K_X + D$ is defined precisely because $K_X + D$ is $\mathbb{R}$-Cartier). By construction, (Y, D_Y) is a log pair, as well. The principle is that the singularities of the pairs (X,D) and (Y,D_Y) are (almost) equivalent.

We note that the definition of D_Y is independent of the choice of K_X. By applying f_* to (3.2), we also see that $f_*(D_Y) = D$. In other words, for every prime divisor T on X, the coefficient of T in D is equal to the coefficient of the proper transform $\widetilde{T}$ in D_Y.

It is clear from definition that if $f\colon Y \to X$ and (X,D) are as above, and if $g\colon Z \to Y$ is another proper birational morphism, with Z normal, then $(D_Y)_Z = D_Z$. In particular, we see that if E is a prime divisor on Y and $\widetilde{E}$ is its proper transform on Z, then the coefficient of E in D_Y is equal to the coefficient of $\widetilde{E}$ in D_Z. It follows that if we define the *log discrepancy* of the pair (X,D) with respect to E as

$$a_E(X,D) := 1 - (\text{the coefficient of } E \text{ in } D_Y),$$

then this invariant only depends on (X,D) and the divisor E over X, but not on Y. For example, if E is a prime divisor on X, then $a_E(X,D) = 1 - \alpha$, where α is the coefficient of E in D. We also note that if $f\colon Z \to X$ is any proper birational morphism, with Z normal, then

$$a_E(X,D) = a_E(Z,D_Z).$$

The divisor in a pair can be zero, in which case we simply write X instead of $(X,0)$. If this is the case, then K_X has to be $\mathbb{Q}$-Cartier (one says that X is $\mathbb{Q}$-*Gorenstein*) and one writes $K_{Y/X}$ for -0_Y; that is,

$$K_{Y/X} = K_Y - f^*(K_X),$$

where, again, we fix the canonical divisors so that $f_*(K_Y) = K_X$. If $f\colon Y \to X$ is a proper birational morphism and (X,D) is a pair such that X is $\mathbb{Q}$-Gorenstein, then $D_Y = f^*(D) - K_{Y/X}$, and therefore we have

$$K_{Y/X} - f^*(D) = \sum_E (a_E(X,D) - 1)E,$$

where the sum runs over all prime divisors on Y.

If both X and Y are smooth, then we have seen in the proof of Lemma 3.1.1 that $K_{Y/X}$ is the effective divisor defined by the morphism of line bundles $f^*(\omega_X) \to \omega_Y$ (hence our current definition is compatible with the one in Lemma B.2.3).

Remark 3.1.6. Note that when X is $\mathbb{Q}$-Gorenstein, the set $\{m \in \mathbb{Z} \mid mK_X \text{ is Cartier}\}$ is a subgroup of $\mathbb{Z}$. Its positive generator is the *index* of X. One says that X is r-Gorenstein if rK_X is Cartier. Note that even if X is 1-Gorenstein, X does not have to be Cohen-Macaulay, hence it might not be Gorenstein (see Example 3.1.7 below). On the other hand, if X is Cohen-Macaulay, then X is Gorenstein if and only if it is 1-Gorenstein (this follows from the fact that $\mathscr{O}_X(K_X)$ is the dualizing sheaf, see Remark 3.1.5).

Example 3.1.7. Let $Y \subset \mathbb{P}^n$ be a smooth projective variety of dimension $d \geq 1$, in a projectively normal embedding, and let $X \subset \mathbb{A}^{n+1}$ be the affine cone over Y. Note that since Y is Cohen-Macaulay, we have X Cohen-Macaulay if and only if $H^i(Y, \mathscr{O}_Y(m)) = 0$ for all m and all i with $1 \leq i \leq d-1$. On the other hand, we claim that X is r-Gorenstein if and only if there is j such that $\omega_Y^r \simeq \mathscr{O}_Y(j)$. Indeed, note that if $U = X \smallsetminus \{0\}$ and $\pi\colon U \to Y$ is the canonical projection, then π is smooth, hence $\omega_U \simeq \pi^*(\omega_Y) \otimes \Omega_{U/Y}$. On the other hand, $\Omega_{U/Y} \simeq \mathscr{O}_U$ (it is enough to check this when $Y = \mathbb{P}^n$ and use the fact that $\mathrm{Pic}(\mathbb{A}^{n+1} \smallsetminus \{0\}) = \{0\}$), hence $\omega_U \simeq \pi^*(\omega_Y)$. Therefore we obtain

$$H^0(X, \mathscr{O}_X(rK_X)) = H^0(U, \pi^*(\omega_Y^r)) \simeq \bigoplus_{m \in \mathbb{Z}} H^0(Y, \omega_Y^r \otimes \mathscr{O}_Y(m)). \tag{3.3}$$

Since $H^0(X, \mathscr{O}_X(rK_X))$ is a graded module over the homogeneous coordinate ring of Y, it is locally free if and only if it is free, and by (3.3), this is the case if and only if $\omega_Y^r \simeq \mathscr{O}_Y(j)$ for some j.

For example, if Y is an abelian variety of dimension $d \geq 2$ in a projectively normal embedding, we see that X is 1-Gorenstein, but it is not Cohen-Macaulay since $H^1(X, \mathscr{O}_X) \neq 0$.

Remark 3.1.8. If $g\colon Z \to Y$ and $f\colon Y \to X$ are proper birational morphisms of $\mathbb{Q}$-Gorenstein varieties, then $K_{Z/X} = g^*(K_{Y/X}) + K_{Z/Y}$. Indeed, note first that if K_Y and

K_Z are chosen on Y and Z, respectively, such that $f_*(K_Y) = K_X$ and $(f \circ g)_*(K_Z) = K_X$, then as observed in Remark 3.1.3, we have $g_*(K_Z) = K_Y$. If we pull-back by g the defining relation $K_Y = f^*(K_X) + K_{Y/X}$, we obtain

$$K_Z = g^*(K_Y) + K_{Z/Y} = (f \circ g)^*(K_X) + f^*(K_{Y/X}) + K_{Z/Y},$$

which implies the claimed equality by definition of $K_{Z/X}$.

Example 3.1.9. Let $f\colon Y \to X$ be a proper birational morphism of normal varieties and suppose that $H \subset X$ is a normal, irreducible, effective Cartier divisor, such that the proper transform $\widetilde{H}$ of H is a normal effective Cartier divisor on Y. Suppose also that D is a $\mathbb{Q}$-divisor on X whose support does not contain H, and such that $K_X + D$ is $\mathbb{Q}$-Cartier. It follows from Remark 3.1.4 that in this case $K_H + D|_H$ is $\mathbb{Q}$-Cartier. If we write $f^*(H) = \widetilde{H} + F$, then

$$(D|_H)_{\widetilde{H}} = (D_Y + F)|_{\widetilde{H}}. \tag{3.4}$$

In particular, if X is $\mathbb{Q}$-Gorenstein and we take $D = 0$, we obtain

$$K_{\widetilde{H}/H} = (K_{Y/X} - F)|_{\widetilde{H}}. \tag{3.5}$$

In order to check (3.4), note first that $\widetilde{H}$ does not appear in either F or D_Y (since F is f-exceptional and H does not appear in D). Therefore the right-hand side of (3.4) is well-defined. Suppose now that m is a positive integer such that $m(K_X + D)$ is Cartier. If follows from Remark 3.1.4 that

$$\mathscr{O}_H(mK_H + mD|_H) \simeq \mathscr{O}_X(mK_X + mD + mH)|_H \text{ and}$$

$$\mathscr{O}_{\widetilde{H}}(mK_{\widetilde{H}} + m(D_Y)|_{\widetilde{H}}) \simeq \mathscr{O}_Y(mK_Y + mD_Y + m\widetilde{H})|_{\widetilde{H}}.$$

We deduce that we have

$$K_{\widetilde{H}} + (D_Y + F)|_{\widetilde{H}} \sim_{\mathbb{Q}} g^*(K_H + D|_F),$$

where $g\colon \widetilde{H} \to H$ is the restriction of f. Therefore in order to prove (3.4), it is enough to show that the proper transform on $\widetilde{H}$ of every prime divisor on H has the same coefficient in $K_{\widetilde{H}} + (D_Y + F)|_{\widetilde{H}}$ and $g^*(K_H + D|_F)$. Since this is an assertion that can be checked in codimension 1 on H, we may assume that H and $\widetilde{H}$ are smooth, and after replacing X by an open neighborhood of H we may assume that also X and Y are smooth.

In this case it is enough to show that we have the equality (3.5), and we do this using the explicit description of $K_{Y/X}$ and $K_{\widetilde{H}/H}$ is terms of the Jacobians of the maps f and g. Suppose we have local coordinates $y_1, \ldots, y_n$ on Y at a point $P \in \widetilde{H}$ and $x_1, \ldots, x_n$ on X at $f(P)$, such that $\widetilde{H}$ is defined by (y_1) and H is defined by (x_1). If $f^*(x_i) = \phi_i$, then $K_{Y/X}$ is defined at P by $A = \det(\partial\phi_i/\partial y_j)_{1 \le i,j \le n}$. Furthermore, if $\psi_i = \phi_i|_{y_1=0}$, then $K_{\widetilde{H}/H}$ is defined at P by $B = \det(\partial\psi_i/\partial y_j)_{2 \le i,j \le n}$. On the other hand, we may write $\phi_1 = y_1 u$, where the ideal (u) defines F. Since $\partial\phi_1/\partial y_1|_{y_1=0} =$

$u|_{y_1=0}$ and $\partial\phi_1/\partial y_i|_{y_1=0} = 0$ for $2 \le i \le n$, it follows that $A|_{y_1=0} = B \cdot u|_{y_1=0}$, which gives the equality (3.5).

Example 3.1.10. Let (X_1, Δ_1) and (X_2, Δ_2) be two log pairs. We consider the product $X = X_1 \times X_2$, with the canonical projections $p_i \colon X \to X_i$, for $i = 1, 2$. Note that for every $\mathbb{R}$-divisor Γ_i on X_i, we may consider the pull-back $p_i^*(\Gamma_i) \in \mathrm{Div}(X)_{\mathbb{R}}$, and this is $\mathbb{R}$-Cartier if Γ_i is. In particular, it is easy to check using the definition that we may take

$$K_X = p_1^*(K_{X_1}) + p_2^*(K_{X_2}).$$

We therefore obtain a log pair (X, Δ), where $\Delta = p_1^*(\Delta_1) + p_2^*(\Delta_2)$. If $f_i \colon Y_i \to X_i$ is a log resolution of (X_i, Δ_i) for $i = 1, 2$, it is straightforward to check that $f = f_1 \times f_2 \colon Y = Y_1 \times Y_2 \to X$ is a log resolution of (X, Δ) and $\Delta_Y = q_1^*((\Delta_1)_{Y_1}) + q_2^*((\Delta_2)_{Y_2})$, where $q_i \colon Y \to Y_i$, for $i = 1, 2$, are the canonical projections.

3.1.4 Log canonical and klt singularities

We now introduce some important classes of singularities for birational geometry. We begin with two such classes that will play a prominent role in what follows, and leave for later the discussion of other classes that will feature less in the next chapters.

Definition 3.1.11. Let (X, D) be a log pair.

i) The pair (X, D) is *log canonical* if for every proper birational morphism $f \colon Y \to X$, with Y normal, all coefficients of D_Y are ≤ 1.
ii) The pair (X, D) is *Kawamata log canonical*[1] if for every proper birational morphism $f \colon Y \to X$, with Y normal, all coefficients of D_Y are < 1.

In terms of log discrepancies, we see that (X, D) is klt (log canonical) if and only if $a_E(X, D) > 0$ (respectively, $a_E(X, D) \ge 0$) for every divisor E over X. It is clear from definition that given a pair (X, D) and a proper birational morphism $f \colon Y \to X$ with Y normal, we have (X, D) log canonical (klt) if and only if (Y, D_Y) is log canonical (klt).

The conditions in Definition 3.1.11 involve *all* divisors over X. The key fact that makes them checkable is that they can be tested on the coefficients of D_Y on a log resolution Y. Recall that a log resolution of (X, D), with $D = \sum_{i=1}^r a_i D_i$, is a projective birational morphism $f \colon Y \to X$, with Y smooth, such that $\mathrm{ExcDiv}(f) + \sum_{i=1}^r \widetilde{D}_i$ has simple normal crossings, where the $\widetilde{D}_i$ are the proper transforms of the D_i on Y. For details about log resolutions, we refer to Section B.3.

Theorem 3.1.12. *If (X, D) is a log pair and $f \colon Y \to X$ is a log resolution of (X, D), then (X, D) is log canonical (klt) if and only if all coefficients of D_Y are ≤ 1 (respectively, < 1).*

[1] Following the literature, we will abbreviate this as *klt*.

An important special case of the theorem is the following: if X is a smooth variety and $D=\sum_{i=1}^r a_iD_i$ is a simple normal crossing divisor on X, then (X,D) is log canonical (klt) if and only if $a_i\leq 1$ (respectively, $a_i<1$) for $1\leq i\leq r$. The key for the proof of Theorem 3.1.12 is the following estimate for log discrepancies of simple normal crossing pairs, which we will use again later.

Lemma 3.1.13. *Let (X,D) be a log pair, $f\colon Y\to X$ a log resolution of (X,D), and $E=E_1+\ldots+E_r$ a simple normal crossing divisor on Y, containing all components of D_Y. If F is a divisor over Y and $E_1,\ldots,E_s$ are the components of E that contain $c_X(F)$, then*

$$a_F(X,D)\geq\sum_{i=1}^s \mathrm{ord}_F(E_i)\cdot a_{E_i}(X,D)+(\mathrm{codim}_Y(c_Y(F))-s), \tag{3.6}$$

with equality if F is the exceptional divisor on the blow-up of Y along a connected component of $\cap_{i=1}^s E_i$.

Proof. Suppose first that $\mathrm{codim}_Y(c_Y(F))=s$. We consider a proper birational morphism $g\colon Z\to Y$, with Z smooth, such that F is a smooth prime divisor on Z. If we write $D_Y=\sum_{i=1}^r \alpha_iE_i$, then

$$a_F(X,D)=a_F(Y,D_Y)=1+\mathrm{ord}_F(K_{Z/Y})-\sum_{i=1}^r\alpha_i\cdot\mathrm{ord}_F(E_i).$$

Since $a_{E_i}(X,D)=a_{E_i}(Y,D_Y)=1-\alpha_i$, the inequality (3.6) is equivalent to

$$\mathrm{ord}_F(K_{Z/Y})\geq -1+\sum_{i=1}^s\mathrm{ord}_F(E_i). \tag{3.7}$$

If F is the exceptional divisor on the blow-up of X along a connected component of $\cap_{i=1}^s E_i$, it follows from Example B.2.4 that we have equality in (3.7), hence in (3.6) (note that in this case $\mathrm{ord}_F(E_i)=1$ for $1\leq i\leq s$).

We choose coordinates $y_1,\ldots,y_n$ in an affine open neighborhood U of the generic point of $c_Y(F)$, such that E_i is defined in U by (y_i) for $1\leq i\leq s$. We also choose coordinates $z_1,\ldots,z_n$ in some affine open subset V in Z that meets F, such that F is defined in V by (z_1). If $b_i=\mathrm{ord}_F(E_i)$ for $1\leq i\leq s$, it follows that for every such i we can write $f^*(y_i)=z_1^{b_i}h_i$ for some $h_i\in\mathscr{O}_Z(V)$. Therefore $f^*(dy_i)=b_iz_1^{b_i-1}h_idz_i+z_1^{b_i}dh_i$. It is then clear that

$$f^*(dy_1\wedge\ldots\wedge dy_n)=z_1^{-1+\sum_{i=1}^s b_i}\eta\quad\text{for some}\quad \eta\in H^0(V,\omega_Z).$$

It follows from the definition of $K_{Z/Y}$ that $\mathrm{ord}_F(K_{Z/Y})\geq -1+\sum_{i=1}^s b_i$.

Suppose now that $c:=\mathrm{codim}_Y(c_Y(F))>s$ (note that $c\geq s$, since E has simple normal crossings). After possibly replacing Y by an affine open subset meeting $c_Y(F)$, we may choose divisors $E_{r+1},\ldots,E_{r+c-s}$ containing $c_Y(F)$ and such that $E'=\sum_{i=1}^{r+c-s}E_i$ has simple normal crossings. Applying what we have already proved to E', and noting that

$$\mathrm{ord}_F(E_i) \geq 1 \quad \text{and} \quad a_{E_i}(Y, D_Y) = 1 \quad \text{for} \quad r+1 \leq i \leq c-s,$$

we obtain (3.6). This completes the proof of the lemma. ☐

Proof of Theorem 3.1.12. The "only if" part follows from definition. For the converse, suppose that F is a divisor over Y and let us write $D_Y = \sum_{i=1}^r a_i E_i$. It follows from Lemma 3.1.13 that

$$a_F(X,D) \geq \sum_{i=1}^{r} (1-a_i) \cdot \mathrm{ord}_F(E_i), \tag{3.8}$$

with the inequality being strict if $\mathrm{ord}_F(E_i) = 0$ for all i. It is then clear that if $a_i \leq 1$ for all i, then $a_F(X,D) \geq 0$, and that if $a_i < 1$ for all i, then $a_F(X,D) > 0$. ☐

For future reference, we record the following consequence of Lemma 3.1.13.

Corollary 3.1.14. *If Y is a smooth variety and F is a divisor over Y, with center Z, then $a_F(Y) \geq \mathrm{codim}_Y(Z)$.*

Proof. Let $r = \mathrm{codim}_Y(Z)$. After possibly replacing Y by an open subset intersecting Z, we may assume that Z is smooth and that we have a simple normal crossing divisor $E = E_1 + \ldots + E_r$ such that $Z = E_1 \cap \ldots \cap E_r$. By applying Lemma 3.1.13 with $X = Y$ and $D = 0$, we obtain the formula in the corollary. ☐

Remark 3.1.15. It is easy to see that the requirement for a log pair (X,D) to satisfy $a_E(X,D) \geq 0$ for all E is the weakest of its kind. More precisely, if $\dim(X) \geq 2$ and there is a divisor E over X such that $a_E(X,D) < 0$, then there is a sequence $(E_m)_{m\geq 1}$ of divisors over X with $\lim_{m\to\infty} a_{E_m}(X,D) = -\infty$. Indeed, let $f\colon Y \to X$ be a log resolution of (X,D), such that E appears as a smooth prime divisor on Y. Since $\dim(Y) \geq 2$, we may choose (after possibly restricting to an open subset) another smooth divisor F on Y that has simple normal crossings with D_Y and such that $E \cap F$ is nonempty, smooth, and connected. Let Y_1 be the blow-up of Y along $E \cap F$, with exceptional divisor F_1, and let E_1 be the proper transform of E on Y_1. We repeat this: given E_m and F_m on Y_m, we let Y_{m+1} be the blow-up of Y_m along $E_m \cap F_m$, with exceptional divisor F_{m+1}, and let E_{m+1} be the proper transform of E_m on Y_{m+1}. It follows from Lemma 3.1.13 that

$$a_{F_m}(X,D) = a_{F_{m-1}}(X,D) + a_E(X,D),$$

and it follows by induction on m that $a_{F_m}(X,D) = m \cdot a_E(X,D) + a_F(X,D)$ for all m. Therefore $\lim_{m\to\infty} a_{E_m}(X,D) = -\infty$.

We now give three examples. In each of these examples, we consider a divisor D in a smooth variety X and want to determine for what q the pair (X, qD) is log canonical or klt.

Example 3.1.16. Suppose that $D = V(f)$, where $f \in k[x_1,\ldots,x_n]$ is a homogeneous polynomial of degree d that has an isolated singularity at 0. We consider the pair $(\mathbb{A}^n, qD)$. If $f\colon Y \to \mathbb{A}^n$ is the blow-up of the origin, with exceptional divisor E, and if $\widetilde{D}$ is the proper transform of D, then the intersection $\widetilde{D} \cap E \subset E \simeq \mathbb{P}^{n-1}$ is identified to the projective hypersurface $H \subset \mathbb{P}^{n-1}$ defined by f. By assumption, this is smooth, hence $\widetilde{D}$ is smooth and intersects E transversely. Therefore f gives a log resolution of $(\mathbb{A}^n, qD)$ and

$$(qD)_Y = qf^*(D) - K_{Y/\mathbb{A}^n} = q\widetilde{D} + (qd - n + 1)E.$$

We conclude that $(\mathbb{A}^n, qD)$ is log canonical (klt) if and only if $q \le \min\{1, n/d\}$ (respectively, $q < \min\{1, n/d\}$).

Example 3.1.17. Let X be a smooth surface and $C \subset X$ an irreducible curve that has a unique singular point P, which is a node, that is, the tangent cone at P consists of two distinct lines, each with multiplicity 1. The blow-up $f\colon Y \to X$ of X at P, with exceptional divisor E, gives a log resolution of (X, qC) and $(qC)_Y = qf^*(C) - K_{Y/X} = q\widetilde{C} + (2q-1)E$, where $\widetilde{C}$ is the proper transform of C. It follows that (X, qC) is log canonical (klt) if and only if $q \le 1$ (respectively, $q < 1$).

Example 3.1.18. Let $X = \mathbb{A}^2 = \mathrm{Spec}(k[x,y])$ and $D = V(f)$, where $f = x^2 + y^3$. With a slight abuse of notation, we use the same letter to denote both a divisor and its proper transform on various blow-ups, making sure we always specify which variety we consider. Let $f_1\colon X_1 \to X$ be the blow-up at the origin, with exceptional divisor E_1. Both curves D and E_1 on X_1 are smooth, but they do not intersect transversely: in the chart with coordinates $x_1 = x/y$ and $y_1 = y$, the curves D and E_1 are defined, respectively, by $(y_1 + x_1^2)$ and (y_1), respectively. Let $f_2\colon X_2 \to X_1$ be the blow-up of X_1 at the unique intersection point of D and E_1, with exceptional divisor E_2. On X_2 we have three smooth curves D, E_1, E_2, all intersecting in one point. We need to blow-up one more time: if $f_3\colon X_3 \to X_2$ is the blow-up of the intersection point, with exceptional divisor E_3, then on X_3 the divisor $D + E_1 + E_2 + E_3$ has simple normal crossings. Therefore $f = f_1 \circ f_2 \circ f_3$ is a log resolution of (X, D). An easy computation gives

$$f^*(D) = D + 2E_1 + 3E_2 + 6E_3 \quad \text{and} \quad K_{X_3/X} = E_1 + 2E_2 + 4E_3,$$

where the second formula follows by a repeated application of Remark 3.1.8. It is straightforward to deduce that $(\mathbb{A}^2, qD)$ is klt (log canonical) if and only if $q < 5/6$ (respectively, $q \le 5/6$).

The following example assumes familiarity with toric varieties. We refer to [Ful93] for the basic facts and notation concerning toric varieties.

Example 3.1.19. Suppose that $X = X(\Delta)$ is a toric variety corresponding to the fan Δ. Recall that X is normal by definition. Let $D_1,\ldots,D_d$ be the prime toric divisors, corresponding to the 1-dimensional cones in Δ. If X is smooth, then $\omega_X \simeq \mathscr{O}_X(-D_1 - \ldots - D_d)$. It follows that for every toric variety X, a canonical divisor is given by $K_X = -D_1 - \ldots - D_d$.

Consider a toric $\mathbb{R}$-divisor $D = a_1D_1 + \ldots + a_dD_d$. The $\mathbb{R}$-divisor $K_X + D$ is $\mathbb{R}$-Cartier if and only if there is a piecewise linear function α_D on the support $|\Delta|$ of the fan such that $\alpha_D(v_i) = 1 - a_i$ for all i, where v_i is the primitive generator of the ray corresponding to D_i. Suppose now that this is the case. It is known that there is a toric resolution of singularities $f\colon Y \to X$, corresponding to a fan Δ_Y refining Δ. Note that the sum of the prime toric divisors on Y has simple normal crossings. Therefore f gives a log resolution of (X,D). It then follows from Theorem 3.1.12 that in order to check whether (X,D) is klt or log canonical it is enough to consider log discrepancies with respect to prime toric divisors on such varieties Y. Each such divisor corresponds to a primitive nonzero lattice element in $|\Delta|$, and if E_v is the divisor corresponding to v, then it follows from definition that $a_{E_v}(X,D) = \alpha_D(v)$. Therefore the pair (X,D) is klt (log canonical) if and only if $\alpha_D \geq 0$ (respectively, $\alpha_D > 0$) on $|\Delta| \smallsetminus \{0\}$. Since α_D is linear on each cone in Δ, it is enough to check this condition on the primitive ray generators. We conclude that (X,D) is klt (log canonical) if and only if $a_i < 1$ for all i (respectively, $a_i \leq 1$ for all i). In other words, the behavior is as if (X,D) had simple normal crossings.

Example 3.1.20. Given two pairs (X_1,Δ_1) and (X_2,Δ_2), we consider the pair (X,Δ), with $X = X_1 \times X_2$, as in Example 3.1.10. It follows from that example that (X,Δ) is log canonical or klt if and only if both (X_1,Δ_1) and (X_2,Δ_2) are log canonical or klt, respectively.

3.1.5 Log discrepancy for triples

One reason for considering log pairs (X,D), as opposed to just normal varieties, is that the divisor K_X might not be $\mathbb{Q}$-Cartier, hence its pull-back might not be defined. On the other hand, even when working on a $\mathbb{Q}$-Gorenstein variety, it turns out that the classes of singularities defined in the previous subsection have intrinsic interest for understanding singularities of divisors. With this in mind, once we allow divisors, it is natural and often useful to also allow subschemes of higher codimension. We now introduce the most general objects we will be concerned with.

Definition 3.1.21. For a variety X, we will consider the $\mathbb{R}$-vector space with basis the proper closed subschemes of X. Suppose that $\mathscr{Z} = \sum_{i=1}^r q_iZ_i$ is an element of this vector space. The *support* $\mathrm{Supp}(\mathscr{Z})$ of $\mathscr{Z}$ is the closed subset of X given by the union of the Z_i for which q_i is nonzero. Furthermore, $\mathscr{Z}$ is *effective* if all q_i are nonnegative. If $f\colon W \to X$ is a morphism of varieties whose image is not contained in any of the Z_i, then we define $f^{-1}(\mathscr{Z}) := \sum_{i=1}^r f^{-1}(\mathscr{Z}_i)$.

Definition 3.1.22. A *log triple* $(X,D,\mathscr{Z})$ consists of a normal variety X, an $\mathbb{R}$-divisor D on X such that $K_X + D$ is $\mathbb{R}$-Cartier, and an element $\mathscr{Z} = \sum_{i=1}^r q_iZ_i$ of the $\mathbb{R}$-vector space with basis the proper closed subschemes of X. The triple is *effective* if both D and $\mathscr{Z}$ are effective and it is *rational* if the coefficients of both D and $\mathscr{Z}$ are in $\mathbb{Q}$. If $\mathfrak{a}_i$ is the ideal of Z_i, we sometimes write the above triple as

$(X,D,\mathfrak{a}_1^{q_1}\ldots\mathfrak{a}_r^{q_r})$. Note that for every such log triple $(X,\Delta,\mathscr{Z})$, we have a log pair (X,D) to which we may apply our previous considerations. A triple as above with $\mathscr{Z}=0$ is just a log pair, that we write as before as (X,D). We write a triple for which $D=0$ as $(X,\mathscr{Z})$, and we call it a *higher-codimension pair*. Note that in this case X has to be $\mathbb{Q}$-Gorenstein.

If $(X,D,\mathscr{Z})$ is a log triple and E is a divisor over X, then the *log discrepancy* of $(X,D,\mathscr{Z})$ with respect to E is

$$a_E(X,D,\mathscr{Z}) := a_E(X,D) - \mathrm{ord}_E(\mathscr{Z}),$$

where if $\mathscr{Z}=\sum_{i=1}^r q_iZ_i$, we put $\mathrm{ord}_E(\mathscr{Z})=\sum_{i=1}^r q_i\cdot\mathrm{ord}_E(Z_i)$.

If $(X,D,\mathscr{Z})$ is a log triple and $f\colon Y\to X$ is a proper birational morphism, with Y normal, then we obtain a log triple $(Y,D_Y,f^{-1}(\mathscr{Z}))$, where if $\mathscr{Z}=\sum_{i=1}^r q_iZ_i$, we put $f^{-1}(\mathscr{Z}):=\sum_{i=1}^r q_if^{-1}(Z_i)$. It is clear that for every divisor E over X we have $a_E(X,D,\mathscr{Z})=a_E(Y,D_Y,f^{-1}(\mathscr{Z}))$.

In what follows, we put an equivalence relation on the set of triples on a fixed variety, by identifying $(X,D,\mathscr{Z})$ and $(X,D',\mathscr{Z}')$ if $a_E(X,D,\mathscr{Z})=a_E(X,D',\mathscr{Z}')$ for all divisors E over X. We do not delve on this equivalence relation, but only mention a few points:

1) Given a log triple $(X,D,\mathscr{Z})$, with $\mathscr{Z}=q_1Z_1+\ldots+q_rZ_r$, if Z_1 is defined by the ideal I_{Z_1} and the closed subscheme Z_1' is defined by $I_{Z_1}^m$, for a positive integer m, then we identify $(X,D,\mathscr{Z})$ and $(X,D,\mathscr{Z}')$, where $\mathscr{Z}'=\frac{q_1}{m}Z_1'+q_2Z_2+\cdots+q_rZ_r$.
2) If $(X,D,\mathscr{Z})$ is a log triple with $\mathscr{Z}=\sum_{i=1}^r q_iZ_i$ and $q_1=q_2$, then we identify $(X,D,\mathscr{Z})$ with $(X,D,\mathscr{Z}')$, where $\mathscr{Z}'=q_1Z_1'+q_3Z_3+\ldots+q_rZ_r$, where Z_1' is defined by the product of the ideals defining Z_1 and Z_2.
3) If $(X,D,\mathscr{Z})$ is a log triple with $\mathscr{Z}=\sum_{i=1}^r q_iZ_i$ such that each Z_i is an effective Cartier divisor, then we identify this triple with the log pair $(X,D+\mathscr{Z})$.
4) If $(X,D,\mathscr{Z})$ is a log triple such that we can write $D=\sum_{i=1}^s a_iD_i$, for effective Cartier divisors D_i and $a_i\in\mathbb{R}$, then we identify $(X,D,\mathscr{Z})$ with the higher-codimension pair $(X,D+\mathscr{Z})$.

Remark 3.1.23. By using the above identifications, we see that if X is $\mathbb{Q}$-Gorenstein, then every effective rational triple $(X,D,\mathscr{Z})$ can be identified to a pair $(X,q\cdot Z)$, where Z is a closed subscheme of X and q is a nonnegative rational number.

Definition 3.1.24. As in the case of log pairs, we say that a log triple $(X,D,\mathscr{Z})$ is *log canonical* (*klt*) if $a_E(X,D,\mathscr{Z})\geq 0$ (respectively, $a_E(X,D,\mathscr{Z})>0$) for every divisor E over X.

It follows from definition that if $(X,D,\mathscr{Z})$ is a log triple and $f\colon Y\to X$ is a proper birational morphism, with Y normal, then $(X,D,\mathscr{Z})$ is log canonical or klt if and only if $(Y,D_Y,f^{-1}(\mathscr{Z}))$ has the same property. In particular, if $\mathscr{Z}=\sum_{i=1}^r q_iZ_i$ and f factors through the blow-up of X along Z_i for every i, then each $f^{-1}(Z_i)$ is an effective Cartier divisor, hence we may consider $f^{-1}(\mathscr{Z})$ as an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor. Therefore we identify the pair $(Y,D_Y,f^{-1}(\mathscr{Z}))$ with the log pair $(Y,D_Y+f^{-1}(\mathscr{Z}))$.

This allows us to reduce many of the formal aspects concerning triples to the case of log pairs.

A *log resolution* of a log triple $(X,D,\mathscr{Z})$ is a projective, birational morphism $f\colon Y \to X$, with Y smooth, and which satisfies the following conditions:

i) If $\mathscr{Z} = \sum_{i=1}^{r} q_i Z_i$, then each $f^{-1}(Z_i)$ is an effective divisor.
ii) If $D = \sum_{j=1}^{s} a_j D_j$, and $\widetilde{D_j}$ is the proper transform of D_j on Y, then the divisor $\mathrm{ExcDiv}(f) + \sum_{i=m}^{r} f^{-1}(Z_i) + \sum_{j=1}^{s} \widetilde{D_j}$ has simple normal crossings.

It follows from Remark B.3.11 that log resolutions for log triples exist. Note that if $f\colon Y \to X$ is a log resolution of $(X,D,\mathscr{Z})$, then $D_Y + f^{-1}(\mathscr{Z})$ is a simple normal crossings divisor. Theorem 3.1.12 implies that one can check whether a triple $(X,D,\mathscr{Z})$ is log canonical or klt using a log resolution.

Corollary 3.1.25. *If $f\colon Y \to X$ is a log resolution of the log triple $(X,D,\mathscr{Z})$ and we write $D_Y + f^{-1}(\mathscr{Z}) = \sum_{i=1}^{r} \alpha_i E_i$, then (X,D) is log canonical (klt) if and only if all $\alpha_i \leq 1$ (respectively, $\alpha_i < 1$) for all i.*

Proof. The triple $(X,D,\mathscr{Z})$ is log canonical or klt if and only if the triple $(Y, D_Y + f^{-1}(\mathscr{Z}))$ has the same property. Therefore the assertion in the corollary follows from Theorem 3.1.12. $\square$

3.1.6 Plt, canonical, and terminal pairs

We now introduce a few other classes of singularities that have traditionally played an important role in the minimal model program. They are defined in terms of log discrepancies for exceptional divisors over X. In order to give a uniform definition, it is convenient to introduce the *exceptional log discrepancy* of a triple $(X,D,\mathscr{Z})$, defined by

$$\mathrm{LogDiscrep}(X,D,\mathscr{Z}) = \inf\{a_E(X,D,\mathscr{Z}) \mid E \text{ exceptional divisor over } X\}.$$

Definition 3.1.26. Let $(X,D,\mathscr{Z})$ be a log triple.

i) $(X,D,\mathscr{Z})$ is *purely log terminal*[2] if $\mathrm{LogDiscrep}(X,D,\mathscr{Z}) > 0$.
ii) $(X,D,\mathscr{Z})$ is *canonical* if $\mathrm{LogDiscrep}(X,D,\mathscr{Z}) \geq 1$.
iii) $(X,D,\mathscr{Z})$ is *terminal* if $\mathrm{LogDiscrep}(X,D,\mathscr{Z}) > 1$.

Note that if X is 1-Gorenstein and D and $\mathscr{Z}$ have integer coefficients, then $(X,D,\mathscr{Z})$ is plt if and only if it is canonical. We also note that if $\dim X = 1$, then there are no exceptional divisors over X, hence the above conditions are vacuous.

Remark 3.1.27. It follows from Remark 3.1.15 that if $\dim X \geq 2$ and there is a divisor E over X (exceptional or not) such that $a_E(X,D,\mathscr{Z}) < 0$, then there is a sequence of

[2] This is abbreviated as *plt*.

exceptional divisors $(E_m)_{m\geq 1}$ over X with $\lim_{m\to\infty} a_{E_m}(X,D,\mathscr{Z}) = -\infty$. Therefore $\mathrm{LogDiscrep}(X,D,\mathscr{Z}) = -\infty$. In particular, this implies that if $(X,D,\mathscr{Z})$ is plt, then $(X,D,\mathscr{Z})$ is log canonical.

Remark 3.1.28. Note that unlike in the case of klt triples, a triple can be plt and have a divisor E with $a_E(X,D,\mathscr{Z}) = 0$. In this case, E has to be a prime divisor on X. Furthermore, if E_1 and E_2 are two such divisors, and for example E_1 is normal and Cartier, then $E_1 \cap E_2 = \emptyset$. Indeed, otherwise $E_1 \cap E_2$ has codimension 2 in X, and each generic point of $E_1 \cap E_2$ lies in the smooth locus of E_1, hence also in the smooth loci of X and E_2 (since E_1 is normal). After restricting to a suitable open subset, we may assume that X is smooth and E_1, E_2 are smooth, and meeting transversely. If E is the exceptional divisor on the blow-up of a connected component of $E_1 \cap E_2$, then $a_E(X,\Delta,\mathscr{Z}) = a_{E_1}(X,\Delta,\mathscr{Z}) + a_{E_2}(X,\Delta,\mathscr{Z}) = 0$, hence $(X,\Delta,\mathscr{Z})$ cannot be plt.

Remark 3.1.29. We have the following implications between the classes of singularities that we introduced so far:

$$\text{terminal} \Rightarrow \text{canonical} \Rightarrow \text{plt} \Rightarrow \text{log canonical},$$

where for the last implication we need $\dim X \geq 2$.

We show that as in the case of log canonical and klt singularities, one can check whether a triple $(X,D,\mathscr{Z})$ is plt, canonical, or terminal just by checking a log resolution. More generally, $\mathrm{LogDiscrep}(X,D,\mathscr{Z})$ can be computed on a log resolution of $(X,D,\mathscr{Z})$.

Theorem 3.1.30. *If $f\colon Y \to X$ is a log resolution of the log triple $(X,D,\mathscr{Z})$, with $\dim X \geq 2$, and we write $D_Y + f^{-1}(\mathscr{Z}) = \sum_{i=1}^r \alpha_i E_i$ (where we assume that all f-exceptional divisors on Y appear amongst the E_i), then the following hold:*

i) $\mathrm{LogDiscrep}(X,D,\mathscr{Z}) \geq 0$ *if and only if* $\mathrm{LogDiscrep}(X,D,\mathscr{Z}) \neq -\infty$*, which is the case if and only if $\alpha_i \leq 1$ for all i.*

ii) If $\alpha_i \leq 1$ for all i, then

$$\mathrm{LogDiscrep}(X,D,\mathscr{Z}) = \min\{2, \min_{i\in I}(1-\alpha_i), \min_{i\notin I}(2-\alpha_i), \min_{(i,j)\in J}(2-\alpha_i-\alpha_j)\}, \tag{3.9}$$

where I is the set of those i such that E_i is f-exceptional and J is the set of those pairs (i,j) with $i \neq j$ and $E_i \cap E_j \neq \emptyset$.

Proof. It follows from Corollary 3.1.25 that $(X,D,\mathscr{Z})$ is log canonical if and only if $\alpha_i \leq 1$ for all i. It is clear that if $(X,D,\mathscr{Z})$ is log canonical, then we have $\mathrm{LogDiscrep}(X,D,\mathscr{Z}) \geq 0$. On the other hand, we have seen in Remark 3.1.27 that if $(X,D,\mathscr{Z})$ is not log canonical, then $\mathrm{LogDiscrep}(X,D,\mathscr{Z}) = -\infty$. This proves the assertion in i).

Suppose now that $\alpha_i \leq 1$ for all i, and let $\tau = \mathrm{LogDiscrep}(X,D,\mathscr{Z})$ and τ' be the right-hand side of (3.9). If F is the exceptional divisor of the blow-up along a codimension 2 smooth subvariety not contained in either of the E_i, then

$a_E(X,D,\mathscr{Z})=2$, hence $\tau \leq 2$. It is clear from definition that $\tau \leq 1-\alpha_i$ for all $i \in I$. Furthermore, given any i, if T is the exceptional divisor of the blow-up along a smooth, codimension 1 subvariety of E_i not contained in any other E_j, then $a_T(X,D,\mathscr{Z}) = 2-\alpha_i$. Suppose now that E_i and E_j are two distinct divisors that intersect and F is the exceptional divisor on the blow-up along a connected component of $E_i \cap E_j$. It follows from Lemma 3.1.13 that $a_F(X,D,\mathscr{Z}) = 2-\alpha_i-\alpha_j$, hence $\tau \leq 2-\alpha_i-\alpha$. By putting all these together we have $\tau \leq \tau'$.

In order to prove the reverse inequality, let G be an arbitrary exceptional divisor over X, and suppose that $c_Y(G)$ is contained in s of the E_i. It follows from Lemma 3.1.13 that

$$a_G(X,D,\mathscr{Z}) \geq \sum_{i=1}^{r} \mathrm{ord}_F(E_i)\cdot(1-\alpha_i)+(\mathrm{codim}_Y(c_Y(G))-s). \tag{3.10}$$

If $c_Y(F) \subseteq E_i$ for some f-exceptional E_i, then (3.10) implies $a_G(X,D,\mathscr{Z}) \geq 1-\alpha_i \geq \tau'$. Suppose now that this is not the case. After possibly reordering the E_i, we may assume that $c_Y(G) \subseteq E_i$ if and only if $1 \leq i \leq s$. If $s \geq 2$, then (3.10) implies

$$a_G(X,D,\mathscr{Z}) \geq \sum_{i=1}^{s}(1-\alpha_i) \geq 2-\alpha_i-\alpha_j \geq \tau'.$$

If $s=1$ and $\mathrm{codim}_Y(c_Y(G)) \geq 2$, then (3.10) gives $a_G(X,D,\mathscr{Z}) \geq 2-\alpha_i \geq \tau'$. Since G cannot be equal to one of the non-exceptional E_i, and all f-exceptional divisors on Y appear amongst the E_i, the only left case to consider is when $s=0$ and $\mathrm{codim}_Y(c_Y(G)) \geq 2$, when (3.10) implies $a_G(X,D,\mathscr{Z}) \geq 2$. Therefore $\tau \geq \tau'$, which completes the proof of the theorem. □

Remark 3.1.31. Given a triple $(X,D,\mathscr{Z})$, one can always find a log resolution $f\colon Y \to X$ of this triple such that no two proper transforms of distinct prime divisors that appear in D or in the support of the schemes in $\mathscr{Z}$ intersect on Y. Indeed, given any log resolution, we consider an intersection of such proper transforms that has smallest possible dimension and blow it up. Then either the smallest such dimension goes up, or it stays the same, but the number of subsets of proper transforms with a nonempty intersection of precisely this dimension goes down. After finitely many such steps, we achieve a log resolution with the desired property.

Given such a log resolution, it is worth spelling out the conditions for $(X,D,\mathscr{Z})$ to be plt, canonical, and terminal, as follow from Theorem 3.1.30. If $D_Y + f^{-1}(\mathscr{Z}) = \sum_{i=1}^{r} \alpha_i E_i$, then $(X,D,\mathscr{Z})$ is plt if and only if $\alpha_i \leq 1$ for all i, with strict inequality if E_i is exceptional. The pair $(X,D,\mathscr{Z})$ is canonical (terminal) if and only if $\alpha_i \leq 0$ ($\alpha_i < 0$) if E_i is exceptional and $\alpha_i \leq 1$ ($\alpha_i < 1$) if E_i is not exceptional.

An important case is that of a $\mathbb{Q}$-Gorenstein variety X. In this case, if $f\colon Y \to X$ is a log resolution, then X is canonical if and only if $K_{Y/X}$ is effective, and it is terminal if $K_{Y/X}$ is effective and its support is $\mathrm{ExcDiv}(f)$. Note that smooth varieties are terminal. More generally, if X has a *small resolution*, that is, a resolution of singularities such that the exceptional locus has codimension ≥ 2, then X has terminal singularities.

Example 3.1.32. Let $f \in k[x_1,\ldots,x_n]$, with $n \geq 3$, be a nonzero homogeneous polynomial of degree d, such that $H = V(f) \subset \mathbb{A}^n$ has an isolated singularity at 0. Since H is Cohen-Macaulay, being a hypersurface, and its singular locus has codimension ≥ 2, it follows that H is normal. If $f\colon Y \to \mathbb{A}^n$ is the blow-up at 0, with exceptional divisor E, we have seen in Example 3.1.16 that f is a log resolution of $(\mathbb{A}^n, H)$ and we have $K_{Y/\mathbb{A}^n} = (n-1)E$ and $f^*(H) = \widetilde{H} + dE$, where $\widetilde{H}$ is the proper transform of H. Therefore the induced map $g\colon \widetilde{H} \to H$ is a log resolution of H. It follows from Example 3.1.9 that if $E_1 = E|_H$, then $K_{\widetilde{H}/H} = (n-1-d)E_1$. Therefore H has terminal singularities if and only if $d \leq n-2$, canonical singularities if and only if $d \leq n-1$, and log canonical singularities if and only if $d \leq n$.

Example 3.1.33. We now consider the condition for a toric variety to have canonical or terminal singularities. For this, we rely on the discussion in Example 3.1.19. Suppose that $X = X(\Delta)$ is a $\mathbb{Q}$-Gorenstein toric variety and $\alpha\colon |\Delta| \to \mathbb{R}$ is the piecewise linear function such that $\alpha(v_i) = 1$ for every primitive ray generator v_i. Note that a prime toric divisor over X corresponding to the primitive lattice element v is exceptional if and only if v does not lie on any ray (or equivalently, $v \neq v_i$ for every i). Therefore X has canonical singularities if and only if for every cone $\sigma \in \Delta$, there are no lattice points in the relative interior of the simplex $\sigma_0 = \{v \in \sigma \mid \alpha(v) \leq 1\}$. Similarly, X has terminal singularities if and only if for every cone $\sigma \in \Delta$, the only lattice points in σ_0 are 0 and the v_i.

Example 3.1.34. We show that in dimension 2, terminal singularities are smooth and canonical singularities are rational double points. Suppose that X is a normal surface with canonical singularities. Since the singular locus is zero-dimensional, we may assume that $X_{\rm sing} = \{P\}$. Let $f\colon Y \to X$ be a resolution of singularities that is an isomorphism over $X \smallsetminus \{P\}$. After possibly contracting the (-1)-curves in the fiber over P, we may assume that f is minimal, that is, there are no curves $C \subseteq f^{-1}(P)$ with $C \simeq \mathbb{P}^1$ and $(C^2) = -1$. Let $C_1,\ldots,C_m$ be the curves in the fiber over P.

We claim that since $K_{Y/X}$ is effective, we must have $K_{Y/X} = 0$. By Corollary 1.6.36, since $K_{Y/X}$ is effective and f-exceptional, it is enough to show that $K_{Y/X}$ is f-nef, that is, $(K_Y \cdot C_i) \geq 0$ for all i. Note that by adjunction, we have $2p_a(C_i) - 2 = (K_Y \cdot C_i) + (C_i^2)$ and $(C_i^2) < 0$ by Proposition 1.6.35. Since $p_a(C_i) \geq 0$ and we cannot have both $p_a(C_i) = 0$ and $(C_i^2) = -1$, it follows that $(K_Y \cdot C_i) \geq 0$, which implies our assertion.

We conclude that X has canonical singularities if and only if $K_{Y/X} = 0$. In particular, X has terminal singularities if and only if $\dim f^{-1}(P) = 0$, which is the case if and only if X is smooth. Furthermore, the above computation shows that if X has canonical singularities, then $2p_a(C_i) - 2 = (C_i^2) < 0$. Therefore $p_a(C_i) = 0$, hence $C_i \simeq \mathbb{P}^1$, and $(C_i^2) = -2$ (one says that C_i is a (-2)-*curve*). It is also easy to see that $(C_i \cdot C_j)$ is either 0 or 1 if $i \neq j$. Indeed, recall that by Proposition 1.6.35 we have $((aC_i + bC_j)^2) < 0$ for all $(a,b) \in \mathbb{R}^2 \smallsetminus \{(0,0)\}$. Therefore $(C_i \cdot C_j)^2 < (C_i^2)(C_j^2) = 4$, which implies our assertion. Therefore any two of the C_i meet transversely. A similar argument shows that there are no three of the C_i meeting in a point: if i,j,k are pairwise distinct and C_i, C_j, C_k meet in a common point, then $((C_i + C_j + C_k)^2) = 0$, a contradiction. We conclude that $C_1 + \ldots + C_m$ is a simple normal crossings divisor. Arguing in the same way, one sees that the intersection graph[3] is a tree. One can show that unless X is smooth, such a resolution exists if and only if (X, P) is a *rational double point*, that is, a rational singularity (in the sense of Section 3.3), such that $\mathscr{O}_{X,P}$ is isomorphic to the local ring of a hypersurface of multiplicity 2. Furthermore, the possible intersection graphs are given by the Dynkin diagrams $(A_n)_{n\geq 1}$, $(D_n)_{n\geq 4}$, and (E_6), (E_7), and (E_8). Each such diagram corresponds to the case when the completion $\widehat{\mathscr{O}_{X,P}}$ is isomorphic to the completion of the ring of the corresponding rational double point:

(A_n) $\quad X = V(x^2 + y^2 + z^{n+1}) \subset \mathbb{A}^3$,
(D_n) $\quad X = V(x^2 + y^2 z + z^{n-1}) \subset \mathbb{A}^3$,
(E_6) $\quad X = V(x^2 + y^3 + z^4) \subset \mathbb{A}^3$,
(E_7) $\quad X = V(x^2 + y^3 + yz^3) \subset \mathbb{A}^3$,
(E_8) $\quad X = V(x^2 + y^3 + z^5) \subset \mathbb{A}^3$.

We refer to [Băd01, Chap. 3] for a discussion of rational double points on surfaces.

In practice, the notions of *canonical* and *terminal* singularities are used almost exclusively for varieties, rather than pairs or triples. Terminal singularities are important since these are the singularities of the minimal models that we now introduce (it has been realized early on that one cannot just consider smooth minmal models).

Definition 3.1.35. Let S be a fixed variety. A projective variety X over S is a *minimal model* if it is normal, has terminal singularities, and K_X is nef over S.

The relevance of this notion comes from the following application of the negativity lemma, showing that $\mathbb{Q}$-factorial minimal models are indeed minimal amongst birational models that are normal, $\mathbb{Q}$-factorial, and with terminal singularities.

Proposition 3.1.36. *If $f\colon X \to Y$ is a birational morphism of normal projective varieties over a variety S, with Y being terminal and $\mathbb{Q}$-factorial and K_X being nef over S, then f is an isomorphism.*

Proof. Since K_X is nef over S, it is in particular f-nef. We can write $K_X = f^*(K_Y) + K_{X/Y}$, hence $K_{X/Y}$ is f-nef, too. Since it is also effective and f-exceptional, it is 0 by Corollary 1.6.36. Furthermore, since every f-exceptional divisor has positive coefficient in $K_{X/Y}$, we deduce that $\mathrm{codim}_X(\mathrm{Exc}(f)) \geq 2$.

On the other hand, it follows from Lemma 2.2.4 that since Y is $\mathbb{Q}$-factorial, there is an effective f-exceptional divisor F on X such that $-F$ is f-ample. We have seen that there are no f-exceptional divisors, hence $F = 0$. This implies that f is finite, and since it is also birational and Y is normal, it follows that it is an isomorphism. $\square$

[3] This is the graph with vertices $1, 2, \ldots, m$, and such that i and j are joined by an edge if C_i and C_j intersect.

A similar argument shows that any two birational minimal models are isomorphic in codimension 1,

Proposition 3.1.37. *If $\phi\colon X \dashrightarrow Y$ is a birational map between two minimal models over a variety S, then ϕ is an isomorphism in codimension 1, that is, there are open subsets $U \subseteq X$ and $V \subseteq Y$, with $\mathrm{codim}_X(X \smallsetminus U) \geq 2$ and $\mathrm{codim}_Y(Y \smallsetminus V) \geq 2$, such that f induces an isomorphism $U \simeq V$.*

Proof. Let $X_0 \subseteq X$ and $Y_0 \subseteq Y$ be the largest open subsets on which ϕ and, respectively, ϕ^{-1} are defined. Since both X and Y are normal, we have $\mathrm{codim}_X(X \smallsetminus X_0) \geq 2$ and $\mathrm{codim}_Y(Y \smallsetminus Y_0) \geq 2$. If

$$U = \{x \in X_0 \mid \phi(x) \in Y_0\} \quad \text{and} \quad V = \{y \in Y_0 \mid \phi^{-1}(y) \in X_0\},$$

it is clear that ϕ induces an isomorphism $U \simeq V$. Therefore it is enough to prove that $\mathrm{codim}_X(X \smallsetminus U) \geq 2$ and $\mathrm{codim}_Y(Y \smallsetminus V) \geq 2$.

For this, it is enough to show that $\mathrm{codim}_Y(\phi(E \cap X_0)) = 1$ for every prime divisor E on X. Indeed, if this is the case, since $\mathrm{codim}_Y(Y \smallsetminus Y_0) \geq 2$, it follows that E intersects U. From the fact that this holds for all E, we deduce that $\mathrm{codim}_X(X \smallsetminus U) \geq 2$, and by symmetry $\mathrm{codim}_Y(Y \smallsetminus V) \geq 2$.

Consider a normal variety W with projective, birational morphisms $f\colon W \to X$ and $g\colon W \to Y$ such that $\phi = g \circ f^{-1}$ (for example, one can take W to be the normalization of the closure in $X \times Y$ of the graph of $\phi\colon X_0 \to Y$). Let $\widetilde{E}$ be the proper transform of E on W. If $\mathrm{codim}_Y(\phi(E \cap X_0)) \geq 2$, it follows that $\widetilde{E}$ is g-exceptional, and since Y has terminal singularities, $\widetilde{E}$ appears with a positive coefficient α_E in $K_{W/Y}$. Note that

$$K_{W/Y} - K_{W/X} \sim_{\mathbb{Q}} f^*(K_X) - g^*(K_Y)$$

is g-nef, since K_X being nef over S implies that $f^*(K_X)$ is nef over S, hence over Y. On the other hand, $K_{W/X}$ is effective, since X has terminal singularities. Therefore $g_*(K_{W/X} - K_{W/Y}) = g_*(K_{W/X})$ is effective, and we conclude from Corollary 1.6.36 that $K_{W/X} - K_{W/Y}$ is effective. This contradicts the fact that the coefficient of E in $K_{W/X} - K_{W/Y}$ is $-\alpha_E < 0$. Therefore $\mathrm{codim}_Y(\phi(E \cap X_0)) = 1$, which completes the proof of the theorem. ☐

A fundamental problem in birational geometry is the following

Conjecture 3.1.38 (Minimal model conjecture). Every smooth projective variety X such that $H^0(X, \omega_X^m) \neq 0$ for some positive integer m, is birational to a minimal model.

A recent breakthrough in birational geometry has been the proof due to Birkar, Cascini, Hacon and M^cKernan [BCHM10] of the above conjecture for varieties of general type (a smooth projective variety X is of *general type* if ω_X is a big line bundle).

Canonical singularities are relevant for several reasons. First, they are related to rational singularities (see Section 3.3). Second, they guarantee that the Grauert–Rimenschneider sheaf is canonically isomorphic to the dualizing sheaf. More precisely, we have the following characterization of canonical singularities.

Proposition 3.1.39. *If X is a normal, $\mathbb{Q}$-Gorenstein variety, then X has canonical singularities if and only if for every proper birational morphism $f\colon Y \to X$, with Y normal, the inclusion $f_*\mathscr{O}_Y(mK_Y) \hookrightarrow \mathscr{O}_X(mK_X)$ is an isomorphism for all positive integers m. Furthermore, if f is a log resolution of X, then it is enough to check the condition for this f and one value of m that is divisible by the index of X.*

Proof. Recall that by Remark 3.1.2, we always have an inclusion $f_*\mathscr{O}_Y(mK_Y) \hookrightarrow \mathscr{O}_X(mK_X)$ of subsheaves of the function field. Suppose first that X has canonical singularities, hence $K_{Y/X}$ is an effective f-exceptional $\mathbb{Q}$-divisor and we have $K_Y = f^*(K_X) + K_{Y/X}$. We need to show that if ϕ is a nonzero rational function such that $\mathrm{div}_X(\phi) + mK_X$ is effective on some open subset V of X, then $\mathrm{div}_Y(\phi) + mK_Y$ is effective on $f^{-1}(V)$. This follows from the fact that

$$\mathrm{div}_Y(\phi) + mK_Y = f^*(\mathrm{div}_X(\phi) + mK_X) + mK_{Y/X}$$

is the sum of two divisors, both of them effective on $f^{-1}(V)$.

Conversely, suppose that we have a log resolution $f\colon Y \to X$ of X and a positive integer m such that mK_X is Cartier and $\mathscr{O}_X(mK_X) = f_*\mathscr{O}_Y(mK_Y)$. Since $mK_Y = f^*(mK_X) + mK_{Y/X}$, it follows that $f_*\mathscr{O}_Y(mK_Y) = \mathscr{O}_X(mK_X) \cdot f_*\mathscr{O}_Y(mK_{Y/X})$. Therefore $f_*\mathscr{O}_Y(mK_{Y/X}) = \mathscr{O}_X$. Since 1 gives a section of $\mathscr{O}_X$, it follows that it also gives a section of $f_*\mathscr{O}_Y(mK_{Y/X})$, hence $mK_{Y/X}$ is effective. It follows from Theorem 3.1.30 that X has canonical singularities. □

Corollary 3.1.40. *If X is a variety with canonical singularities, then $\omega_X^{\mathrm{GR}} = \mathscr{O}_X(K_X)$.*

Another reason why canonical singularities are important is that they appear on *canonical models* of varieties of general type. A *canonically polarized variety Y* is a projective normal variety Y, with canonical singularities, such that K_Y is ample. One has the following result due to Reid [Rei87].

Theorem 3.1.41. *A smooth projective variety of general type X is birationally equivalent to a canonically polarized variety Y if and only if the canonical ring $R(X,\omega_X) := \oplus_{m\geq 0} H^0(X,\omega_X^m)$ is finitely generated. In this case $Y \simeq \mathrm{Proj}(R(X,\omega_X))$.*

A fundamental result of [BCHM10] is that indeed, the canonical ring $R(X,\omega_X)$ is finitely generated for every smooth projective variety X. When X is of general type, the variety $\mathrm{Proj}(R(X,\omega_X))$ is the *canonical model* of X.

3.2 Shokurov-Kollár connectedness theorem

Let $(X,D,\mathscr{Z})$ be a rational triple and $f\colon Y \to X$ a log resolution of this triple. We write as usual $K_Y + D_Y = f^*(K_X + D_X)$ with $K_X = f_*(K_Y)$. We can uniquely write

$$\lfloor D_Y + f^{-1}(\mathscr{Z}) \rfloor = A - B,$$

with A and B effective divisors, without common components. Note that the triple $(X,D,\mathscr{Z})$ is klt if and only if $A = 0$. In general, we introduce the following locus.

Definition 3.2.1. The *non-klt locus* of $(X,D,\mathscr{Z})$ is the set

$$\mathrm{Nklt}(X,D,\mathscr{Z}) := f(\mathrm{Supp}(A)) \subseteq X.$$

It follows from definition that $\mathrm{Nklt}(X,D,\mathscr{Z})$ is the smallest closed subset of X such that if $U = X \smallsetminus \mathrm{Nklt}(X,D,\mathscr{Z})$, then the triple $(U, D|_U, \mathscr{Z}|_U)$ is klt. This implies that $\mathrm{Nklt}(X,D,\mathscr{Z})$ does not depend on the choice of log resolution. We also note that one can equivalently describe the non-klt locus by

$$\mathrm{Nklt}(X,D,\mathscr{Z}) = \bigcup_{E;a_E(X,D,\mathscr{Z})<0} c_X(E),$$

where the union is over all divisors E over X with $a_E(X,D,\mathscr{Z}) < 0$.

The following important connectedness theorem was first discovered in dimension 2 by Shokurov [Sho92], and then established in all dimensions by Kollár [Kol92, Chapter 17].

Theorem 3.2.2. *With the above notation, if the triple $(X,D,\mathscr{Z})$ is effective, then all fibers of the induced map $\mathrm{Supp}(A) \to \mathrm{Nklt}(X,D,\mathscr{Z})$ are connected. In particular, A is connected in a neighborhood of any fiber of f.*

We will deduce this from the following more general version.

Theorem 3.2.3. *Let $g\colon X \to W$ be a projective surjective morphism, with X a normal variety, and F a $\mathbb{Q}$-divisor on X such that the following hold:*

i) $-(K_X+F)$ is $\mathbb{Q}$-Cartier, and it is g-big and g-nef.
ii) There is an effective Cartier divisor G on X such that $g_*\mathscr{O}_X(G) = \mathscr{O}_W$ and $F+G$ is effective.

In this case the induced map $\mathrm{Nklt}(X,F) \to W$ has connected fibers. In particular, $\mathrm{Nklt}(X,F)$ is connected in the neighborhood of any fiber of g.

Proof. Let $f\colon Y \to X$ be a log resolution of the pair (X,F) and $h = g \circ f$. We write as usual $K_Y + F_Y = f^*(K_X+F)$, hence by assumption we have that $-(K_Y+F_Y)$ is h-big and h-nef. Let us write $\lfloor F_Y \rfloor = A - B$, where A and B are effective divisors, with no common component. Since $F_Y - \lfloor F_Y \rfloor$ has simple normal crossings, it follows from the relative vanishing theorem (see Theorem 2.6.1) that $R^1h_*\mathscr{O}_Y(B-A) = 0$.

We consider the commutative diagram

$$\begin{tikzcd}
 & & \mathscr{O}_Y \arrow[r] \arrow[d] & \mathscr{O}_A \arrow[d] & \\
0 \arrow[r] & \mathscr{O}_Y(B-A) \arrow[r] & \mathscr{O}_Y(B) \arrow[r] & \mathscr{O}_A(B) \arrow[r] & 0.
\end{tikzcd}$$

Applying h_* and using the above vanishing, we see that the induced morphism $h_*\mathscr{O}_Y(B) \to h_*\mathscr{O}_A(B)$ is surjective.

On the other hand, since $F+G$ is effective, it follows that there is an effective f-exceptional divisor G' such that $B \leq f^*(G)+G'$, which gives using Lemma B.2.5 and the hypothesis on G

$$h_*\mathscr{O}_Y(B) \subseteq g_*(f_*\mathscr{O}_Y(f^*(G)+G')) = g_*\mathscr{O}_X(G) = \mathscr{O}_W.$$

Therefore the natural morphism $\mathscr{O}_W \to h_*\mathscr{O}_Y(B)$ is an isomorphism, hence the morphism $\phi\colon \mathscr{O}_W \to h_*\mathscr{O}_A(B)$ is surjective. Note that $\mathscr{O}_A(B)$ is a line bundle on A. If the fiber of $\mathrm{Supp}(A) \to W$ over some $w \in W$ is disconnected, then the theorem on formal functions (see [Har77, Theorem 11.1]) implies that the local ring $\widehat{\mathscr{O}_{W,w}}$ has a quotient that decomposes nontrivially as the direct sum of two modules. This is a contradiction, proving that the map $\mathrm{Supp}(A) \to W$ has connected fibers. In particular, the induced map $\mathrm{Nklt}(X,F) = f(\mathrm{Supp}(A)) \to W$ has connected fibers. □

Proof of Theorem 3.2.2. We use the notation introduced before the statement of Theorem 3.2.2. We apply Theorem 3.2.3 with $g = f$ and $F = D_Y + f^{-1}(\mathscr{Z})$. Since f is birational, every divisor on Y is f-big. Moreover, $-(K_Y+F) = -f^*(K_X + D) - f^{-1}(\mathscr{Z})$ is f-nef by Lemma 3.2.4 below. We have $\lfloor F \rfloor = A - B$ and by definition $\mathrm{Nklt}(Y,F) = \mathrm{Supp}(A)$. Since the coefficients of both D and $\mathscr{Z}$ are nonnegative, it follows that every prime divisor on Y that appears with negative coefficient in F is f-exceptional. Therefore we can find an effective f-exceptional divisor G such that $F+G$ is effective. Since f is birational, we have $f_*\mathscr{O}_Y(G) = \mathscr{O}_X$ by Lemma B.2.5. We can thus apply Theorem 3.2.3 to conclude that the map $\mathrm{Supp}(A) \to f(\mathrm{Supp}(A)) = \mathrm{Nklt}(X,D,\mathscr{Z})$ has connected fibers. □

Lemma 3.2.4. *If $f\colon Y \to X$ is a projective, birational morphism of varieties and Z is a closed subscheme of X such that $f^{-1}(Z)$ is an effective Cartier divisor, then $-f^{-1}(Z)$ is f-nef.*

Proof. It follows from the universal property of the blow-up (see [Har77, Proposition 7.14]) that we can factor f as $g \circ h$, where $g\colon \widetilde{X} \to X$ is the blow-up of X along Z. If E is the effective Cartier divisor on $\widetilde{X}$ such that $g^{-1}(Z) = E$, then $-E$ is g-ample, which implies that $-f^{-1}(Z) = h^*(-E)$ is f-nef. □

The connectedness result in Theorem 3.2.2 is very useful when studying restriction properties of pairs. We now introduce this setting and give the first results in this direction. We will return to this circle of ideas several times later in the book.

Let $(X,D,\mathscr{Z})$ be a rational triple and suppose that H is an irreducible normal Cartier divisor on X which is not contained in $\mathrm{Supp}(D) \cup \mathrm{Supp}(\mathscr{Z})$. We have seen in Remark 3.1.4 that in this case we have an induced divisor $D|_H$ on H such that $K_H + D|_H$ is $\mathbb{Q}$-Cartier. If $\mathscr{Z} = \sum_i q_i Z_i$, we also put $\mathscr{Z}|_H = \sum_i q_i Z_i|_H$. The adjunction formula suggests that in a neighborhood of H, the singularities of the two triples $(X, D+H, \mathscr{Z})$ and $(H, D|_H, \mathscr{Z}|_H)$ are related. In this setting one talks about *adjunction* when deducing properties of $(H, D|_H, \mathscr{Z}|_H)$ from those of $(X, D+H, \mathscr{Z})$ and about *inversion of adjunction* when going in the reverse direction.

Let $f\colon Y \to X$ be a log resolution of $(X, D+H, \mathscr{Z})$. If $\widetilde{H}$ is the proper transform of H, then by assumption $\widetilde{H}$ is smooth and it is easy to see that the induced morphism

$g\colon \widetilde{H} \to H$ is a log resolution of $(H, D|_H, \mathscr{Z}_H)$. We have seen in Remark 3.1.9 that if we write $f^*(H) = \widetilde{H} + F$, then $\widetilde{H} \not\subseteq \mathrm{Supp}(F)$ and

$$(D|_H)_{\widetilde{H}} = (D_Y + F)|_{\widetilde{H}}.$$

Moreover, it is clear that $\widetilde{H} \not\subseteq \mathrm{Supp}(f^{-1}(\mathscr{Z}))$ and $f^{-1}(\mathscr{Z})|_{\widetilde{H}} = g^{-1}(\mathscr{Z}|_H)$. We also note that for every prime divisor $E \neq \widetilde{H}$ that appears in $\mathrm{Supp}(D_Y) \cup \mathrm{Supp}(f^{-1}(\mathscr{Z}))$, the intersection $E \cap \widetilde{H}$ is smooth, though possibly disconnected. We conclude that if $E \cap \widetilde{H}$ is nonempty, then for every irreducible component E_0 of $E \cap \widetilde{H}$, we have

$$a_E(X, D + H, \mathscr{Z}) = a_{E_0}(H, D|_H, \mathscr{Z}|_H). \tag{3.11}$$

Note also that $a_{\widetilde{H}}(X, H + D, \mathscr{Z}) = 0$. For example, the above discussion gives the following adjunction statement.

Proposition 3.2.5. *With the above notation, if the triple $(X, D + H, \mathscr{Z})$ is log canonical, then the triple $(H, D|_H, \mathscr{Z}|_H)$ is log canonical. Similarly, if we have $a_E(X, D + H, \mathscr{Z}) > 0$ for every divisor E over X different from $\widetilde{H}$, then $(H, D|_H, \mathscr{Z}|_H)$ is klt.*

We can do better if we start with a rational triple $(X, D, \mathscr{Z})$ and let H be general in a base-point free linear system. Note that in this case H is automatically normal by Bertini. It is not necessarily irreducible, but for the discussion that follows this is not important: we can simply consider separately each irreducible component. Therefore, for the ease of notation, we keep the assumption that H is irreducible. Let $f\colon Y \to X$ be a log resolution of $(X, D, \mathscr{Z})$. Since $f^*(H)$ is again a general member of a base-point free linear system, it follows from Kleiman's version of Bertini's theorem that $f^*(H)$ is again smooth and has simple normal crossings with the divisors contained in $\mathrm{Supp}(D_Y) \cup \mathrm{Supp}(f^{-1}(\mathscr{Z})) \cup \mathrm{Exc}(f)$. In particular, we see that in this case $f^*(H) = \widetilde{H}$. Moreover, f is a log resolution of $(X, D + H, \mathscr{Z})$ and the induced morphism $g\colon \widetilde{H} \to H$ is a log resolution of $(H, D|_H, \mathscr{Z}|_H)$. Note that if $E \neq \widetilde{H}$ is a prime divisor that appears in $\mathrm{Supp}(D_Y) \cup \mathrm{Supp}(f^{-1}(\mathscr{Z}))$ such that $\dim(c_X(E)) = 0$, we have $E \cap \widetilde{H} = \emptyset$ (recall that H is general in a base-point free linear system). On the other hand, if $E \cap \widetilde{H} \neq \emptyset$, then for every irreducible component E_0 of $E \cap \widetilde{H}$, we have

$$a_E(X, D + H, \mathscr{Z}) = a_E(X, D, \mathscr{Z}) = a_{E_0}(H, D|_H, \mathscr{Z}|_H). \tag{3.12}$$

We thus obtain the following version of the above adjunction statement.

Proposition 3.2.6. *If the triple $(X, D, \mathscr{Z})$ is klt (log canonical) outside a finite set of points, then for a general member H of a base-point free linear system on X, the triple $(H, D|_H, \mathscr{Z}|_H)$ is klt (log canonical).*

Inversion of adjunction is more subtle. We begin with the following application of the Shokurov-Kollár connectedness theorem.

Corollary 3.2.7. *Let $(X, D, \mathscr{Z})$ be an effective rational pair and H an irreducible, normal Cartier divisor on X, not contained in* $\mathrm{Supp}(D) \cup \mathrm{Supp}(\mathscr{Z})$. *If there is a* *prime divisor E over X different from the proper transform of H such that $a_E(X,D+H,\mathscr{Z})\leq 0$ and $c_X(E)\cap H\neq\emptyset$, then for every irreducible component W of $c_X(E)\cap H$, there exists a divisor E_0 over H such that $W\subseteq c_H(E_0)$ and $a_{E_0}(H,D|_H,\mathscr{Z}|_H)\leq 0$.*

Proof. Let $f\colon Y\to X$ be a log resolution of $(X,D+H,\mathscr{Z})$ such that E is a divisor on Y and let $\widetilde{H}$ be the proper transform of H. As in Theorem 3.2.2, we write

$$\lfloor (D+H)_Y+f^{-1}(\mathscr{Z})\rfloor = A-B.$$

Note that both E and $\widetilde{H}$ are contained in the support of A.

Let η_W be the generic point of W. Since $\eta_W\in f(E)\cap f(\widetilde{H})$ and $f^{-1}(\eta_W)\cap \mathrm{Supp}(A)$ is connected by Theorem 3.2.2, it follows that there is a prime divisor E' in $\mathrm{Supp}(A)$, with $E'\neq\widetilde{H}$ and $E'\cap\widetilde{H}\cap f^{-1}(\eta_W)\neq\emptyset$. We deduce from (3.11) that if E_0 is a connected component of $E'\cap\widetilde{H}$ that intersects $f^{-1}(\eta_W)$, then

$$a_{E_0}(H,D|_H,\mathscr{Z}|_H)=a_E(X,D+H,\mathscr{Z})\leq 0.$$

Since $W\subseteq f(E_0)$, this completes the proof of the corollary. □

In particular, we obtain the following version of inversion of adjunction. Note that in this case, we have to restrict to effective triples.

Corollary 3.2.8. *Let $(X,D,\mathscr{Z})$ be an effective rational pair and H an irreducible, normal effective Cartier divisor on X, not contained in $\mathrm{Supp}(D)\cup\mathrm{Supp}(\mathscr{Z})$. If $(H,D|_H,\mathscr{Z}|_H)$ is klt, then for every divisor E over X different from the proper transform of H and with $c_X(E)\cap H\neq\emptyset$, we have $a_E(X,D+H,\mathscr{Z})>0$. In particular, $(X,D+H,\mathscr{Z})$ is plt in some neighborhood of H.*

The following consequence of Corollary 3.2.7 is useful in the study of singularities of rational maps.

Corollary 3.2.9. *Suppose that the rational effective triple $(X,D,\mathscr{Z})$ is not terminal, and let E be an exceptional divisor over X such that $a_E(X,D,\mathscr{Z})\leq 1$. If H is a normal, irreducible, effective Cartier divisor on X such that $c_X(E)\subseteq H$, then $(H,D|_H,\mathscr{Z}|_H)$ is not log terminal around any point of $c_X(E)$.*

Proof. Note that since E is exceptional, E is different from the proper transform of H. Since $c_X(E)\subseteq H$, we have

$$a_E(X,D+H,\mathscr{Z})\leq a_E(X,D,\mathscr{Z})-1\leq 0$$

and the assertion follows from Corollary 3.2.7. □

3.3 Rational singularities

In this section we discuss rational singularities. This class of singularities has a longer history than the singularities of pairs discussed in Section 3.1, going back in the case of surfaces to [Art66]. The definition is of a cohomological nature and as a result, the proofs of the main results rely on Grothendieck's duality theorem. For the benefit of the reader, we first prove these results in the global setting, following [KM98]. The advantage in this case is that the proofs become more elementary, only making use of Serre duality. For the brave reader we then return and reprove the results in the general setting.

In this section we work over an algebraically closed field k of characteristic 0. The hypothesis on the characteristic is important, since we will make use of vanishing theorems. Let X be a variety and $f\colon Y \to X$ a resolution of singularities.

Definition 3.3.1. The resolution f is *rational* if $f_*(\mathscr{O}_Y) = \mathscr{O}_X$ (that is, X is normal) and $R^i f_*(\mathscr{O}_Y) = 0$ for $i > 0$. We say that X has *rational singularities* if every resolution of singularities of X is rational.

The starting point in the study of rational singularities is the following characterization of rational resolutions going back to [KKMSD73, p.50]. As we have mentioned, we first state and prove the results in the global setting.

Theorem 3.3.2. *Let $f\colon Y \to X$ be a resolution of singularities of a normal projective variety X. The following are equivalent:*

i) The resolution f is rational.
ii) X is Cohen–Macaulay and the canonical morphism $f_*(\omega_Y) \to \omega_X$ is an isomorphism.

We recall that if $f\colon Y \to X$ is a projective, birational morphism between n-dimensional normal varieties, then we have a "trace map", a canonical injective morphism $t_{Y/X}\colon f_*\mathscr{O}(K_Y) \hookrightarrow \mathscr{O}(K_X)$ (see Remark 3.1.2). By identifying the sheaves corresponding to the canonical divisors to the dualizing sheaves (see Remark 3.1.5), we can interpret this inclusion as a map $f_*\omega_Y^\circ \hookrightarrow \omega_X^\circ$. Suppose now that both X and Y are Cohen-Macaulay projective varieties, hence we may write ω_X and ω_Y instead of ω_X° and ω_Y°, respectively. In this case, the trace map $f_*\omega_Y \hookrightarrow \omega_X$ is compatible with Serre duality, in the sense that for every line bundle $\mathscr{M}$ on X and every i, the following diagram

$$\begin{tikzcd}
H^i(X, f_*(\omega_Y)\otimes\mathscr{M}) \arrow[r, "\alpha"'] \arrow[dr, "\gamma"'] & H^i(Y, \omega_Y\otimes f^*(\mathscr{M})) \arrow[r, "\sim", "\beta"'] & H^{n-i}(Y, f^*(\mathscr{M})^{-1})^\vee \arrow[d, "\phi"] \\
 & H^i(X, \omega_X\otimes\mathscr{M}) \arrow[r, "\sim", "\delta"'] & H^{n-i}(X, \mathscr{M}^{-1})^\vee
\end{tikzcd} \tag{3.13}$$

is commutative, where β and δ are the isomorphisms provided by Serre duality, γ is induced by $t_{Y/X}$, ϕ is the dual of the pull-back map in cohomology, and α is an edge map corresponding to the Leray spectral sequence

$$E_2^{p,q} = H^p(X, R^q f_*(\omega_Y\otimes f^*(\mathscr{M})) \simeq H^p(X, R^q f_*(\omega_Y)\otimes\mathscr{M}) \Rightarrow H^{p+q}(Y, \omega_Y\otimes f^*(\mathscr{M})).$$

Note that if Y is smooth, then the Grauert–Riemenschneider theorem implies that in the above spectral sequence we have $E_2^{p,q}=0$ unless $q=0$, hence α is an isomorphism as well.

We will also make use of the following lemma (see [Har77, Theorem III.7.6] and its proof).

Lemma 3.3.3. *If Z is a projective scheme and $\mathscr{M}$ is an ample line bundle on Z, then Z is equidimensional and Cohen–Macaulay if and only if $H^i(Z,\mathscr{M}^j)=0$ for all $j \ll 0$ and all $i<\dim(Z)$.*

Proof of Theorem 3.3.2. We pick an ample line bundle $\mathscr{L}$ on X. For every integer m, we consider the Leray spectral sequence

$$E_2^{p,q}=H^p(X,R^qf_*(\mathscr{O}_Y)\otimes\mathscr{L}^{-m})\Rightarrow H^{p+q}(Y,f^*\mathscr{L}^{-m}). \tag{3.14}$$

We first show that ii) $\Rightarrow$ i). Therefore suppose that X is Cohen–Macaulay and the canonical morphism $t_{Y/X}\colon f_*(\omega_Y)\to\omega_X$ is an isomorphism. We argue by induction on $n=\dim(X)$. If $n=1$, then X is smooth and f is an isomorphism, hence it is clearly rational. Suppose now that $n\geq 2$ and let $H\subset X$ be a general member of a very ample linear system on X. By Bertini's theorem, we have that H is normal and irreducible and $\widetilde{H}=f^*(H)$ is smooth and equal to the proper transform of H (see, for example, the discussion after Proposition 3.2.6). Since H is general, it intersects the open subset over which f is an isomorphism, hence the induced morphism $g\colon\widetilde{H}\to H$ is a resolution of singularities. We have a commutative diagram

$$\begin{tikzcd}
f_*(\omega_Y(\widetilde{H})) \arrow[r] \arrow[d, "t_{Y/X}\otimes\mathscr{O}_X(H)"'] & g_*(\omega_{\widetilde{H}}) \arrow[d, "t_{\widetilde{H}/H}"] \\
\omega_X(H) \arrow[r] & \omega_H
\end{tikzcd}$$

in which the horizontal maps are induced by the adjunction isomorphisms. Since the bottom horizontal map is surjective and $t_{Y/X}$ is an isomorphism, we conclude that $t_{\widetilde{H}/H}$ is surjective, hence an isomorphism. Since H is also Cohen–Macaulay, we conclude by induction that g is a rational resolution of H, hence $R^if_*(\mathscr{O}_{\widetilde{H}})=0$ for every $i\geq 1$.

Using the exact sequence

$$0\to\mathscr{O}_Y(-\widetilde{H})\to\mathscr{O}_Y\to\mathscr{O}_{\widetilde{H}}\to 0,$$

we obtain an exact sequence

$$R^if_*(\mathscr{O}_Y(-\widetilde{H}))\simeq R^if_*(\mathscr{O}_Y)\otimes\mathscr{O}_X(-H)\to R^if_*(\mathscr{O}_Y)\to R^if_*(\mathscr{O}_{\widetilde{H}})=0.$$

It follows from Nakayama's lemma that $\mathrm{Supp}(R^if_*(\mathscr{O}_Y))$ is disjoint from H for all $i>0$. In particular, since H is ample, we conclude that

$$\dim\mathrm{Supp}(R^if_*(\mathscr{O}_Y))\leq 0 \text{ for } i>0. \tag{3.15}$$

Therefore in order to show that $R^i f_*(\mathscr{O}_Y) = 0$ for $i > 0$, it is enough to show that $H^0(X, R^i f_*(\mathscr{O}_Y) \otimes \mathscr{L}^{-m}) = 0$ for $m \gg 0$. Moreover, (3.15) implies that in the spectral sequence (3.14) we have $E_2^{p,q} = 0$ whenever $p > 0$ and $q > 0$. It follows that for every $i \geq 0$ we have an exact sequence

$$0 \to E_\infty^{i,0} \to H^i(Y, f^*\mathscr{L}^{-m}) \to E_\infty^{0,i} \to 0. \tag{3.16}$$

In addition, if a map $d_r \colon E_r^{p,q} \to E_r^{p+r,q-r+1}$ in the spectral sequence is nonzero, with $r \geq 2$, then $p = 0$ and $r = q+1$. Therefore for every i we have an exact sequence

$$0 \to E_\infty^{0,i} \to E_2^{0,i} \stackrel{d_{i+1}}{\to} E_2^{i+1,0} \to E_\infty^{i+1,0} \to 0. \tag{3.17}$$

On the other hand, since $\mathscr{L}$ is ample, we obtain using Serre duality and asymptotic Serre vanishing on X (recall that X is Cohen–Macaulay)

$$E_2^{p,0} = H^p(X, \mathscr{L}^{-m}) \simeq H^{n-p}(X, \omega_X \otimes \mathscr{L}^m)^\vee = 0 \text{ for } p < n \text{ and } m \gg 0.$$

Using Serre duality on Y and the Kawamata–Viehweg vanishing theorem (note that $f^*(\mathscr{L})$ is big and nef), we obtain

$$H^i(Y, f^*\mathscr{L}^{-m}) \simeq H^{n-i}(Y, \omega_Y \otimes f^*\mathscr{L}^m)^\vee = 0 \text{ for } i < n \text{ and } m \gg 0.$$

Therefore the exact sequence (3.16) implies that for $m \gg 0$ we have $E_\infty^{0,i} = 0 = E_\infty^{i,0}$ for all $i < n$. Since we also have $E_2^{i,0} = 0$ for $i < n$, the exact sequence (3.17) implies that $E_2^{0,i} = 0$ for $i+1 < n$. As we have seen, this implies $R^i f_*(\mathscr{O}_Y) = 0$ for $0 < i < n-1$. Moreover, we clearly have $R^n f_*(\mathscr{O}_Y) = 0$ since all fibers of f have dimension $< n$. By taking $i = n$ in (3.16) we obtain $E_\infty^{0,n} = H^n(Y, f^*\mathscr{L}^{-m})$ and by taking $i = n-1$ in (3.17), we obtain for $m \gg 0$

$$H^0(X, R^{n-1} f_*(\mathscr{O}_Y) \otimes \mathscr{L}^{-m}) = E_2^{0,n-1} = \ker\left(H^n(X, \mathscr{L}^{-m}) \to H^n(Y, f^*\mathscr{L}^{-m})\right).$$

By Serre duality and the Grauert-Riemenschneider theorem (see the commutative diagram (3.13)), the dual of the right-hand side in the above formula is isomorphic to the cokernel of the map $\phi \colon H^0(X, f_*(\omega_Y) \otimes \mathscr{L}^m) \to H^0(X, \omega_X \otimes \mathscr{L}^m)$ induced by $t_{Y/X}$. Since $t_{Y/X}$ is an isomorphism, ϕ is an isomorphism and therefore it has trivial cokernel. We thus deduce that $R^{n-1} f_*(\mathscr{O}_Y) = 0$, which completes the proof of ii) $\Rightarrow$ i).

Conversely, suppose that f is a rational resolution. In this case the spectral sequence (3.14) has $E_2^{p,q} = 0$ for $q \neq 0$, hence

$$H^i(X, \mathscr{L}^{-m}) \simeq H^i(Y, f^*\mathscr{L}^{-m}) \tag{3.18}$$

for every i and every m. By Serre duality and the Kawamata–Viehweg vanishing theorem, we have

$$H^i(Y, f^*\mathscr{L}^{-m}) \cong H^{n-i}(Y, \omega_Y \otimes f^*\mathscr{L}^m)^\vee = 0 \text{ for } i < n \text{ and } m \geq 1.$$

Therefore X is Cohen–Macaulay by Lemma 3.3.3.

Since $\mathscr{L}$ is ample, in order to prove that the injective map $t_{Y/X}\colon f_*(\omega_Y) \hookrightarrow \omega_X$ is an isomorphism, it is enough to show that $H^0(X, \mathrm{Coker}(t_{Y/X}) \otimes \mathscr{L}^m) = 0$ for $m \gg 0$. Moreover, we have an exact sequence

$$0 \to H^0(X, f_*(\omega_Y) \otimes \mathscr{L}^m) \xrightarrow{\iota} H^0(X, \omega_X \otimes \mathscr{L}^m) \to H^0(X, \mathrm{Coker}(t_{Y/X}) \otimes \mathscr{L}^m) \to 0$$

and the dual of ι corresponds by Serre duality and the Grauert–Riemenschneider theorem (see the commutative diagram (3.13)) to the map

$$H^0(X, \mathscr{L}^{-m}) \to H^0(Y, f^*\mathscr{L}^{-m}).$$

This is an isomorphism by (3.18). It follows that ι is an isomorphism, hence $t_{Y/X}$ is an isomorphism. This completes the proof of i) $\Rightarrow$ ii). $\square$

Corollary 3.3.4. *Let $f\colon Y \to X$ be a resolution of singularities of a normal projective variety X. If $\mathscr{L}$ is an ample line bundle on X such that the natural map*

$$H^i(X, \mathscr{L}^{-m}) \to H^i(Y, f^*\mathscr{L}^{-m})$$

is injective for every i and all $m \gg 0$, then f is a rational resolution.

Proof. Let $n = \dim(X)$. By Serre duality and the Kawamata–Viehweg vanishing theorem, we have

$$H^i(Y, f^*\mathscr{L}^{-m}) \simeq H^{n-i}(Y, \omega_Y \otimes f^*\mathscr{L}^m)^\vee = 0 \text{ for } i < n \text{ and } m \geq 1,$$

hence the injectivity hypothesis implies $H^i(X, \mathscr{L}^{-m}) = 0$ for $i < n$ and $m \geq 1$. Therefore X is Cohen–Macaulay by Lemma 3.3.3. Moreover, we see as in the last part of the proof of Theorem 3.3.2 that the kernel of the map $H^n(X, \mathscr{L}^{-m}) \to H^n(Y, f^*\mathscr{L}^{-m})$ (which is trivial by our assumption) is dual to the cokernel of the inclusion $H^0(Y, f_*(\omega_Y) \otimes \mathscr{L}^m) \to H^0(X, \omega_X \otimes \mathscr{L}^m)$. Since $\mathscr{L}$ is ample, we conclude that the natural map $f_*(\omega_Y) \to \omega_X$ is an isomorphism and we deduce using Theorem 3.3.2 that f is a rational resolution. $\square$

Remark 3.3.5. Note that the converse of the assertion in Corollary 3.3.4 is clearly true: if $f\colon Y \to X$ is a rational resolution, then for every line bundle $\mathscr{L}$ on X, every i and m, the natural map

$$H^i(X, \mathscr{L}^{-m}) \to H^i(Y, f^*\mathscr{L}^{-m})$$

is an isomorphism. Indeed, the assumption implies that in the Leray spectral sequence for f and $f^*(\mathscr{L}^{-m})$ we have $E_2^{p,q} = 0$ for $q \neq 0$, which implies our assertion.

Corollary 3.3.6. *A projective variety X has rational singularities if there exists one rational resolution of singularities of X. In particular, a smooth projective variety has rational singularities.*

Proof. We need to show that if $f\colon Y \to X$ and $f'\colon Y' \to X$ are two resolutions of singularities, then one is rational if and only if the other one is. Since any two resolutions can be dominated by a third one, we may assume that there is a projective, birational morphism $g\colon Y' \to Y$ such that $f' = f \circ g$. Since X has a rational resolution, it follows that X is normal. Moreover, since Y and Y' are both smooth, the natural map $t_{Y'/Y}\colon g_*(\omega_{Y'}) \to \omega_Y$ is an isomorphism (see Corollary B.2.6). Since the composition

$$f_*(g_*(\omega_{Y'})) \xrightarrow{f_*(t_{Y'/Y})} f_*(\omega_Y) \xrightarrow{t_{Y/X}} \omega_X$$

is equal to $t_{Y'/X}$, it follows that $t_{Y/X}$ is an isomorphism if and only if $t_{Y'/X}$ is an isomorphism. Therefore f is rational if and only if f' is rational by Theorem 3.3.2. $\square$

As an application, we prove the following theorem of Elkik [Elk81].

Theorem 3.3.7. *Let (X,D) be a rational effective pair, with X projective. If (X,D) is klt, then X has rational singularities.*

Proof. Note that X is by assumption normal. Let $f\colon Y \to X$ be a log resolution of (X,D). We write as usual $K_Y + D_Y = f^*(K_X + D)$ and put $E = \lceil -D_Y \rceil$. Since (X,D) is kit, it follows that E is effective. On the other hand, since D is effective, it follows that E is f-exceptional. Therefore the natural map $\mathscr{O}_X \to f_*\mathscr{O}_Y(E)$ is an isomorphism (see Lemma B.2.5).

If we write $E = -D_Y + \Delta'$, then Δ' has simple normal crossings, since f is a log resolution of (X,D). Since $-f^*(K_X + D)$ is f-nef and f-big, we may apply Theorem 2.6.1 to conclude $R^i f_*(\mathscr{O}_Y(E)) = 0$ for $i \geq 1$. We deduce that if $\mathscr{L}$ is any ample line bundle on X, then the Leray spectral sequence

$$E_2^{p,q} = H^p(X, R^q f_*(\mathscr{O}_Y(E)) \otimes \mathscr{L}^{-m}) \Rightarrow H^{p+q}(Y, \mathscr{O}_Y(E) \otimes f^*\mathscr{L}^{-m})$$

has $E_2^{p,q} = 0$ for $q \neq 0$. In particular, the canonical morphism

$$H^i(X, f_*(\mathscr{O}_Y(E)) \otimes \mathscr{L}^{-m}) \to H^i(Y, \mathscr{O}_Y(E) \otimes f^*\mathscr{L}^{-m})$$

is an isomorphism for every i and m. Consider the commutative diagram

$$\begin{tikzcd}
H^i(X,\mathscr{L}^{-m}) \arrow[r, "\alpha"] \arrow[d, equal] & H^i(Y, f^*\mathscr{L}^{-m}) \arrow[d] & \\
H^i(X, f_*(\mathscr{O}_Y(E)) \otimes \mathscr{L}^{-m}) \arrow[r, "\beta"] & H^i(Y, f^*f_*(\mathscr{O}_Y(E)) \otimes f^*\mathscr{L}^{-m}) \arrow[r, "\gamma"] & H^i(Y, \mathscr{O}_Y(E) \otimes f^*\mathscr{L}^{-m}).
\end{tikzcd}$$

As we have seen, the composition $\gamma \circ \beta$ is an isomorphism, hence β is injective and we conclude that α is injective. Therefore f is a rational resolution by Corollary 3.3.4, and thus X has rational singularities by Corollary 3.3.6. $\square$

Corollary 3.3.8. *If X is a normal projective variety such that K_X is Cartier, then X has rational singularities if and only if X has canonical singularities.*

Proof. Note that since K_X is Cartier, X has canonical singularities if and only if X has klt singularities. If this is the case, then X has rational singularities by Theorem 3.3.7. Conversely, suppose that X has rational singularities. If $f\colon Y \to X$ is a log resolution, then Theorem 3.3.2 implies that the canonical morphism $t_{Y/X}\colon f_*\mathscr{O}_Y(K_Y) \to \mathscr{O}_X(K_X)$ is an isomorphism. It follows from Proposition 3.1.39 that in this case X has canonical singularities. $\square$

Similar arguments also give the following result of Kollár [Kol97].

Theorem 3.3.9. *Let $g\colon X' \to X$ be a surjective morphism between normal projective varieties. If X' has rational singularities and $R^i g_*\mathscr{O}_{X'} = 0$ for $i > 0$, then X has rational singularities.*

Proof. Note first that the canonical injective map $\mathscr{O}_X \to g_*\mathscr{O}_{X'}$ splits. Indeed, if $g = g_2 \circ g_1\colon X' \to Z \to X$ is the Stein factorization of g, then $g_*\mathscr{O}_{X'} = (g_2)_*\mathscr{O}_Z$. Since X is normal, the trace map for the function field extension $K(X) \hookrightarrow K(Z)$ induces a morphism $(g_2)_*(\mathscr{O}_Z) \to \mathscr{O}_X$ and multiplying this by $\frac{1}{d}$, where $d = \deg(Z/X)$, gives a splitting of $\mathscr{O}_X \hookrightarrow (g_2)_*(\mathscr{O}_Z) = g_*(\mathscr{O}_{X'})$.

We consider a commutative diagram

$$\begin{tikzcd} Y' \arrow[r, "f'"] \arrow[d, "h"'] & X' \arrow[d, "g"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

in which both f and f' are resolutions of singularities (for example, construct first a resolution f and then let $Y' \to W$ be a resolution of singularities of the unique irreducible component W of $Y \times_X X'$ which maps birationally onto X'). Let $p = g \circ f' = f \circ h\colon Y' \to X$. We fix an ample line bundle $\mathscr{L}$ on X and for every i and m we consider the commutative diagram

$$\begin{tikzcd} H^i(X, \mathscr{L}^{-m}) \arrow[r, "\gamma"] \arrow[d, "\beta"'] & H^i(Y, f^*\mathscr{L}^{-m}) \arrow[d] \\ H^i(X', g^*\mathscr{L}^{-m}) \arrow[r, "\alpha"] & H^i(Y', p^*\mathscr{L}^{-m}). \end{tikzcd}$$

Since f' is a rational resolution, the Leray spectral sequence for f' and $p^*\mathscr{L}^{-m}$ satisfies $E_2^{p,q} = 0$ for all $q \neq 0$ and therefore α is an isomorphism. Similarly, since $R^j g_*(\mathscr{O}_{X'}) = 0$ for all $j > 0$, the Leray spectral sequence for g and $g^*\mathscr{L}^{-m}$ satisfies $E_2^{p,q} = 0$ for $q \neq 0$, which implies that the canonical morphism

$$H^i(X, g_*(g^*\mathscr{L}^{-m})) = H^i(X, g_*(\mathscr{O}_{X'}) \otimes \mathscr{L}^{-m}) \to H^i(X', g^*\mathscr{L}^{-m})$$

is an isomorphism. On the other hand, the canonical morphism

$$H^i(X, \mathscr{L}^{-m}) \to H^i(X, g_*(g^*\mathscr{L}^{-m}))$$

is injective since the inclusion $\mathscr{O}_X \hookrightarrow g_*(\mathscr{O}_{X'})$ is split. Therefore the composition of these two maps, which is equal to β, is injective. We deduce from the above commutative diagram that γ is injective. It follows by Corollary 3.3.4 that f is a rational resolution and thus X has rational singularities by Corollary 3.3.6. $\square$

Corollary 3.3.10. *If $g\colon X' \to X$ is a finite surjective morphism between two normal projective varieties and X' has rational singularities, then X has rational singularities.*

Proof. Since g is finite, we have $R^i g_*(\mathscr{O}_{X'}) = 0$ for all $i > 0$, hence we may apply Theorem 3.3.9. $\square$

As we have promised, we now turn to the proofs of Theorems 3.3.2, 3.3.7, 3.3.9, and Corollary 3.3.6 for not-necessarily-projective varieties. TO BE WRITTEN.

Remark 3.3.11. Suppose that X is covered by the images of a family of étale maps $\phi_i\colon U_i \to X$. If $f\colon Y \to X$ is a resolution of singularities, then each $f_i\colon V_i = Y \times_X U_i \to U_i$ is a resolution of singularities. By flat base-change $R^j(f_i)_*(\mathscr{O}_{V_i}) \simeq \phi_i^*(R^j f_*(\mathscr{O}_Y))$. Moreover, since the map $\sqcup_i U_i \to X$ is faithfully flat, we see that a coherent sheaf $\mathscr{M}$ on X is 0 if and only if all $\phi_i^*(\mathscr{M})$ are 0. This implies that X has rational singularities if and only if each U_i has rational singularities.

For example, recall that a variety X has *quotient singularities* if there is such a cover with each U_i isomorphic to Y_i/G_i, where Y_i is a smooth quasiprojective variety and G_i is a finite group acting on Y_i. In particular, we see that there is a finite surjective morphism $Y_i \to U_i$, hence U_i has rational singularities by the local version of Corollary 3.3.9. We conclude that X has rational singularities.

3.4 Log canonical thresholds

3.4.1 Definition and examples

3.4.2 First properties of log canonical thresholds

3.4.3 Semicontinuity of log canonical thresholds

3.4.4 Log canonical thresholds and Hilbert–Samuel multiplicity

3.5 Log canonical centers

3.6 $\mathfrak{m}$-adic semicontinuity of log canonical thresholds

3.7 ACC for log canonical thresholds on smooth varieties

3.8 Minimal log discrepancies

Chapter 4
Multiplier ideals

4.1 Multiplier ideals

In this section we introduce the multiplier ideal of a triple and prove its basic properties. In particular, we prove the two main vanishing results that involve multiplier ideals, the local vanishing theorem and Nadel's vanishing theorem. Our presentation is heavily inspired from that in [Laz04b, Chap. 9].

4.1.1 Definition and first properties

Definition 4.1.1. Let $(X,\Delta,\mathscr{Z})$ be a log triple. Given a log resolution $f\colon Y\to X$ of this triple, we consider the triple $(Y,\Delta_Y,f^{-1}(\mathscr{Z}))$. The *multiplier ideal* of $(X,\Delta,\mathscr{Z})$ is

$$\mathscr{J}(X,\Delta,\mathscr{Z}) := f_*\mathscr{O}_Y(-\lfloor \Delta_Y + f^{-1}(\mathscr{Z})\rfloor).$$

Note that if K_X is Cartier, then $K_{Y/X}$ is integral, hence we can also write

$$\mathscr{J}(X,\Delta,\mathscr{Z}) = f_*\mathscr{O}_Y(K_{Y/X} - \lfloor f^*(\Delta) + f^{-1}(\mathscr{Z})\rfloor).$$

If instead of a log triple we have either a log pair (X,Δ) or a higher codimension pair $(X,\mathscr{Z})$, then we simply write $\mathscr{J}(X,\Delta)$ or $\mathscr{J}(X,\mathscr{Z})$, respectively. If a triple is written as $(X,\Delta,\mathfrak{a}_1^{q_1}\dots\mathfrak{a}_r^{q_r})$, then the corresponding multiplier ideal is written as $\mathscr{J}(X,\Delta,\mathfrak{a}_1^{q_1}\dots\mathfrak{a}_r^{q_r})$.

Remark 4.1.2. In general, the multiplier ideal is not an ideal of $\mathscr{O}_X$, but a fractional ideal. On the other hand, if the triple is effective, then the only divisors in $\Delta_Y + f^{-1}(\mathscr{Z})$ that appear with negative coefficient are exceptional. Therefore there is an effective exceptional divisor F on Y such that $\mathscr{J}(X,\Delta,\mathscr{Z}) \subseteq f_*\mathscr{O}_Y(F) = \mathscr{O}_X$. We conclude that in this case the multiplier ideal is indeed an ideal in $\mathscr{O}_X$. We also note that in general, the multiplier ideal is nonzero.

Since the definition of the multiplier ideal involves the choice of a log resolution, we first need to show that this notion is well-defined.

Theorem 4.1.3. *Given a log triple $(X,\Delta,\mathscr{Z})$, the multiplier ideal $\mathscr{J}(X,\Delta,\mathscr{Z})$ is independent of the choice of a log resolution in its definition.*

Proof. Since any two log resolutions can be dominated by a third one, it is enough to consider two morphisms $W \xrightarrow{g} Y \xrightarrow{f} X$, such that both f and $f\circ g$ give log resolutions for $(X,\Delta,\mathscr{Z})$, and show that in this case

$$f_*\mathscr{O}_Y(-\lfloor \Delta_Y + f^{-1}(\mathscr{Z})\rfloor) = f_*g_*\mathscr{O}_W(-\lfloor \Delta_W + g^{-1}(f^{-1}(\mathscr{Z}))\rfloor).$$

Let $A = \Delta_Y + f^{-1}(\mathscr{Z})$. Since $\Delta_W + g^{-1}(f^{-1}(\mathscr{Z})) = g^*(A) - K_{W/Y}$, we see that it is enough to show that

$$\mathscr{O}_Y(-\lfloor A\rfloor) = g_*\mathscr{O}_W(-\lfloor g^*(A) - K_{W/Y}\rfloor).$$

Let us write $A = \lfloor A\rfloor + F$. Note that F is a divisor with simple normal crossings and with $\lfloor F\rfloor = 0$, hence the pair (Y,F) is klt by Theorem 3.1.12. Therefore the divisor $G := \lceil K_{W/Y} - g^*(F)\rceil$ is effective, and since F is effective, we deduce that G is g-exceptional. Therefore $g_*\mathscr{O}_W(G) = \mathscr{O}_Y$, and using the projection formula, we conclude that

$$g_*(-\lfloor g^*(A) - K_{W/Y}\rfloor) = g_*\mathscr{O}_W(-g^*(\lfloor A\rfloor) + G) = \mathscr{O}_Y(-\lfloor A\rfloor).$$

This completes the proof of the theorem. $\square$

Remark 4.1.4. Let $(X,\Delta,\mathscr{Z})$ be a log triple and $f\colon Y \to X$ be a log resolution of this triple. If $V \subseteq X$ is an affine open subset and ϕ is a nonzero rational function, then $\phi \in H^0(V, \mathscr{J}(X,\Delta,\mathscr{Z}))$ if and only if

$$\mathrm{ord}_E(\phi) > \mathrm{ord}_E(\Delta_Y) + \mathrm{ord}_E(\mathscr{Z}) - 1$$

for all divisors E on Y such that $E \cap f^{-1}(V) \neq \emptyset$. Furthermore, it follows from Theorem 4.1.3 that one can equivalently put this condition for all log resolutions. Therefore $\phi \in H^0(V, \mathscr{J}(X,\Delta,\mathscr{Z}))$ if and only if

$$\mathrm{ord}_E(\phi) + a_E(X,\Delta,\mathscr{Z}) > 0 \text{ for all divisors } E \text{ over } X, \text{ with } c_X(E)\cap V \neq \emptyset.$$

This is the case if and only if the restriction of the triple $(X,\Delta - \mathrm{div}_X(\phi),\mathscr{Z})$ to V is klt. Note that even if the triple $(X,\Delta,\mathscr{Z})$ is effective, the triple that appears in this condition is not, in general, effective.

In particular, we see that $\mathscr{O}_X \subseteq \mathscr{J}(X,D,\mathscr{Z})$ if and only if the triple $(X,\Delta,\mathscr{Z})$ is klt. The largest open subset W of X on which $\mathscr{O}_X \subseteq \mathscr{J}(X,\Delta,\mathscr{Z})$ thus coincides with the largest open subset on which the restriction of $(X,\Delta,\mathscr{Z})$ is klt. If $(X,\Delta,\mathscr{Z})$ is an effective pair, one can therefore describe this latter open subset as the complement of $\mathrm{Supp}(\mathscr{O}_X/\mathscr{J}(X,\Delta,\mathscr{Z}))$.

One can think of the multiplier ideal $\mathscr{J}(X,\Delta,\mathscr{Z})$ as measuring the singularities of the triple $(X,\Delta,\mathscr{Z})$. The above remark suggests that in this respect larger multiplier ideals to correspond to "better singularities".

Example 4.1.5. If $(X,\Delta,\mathscr{Z})$ is a log triple and A is a Cartier divisor, then

$$\mathscr{J}(X,\Delta+A,\mathscr{Z})=\mathscr{O}_X(-A)\cdot\mathscr{J}(X,\Delta,\mathscr{Z}).$$

This follows from the definition and the projection formula.

Remark 4.1.6. Given a log pair (X,Δ), there is a nonzero ideal J on X such that

$$J\cdot\mathfrak{a}\subseteq\mathscr{J}(X,\Delta,\mathfrak{a})\text{ for all nonzero ideals }\mathfrak{a}\subseteq\mathscr{O}_X.$$

Indeed, this follows from definition, by taking for example $J=\mathscr{J}(X,\Delta)\cap\mathscr{O}_X$.

Example 4.1.7 (Multiplier ideal of a smooth subvariety). If X is a smooth variety and $Z\hookrightarrow X$ is a smooth subvariety of codimension r, defined by the ideal I_Z, then for $q\in\mathbb{R}_{\geq 0}$, we have

$$\mathscr{J}(X,qZ)=\begin{cases}\mathscr{O}_X, & \text{if } q<r;\\ I_Z^{\lfloor q\rfloor-r+1}, & \text{if } q\geq r.\end{cases}$$

Indeed, the blow-up $f\colon Y\to X$ along Z is a log resolution of (X,Z), and the above formula follows from the fact that if E is the exceptional divisor of f, then $f_*\mathscr{O}_Y(-jE)=I_Z^j$ for all $j\geq 0$ (see Lemma 2.5.14).

Example 4.1.8 (Multiplier ideal of a nodal curve). If X is a smooth surface and $C\subset X$ is a curve with at most nodes as singularities, then for every $q\in\mathbb{R}_{\geq 0}$, we have

$$\mathscr{J}(X,qC)=\mathscr{O}_X(-\lfloor q\rfloor C). \tag{4.1}$$

Indeed, let $f\colon Y\to X$ be the blow-up of X at the nodes of C. Note that f is a log resolution of (X,C). If $E=E_1+\ldots+E_m$ is the exceptional divisor of f and $\widetilde{C}$ is the proper transform of C, then

$$(qC)_Y=qf^*(C)-K_{Y/X}=q\widetilde{C}+\sum_{i=1}^m(2q-1)E_i.$$

The formula in (4.1) is clear for $0\leq\alpha<1$, and the general case then follows from Example 4.1.5.

Example 4.1.9 (Multiplier ideal of a cuspidal curve). Suppose that $X=\mathbb{A}^2=\operatorname{Spec}(k[x,y])$ and $D=V(f)\subset X$, where $f=x^2+y^3$. For $q\in\mathbb{R}_{\geq 0}$, the multiplier ideal of (X,qD) is given by

$$\mathscr{J}(X,qD)=\begin{cases}\mathscr{O}_X, & \text{if } q<\frac{5}{6};\\ (x,y), & \text{if } \frac{5}{6}\leq q<1;\\ f^m\cdot\mathscr{J}(X,(q-m)D), & \text{if } m\leq q<m+1,\ m\in\mathbb{Z}_{>0}.\end{cases}$$

In order to check this, we use the log resolution $f\colon Y\to X$ described in Example 3.1.18, as well as the notation in that example. Since $(X, q\cdot D)$ is klt for $0\leq q<\frac{5}{6}$, we deduce that $\mathscr{J}(X,qD)=\mathscr{O}_X$ for q in this range. On the other hand, if $\frac{5}{6}\leq q<1$, then

$$(qD)_Y = qf^*(D)-K_{Y/X} = q\widetilde{D}+(2q-1)E_1+(3q-2)E_2+(6q-4)E_3,$$

hence

$$f_*\mathscr{O}_Y(-\lfloor (qD)_Y\rfloor) = f_*\mathscr{O}_Y(-E_3) = (x,y).$$

The fact that $\mathscr{J}(X,qD)=\mathscr{O}_X(-mD)\cdot\mathscr{J}(X,(q-m)D)$ for $m\leq q<m+1$ is a consequence of Example 4.1.5.

Example 4.1.10 (Multiplier ideal of a cone over a smooth hypersurface). Let $X=\mathbb{A}^n$ and $D=V(f)$, where $f\in k[x_1,\ldots,x_n]$ is a homogeneous polynomial of degree d, with an isolated singularity at 0. In order to compute $\mathscr{J}(X,qD)$, we use the log resolution in Example 3.1.16 and the computations therein. In particular, we see that $\mathscr{J}(X,qD)=\mathscr{O}_X$ if $0\leq q<\min\{1,n/d\}$. Suppose now that $d>n$ and let us show that for every i, with $0\leq i\leq d-n-1$, we have

$$\mathscr{J}(X,qD)=(x_1,\ldots,x_n)^{i+1} \text{ if } \frac{n+i}{d}\leq q<\frac{n+i+1}{d}.$$

With the notation in Example 3.1.16, recall that $(qD)_Y=q\widetilde{D}+(qd-n+1)E$, hence our condition on q implies $\lfloor (qD)_Y\rfloor=(i+1)E$, hence

$$\mathscr{J}(X,D,\mathscr{Z}) = f_*\mathscr{O}_Y(-(i+1)E) = (x_1,\ldots,x_n)^{i+1}.$$

Example 4.1.11. One can often write the multiplier ideal of a log triple as the multiplier ideal of a log pair arguing as follows. Suppose that we have an effective log triple $(X,\Delta,\mathscr{Z})$, with $\mathscr{Z}=\sum_{i=1}^r q_iZ_i$. We assume that for every i there is a finite-dimensional linear system $V_i\subseteq H^0(X,\mathscr{L}_i)$ such that Z_i is the base locus of V_i (for example, if X is affine, we may take $\mathscr{L}_i=\mathscr{O}_X$ and V_i to be spanned by a system of generators of the ideal defining Z_i). For every i, we choose $r_i>q_i$ and let $h_{i,1},\ldots,h_{i,r_i}$ be general elements of V_i. We claim that if $D_{i,j}$ is the effective Cartier divisor defined by $h_{i,j}$ and $\Gamma=\sum_{i=1}^r\frac{q_i}{r_i}\cdot\sum_{j=1}^{r_i}D_{i,j}$, then

$$\mathscr{J}(X,\Delta,\mathscr{Z})=\mathscr{J}(X,\Delta+\Gamma). \tag{4.2}$$

Indeed, suppose that $f\colon Y\to X$ is a log resolution of $(X,\Delta,\mathscr{Z})$ and write $f^{-1}(Z_i)=E_i$. For every i and j, we can write $f^*(D_{i,j})=E_i+F_{i,j}$, and the genericity hypothesis on the $h_{i,j}$ together with Kleiman's version of Bertini's theorem imply that all $F_{i,j}$ are smooth (possibly disconnected), having no common components with the divisors in $\Delta_Y+f^{-1}(\mathscr{Z})$, and in fact, such that f is a log resolution of $(X,\Delta+\Gamma)$. Furthermore, we have

$$\lfloor (\Delta+\Gamma)_Y \rfloor = \lfloor \Delta_Y + f^*(\Gamma) \rfloor = \lfloor \Delta_Y + f^{-1}(\mathscr{Z}) + \sum_{i=1}^{r} \frac{q_i}{r_i} \sum_{j=1}^{r_i} F_{i,j} \rfloor = \lfloor \Delta_Y + f^{-1}(\mathscr{Z}) \rfloor.$$

The equality in (4.2) then follows from the definition of multiplier ideals.

Example 4.1.12. Let $\mathfrak{a} \subseteq k[x_1,\ldots,x_n]$ be an ideal generated by monomials. For every $u=(u_1,\ldots,u_n) \in \mathbb{Z}^n_{\geq 0}$, we put $x^u = x_1^{u_1}\cdots x_n^{u_n}$. We denote by $\langle \cdot,\cdot \rangle$ the canonical pairing between $N=\mathbb{Z}^n$ and its dual $M=\mathbb{Z}^n$. The *Newton polyhedron* of $\mathfrak{a}$ is

$$P(\mathfrak{a}) := \mathrm{conv}\{u \in \mathbb{Z}^n_{\geq 0} \mid x^u \in \mathfrak{a}\}.$$

It is a result due to Howald [How01] that

$$\mathscr{J}(\mathbb{A}^n,\mathfrak{a}^q) = (x^u \mid u \in \mathbb{Z}^n_{\geq 0}, u+e \in \mathrm{Int}(qP(\mathfrak{a}))), \tag{4.3}$$

where $e=(1,\ldots,1)$.

In order to prove (4.3) we use some basic facts of toric geometry, as in Example 3.1.19. We consider $\mathbb{A}^n$ with the standard structure of toric variety corresponding to the lattice N and to the cone $\mathbb{R}^n_{\geq 0} \subseteq N_{\mathbb{R}}$. Let $f\colon W \to \mathbb{A}^n$ be the normalized blow-up of X along $\mathfrak{a}$. Since $\mathfrak{a}$ is a monomial ideal, the action of the torus $T=T_N$ on $\mathbb{A}^n$ has an induced action on W that makes W a toric variety and f a toric morphism. If $g\colon Y \to W$ is a projective birational morphism induced by a fan refinement, such that Y is a smooth toric variety, then $\mathfrak{a}\cdot\mathscr{O}_Y = \mathscr{O}_Y(-E)$ for a toric divisor E. We see that $f \circ g$ is a log resolution of $(X,\mathfrak{a})$. Since $K_{Y/\mathbb{A}^n} - \lfloor qE \rfloor$ is a toric divisor on Y, it follows that the multiplier ideal $\mathscr{J}(\mathbb{A}^n,\mathfrak{a}^q)$ is preserved by the torus action, hence it is generated by monomials.

Therefore it is enough to check that a monomial x^u lies in $\mathscr{J}(\mathbb{A}^n,\mathfrak{a}^q)$ if and only if $u+e \in \mathrm{Int}(qP(\mathfrak{a}))$. Recall that each prime divisor D on Y corresponds to a primitive ray generator $v \in \mathbb{Z}^n_{\geq 0}$ for the fan of Y. Furthermore, each primitive nonzero element $v \in \mathbb{Z}^n_{\geq 0}$ corresponds to a divisor on some variety Y as above. It follows from definition that

$$\mathrm{ord}_D(\mathfrak{a}) = \min\{\langle w,v \rangle \mid w \in P(\mathfrak{a}\})$$

and we have seen in Example 3.1.19 that $\mathrm{ord}_D(K_{Y/\mathbb{A}^n}) = \langle e,v \rangle$. It follows that $x^u \in \mathscr{J}(\mathbb{A}^n,\mathfrak{a}^q)$ if and only if

$$\langle u+e,v \rangle > \min\{\langle w,v \rangle \mid w \in qP(\mathfrak{a})\}$$

for all primitive $v \in \mathbb{Z}^n_{\geq 0}$. This is the case if and only if $u+e \in \mathrm{Int}(qP(\mathfrak{a}))$.

Example 4.1.13. Suppose, for example, that $\mathfrak{a} = (x_1^{a_1},\ldots,x_n^{a_n}) \subset k[x_1,\ldots,x_n]$, for positive integers $a_1,\ldots,a_n$. The Newton polyhedron of $\mathfrak{a}$ is given by

$$P(\mathfrak{a}) = \left\{ u \in \mathbb{R}^n_{\geq 0} \mid \frac{u_1}{a_1} + \ldots + \frac{u_n}{a_n} \geq 1 \right\}.$$

It follows from Example 4.1.12 that

$$\mathscr{J}(\mathbb{A}^n,\mathfrak{a}^q)=\left(x^u \mid u\in\mathbb{Z}^n_{\geq 0}, \frac{u_1+1}{a_1}+\ldots+\frac{u_n+1}{a_n}>q\right).$$

In particular, we see that

$$(\mathbb{A}^n,\mathfrak{a}^q) \text{ is klt if and only if } q<\sum_{i=1}^n\frac{1}{a_i}.$$

Example 4.1.14. Let $f=\sum_{i=1}^n x_i^{a_i}\in k[x_1,\ldots,x_n]$, for positive integers $a_1,\ldots,a_n$. Note that if $\lambda_1,\ldots,\lambda_n\in k$ are nonzero, then there is an automorphism of $k[x_1,\ldots,x_n]$ that takes f to $\sum_{i=1}^n\lambda_i x_i^{a_i}$. It follows from Example 4.1.11 that for every $q<1$, we have

$$\mathscr{J}(X,f^q)=\mathscr{J}(\mathbb{A}^n,(x_1^{a_1},\ldots,x_n^{a_n})^q).$$

For example, it follows that (X,f^q) is klt if and only if $q<\min\left\{1,\sum_{i=1}^n\frac{1}{q_i}\right\}$.

Example 4.1.15. Let $(X_1,\Delta_1,\mathscr{Z}_1)$ and $(X_2,\Delta_2,\mathscr{Z}_2)$ be two log triples. Consider, as in Example 3.1.10 $X=X_1\times X_2$, with canonical projections $p_i\colon X\to X_i$, for $i=1,2$. We also consider $\Delta=p_1^*(\Delta_1)+p_2^*(\Delta_2)$ and $\mathscr{Z}=p_1^{-1}(\mathscr{Z})+p_2^{-1}(\mathscr{Z})$, so that we have a log triple $(X,\Delta,\mathscr{Z})$. If $f_i\colon Y_i\to X_i$ is a log resolution of $(X_i,\Delta_i,\mathscr{Z}_i)$ for $i=1,2$, then $f=f_1\times f_2\colon Y=Y_1\times Y_2\to X$ is a log resolution of $(X,\Delta,\mathscr{Z})$. If $q_i\colon Y\to Y_i$, for $i=1,2$ are the canonical projections, then

$$\lfloor\Delta_Y+f^{-1}(\mathscr{Z})\rfloor=p_1^*(\lfloor(\Delta_1)_{Y_1}+f_1^{-1}(\mathscr{Z}_1)\rfloor)+p_2^*(\lfloor(\Delta_2)_{Y_2}+f_2^{-1}(\mathscr{Z}_2)\rfloor).$$

Using the Künneth formula, we deduce

$$\mathscr{J}(X,\Delta,\mathscr{Z})=\mathscr{J}(X_1,\Delta_1,\mathscr{Z}_1)\cdot\mathscr{O}_X+\mathscr{J}(X_2,\Delta_2,\mathscr{Z}_2)\cdot\mathscr{O}_X.$$

Proposition 4.1.16. *If $(X,\Delta,\mathscr{Z})$ is a log triple and $g\colon W\to X$ is any projective, birational morphism, with W normal, then*

$$\mathscr{J}(X,\Delta,\mathscr{Z})=g_*\mathscr{J}(W,\Delta_W,g^{-1}(\mathscr{Z})).$$

Proof. Let $f\colon Y\to W$ be such that $g\circ f$ is a log resolution of $(X,\Delta,\mathscr{Z})$, in which case f is a log resolution of $(W,\Delta_W,g^{-1}(\mathscr{Z}))$. If we compute $\mathscr{J}(X,\Delta,\mathscr{Z})$ and $\mathscr{J}(W,\Delta_W,g^{-1}(\mathscr{Z}))$ using $g\circ f$ and f, respectively, we obtain

$$g_*\mathscr{J}(W,\Delta_W,g^{-1}(\mathscr{Z}))=g_*f_*\mathscr{J}(Y,\Delta_Y,(g\circ f)^{-1}(\mathscr{Z}))=\mathscr{J}(X,\Delta,\mathscr{Z}).$$

□

The following proposition gives some monotonicity properties of multiplier ideals.

Proposition 4.1.17. *Suppose that $(X,\Delta,\mathscr{Z})$ is a log triple, Δ' is an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor, and $\mathscr{Z}'$ is an effective linear combination of proper closed subschemes of X. In this case, we have*

$$\mathscr{J}(X,\Delta+\Delta',\mathscr{Z}+\mathscr{Z}') \subseteq \mathscr{J}(X,\Delta,\mathscr{Z}). \tag{4.4}$$

In particular, given a log triple $(X,\Delta,\mathfrak{a}^q)$, then for every $q' \geq q$, we have

$$\mathscr{J}(X,\Delta,\mathfrak{a}^{q'}) \subseteq \mathscr{J}(X,\Delta,\mathfrak{a}^q). \tag{4.5}$$

Similarly, if $\mathfrak{b}$ is another ideal such that $\mathfrak{a} \subseteq \mathfrak{b}$, then

$$\mathscr{J}(X,\Delta,\mathfrak{a}^q) \subseteq \mathscr{J}(X,\Delta,\mathfrak{b}^q) \tag{4.6}$$

for every $q \in \mathbb{R}_{\geq 0}$.

Proof. In order to prove the first assertion, consider a log resolution $f\colon Y \to X$ of both $(X,\Delta,\mathscr{Z})$ and $(X,\Delta+\Delta',\mathscr{Z}+\mathscr{Z}')$. Since

$$(\Delta+\Delta')_Y + f^{-1}(\mathscr{Z}+\mathscr{Z}') - \big(\Delta_Z + f^{-1}(\mathscr{Z})\big) = f^*(\Delta') + f^{-1}(\mathscr{Z}')$$

is an effective divisor, it follows that we have an inclusion of sheaves on Y

$$\mathscr{O}_Y(-\lfloor(\Delta+\Delta')_Z + f^{-1}(\mathscr{Z}+\mathscr{Z}')\rfloor) \hookrightarrow \mathscr{O}_Y(-\lfloor\Delta_Y + f^{-1}(\mathscr{Z})\rfloor),$$

and applying f_* gives the inclusion in (4.4). The inclusion in (4.5) is a special case. For the last assertion, consider a log resolution $g\colon W \to X$ of $(X,\Delta,\mathfrak{a}\cdot\mathfrak{b})$. In this case, if $\mathfrak{a}\cdot\mathscr{O}_W = \mathscr{O}_W(-E)$ and $\mathfrak{b}\cdot\mathscr{O}_W = \mathscr{O}_W(-F)$, then $E-F$ is an effective divisor, and the inclusion in (4.6) follows as above by applying f_* to the corresponding inclusion of sheaves on W. $\square$

We now show that multiplier ideals are unchanged by a small increase in the coefficients.

Proposition 4.1.18. *Given a log triple $(X,\Delta,\mathscr{Z})$, with $\mathscr{Z} = \sum_{i=1}^r q_i Z_i$, there is $\varepsilon > 0$ such that*

$$\mathscr{J}(X,\Delta,\mathscr{Z}) = \mathscr{J}(X,\Delta,\mathscr{Z}')$$

whenever $\mathscr{Z}' = \sum_{i=1}^r q_i' Z_i$, with $q_i \leq q_i' \leq q_i + \varepsilon$ for all i.

Proof. Let $f\colon Y \to X$ be a log resolution of $(X,\Delta,\mathscr{Z})$. The assertion in the proposition follows from the fact that

$$\lfloor\Delta_Y + \sum_{i=1}^r q_i f^{-1}(Z_i)\rfloor = \lfloor\Delta_Y + \sum_{i=1}^r q_i' f^{-1}(Z_i)\rfloor$$

if $0 \leq q_i' - q_i \ll 1$ for all i. $\square$

4.1.2 Nadel vanishing theorem

The following theorem is behind many of the applications of multiplier ideals. As we will see later, it allows us in particular to translate Kawamata–Viehweg vanishing on a log resolution as a vanishing theorem on the original variety, involving the twist by a multiplier ideal.

Theorem 4.1.19 (Relative vanishing). *Let $(X,\Delta,\mathscr{Z})$ be a rational log triple, with $\mathscr{Z}$ effective. If $f\colon Y \to X$ is a log resolution of $(X,\Delta,\mathscr{Z})$, then*

$$R^i f_*\mathscr{O}_Y(-\lfloor \Delta_Y + f^{-1}(\mathscr{Z})\rfloor) = 0 \textit{ for all } i \geq 1.$$

Proof. We can write

$$-\lfloor \Delta_Y + f^{-1}(\mathscr{Z})\rfloor = \lceil -\Delta_Y - f^{-1}(\mathscr{Z})\rceil = K_Y + \lceil -f^*(K_X+\Delta) - f^{-1}(\mathscr{Z})\rceil.$$

Note that the divisor

$$\lceil -f^*(K_X+\Delta) - f^{-1}(\mathscr{Z})\rceil + f^*(K_X+\Delta) + f^{-1}(\mathscr{Z}) = -\lfloor \Delta_Y + f^{-1}(\mathscr{Z})\rfloor + \Delta_Y + f^{-1}(\mathscr{Z})$$

has simple normal crossings by the assumption that f is a log resolution of $(X,\Delta,\mathscr{Z})$. Since f is birational, every divisor on Y is f-big. Moreover, $f^*(K_X+\Delta)$ is f-trivial and $-f^{-1}(\mathscr{Z})$ is f-nef by Lemma 3.2.4. Therefore $-f^*(K_X+\Delta)-f^{-1}(\mathscr{Z})$ is f-big and f-nef and the assertion in the theorem follows from Theorem 2.6.1. □

We can now deduce the following generalization of the Kawamata–Viehweg vanishing theorem. It is an algebraic version of a theorem due to Nadel in the analytic setting, but in this algebraic framework it first appeared in the work of Esnault and Viehweg.

Theorem 4.1.20 (Nadel). *Let $(X,\Delta,\mathscr{Z})$ be a rational log triple, with X a projective variety. Suppose that $\mathscr{Z} = \sum_{j=1}^r q_j Z_j$, with $q_j \in \mathbb{Q}_{\geq 0}$, and for every j we have a Cartier divisor A_j on X such that $I_{Z_j} \otimes \mathscr{O}_X(A_j)$ is globally generated, where I_{Z_j} is the ideal defining Z_j. If A is a Cartier divisor such that $A-(K_X+\Delta)-\sum_{j=1}^r q_j A_j$ is big and nef, then*

$$H^i(X, \mathscr{J}(X,\Delta,\mathscr{Z}) \otimes \mathscr{O}_X(A)) = 0 \textit{ for all } i \geq 1.$$

Proof. Let $f\colon Y \to X$ be a log resolution of $(X,\Delta,\mathscr{Z})$. If $E_j = f^{-1}(Z_j)$, the hypothesis on A_j implies that $\mathscr{O}_Y(f^*(A_j)-E_j)$ is globally generated. In particular, $f^*(A_j)-E_j$ is nef for every j. Let $B = \Delta_Y + f^{-1}(\mathscr{Z})$. It follows from the definition and the projection formula that

$$\mathscr{J}(X,\Delta,Z)\otimes \mathscr{O}_X(A) \simeq f_*\mathscr{O}_Y(f^*(A) - \lfloor B\rfloor).$$

Furthermore, the projection formula and Theorem 4.1.19 imply

$$R^p f_*\mathscr{O}_Y(f^*(A)-\lfloor B\rfloor) = 0 \text{ for all } p \geq 1.$$

We deduce using the Leray spectral sequence that

$$H^i(X, \mathscr{J}(X,\Delta,Z)\otimes \mathscr{O}_X(A)) \simeq H^i(Y, \mathscr{O}_Y(f^*(A)-\lfloor B\rfloor)). \tag{4.7}$$

On the other hand, we can write

$$f^*(A) - \lfloor B \rfloor = K_Y + \lceil f^*(A) - (K_Y + \Delta_Y) - f^{-1}(\mathscr{Z}) \rceil$$
$$= K_Y + \lceil f^*(A - (K_X + \Delta) - \sum_{j=1}^{r} q_j A_j) + \sum_{j=1}^{r} q_j (f^*(A_j) - E_j) \rceil.$$

The divisor under the round-up sign is big and nef, as the sum of the pull-back via f of a big and nef divisor with a nef divisor. Furthermore, the divisor

$$B - \lfloor B \rfloor = \Delta_Y + f^{-1}(\mathscr{Z}) + \lceil -\Delta_Y - f^{-1}(\mathscr{Z}) \rceil$$

has simple normal crossings, since f is a log resolution of $(X, \Delta, \mathscr{Z})$. Therefore the desired vanishings follow from (4.7) and the Kawamata–Viehweg vanishing theorem. □

Remark 4.1.21. It follows from the proof of Theorem 4.1.20 that if we can find a log resolution $f\colon Y \to X$ of $(X, \Delta, \mathscr{Z})$ such that $f^*(A_j) - E_j$ is big for some j with $q_j > 0$, then the same vanishings hold if we only assume that $A - (K_X + \Delta) - \sum_{j=1}^{r} q_j A_j$ is nef, instead of big and nef.

Corollary 4.1.22. *Under the hypothesis of Theorem 4.1.20, if H is a Cartier divisor on X such that $\mathscr{O}_X(H)$ is ample and globally generated, then the sheaf $\mathscr{J}(X, \Delta, \mathscr{Z}) \otimes \mathscr{O}_X(A + mH)$ is globally generated for every $m \geq \dim(X)$.*

Proof. It follows from Theorem 4.1.20 that

$$H^i(X, \mathscr{J}(X, \Delta, Z) \otimes \mathscr{O}_X(A + (m-i)H)) = 0 \text{ for all } i \geq 1.$$

Therefore the sheaf $\mathscr{J}(X, \Delta, Z) \otimes \mathscr{O}_X(A + mH)$ is 0-regular with respect to $\mathscr{O}_X(H)$, hence globally generated by Theorem 2.4.3. □

Another consequence of vanishing theorems is the following non-vanishing result.

Corollary 4.1.23. *Under the hypothesis of Theorem 4.1.20, if A' is a big and nef Cartier divisor on X, then there is i, with $0 \leq i \leq n = \dim(X)$, such that*

$$H^0(X, \mathscr{J}(X, \Delta, \mathscr{Z}) \otimes \mathscr{O}_X(A + iA')) \neq 0.$$

Proof. It follows from Theorem 4.1.20 that

$$Q(i) := h^0(X, \mathscr{J}(X, \Delta, Z) \otimes \mathscr{O}_X(A + iA')) = \chi(X, \mathscr{J}(X, \Delta, Z) \otimes \mathscr{O}_X(A + iA'))$$

for every $i \geq 0$. We deduce using Proposition 1.2.1 that $Q(i)$ is a polynomial in i of degree $\leq n$. If it vanishes for $(n+1)$ values of i, then it is identically zero. On the other hand, since $\mathscr{J}(X, \Delta, \mathscr{Z})$ is a nonzero fractional ideal and A' is big, we deduce from Lemma 1.4.17 that $Q(i) \geq Ci^n$ for some $C > 0$ and all $i \gg 0$. This gives a contradiction, and thus proves the assertion in the corollary. □

4.2 Asymptotic multiplier ideals

Some of the most powerful applications of multiplier ideals come from an asymptotic version of such ideals that we now describe. We refer to [Laz04b, Chap. 10] for a more detailed introduction to this topic.

4.2.1 *Multiplier ideals for graded sequences*

An asymptotic multiplier ideal is associated to a graded sequence of ideals, in the sense of Definition 1.7.1. We show that given a graded sequence of ideals, one can use the Noetherian property to select a multiplier ideal from those associated to the different elements of the sequence. This is based on the following simple lemma.

Lemma 4.2.1. *If (X,Δ) is a log pair and $\mathfrak{a}_\bullet$ is a nonzero graded sequence of ideals on X, then for every $m,p \geq 1$ such that $\mathfrak{a}_m \neq 0$, and every $\lambda \in \mathbb{R}_{\geq 0}$, we have the following inclusion of multiplier ideals*

$$\mathscr{J}(X,\Delta,\mathfrak{a}_m^{\lambda/m}) \subseteq \mathscr{J}(X,\Delta,\mathfrak{a}_{mp}^{\lambda/mp}).$$

Proof. It follows from definition that

$$\mathscr{J}(X,\Delta,\mathfrak{a}_m^{\lambda/m}) = \mathscr{J}(X,\Delta,(\mathfrak{a}_m^p)^{\lambda/pq}).$$

On the other hand, the defining property of a graded sequence implies $\mathfrak{a}_m^p \subseteq \mathfrak{a}_{mp}$, hence Proposition 4.1.17 gives

$$\mathscr{J}(X,\Delta,(\mathfrak{a}_m^p)^{\lambda/pq}) \subseteq \mathscr{J}(X,\Delta,(\mathfrak{a}_{mp})^{\lambda/pq}).$$

We thus have the inclusion in the lemma. □

Corollary 4.2.2. *If (X,Δ) is a log pair and $\mathfrak{a}_\bullet$ is a nonzero graded sequence of ideals on X, then for every $\lambda \in \mathbb{R}_{\geq 0}$, there is a positive integer q with $\mathfrak{a}_q \neq 0$ such that for every m with $\mathfrak{a}_m \neq 0$ we have $\mathscr{J}(X,\Delta,\mathfrak{a}_m^{\lambda/m}) \subseteq \mathscr{J}(X,\Delta,\mathfrak{a}_q^{\lambda/q})$, with equality if m is divisible by q.*

Proof. For every m such that $\mathfrak{a}_m \neq 0$, the multiplier ideal $\mathscr{J}(X,\Delta,\mathfrak{a}_m^{\lambda/m})$ is contained in the fractional ideal $\mathscr{J}(X,\Delta)$. The Noetherian property of this fractional ideal implies that the set

$$\{\mathscr{J}(X,\Delta,\mathfrak{a}_m^{\lambda/m}) \mid \mathfrak{a}_m \neq 0\}$$

contains a maximal element $J = \mathscr{J}(X,\Delta,\mathfrak{a}_q^{\lambda/q})$. On the other hand, Lemma 4.2.1 implies that for every m such that $\mathfrak{a}_m \neq 0$, we have

$$\mathscr{J}(X,\Delta,\mathfrak{a}_m^{\lambda/m}) \subseteq \mathscr{J}(X,\Delta,\mathfrak{a}_{mq}^{\lambda/mq}) \text{ and } \mathscr{J}(X,\Delta,\mathfrak{a}_q^{\lambda/q}) \subseteq \mathscr{J}(X,\Delta,\mathfrak{a}_{mq}^{\lambda/mq}).$$

The maximality of J implies that $J = \mathscr{J}(X,\Delta,\mathfrak{a}_{mq}^{\lambda/mq})$ and therefore $\mathscr{J}(X,\Delta,\mathfrak{a}_m^{\lambda/m}) \subseteq J$. This completes the proof of the corollary. $\square$

Definition 4.2.3. If (X,Δ) is a log pair and $\mathfrak{a}_\bullet$ is a nonzero graded sequence of ideals on X, then for every $\lambda \in \mathbb{R}_{\geq 0}$, the *asymptotic multiplier ideal* $\mathscr{J}(X,\Delta,\mathfrak{a}_\bullet^\lambda)$ is the unique maximal element of the set of multiplier ideals $\mathscr{J}(X,\Delta,\mathfrak{a}_m^{\lambda/m})$, for m such that $\mathfrak{a}_m \neq 0$. Note that this is, in general, a fractional ideal, but it is an ideal in $\mathscr{O}_X$ whenever Δ is effective.

Definition 4.2.4. An important special case of the previous definition is the following. If (X,Δ) is a log pair and $\mathscr{L}$ is a line bundle on X such that $h^0(X,\mathscr{L}^m) \geq 1$ for some positive integer m, then we put

$$\mathscr{J}(X,\Delta,\lambda\cdot \parallel \mathscr{L} \parallel) := \mathscr{J}(X,\Delta,\mathfrak{a}_\bullet^\lambda),$$

where $\mathfrak{a}_\bullet$ is the graded sequence of base-loci ideals corresponding to $\mathscr{L}$. More generally, if $V_\bullet$ is a graded linear series corresponding to a line bundle $\mathscr{L}$, such that $V_m \neq 0$ for some m, and $\mathfrak{a}_\bullet$ is the corresponding graded sequence of ideals, then we put

$$\mathscr{J}(X,\Delta,\lambda\cdot \parallel V_\bullet \parallel) := \mathscr{J}(X,\Delta,\mathfrak{a}_\bullet^\lambda).$$

Remark 4.2.5. If (X,Δ) is a log pair and $\mathscr{L}$ is a line bundle on X such that $h^0(X,\mathscr{L}^m) \geq 1$ for some positive integer m, then

$$\mathscr{J}(X,\Delta,\lambda\cdot \parallel \mathscr{L}^q \parallel) = \mathscr{J}(X,\Delta,\lambda q\cdot \parallel \mathscr{L} \parallel) \tag{4.8}$$

for every positive integer q. Indeed, if $\mathfrak{a}_m$ is the ideal defining the base-locus of $\mathscr{L}^m$, then for m divisible enough, both ideals in (4.8) are equal to $\mathscr{J}(X,\Delta,\mathfrak{a}_{mq}^{\lambda/m})$.

Definition 4.2.6. If (X,Δ) is a log pair, and M is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X such that $h^0(X,\mathscr{O}_X(mM)) \geq 1$ for m sufficiently divisible, then for every $\lambda \in \mathbb{R}_{\geq 0}$, we put

$$\mathscr{J}(X,\Delta,\lambda\cdot \parallel M \parallel) := \mathscr{J}(X,\Delta,(\lambda/m)\cdot \parallel \mathscr{O}_X(mM) \parallel),$$

where m is a positive integer that is divisible enough. It follows from Remark 4.2.5 that the definition is independent of m, and furthermore, if $\lambda' \in \mathbb{Q}_{\geq 0}$, then

$$\mathscr{J}(X,\Delta,\lambda\cdot \parallel \lambda' M \parallel) = \mathscr{J}(X,\Delta,\lambda\lambda'\cdot \parallel M \parallel).$$

4.2.2 Basic properties of asymptotic multiplier ideals

We now deduce the basic properties of multiplier ideals of graded sequences from the corresponding properties of multiplier ideals associated to triples.

Proposition 4.2.7. *Let (X,Δ) be a log pair and $\mathfrak{a}_\bullet$ and $\mathfrak{b}_\bullet$ nonzero graded sequences of ideals on X.*

i) We have
$$\mathscr{J}(X,\Delta,\mathfrak{a}_\bullet^\lambda) \subseteq \mathscr{J}(X,\Delta,\mathfrak{a}_\bullet^\mu)$$
for all $\lambda,\mu \in \mathbb{R}_{\geq 0}$ *with* $\lambda \geq \mu$.

ii) For every $\lambda \in \mathbb{R}_{\geq 0}$*, there is* $\varepsilon > 0$ *such that*
$$\mathscr{J}(X,\Delta,\mathfrak{a}_\bullet^\lambda) = \mathscr{J}(X,\Delta,\mathfrak{a}_\bullet^{\lambda'})$$
for all λ' *with* $\lambda \leq \lambda' \leq \lambda+\varepsilon$.

iii) If $\mathfrak{c}$ *is a nonzero ideal on* X *and* r *is a non-negative integer such that* $\mathfrak{c}\cdot\mathfrak{a}_m \subseteq \mathfrak{b}_{m+r}$ *for all* $m \gg 0$*, then*
$$\mathscr{J}(X,\Delta,\mathfrak{a}_\bullet^\lambda) \subseteq \mathscr{J}(X,\Delta,\mathfrak{b}_\bullet^\lambda)$$
for all $\lambda \in \mathbb{R}_{\geq 0}$.

Proof. In order to prove i), we choose m divisible enough and use Proposition 4.1.17 to get
$$\mathscr{J}(X,\Delta,\mathfrak{a}_\bullet^\lambda) = \mathscr{J}(X,\Delta,\mathfrak{a}_m^{\lambda/m}) \subseteq \mathscr{J}(X,\Delta,\mathfrak{a}_m^{\mu/m}) = \mathscr{J}(X,\Delta,\mathfrak{a}_\bullet^\mu).$$

For ii), let m be such that $\mathscr{J}(X,\Delta,\mathfrak{a}_\bullet^\lambda) = \mathscr{J}(X,\Delta,\mathfrak{a}_m^{\lambda/m})$. It follows from Proposition 4.1.18 that there is $\varepsilon > 0$ such that
$$\mathscr{J}(X,\Delta,\mathfrak{a}_m^{\lambda/m}) = \mathscr{J}(X,\Delta,\mathfrak{a}_m^{(\lambda+\varepsilon)/m}) \subseteq \mathscr{J}(X,\Delta,\mathfrak{a}_\bullet^{\lambda+\varepsilon}).$$
Using also i), we conclude that $\mathscr{J}(X,\Delta,\mathfrak{a}_\bullet^\lambda) = \mathscr{J}(X,\Delta,\mathfrak{a}_\bullet^\mu)$ if $\lambda \leq \mu \leq \lambda+\varepsilon$.

In order to prove iii), we choose m such that $\mathscr{J}(X,\Delta,\mathfrak{a}_\bullet) = \mathscr{J}(X,\Delta,\mathfrak{a}_m^{\lambda/m})$. It follows from Proposition 4.1.18 that if $q \gg 0$, then
$$\mathscr{J}(X,\Delta,\mathfrak{a}_m^{\lambda/m}) = \mathscr{J}(X,\Delta,\mathfrak{c}^{\lambda/mq}\mathfrak{a}_m^{\lambda/m}) = \mathscr{J}(X,\Delta,(\mathfrak{c}\mathfrak{a}_m^q)^{\lambda/mq}) \subseteq \mathscr{J}(X,\Delta,(\mathfrak{c}\mathfrak{a}_{mq})^{\lambda/mq}).$$
On the other hand, it follows from hypothesis that for $q \gg 0$ we have
$$\mathscr{J}(X,\Delta,(\mathfrak{c}\mathfrak{a}_{mq})^{\lambda/mq}) \subseteq \mathscr{J}(X,\Delta,\mathfrak{b}_{mq+r}^{\lambda/mq}) \subseteq \mathscr{J}(X,\Delta,\mathfrak{b}_\bullet^{\lambda(mq+r)/mq}).$$
Furthermore, we deduce from ii) that for $q \gg 0$, we have
$$\mathscr{J}(X,\Delta,\mathfrak{b}_\bullet^{\lambda(mq+r)/mq}) = \mathscr{J}(X,\Delta,\mathfrak{b}_\bullet^\lambda).$$
By combining these facts, we obtain the assertion in iii). □

Theorem 4.2.8. *Let* (X,Δ) *be a rational log pair, with* X *projective, and* M *a* $\mathbb{Q}$*-Cartier* $\mathbb{Q}$*-divisor on* X *such that* $h^0(X,\mathscr{O}_X(mM)) \geq 1$ *for positive integers* m *that are divisible enough.*

1) If $\lambda \in \mathbb{R}_{\geq 0}$ *and* A *is a Cartier divisor on* X *such that* $A-(K_X+\Delta)-\lambda M$ *is big and nef, then*
$$H^i(X,\mathscr{J}(X,\Delta,\lambda\cdot\parallel M\parallel)\otimes\mathscr{O}_X(A)) = 0 \text{ for all } i \geq 1. \tag{4.9}$$

2) If $\lambda > 0$ and M is big, then it is enough to assume that $A-(K_X+\Delta)-\lambda M$ is just nef, in order to have the vanishing in (4.9). In particular, if both $K_X+\Delta$ and M are Cartier divisors, then

$$H^i(X, \mathscr{J}(X,\Delta,\| M \|) \otimes \mathscr{O}_X(K_X+\Delta+M)) = 0 \text{ for all } i \geq 1. \tag{4.10}$$

More generally, we have the following variant, that applies to graded linear series.

Theorem 4.2.9. *Let (X,Δ) be a rational log pair, with X projective, D a Cartier divisor on X, and $V_\bullet$ a graded linear series corresponding to $\mathscr{O}_X(D)$, such that $V_m \neq 0$ for some positive integer m.*

1) If $\lambda \in \mathbb{Q}_{\geq 0}$ and A is a Cartier divisor on X such that $A-(K_X+\Delta)-\lambda D$ is big and nef, then

$$H^i(X, \mathscr{J}(X,\Delta,\lambda\cdot \| V_\bullet \|) \otimes \mathscr{O}_X(A)) = 0 \text{ for all } i \geq 1. \tag{4.11}$$

2) If $\lambda \in \mathbb{Q}_{>0}$ and, in addition, some V_m defines a rational map that is birational onto its image, then the vanishing in (4.11) holds if we only assume that $A-(K_X+\Delta)-\lambda D$ is nef. In particular, if $K_X+\Delta$ is a Cartier divisor, then

$$H^i(X, \mathscr{J}(X,\Delta,m\cdot \| V_\bullet \|) \otimes \mathscr{O}_X(K_X+\Delta+mD)) = 0 \text{ for all } i,m \geq 1. \tag{4.12}$$

Proof of Theorems 4.2.8 and 4.2.9. Suppose first that we are in the setting of Theorem 4.2.9. Let $\mathfrak{a}_p$ denote the ideal defining the base locus of V_p. Suppose that p is divisible enough, such that

$$\mathscr{J}(X,\Delta,\lambda\cdot \| V_\bullet \|) = \mathscr{J}(X,\Delta,\mathfrak{a}_p^{\lambda/p}). \tag{4.13}$$

Since $\mathfrak{a}_p \otimes \mathscr{O}_X(pD)$ is globally generated by assumption, the vanishing in (4.11) follows from Theorem 4.1.20.

Suppose now that V_m defines a map $\phi_m : X \dashrightarrow \mathbb{P}^{n_m}$ that is birational onto image. Note first that the same holds for each V_{mq}, for $q \geq 1$. Indeed, suppose that W_{mq} is the subspace of V_{mq} generated by the degree q monomials in the sections in V_m. In this case, we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, dashed, "\phi_m"] \arrow[d, dashed, "\phi_{mq}"'] & \mathbb{P}^{N_m} \arrow[d, hook, "\nu_q"] \\ \mathbb{P}^{N_{mq}} \arrow[r, dashed, "\pi"] & \mathbb{P}^{N}, \end{tikzcd}$$

in which ν_q is a Veronese embedding, ϕ_{md} is the map defined by W_{mq}, and π is a linear projection. Since ϕ_m is birational onto image, it follows that ϕ_{mq} is birational onto image.

We choose q divisible enough, such that (4.13) holds for $p = mq$. Let $f : Y \to X$ be a log resolution of $(X,\Delta,\mathfrak{a}_p)$. If $\mathfrak{a}_p \cdot \mathscr{O}_Y = \mathscr{O}_Y(-E)$, it follows that we can identify

V_p to a linear subspace in $H^0(Y, \mathscr{O}_Y(pf^*(D)-E))$, and the rational map it defines is $\phi_p \circ f$. In particular, $pf^*(D)-E$ is a big divisor. Since $A-(K_X+\Delta)-\lambda D$ is nef, the vanishings in (4.11) follow (see Remark 4.1.21).

Suppose now that we are in the setting of Theorem 4.2.8. If ℓ is a positive integer such that $D = \ell M$ is a Cartier divisor and we take $V_q = H^0(X, \mathscr{O}_X(qD))$, then

$$\mathscr{J}(X, \Delta, \lambda \cdot \parallel M \parallel) = \mathscr{J}(X, \Delta, (\lambda/\ell) \cdot \parallel V_\bullet \parallel),$$

and the assertions in Theorem 4.2.8 follow from those in Theorem 4.2.9. □

In this case, too, we can use the vanishing results in Theorem 4.2.8 in combination with Castelnuovo-Mumford regularity to obtain global generation results.

Corollary 4.2.10. *Let (X, Δ) be a rational log pair, with X projective, M a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X, and A, H two Cartier divisors on X, with $\mathscr{O}_X(H)$ ample and globally generated. If one of the following two conditions holds:*

a) $H^0(X, \mathscr{O}_X(mM)) \neq 0$ for some m such that mM is Cartier, and $A-(K_X+\Delta)-\lambda M$ is big and nef, for some $\lambda \in \mathbb{Q}_{\geq 0}$, or
b) M is big and $A-(K_X+\Delta)-\lambda M$ is nef, for some $\lambda \in \mathbb{Q}_{>0}$,

then $\mathscr{J}(X, \Delta, \lambda \cdot \parallel M \parallel) \otimes \mathscr{O}_X(A+jH)$ is globally generated for every $j \geq n =$ $\dim(X)$.

4.2.3 *Asymptotic multiplier ideals of big and pseudo-effective divisors*

We begin by showing that numerically equivalent big line bundles have the same asymptotic multiplier ideals.

Proposition 4.2.11. *If (X, Δ) is a log pair, with X projective, and D, D' are big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors such that $D'-D$ is nef, then*

$$\mathscr{J}(X, \Delta, \lambda \cdot \parallel D \parallel) \subseteq \mathscr{J}(X, \Delta, \lambda \cdot \parallel D' \parallel) \textit{ for every } \lambda \in \mathbb{R}_{\geq 0}.$$

In particular, if D and D' are numerically equivalent, then

$$\mathscr{J}(X, \Delta, \lambda \cdot \parallel D \parallel) = \mathscr{J}(X, \Delta, \lambda \cdot \parallel D' \parallel) \textit{ for every } \lambda \in \mathbb{R}_{\geq 0}.$$

Proof. The first assertion follows from Proposition 4.2.7iii) and Lemma 1.7.16. The second assertion is an immediate consequence of the first one. □

Remark 4.2.12. If (X, Δ) is a log pair and D, D' are Cartier divisors on X, with D' big and $D'-D$ nef, and $\mathfrak{a}_\bullet$, $\mathfrak{a}'_\bullet$ are the corresponding graded sequences of base-loci ideals, then there is a nonzero ideal $\mathfrak{c} \subseteq \mathscr{O}_X$ such that

$$\mathfrak{c} \cdot \mathscr{J}(X, \Delta, \mathfrak{a}_\bullet^m) \subseteq \mathfrak{a}'_m \text{ for } m \gg 0. \tag{4.14}$$

Indeed, let H be a very ample Cartier divisor and $n=\dim(X)$. Since D' is big, it follows from Lemma 1.4.14 that there is a positive integer ℓ and an effective Cartier divisor G such that

$$\ell D' - (K_X+\Delta) - nH - G \text{ is ample.}$$

In this case, it follows from Corollary 4.2.10 that for every $m>\ell$

$$\mathscr{J}(X,\Delta,(m-\ell)\cdot \parallel D\parallel)\otimes \mathscr{O}_X(mD'-G) \text{ is globally generated.}$$

We deduce that if $\mathfrak{b}_m$ is the ideal defining the base-locus of $\mathscr{O}_X(mD'-G)$, then

$$\mathscr{O}_X(-G)\cdot \mathscr{J}(X,\Delta,m\cdot \parallel D\parallel)\subseteq \mathscr{O}_X(-G)\cdot \mathscr{J}(X,\Delta,(m-\ell)\cdot \parallel D\parallel)\subseteq \mathscr{O}_X(-G)\cdot \mathfrak{b}_m\subseteq \mathfrak{a}'_m$$

for every $m>\ell$. We thus obtain (4.14) by taking $\mathfrak{c}=\mathscr{O}_X(-G)$.

In particular, we obtain a stronger version of Lemma 1.7.16 when X is a normal variety. Indeed, let us choose Δ such that (X,Δ) is a log pair (for example, we may take $\Delta=-K_X$). In this case, it follows from Remark 4.1.6 that there is a nonzero ideal J on X such that

$$J\cdot \mathfrak{a}_m\subseteq \mathscr{J}(X,\Delta,\mathfrak{a}_m)\subseteq \mathscr{J}(X,\Delta,\mathfrak{a}_\bullet^m).$$

By combining this with (4.14), we conclude that $J\cdot\mathfrak{c}\cdot\mathfrak{a}_m\subseteq\mathfrak{a}'_m$ for all $m\gg 0$.

We now describe an application of Corollary 4.2.10 due to Hacon. The goal is to associate a version of asymptotic multiplier ideal to every pseudo-effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor, by adding a small ample divisor. More precisely, suppose that (X,Δ) is a log pair and $D\in \mathrm{CDiv}(X)_{\mathbb{R}}$ is pseudo-effective. We consider various ample $A\in\mathrm{CDiv}(X)_{\mathbb{R}}$ such that $D+A$ is a $\mathbb{Q}$-divisor (note that this is automatically big). For such A, we consider $\mathscr{J}(X,\Delta,\lambda\cdot\parallel D+A\parallel)$ for $\lambda\in\mathbb{R}_{\geq 0}$.

Proposition 4.2.13. *Let (X,Δ) be a log pair, with X projective, D a pseudo-effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor, and $\lambda\in\mathbb{R}_{\geq 0}$. Among all fractional ideals of the form $\mathscr{J}(X,\Delta,\lambda\cdot\parallel D+A\parallel)$, where A varies over the ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisors such that $D+A\in\mathrm{CDiv}(X)_{\mathbb{Q}}$, there is one contained in all others. Furthermore, there is an open neighborhood $\mathscr{U}_\lambda$ of the origin in $\mathrm{N}^1(X)_{\mathbb{R}}$ such that*

$$\mathscr{J}_+(X,\Delta,\lambda\cdot\parallel D\parallel)=\mathscr{J}(X,\Delta,\lambda\cdot\parallel D+A\parallel)$$

for every $A\in\mathrm{CDiv}(X)_{\mathbb{R}}$ ample, whose numerical class lies in $\mathscr{U}$, and with $D+A\in\mathrm{CDiv}(X)_{\mathbb{R}}$.

Proof. Since the pair (X,Δ) is fixed, in order to simplify the notation, we write $\mathscr{J}(\lambda\cdot\parallel D+A\parallel)$ for $\mathscr{J}(X,\Delta,\lambda\cdot\parallel D+A\parallel)$. Note first that if $A_1,A_2\in\mathrm{CDiv}(X)_{\mathbb{R}}$ are such that both $D+A_1$ and $D+A_2$ are in $\mathrm{CDiv}(X)_{\mathbb{Q}}$ and A_1-A_2 is nef, then Proposition 4.2.11 gives

$$\mathscr{J}(\lambda\cdot\parallel D+A_2\parallel)\subseteq\mathscr{J}(\lambda\cdot\parallel D+A_1\parallel).$$

We choose a very ample Cartier divisor H on X and let $n = \dim(X)$. Suppose that B is a fixed ample Cartier divisor such that $B - (K_X + \Delta) - \lambda D$ is ample. If $A \in \mathrm{CDiv}(X)_{\mathbb{R}}$ is such that $D + A \in \mathrm{CDiv}(X)_{\mathbb{Q}}$ and $B - (K_X + \Delta) - \lambda(D + A)$ is ample, then Corollary 4.2.10 implies that

$$\mathscr{J}(\lambda \cdot \| D + A \|) \otimes \mathscr{O}_X(B + nH)$$

is globally generated (if $\lambda \notin \mathbb{Q}$, then we apply the corollary to some rational $\lambda' > \lambda$ such that $\mathscr{J}(\lambda \cdot \| D + A \|) = \mathscr{J}(\lambda' \cdot \| D + A \|)$ and with $B - (K_X + \Delta) - \lambda'(D + A)$ ample). It follows that $\mathscr{J}(\lambda \cdot \| D + A \|)$ is determined by the linear subspace

$$\begin{aligned} W_A &:= H^0(X, \mathscr{J}(\lambda \cdot \| D + A \|) \otimes \mathscr{O}_X(B + nH)) \\ &\subseteq W = H^0(X, \mathscr{O}_X(B + nH)). \end{aligned}$$

Since W is finite-dimensional, we can find A as above for which W_A is minimal among all such subspaces. We first show that for every $A_1 \in \mathrm{CDiv}(X)_{\mathbb{R}}$ ample such that $D + A_1 \in \mathrm{CDiv}(X)_{\mathbb{Q}}$, we have $\mathscr{J}(\lambda \cdot \| D + A \|) \subseteq \mathscr{J}(\lambda \cdot \| D + A_1 \|)$. Let us choose an ample A_2 such that both $A - A_2$ and $A_1 - A_2$ are ample and lie in $\mathrm{CDiv}(X)_{\mathbb{Q}}$. We have seen that this implies

$$\mathscr{J}(\lambda \cdot \| D + A_2 \|) \subseteq \mathscr{J}(\lambda \cdot \| D + A_1 \|), \ \mathscr{J}(\lambda \cdot \| D + A_2 \|) \subseteq \mathscr{J}(\lambda \cdot \| D + A \|). \tag{4.15}$$

We note that $B - (K_X + \Delta) - \lambda(D + A_2)$ is ample and the second inclusion in (4.15) implies that $W_{A_2} \subseteq W_A$. The minimality in the choice of A implies $W_{A_2} = W_A$ and therefore

$$\mathscr{J}(\lambda \cdot \| D + A_2 \|) = \mathscr{J}(\lambda \cdot \| D + A \|) \subseteq \mathscr{J}(\lambda \cdot \| D + A_1 \|).$$

Suppose now that $\mathscr{U}_\lambda \subseteq \mathrm{N}_1(X)_{\mathbb{R}}$ consists of the classes of those E for which $A - E$ is ample. In this case $\mathscr{U}_\lambda$ is an open neighborhood of the origin and it satisfies the last assertion in the proposition. Indeed, if $A' \in \mathrm{CDiv}(X)_{\mathbb{R}}$ is ample, its class lies in $\mathscr{U}_\lambda$, and $D + A' \in \mathrm{CDiv}(X)_{\mathbb{Q}}$, then

$$\mathscr{J}(\lambda \cdot \| D + A' \|) \subseteq \mathscr{J}(\lambda \cdot \| D + A \|)$$

(this follows since $A - A'$ is ample), while the reverse inclusion follows from the minimality of $\mathscr{J}(\lambda \cdot \| D + A \|)$, which we have proved. $\square$

In the next proposition we collect some basic properties of this version of asymptotic multiplier ideals.

Proposition 4.2.14. *Let (X, Δ) be a log pair, with X projective, $D \in \mathrm{CDiv}(X)_{\mathbb{R}}$ pseudo-effective, and $\lambda \in \mathbb{R}_{\geq 0}$.*

i) If $E \in \mathrm{CDiv}(X)_{\mathbb{R}}$ is numerically equivalent to D, then

$$\mathscr{J}_+(X, \Delta, \lambda \cdot \| D \|) = \mathscr{J}_+(X, \Delta, \lambda \cdot \| E \|).$$

ii) If $\lambda \geq \mu$, then

$$\mathscr{J}_+(X,\Delta,\lambda\cdot \parallel D \parallel) \subseteq \mathscr{J}_+(X,\Delta,\mu\cdot \parallel D \parallel).$$

iii) If $B \in \mathrm{CDiv}(X)_{\mathbb{R}}$ is nef, then

$$\mathscr{J}_+(X,\Delta,\lambda\cdot \parallel D \parallel) \subseteq \mathscr{J}_+(X,\Delta,\lambda\cdot \parallel D+B \parallel).$$

iv) We have $\mathscr{J}_+(X,\Delta,\lambda\cdot \parallel D \parallel) = \mathscr{J}_+(X,\Delta,\parallel \lambda D \parallel)$.

Proof. Note that if $A \in \mathrm{CDiv}(X)_{\mathbb{R}}$ is ample, then we can write $D+A = E+(A+D-E)$ and $A+D-E$ is ample. Therefore the equality in i) follows from definition. In order to prove ii), note that if $A \in \mathrm{CDiv}(X)_{\mathbb{R}}$ is ample and $D+A$ is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor, then we have

$$\mathscr{J}(X,\Delta,\lambda\cdot \parallel D+A \parallel) \subseteq \mathscr{J}(X,\Delta,\mu\cdot \parallel D+A \parallel)$$

by Proposition 4.2.7i). We thus deduce the inclusion in ii) directly from definition.

We now prove iii). Let $A \in \mathrm{CDiv}(X)_{\mathbb{R}}$ be such that $D+B+A$ is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor and

$$\mathscr{J}_+(X,\Delta,\lambda\cdot \parallel D+B \parallel) = \mathscr{J}(X,\Delta,\lambda\cdot \parallel D+B+A \parallel).$$

Since $A+B$ is ample, we may choose $A' \in \mathrm{CDiv}(X)_{\mathbb{R}}$ ample such that $A+B-A'$ is ample and $D+A' \in \mathrm{CDiv}(X)_{\mathbb{Q}}$. In this case, we have

$$\mathscr{J}_+(X,\Delta,\lambda \parallel D \parallel) \subseteq \mathscr{J}(X,\Delta,\lambda \parallel D+A' \parallel) \subseteq \mathscr{J}(X,\Delta,\lambda\cdot \parallel D+B+A \parallel),$$

where the second inclusion follows from Proposition 4.2.11.

In order to prove iv), let $A \in \mathrm{CDiv}(X)_{\mathbb{R}}$ be ample, such that $D+A \in \mathrm{CDiv}(X)_{\mathbb{Q}}$, and $\mathscr{J}_+(X,\Delta,\lambda\cdot \parallel D \parallel) = \mathscr{J}(X,\Delta,\lambda\cdot \parallel D+A \parallel)$. If $\lambda' > \lambda$ is rational and small enough (depending on A), then

$$\mathscr{J}(X,\Delta,\lambda\cdot \parallel D+A \parallel) = \mathscr{J}(X,\Delta,\lambda'\cdot \parallel D+A \parallel) = \mathscr{J}(X,\Delta,\parallel \lambda'(D+A) \parallel).$$

On the other hand, the difference $\lambda'(D+A)-\lambda D = (\lambda'-\lambda)D+\lambda' A$ is ample if $\lambda'-\lambda$ is small enough, hence it follows from definition that

$$\mathscr{J}_+(X,\Delta,\parallel \lambda D \parallel) \subseteq \mathscr{J}(X,\Delta,\parallel \lambda'(D+A) \parallel) = \mathscr{J}_+(X,\Delta,\lambda\cdot \parallel D \parallel).$$

For the reverse inclusion, we choose $F \in \mathrm{CDiv}(X)_{\mathbb{R}}$ ample such that $\lambda D+F \in \mathrm{CDiv}(X)_{\mathbb{Q}}$ and $\mathscr{J}_+(X,\Delta,\parallel \lambda D \parallel) = \mathscr{J}(X,\Delta,\parallel \lambda D+F \parallel)$. Since F is ample, we can choose $F' \in \mathrm{CDiv}(X)_{\mathbb{R}}$ ample such that $F-\lambda F'$ is ample and $D+F' \in \mathrm{CDiv}(X)_{\mathbb{Q}}$. Suppose now that $\mu \in \mathbb{Q}$ is such that $0 < \mu-\lambda \ll 1$, so that

$$(\lambda D+F)-\mu(D+F') = (\lambda-\mu)D+(F-\mu F')$$

is ample. Furthermore, the hypothesis on μ gives

$$\mathscr{J}(X,\Delta,\lambda \parallel D+F' \parallel) = \mathscr{J}(X,\Delta,\mu\cdot \parallel D+F' \parallel) = \mathscr{J}(X,\Delta,\parallel \mu(D+F') \parallel)$$
$$\subseteq \mathscr{J}(X,\Delta,\parallel \lambda D+F \parallel) = \mathscr{J}_+(X,\Delta,\parallel \lambda D \parallel).$$

It then follows from definition that $\mathscr{J}_+(X,\Delta,\lambda\cdot \parallel D \parallel) \subseteq \mathscr{J}_+(X,\Delta,\parallel \lambda D \parallel)$. This completes the proof of iv). □

4.3 Adjoint ideals, the restriction theorem, and subadditivity

In this section we prove one of the central results concerning multiplier ideals, which relates the multiplier ideal on an ambient variety and the multiplier ideal for the restriction to a divisor. As an intermediary for this, we use a variant of multiplier ideals that we now introduce.

4.3.1 Adjoint ideals

There are several notions of adjoint ideals and we now introduce the simplest version. Consider a log triple of the form $(X,S+\Delta,\mathscr{Z})$, where S a prime divisor on X that does not appear in the supports of either Δ or $\mathscr{Z}$. Note that in this case we have $a_S(X,S+\Delta,\mathscr{Z})=0$.

Definition 4.3.1. With the above notation, consider a log resolution $f\colon Y\to X$ of $(X,S+\Delta,\mathscr{Z})$. The *adjoint* ideal $\mathrm{Adj}_S(X,S+\Delta,\mathscr{Z})$ is defined as

$$\mathrm{Adj}_S(X,S+\Delta,\mathscr{Z}) := f_*\mathscr{O}_Y(-\lfloor (S+\Delta)_Y + f^{-1}(\mathscr{Z})\rfloor + \widetilde{S}),$$

where $\widetilde{S}$ is the proper transform of S on Y.

Remark 4.3.2. In general, the adjoint ideal is a (nonzero) fractional ideal. However, when $(X,S+\Delta,\mathscr{Z})$ is an effective pair, then the only divisors that appear in $\lfloor (S+\Delta)_Y + f^{-1}(\mathscr{Z})\rfloor - \widetilde{S}$ with negative coefficients are f-exceptional. Therefore in this case $\mathrm{Adj}_S(X,S+\Delta,\mathscr{Z})$ is an ideal in $\mathscr{O}_X$.

Proposition 4.3.3. *The definition of* $\mathrm{Adj}_S(X,S+\Delta,\mathscr{Z})$ *is independent of the chosen log resolution.*

Proof. Arguing as in the proof of Theorem 4.1.3, we see that it is enough to show that if $f\colon Y\to X$ and $g\colon W\to Y$ are such that both f and $f\circ g$ are log resolutions of $(X,S+\Delta,\mathscr{Z})$, then

$$g_*\mathscr{O}_Z(-\lfloor (S+\Delta)_Z + g^{-1}(f^{-1}(\mathscr{Z}))\rfloor + \widetilde{S}') = \mathscr{O}_Y(-\lfloor (S+\Delta)_Y + f^{-1}(\mathscr{Z})\rfloor + \widetilde{S}), \tag{4.16}$$

where $\widetilde{S}'$ is the proper transform of S on Z. For $A=(S+\Delta)_Y + f^{-1}(\mathscr{Z})$, we can write $A=\widetilde{S}+B+F$, where B is a Cartier divisor, $\lfloor F\rfloor = 0$, and $\widetilde{S}$ does not appear in the support of $B+F$. In this case, the right-hand side of (4.16) is $\mathscr{O}_Y(-B)$. On the other hand, we have

$$\lfloor (S+\Delta)_Z + g^{-1}(f^{-1}(\mathscr{Z}))\rfloor - \widetilde{S}' = g^*(B) + \lfloor g^*(S+F) - K_{W/Y} - \widetilde{S}'\rfloor.$$

Using the projection formula, we see that in order to prove (4.16), it is enough to show that

$$g_*\mathscr{O}_Z(-\lfloor g^*(S+F) - K_{W/Y} - \widetilde{S}'\rfloor) = \mathscr{O}_Y. \tag{4.17}$$

Since $\widetilde{S}+F$ has simple normal crossings, $\lfloor F\rfloor = 0$, and S does not appear in the support of F, it follows from Remark 3.1.31 that the log pair $(Y, \widetilde{S}+F)$ is plt. Therefore $g^*(\widetilde{S}+F) - K_{W/Y} - \widetilde{S}'$ has all coefficients < 1. Moreover, since F is effective, all prime divisors that appear with negative coefficients are g-exceptional. We conclude that $-\lfloor g^*(S+F) - K_{W/Y} - \widetilde{S}'\rfloor$ is an effective g-exceptional divisor, and we obtain (4.17). □

Remark 4.3.4. It follows from the definition of adjoint ideals and the independence of log resolution that for a triple $(X, S+\Delta, \mathscr{Z})$ as in Definition 4.3.1, we have $\mathscr{O}_X \subseteq \mathrm{Adj}_S(X, S+\Delta, \mathscr{Z})$ if and only if $a_E(X, S+\Delta, \mathscr{Z}) \geq 0$ for all divisors E over X, with equality if and only if $E = S$. In particular, we see that in this case $(X, S+\Delta, \mathscr{Z})$ is plt. Furthermore, if S is normal and Cartier, and $(X, S+\Delta, \mathscr{Z})$ is plt, then $\mathscr{O}_X \subseteq \mathrm{Adj}_S(X, S+\Delta, \mathscr{Z})$ in a neighborhood of S. Indeed, in this case every prime divisor $E \neq S$ on X with $a_E(X, S+\Delta, \mathscr{Z}) = 0$ does not intersect S (see Remark 3.1.28).

Remark 4.3.5. If $(X, S+\Delta, \mathscr{Z})$ is a triple as in Definition 4.3.1 and S is $\mathbb{Q}$-Cartier, then for every $\varepsilon > 0$, we have

$$\mathrm{Adj}_S(X, S+\Delta, \mathscr{Z}) \subseteq \mathscr{J}(X, (1-\varepsilon)S+\Delta, \mathscr{Z}). \tag{4.18}$$

Indeed, if $f\colon Y \to X$ is a log resolution of $(X, S+\Delta, \mathscr{Z})$, then

$$\lfloor (S+\Delta)_Y + f^{-1}(\mathscr{Z})\rfloor - \widetilde{S} = \lfloor f^*(S) + \Delta_Y + f^{-1}(\mathscr{Z})\rfloor - \widetilde{S}$$
$$\geq \lfloor (1-\varepsilon)f^*(S) + \Delta_Y + f^{-1}(\mathscr{Z})\rfloor = \lfloor ((1-\varepsilon)S+\Delta)_Y + f^{-1}(\mathscr{Z})\rfloor.$$

By taking the corresponding sheaves and pushing-forward via f, we obtain the inclusion in (4.18).

4.3.2 The restriction theorem

We now turn to one of the most important results concerning multiplier ideals. We will discuss in this section applications of this result to the restriction theorem for multiplier ideals, as well as to vanishing theorems for adjoint ideals. Other implications to extension theorems will be given later.

We fix a rational log triple $(X, S+\Delta, \mathscr{Z})$, where S is a prime divisor on X that is not contained in the support of either Δ or $\mathscr{Z}$. In addition, we assume that $\mathscr{Z}$

is effective and S is normal and a Cartier divisor. Recall that we may consider the restriction $\Delta|_S$ (see Remark 3.1.4) and also $\mathscr{Z}|_S$.

Theorem 4.3.6 (Adjunction sequence). *With the above notation, we have an exact sequence*

$$0 \to \mathscr{J}(X, S+\Delta, \mathscr{Z}) \to \mathrm{Adj}_S(X, S+\Delta, \mathscr{Z}) \to \mathscr{J}(S, \Delta|_S, \mathscr{Z}|_S) \to 0. \tag{4.19}$$

Note that by Example 4.1.5, we can rewrite the first term in the sequence as $\mathscr{J}(X, S+\Delta, \mathscr{Z}) = \mathscr{J}(X, \Delta, \mathscr{Z}) \otimes \mathscr{O}_X(-S)$.

Proof of Theorem 4.3.6. Let $f\colon Y \to X$ be a log resolution of $(X, S+\Delta, \mathscr{Z})$ and write $f^*(S) = \widetilde{S} + F$, where $\widetilde{S}$ is the proper transform of S on Y. Recall that by Example 3.1.9, we have $(\Delta|_S)_{\widetilde{S}} = (F + \Delta_Y)|_{\widetilde{S}}$. Let

$$A = \lfloor (S+\Delta)_Y + f^{-1}(\mathscr{Z}) \rfloor - \widetilde{S} = \lfloor F + \Delta_Y + f^{-1}(\mathscr{Z}) \rfloor$$

and consider the following exact sequence on Y

$$0 \to \mathscr{O}_Y(-A-\widetilde{S}) \to \mathscr{O}_Y(-A) \to \mathscr{O}_Y(-A)|_{\widetilde{S}} \to 0. \tag{4.20}$$

Since we deal with simple normal crossing divisors, restricting to S commutes with rounding-down. Therefore if $g\colon \widetilde{S} \to S$ is the restriction of f, we have

$$A|_{\widetilde{S}} = \lfloor (F+\Delta_Y)|_{\widetilde{S}} + f^{-1}(\mathscr{Z})|_{\widetilde{S}} \rfloor = \lfloor (\Delta|_S)_{\widetilde{S}} + g^{-1}(\mathscr{Z}|_S) \rfloor.$$

Since g is a log resolution of $(S, \Delta|_S, \mathscr{Z}|_S)$, we conclude that $f_*(\mathscr{O}_Y(-A)|_{\widetilde{S}}) = \mathscr{J}(S, \Delta|_S, \mathscr{Z}|_S)$.

On the other hand, it follows from the definition of multiplier ideals that

$$f_*\mathscr{O}_Y(-A-\widetilde{S}) = \mathscr{J}(X, S+\Delta, \mathscr{Z}).$$

Furthermore, Theorem 4.1.19 implies $R^1 f_*\mathscr{O}_Y(-A-\widetilde{S}) = 0$. Therefore by applying f_* to the exact sequence (4.20), the resulting sequence is still exact, and this is precisely the sequence in the theorem. □

We can use the above proof to show that adjoint ideals also satisfy versions of relative vanishing and Nadel vanishing theorems.

Corollary 4.3.7. *Let $(X, S+\Delta, \mathscr{Z})$ be a rational triple, with $\mathscr{Z}$ effective, where S is a prime normal Cartier divisor on X that is not contained in the support of either Δ or $\mathscr{Z}$.*

i) If $f\colon Y \to X$ is a log resolution of $(X, S+\Delta, \mathscr{Z})$, then

$$R^i f_*\mathscr{O}_Y(-\lfloor (S+\Delta)_Y + f^{-1}(\mathscr{Z}) \rfloor + \widetilde{S}) = 0 \textit{ for all } i \geq 1.$$

ii) Suppose that $\mathscr{Z} = \sum_{j=1}^{r} q_j Z_j$*, with each* Z_j *a closed subscheme defined by the ideal* I_{Z_j} *and we have Cartier divisors* A_j *such that* $I_{Z_j} \otimes \mathscr{O}_X(A_j)$ *is globally generated for all j. If X is projective and A is a Cartier divisor such that* $A - (K_X + S + \Delta) - \sum_{j=1}^{r} q_j A_j$ *is big, nef, and its augmented base locus does not contain S, then*

$$H^i(X, \mathrm{Adj}_S(X, S+\Delta, \mathscr{Z}) \otimes \mathscr{O}_X(A)) = 0 \textit{ for all } i \geq 1.$$

Proof. We use the notation in the proof of Theorem 4.3.6. It follows from Theorem 4.1.19 that

$$R^i f_* \mathscr{O}_Y(-A - \widetilde{S}) = 0 \text{ and } R^i f_*(\mathscr{O}_Y(-A)|_{\widetilde{S}}) = 0$$

for all $i \geq 1$. The long exact sequence of higher direct images corresponding to the short exact sequence (4.20) implies the assertion in i).

Similarly, under the assumptions in ii), Theorem 4.1.20 implies

$$H^i(X, \mathscr{J}(X, S+\Delta, \mathscr{Z}) \otimes \mathscr{O}_X(A)) = 0 \text{ and } H^i(S, \mathscr{J}(S, \Delta|_S, \mathscr{Z}|_S) \otimes \mathscr{O}_X(A)|_S) = 0$$

for all $i \geq 1$ (note that since S is not contained in the augmented base locus of $A - (K_X + S + \Delta) - \sum_{j=1}^{r} q_j A_j$, it follows from Remark 1.5.12 that the corresponding restriction to S is big, and clearly also nef). The long exact sequence in cohomology corresponding to the short exact sequence (4.19) gives the assertion in ii). This completes the proof of the corollary. □

As another consequence of Theorem 4.3.6, we obtain the following relation between the multiplier ideal of a triple on X and that of its restriction to a normal Cartier divisor.

Corollary 4.3.8 (Restriction theorem). *Let* $(X, \Delta, \mathscr{Z})$ *be an effective, rational triple, and S a prime divisor on X, which is normal and Cartier, and which is not contained in the support of either* Δ *or* $\mathscr{Z}$*.*

i) We have $\mathrm{Adj}_S(X, S+\Delta, \mathscr{Z}) \cdot \mathscr{O}_S = \mathscr{J}(S, \Delta|_S, \mathscr{Z}|_S)$*.*
ii) In particular, we have $\mathscr{J}(S, \Delta|_S, \mathscr{Z}|_S) \subseteq \mathscr{J}(X, (1-\varepsilon)S + \Delta, \mathscr{Z}) \cdot \mathscr{O}_S$ *for every* $\varepsilon > 0$*.*

Proof. The assertion in i) follows from Theorem 4.3.6 by noting that under our assumptions, $\mathrm{Adj}_S(X, S+\Delta, \mathscr{Z})$ and $\mathscr{J}(S, \Delta|_S, \mathscr{Z}|_S)$ are ideals in $\mathscr{O}_X$ and $\mathscr{O}_S$, respectively, and the corresponding map in the exact sequence in the theorem is induced by restricting sections to S. The inclusion in ii) then follows from the equality in i) and Remark 4.3.5. □

Corollary 4.3.9 (Generic restriction theorem). *With the same assumptions as in Corollary 4.3.8, if S is a general member of a base-point free linear system, then* $\mathscr{J}(X, \Delta, \mathscr{Z}) \cdot \mathscr{O}_S = \mathscr{J}(S, \Delta|_S, \mathscr{Z}|_S)$*.*

Proof. Fix a log resolution $f\colon Y \to X$ of $(X, \Delta, \mathscr{Z})$. If S is general, then f is a log resolution of $(X, S+\Delta, \mathscr{Z})$ and $f^*(S)$ is equal to the proper transform of S. Therefore $\mathscr{J}(X, \Delta, \mathscr{Z}) = \mathrm{Adj}_S(X, S+\Delta, \mathscr{Z})$ and the corollary follows from Corollary 4.3.8. □

Corollary 4.3.10 (Inversion of adjunction). *Let $(X, \Delta, \mathscr{Z})$ be an effective, rational triple, and S a normal, prime divisor on X, which is Cartier, and which is not contained in the support of either Δ or $\mathscr{Z}$. In this case $(S, \Delta|_S, \mathscr{Z}|_S)$ is klt if and only if $(X, S+\Delta, \mathscr{Z})$ is plt in a neighborhood of S. In particular, if $(S, \Delta|_S, \mathscr{Z}|_S)$ is klt, then $(X, \Delta, \mathscr{Z})$ is klt in a neighbourhood of S.*

Proof. Note that if $\mathscr{I}$ is an ideal in $\mathscr{O}_X$, then $\mathscr{I} \cdot \mathscr{O}_S = \mathscr{O}_S$ if and only if $\mathscr{I} = \mathscr{O}_X$ is a neighborhood of S. Therefore the first assertion follows from Corollary 4.3.8 and the fact that $(S, \Delta|_S, \mathscr{Z}|_S)$ is klt if and only if $\mathscr{J}(S, \Delta|_S, \mathscr{Z}|_S) = \mathscr{O}_S$ and $(X, \Delta, \mathscr{Z})$ is plt in a neighborhood of S if and only if $\mathrm{Adj}_S(X, S+\Delta, \mathscr{Z}) = \mathscr{O}_X$ in such a neighhorhood (see Remarks 4.1.4 and 4.3.4). □

Corollary 4.3.11. *Let $(X, \Delta, \mathscr{Z})$ be a rational effective triple, with X a smooth variety. If $Y \hookrightarrow X$ is a smooth closed subvariety of codimension c that is not contained in the support of either Δ or $\mathscr{Z}$, then*

$$\mathscr{J}(Y, \Delta|_Y, \mathscr{Z}|_Y) \subseteq \mathscr{J}(X, \Delta, \mathscr{Z} + (c-\varepsilon)Y) \cdot \mathscr{O}_Y$$

for every $\varepsilon > 0$.

Proof. Since both X and Y are smooth, arguing locally we may assume that X is affine and we have $Y = H_1 \cap \ldots \cap H_c$, for suitable effective Cartier divisors $H_1, \ldots, H_c$. After replacing X by an open neighborhood of Y, we may assume that $H_1 + \ldots + H_c$ is a simple normal crossing divisor. Furthermore, by taking the H_i to be general, we see as in Example 4.1.11 that we may assume that

$$\mathscr{J}(X, \Delta, \mathscr{Z} + (c-\varepsilon)Y) = \mathscr{J}(X, \Delta + \sum_{i=1}^{c} \delta H_i, \mathscr{Z}),$$

where $\delta = \frac{c-\varepsilon}{\varepsilon}$. Let $Y_d = \bigcap_{i=1}^{c-d} H_i$, for $1 \leq d \leq c-1$. Applying Corollary 4.3.8 c times, we obtain

$$\mathscr{J}(Y, \Delta|_Y, \mathscr{Z}|_Y) \subseteq \mathscr{J}(Y_1, \Delta|_{Y_1} + \delta H_n|_{Y_1}, \mathscr{Z}|_{Y_1}) \cdot \mathscr{O}_Y \subseteq \ldots$$

$$\ldots \subseteq \mathscr{J}(Y_{c-1}, \Delta|_{Y_{c-1}} + \sum_{i=1}^{c-1} \delta H_i|_{Y_{c-1}}, \mathscr{Z}|_{Y_{c-1}}) \cdot \mathscr{O}_Y \subseteq \mathscr{J}(X, \Delta, \mathscr{Z} + (c-\varepsilon)Y) \cdot \mathscr{O}_Y.$$

□

Corollary 4.3.12. *If $f\colon W \to X$ is a morphism of smooth varieties and we have a log triple $(X, \Delta, \mathscr{Z})$ such that the image of f is not contained in the support of either Δ or $\mathscr{Z}$, then*

$$\mathscr{J}(W, f^*(\Delta), f^{-1}(\mathscr{Z})) \subseteq \mathscr{J}(X, \Delta, \mathscr{Z}) \cdot \mathscr{O}_W.$$

Proof. Consider the factorization of f as $p \circ g$, where $p\colon W \times X \to X$ is the projection and $g\colon W \hookrightarrow W \times X$ is the graph of f. Therefore it is enough to show that the assertion in the theorem holds for both g and p. For g, this is a consequence of Corollary 4.3.11, while for p, this follows from Example 4.1.15 (in fact, in this case the inclusion is an equality). □

4.3.3 Asymptotic adjoint ideals

We can define asymptotic versions of adjoint ideals in the same way we did it for multiplier ideals. Given a log pair $(X, S+\Delta)$ such that S is a prime divisor not contained in the support of Δ, suppose we have a graded sequence $\mathfrak{a}_\bullet$ such that $\mathfrak{a}_m \cdot \mathscr{O}_S \neq 0$ for some m (hence for all m divisible enough). For every positive integers m and p such that $\mathfrak{a}_m \cdot \mathscr{O}_S \neq 0$, we have

$$\mathrm{Adj}_S(X, S+\Delta, \mathfrak{a}_m^{\lambda/m}) = \mathrm{Adj}_S(X, S+\Delta, (\mathfrak{a}_m^p)^{\lambda/mp}) \subseteq \mathrm{Adj}_S(X, S+\Delta, \mathfrak{a}_{mp}^{\lambda}).$$

Arguing as in the case of multiplier ideals, we see that among the set of ideals

$$\mathrm{Adj}_S(X, S+\Delta, \mathfrak{a}_m^{\lambda/m}), \text{ where } \mathfrak{a}_m \cdot \mathscr{O}_S \neq 0$$

there is a unique smallest one, the *asymptotic adjoint ideal* $\mathrm{Adj}_S(X, S+\Delta, \mathfrak{a}_\bullet^{\lambda})$, equal to $\mathrm{Adj}_S(X, S+\Delta, \mathfrak{a}_m^{\lambda/m})$ for m divisible enough.

In particular, if $\mathscr{L} \in \mathrm{Pic}(X)$ and $\mathfrak{a}_\bullet$ is the corresponding graded sequence of base-loci ideals, we may consider the above definition as long as $S \not\subseteq \mathrm{SB}(\mathscr{L})$. In this case, $\mathrm{Adj}_S(X, S+\Delta, \lambda \cdot \parallel \mathscr{L} \parallel)$ denotes the corresponding asymptotic adjoint ideal. More generally, if $V_\bullet$ is a graded linear series such that $S \not\subseteq \mathrm{Bs}(V_m)$ for some m, then we may define the asymptotic adjoint ideal $\mathrm{Adj}_S(X, S+\Delta, \lambda \cdot \parallel V_\bullet \parallel)$.

As in the case of multiplier ideals, we see that for every positive integer q, we have

$$\mathrm{Adj}_S(X, S+\Delta, \lambda q \cdot \parallel \mathscr{L} \parallel) = \mathrm{Adj}_S(X, S+\Delta, \lambda \cdot \parallel \mathscr{L}^q \parallel).$$

Using this, we define in the obvious way $\mathrm{Adj}_S(X, S+\Delta, \lambda \cdot \parallel D \parallel)$ for $\mathbb{Q}$-divisors D such that $S \not\subseteq \mathrm{SB}(D)$.

The relation between adjoint ideals and multiplier ideals, as well as the vanishing results for adjoint ideals, admit variants in the asymptotic setting.

Corollary 4.3.13. *Let $(X, S+\Delta)$ be a rational log pair, with S a prime normal Cartier divisor on X that is not contained in the support of Δ. If $\mathfrak{a}_\bullet$ is a graded sequence of ideals on X such that $\mathfrak{a}_m \cdot \mathscr{O}_S \neq 0$ for some m, and if $\mathfrak{b}_p = \mathfrak{a}_p|_S$ for all $p \geq 1$, then for every $\lambda \in \mathbb{Q}_{\geq 0}$, there is an exact sequence*

$$0 \to \mathscr{J}(X, S+\Delta, \mathfrak{a}_\bullet^{\lambda}) \to \mathrm{Adj}_S(X, S+\Delta, \mathfrak{a}_\bullet^{\lambda}) \to \mathscr{J}(S, \Delta|_S, \mathfrak{b}_\bullet^{\lambda}) \to 0.$$

Proof. This follows by applying Theorem 4.3.6 with $\mathscr{Z} = \frac{\lambda}{m} \cdot V(\mathfrak{a}_m)$, for m divisible enough, in which case the corresponding adjoint ideal and multiplier ideals are equal to the asymptotic ones in the above statement. □

Corollary 4.3.14. *Let $(X, S+\Delta)$ be a rational log pair, with X projective, and S a prime normal Cartier divisor on X that is not contained in the support of Δ. Suppose that D is a Cartier divisor on X and $V_\bullet$ is a graded linear series corresponding to $\mathscr{O}_X(D)$, such that the base-locus of some V_m does not contain S.*

i) If $\lambda \in \mathbb{Q}_{\geq 0}$ and A is a Cartier divisor such that $A-(K_X+S+\Delta)-\lambda D$ is big and nef and also its restriction to S is big, then

$$H^i(X, \mathrm{Adj}_S(X, S+\Delta, \lambda \cdot \| V_\bullet \|) \otimes \mathscr{O}_X(A)) = 0 \text{ for all } i \geq 1. \tag{4.21}$$

ii) If $\lambda \in \mathbb{Q}_{>0}$ and some V_m gives a rational map that is birational onto its image and whose restriction to S is again birational onto its image[1]*, then for every Cartier divisor A such that $A-(K_X+S+\Delta)-\lambda D$ is nef, the vanishing in (4.21) holds.*

Proof. We use the exact sequence in Corollary 4.3.13, in which we take $\mathfrak{a}_\bullet$ to be the graded sequence of base-loci ideals of $V_\bullet$ (note that the hypothesis implies that $\mathfrak{a}_m \cdot \mathscr{O}_S \neq 0$ for some m). We denote by $W_m \subseteq H^0(S, \mathscr{O}_X(mD)|_S)$ the image of V_m. Note that in case i) the hypothesis says that some W_m is nonzero, while in case ii) it says that some W_m defines a birational map onto image.

It follows from the long exact sequence in cohomology that in order to complete the proof of the corollary, it is enough to note that in both cases i) and ii), the hypotheses guarantee that we can apply Theorem 4.2.9 to deduce that

$$H^i(X, \mathscr{J}(X, S+\Delta, \lambda \cdot \| V_\bullet \|) \otimes \mathscr{O}_X(A)) = 0 \text{ for all } i \geq 1, \text{ and} \tag{4.22}$$

$$H^i(S, \mathscr{J}(S, \Delta|_S, \lambda \cdot \| W_\bullet \|) \otimes \mathscr{O}_X(A)|_S) = 0 \text{ for all } i \geq 1. \tag{4.23}$$

□

4.3.4 Subadditivity

A special property of multiplier ideals in the case of smooth varieties is the following subadditivity theorem, due to Demailly, Ein, and Lazarsfeld [DEL00].

Theorem 4.3.15. *If we consider two effective rational triples $(X, \Delta_1, \mathscr{Z}_1)$ and $(X, \Delta_2, \mathscr{Z}_2)$, where X is a smooth variety, then*

$$\mathscr{J}(X, \Delta_1+\Delta_2, \mathscr{Z}_1+\mathscr{Z}_2) \subseteq \mathscr{J}(X, \Delta_1, \mathscr{Z}_1) \cdot \mathscr{J}(X, \Delta_2, \mathscr{Z}_2). \tag{4.24}$$

[1] This condition is satisfied, for example, if $V_m = H^0(X, \mathscr{O}_X(mD))$ for all $m \geq 1$, and S is not contained in the augmented base locus of D.

Proof. If $p_i\colon X\times X\to X$ are the canonical projections and $\Delta = p_1^*(\Delta_1)+p_2^*(\Delta_2)$ and $\mathscr{Z} = p_1^{-1}(\mathscr{Z}_1)+p_2^{-1}(\mathscr{Z}_2)$, then it follows from Example 4.1.15 that

$$\mathscr{J}(X\times X,\Delta,\mathscr{Z}) = p_1^{-1}\mathscr{J}(X,\Delta_1,\mathscr{Z}_1)+p_2^{-1}\mathscr{J}(X,\Delta_2,\mathscr{Z}_2).$$

On the other hand, it follows from Corollary 4.3.11 applied to the diagonal embedding $X\hookrightarrow X\times X$ that

$$\mathscr{J}(X,\Delta_1+\Delta_2,\mathscr{Z}_1+\mathscr{Z}_2) = \mathscr{J}(X,\Delta|_X,\mathscr{Z}|_X)\subseteq \mathscr{J}(X\times X,\Delta,\mathscr{Z})\cdot\mathscr{O}_X$$
$$= \mathscr{J}(X,\Delta_1,\mathscr{Z}_1)\cdot\mathscr{J}(X,\Delta_2,\mathscr{Z}_2).$$

This completes the proof of the theorem. □

Remark 4.3.16. Eisenstein [Eis] and Takagi [Tak13] have given versions of the subadditivity theorem when X is allowed to be singular. For example, if $\Delta_1=\Delta_2=0$, then one has to multiply the ideal on the left-hand side of (4.24) by the Jacobian ideal of X.

We now give a version of the subadditivity theorem for asymptotic multiplier ideals.

Corollary 4.3.17. *Let X be a smooth variety.*

i) If $\mathfrak{a}_\bullet$ and $\mathfrak{b}_\bullet$ are nonzero graded sequences of ideals on X and we put $\mathfrak{c}_m=\mathfrak{a}_m\cdot\mathfrak{b}_m$ for all $m\geq 1$, then for every $\lambda\in\mathbb{Q}_{\geq 0}$ we have

$$\mathscr{J}(X,\mathfrak{c}_\bullet^\lambda)\subseteq\mathscr{J}(X,\mathfrak{a}_\bullet^\lambda)\cdot\mathscr{J}(X,\mathfrak{b}_\bullet^\lambda).$$

ii) If $\mathfrak{a}_\bullet$ is a nonzero graded sequence of ideals on X, then

$$\mathscr{J}(X,\mathfrak{a}_\bullet^{\lambda+\mu})\subseteq\mathscr{J}(X,\mathfrak{a}_\bullet^\lambda)\cdot\mathscr{J}(X,\mathfrak{a}_\bullet^\mu)$$

for every $\lambda,\mu\in\mathbb{Q}_{\geq 0}$. In particular, $\mathscr{J}(X,\mathfrak{a}_\bullet^{m\lambda})\subseteq\mathscr{J}(X,\mathfrak{a}_\bullet^\lambda)^m$ for all $\lambda\in\mathbb{Q}_{\geq 0}$ and all positive integers m.

Proof. The assertion in i) follows from Theorem 4.3.15 and the fact that if m is divisible enough, then $\mathscr{J}(X,\mathfrak{c}_\bullet^\lambda)=\mathscr{J}(X,\mathfrak{c}_m^{\lambda/m})$, $\mathscr{J}(X,\mathfrak{a}_\bullet^\lambda)=\mathscr{J}(X,\mathfrak{a}_m^{\lambda/m})$, and $\mathscr{J}(X,\mathfrak{b}_\bullet^\lambda)=\mathscr{J}(X,\mathfrak{b}_m^{\lambda/m})$. In order to check the first assertion in i), let m be divisible enough. Using Theorem 4.3.15, we obtain

$$\mathscr{J}(X,\mathfrak{a}_\bullet^{\lambda+\mu})=\mathscr{J}(X,\mathfrak{a}_m^{(\lambda+\mu)/m})=\mathscr{J}(X,\mathfrak{a}_m^{\lambda/m}\cdot\mathfrak{a}_m^{\mu/m})$$
$$\subseteq\mathscr{J}(X,\mathfrak{a}_m^{\lambda/m})\cdot\mathscr{J}(X,\mathfrak{a}_m^{\mu/m})=\mathscr{J}(X,\mathfrak{a}_\bullet^\lambda)\cdot\mathscr{J}(X,\mathfrak{a}_\bullet^\mu).$$

The second assertion in ii) follows easily from the first one by induction on m. □

4.4 Further properties of multiplier ideals

4.5 Kawakita's inversion of adjunction for log canonical pairs

4.6 Analytic approach to multiplier ideals

4.7 Bernstein-Sato polynomials, V-filtrations, and multiplier ideals

Chapter 5
Applications of multiplier ideals

In this chapter we collect several applications of multiplier ideals to geometric problems. Unless explicitly mentioned otherwise, all varieties are assumed to be defined over an algebraically closed field of characteristic zero.

5.1 Asymptotic invariants of divisors, revisited

We now return to the study of asymptotic invariants of linear systems discussed in Section 1.7. Our main goal is to describe, at least on smooth varieties, the non-nef locus of a divisor using the asymptotic invariants. For this, we follow the approach in [ELM+06].

5.1.1 Asymptotic invariants via multiplier ideals

We start by showing that under fairly general assumptions, one recovers the asymptotic order of vanishing along a graded sequence of ideals from the orders of vanishing along the corresponding asymptotic multiplier ideals.

Proposition 5.1.1. *Let (X,Δ) be a log pair and $\mathfrak{a}_\bullet$ a nonzero graded sequence of ideals on X. If $\mathfrak{b}_m = \mathscr{J}(X,\Delta,\mathfrak{a}_\bullet^m)$, then for every divisor E over X, we have*

$$\mathrm{ord}_E(\mathfrak{a}_\bullet) = \lim_{m\to\infty} \frac{\mathrm{ord}_E(\mathfrak{b}_m)}{m}.$$

Proof. Let m_0 be such that $\mathfrak{a}_{m_0} \neq 0$, hence $\mathfrak{a}_{\ell m_0} \neq 0$ for all $\ell \geq 1$. It follows from Remark 4.1.6 that there is a nonzero ideal J on X such that

$$J \cdot \mathfrak{a}_m \subseteq \mathscr{J}(X,\Delta,\mathfrak{a}_m) \subseteq \mathfrak{b}_m$$

for every $m \geq 1$. By taking $m = \ell m_0$, we obtain

$$\frac{\mathrm{ord}_E(\mathfrak{b}_{\ell m_0})}{\ell m_0} \leq \frac{\mathrm{ord}_E(\mathfrak{a}_{\ell m_0})}{\ell m_0} + \frac{\mathrm{ord}_E(J)}{\ell m_0}. \tag{5.1}$$

Moreover, for every i with $1 \leq i \leq m_0$, we have $\mathfrak{b}_{(\ell+1)m_0} \subseteq \mathfrak{b}_{\ell m_0+i}$ by Proposition 4.2.7i), hence

$$\frac{\mathrm{ord}_E(\mathfrak{b}_{\ell m_0+i})}{\ell m_0+i} \leq \frac{\mathrm{ord}_E(\mathfrak{b}_{(\ell+1)m_0})}{(\ell+1)m_0} \cdot \frac{(\ell+1)m_0}{\ell m_0+i}. \tag{5.2}$$

By combining (5.1) and (5.2), we conclude that

$$\limsup_{m\to\infty} \frac{\mathrm{ord}_E(\mathfrak{b}_m)}{m} \leq \lim_{\ell\to\infty} \frac{\mathrm{ord}_E(\mathfrak{a}_{\ell m_0})}{\ell m_0} = \mathrm{ord}_E(\mathfrak{a}_\bullet).$$

On the other hand, given any m, we can write

$$\mathfrak{b}_m = \mathscr{J}(X,\Delta,\mathfrak{a}_\bullet^m) = \mathscr{J}(X,\Delta,\mathfrak{a}_{qm}^{1/q}),$$

where q is divisible enough. It follows from the definition of multiplier ideals that

$$\mathrm{ord}_E(\mathfrak{b}_m) = \mathrm{ord}_E(\mathscr{J}(X,\Delta,\mathfrak{a}_{qm}^{1/q})) > \frac{1}{q}\cdot \mathrm{ord}_E(\mathfrak{a}_{qm}) - a_E(X,\Delta).$$

Therefore

$$\frac{\mathrm{ord}_E(\mathfrak{b}_m)}{m} > \frac{\mathrm{ord}_E(\mathfrak{a}_{qm})}{qm} - \frac{a_E(X,\Delta)}{m} \geq \mathrm{ord}_E(\mathfrak{a}_\bullet) - \frac{a_E(X,\Delta)}{m}$$

for every $m \geq 1$, hence

$$\liminf_{m\to\infty} \frac{\mathrm{ord}_E(\mathfrak{b}_m)}{m} \geq \mathrm{ord}_E(\mathfrak{a}_\bullet).$$

We thus conclude that $\lim_{m\to\infty} \frac{\mathrm{ord}_E(\mathfrak{b}_m)}{m} = \mathrm{ord}_E(\mathfrak{a}_\bullet)$. $\square$

Computing the asymptotic invariants in terms of multiplier ideals is particularly effective in the case of a smooth ambient variety, due to the subadditivity theorem. This implies that the limit in the statement of Proposition 5.1.1 is also a supremum, as follows.

Corollary 5.1.2. *If $\mathfrak{a}_\bullet$ is a nonzero graded sequence of ideals on a smooth variety X and $\mathfrak{b}_m = \mathscr{J}(X,\mathfrak{a}_\bullet^m)$ for every $m \geq 1$, then for every divisor E over X we have*

$$\mathrm{ord}_E(\mathfrak{a}_\bullet) = \sup_{m\geq 1} \frac{\mathrm{ord}_E(\mathfrak{b}_m)}{m}.$$

In particular, we have $\operatorname{ord}_E(\mathfrak{a}_\bullet) = 0$ *if and only if the center* $c_X(E)$ *of* E *is not contained in the zero-locus* $V(\mathfrak{b}_m)$ *of* $\mathfrak{b}_m$ *for any* $m \geq 1$.

Proof. Note that in our setting each $\mathfrak{b}_m$ is a nonzero ideal in $\mathscr{O}_X$ and Corollary 4.3.17 gives $\mathfrak{b}_{p+q} \subseteq \mathfrak{b}_p \cdot \mathfrak{b}_q$ for all $p,q \geq 1$. Therefore we have $\operatorname{ord}_E(\mathfrak{b}_p) + \operatorname{ord}_E(\mathfrak{b}_q) \leq \operatorname{ord}_E(\mathfrak{b}_{p+q})$ for all $p,q \geq 1$, and applying Lemma 1.7.9 with $\alpha_m = -\operatorname{ord}_E(\mathfrak{b}_m)$, we obtain

$$\lim_{m\to\infty} \frac{\operatorname{ord}_E(\mathfrak{b}_m)}{m} = \sup_{m\geq 1} \frac{\operatorname{ord}_E(\mathfrak{b}_m)}{m}.$$

Therefore the first assertion in the corollary follows from Proposition 5.1.1. The last assertion is an immediate consequence, since each $\operatorname{ord}_E(\mathfrak{b}_m)$ is nonnegative. □

5.1.2 Asymptotic invariants of big and pseudo-effective divisors

We can now prove a criterion for the vanishing of the asymptotic invariants of a big Cartier divisor on a smooth variety.

Theorem 5.1.3. *Let* D *be a big Cartier divisor on a smooth projective variety* X. *For every divisor* E *over* X*, the following are equivalent:*

i) $\operatorname{ord}_E(\| D \|) = 0$.
ii) There is M *such that* $\operatorname{ord}_E(|mD|) \leq M$ *for all* $m \gg 0$.
iii) For every ample $A \in \operatorname{CDiv}(X)_{\mathbb{Q}}$*, we have* $c_X(E) \not\subseteq \operatorname{SB}(D+A)$[1].
iv) There is a Cartier divisor G *(that we may assume ample) such that* $c_X(E) \not\subseteq \operatorname{Bs}(|mD+G|)$ *for every* $m \geq 1$.
v) For every $m \geq 1$*, the center* $c_X(E)$ *is not contained in the zero-locus of* $\mathscr{J}(X, m\cdot \| D \|)$.

Proof. Note that the implication i)⇒v) follows from Corollary 5.1.2. Since D is big, it follows from Corollary 4.2.10 that if H is a very ample Cartier divisor on X and $G = K_X + nH$, where $n = \dim(X)$, then

$$\mathscr{J}(X, \| mD \|) \otimes \mathscr{O}_X(mD+G) \text{ is globally generated for every } m \geq 1.$$

It follows that if $c_X(E)$ in the zero-locus of $\mathscr{J}(X, m\cdot \| D \|)$, then $\mathscr{O}_X(mD+G)$ is globally generated at the generic point of $c_X(E)$. We thus see that v) implies iv).

We now show that iv) implies iii). If G is a Cartier divisor as in iv), then for every $A \in \operatorname{CDiv}(X)_{\mathbb{Q}}$, we have $A - \frac{1}{m}G$ ample for $m \gg 0$. Therefore we have

$$\operatorname{SB}(D+A) \subseteq \operatorname{SB}(D + \frac{1}{m}G) \subseteq \operatorname{Bs}(|mD+G|),$$

and iv) implies that $c_X(E)$ is not contained in $\operatorname{SB}(D+A)$.

[1] If either $c_X(E)$ is a point, or the ground field is uncountable, this condition is equivalent to $c_X(E) \not\subseteq \operatorname{B}_-(D)$.

On the other hand, if iii) holds, then for every ample Cartier divisor A, we have $c_X(E) \not\subseteq \mathrm{SB}(D+\frac{1}{m}A)$ for every $m \geq 1$, hence $\mathrm{ord}_E(\| D+\frac{1}{m}A \|) = 0$. It follows from the continuity of the function $\mathrm{ord}_E(\| - \|)$ on the big cone (see Proposition 1.7.18) that $\mathrm{ord}_E(\| D \|) = 0$. We have thus shown that i), iii), iv), and v) are equivalent.

Since ii)⇒i) is trivial from the definition of asymptotic invariants, in order to prove the equivalence of all five conditions, it is enough to show that iv)⇒ii). Let G be as in iv). After possibly adding a suitable ample Cartier divisor, we may assume that G is ample. Since D is big, it follows from Kodaira's lemma that there is $\ell \geq 1$ and an effective divisor F such that $\ell D - G \sim F$. In this case, for every $m > \ell$, we have

$$\mathrm{ord}_E(|mD|) \leq \mathrm{ord}_E(|\ell D - G|) + \mathrm{ord}_E(|(m-\ell)D + G|) \leq \mathrm{ord}_E(F)$$

and therefore ii) holds. This completes the proof of the theorem. □

Remark 5.1.4. The equivalence between i) and ii) in Theorem 5.1.3 holds if we only assume that X is normal, instead of smooth. Indeed, it is enough to consider a resolution of singularities $f\colon Y \to X$ and apply Theorem 5.1.3 for $f^*(D)$, using the fact that $\mathrm{ord}_E(|mD|) = \mathrm{ord}_E(|mf^*(D)|)$ for all $m \geq 1$.

Remark 5.1.5. It is shown in [Mus13] that the equivalences i)-iv) in Theorem 5.1.3 also hold over a field of positive characteristic. Furthermore, they are also equivalent to v), if the asymptotic multiplier ideal is replaced by the so-called asymptotic test ideal.

Corollary 5.1.6. *If X is a smooth variety and $D \in \mathrm{CDiv}(X)_{\mathbb{R}}$ is pseudo-effective, then for every divisor E over X, the following are equivalent:*
i) $\sigma_E(D) = 0$.
ii) For every ample $A \in \mathrm{CDiv}(X)_{\mathbb{R}}$, with $(D+A)$ a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor, we have $c_X(E) \not\subseteq \mathrm{SB}(D+A)$[2].
iii) The center $c_X(E)$ is not contained in the locus defined by $\mathscr{J}_+(X, m\cdot \| D \|)$ for any $m \geq 1$.

Proof. We first prove i)⇒iii). It follows from the definition of $\sigma_E(D)$ that this is 0 if and only if for every ample $B \in \mathrm{CDiv}(X)_{\mathbb{R}}$ (and it is enough to only consider those B such that $(D+B) \in \mathrm{CDiv}(X)_{\mathbb{Q}}$) we have $\mathrm{ord}_E(\| D+B \|) = 0$. Given any $m \geq 1$, we can find $A' \in \mathrm{CDiv}(X)_{\mathbb{R}}$ ample, with $(D+A') \in \mathrm{CDiv}(X)_{\mathbb{Q}}$, such that

$$\mathscr{J}_+(X, m\cdot \| D \|) = \mathscr{J}(X, m\cdot \| D+A' \|).$$

If q is a positive integer such that $q(D+A')$ is a Cartier divisor and $\mathfrak{a}_\bullet$ is the graded sequence of ideals such that $\mathfrak{a}_p$ is the ideal defining the base-locus of $|pq(D+A')|$, then

$$\mathscr{J}(X, m\cdot \| D+A' \|) = \mathscr{J}(X, \mathfrak{a}_\bullet^{m/q}) \supseteq \mathscr{J}(X, \mathfrak{a}_\bullet^{m}). \tag{5.3}$$

[2] If either $c_X(E)$ is a point, or the ground field is uncountable, this condition is equivalent with $c_X(E) \not\subseteq \mathrm{B}_-(D)$.

Since $\mathrm{ord}_E(\| D+A' \|)=0$, it follows from Corollary 5.1.2 that $c_X(E)$ is not contained in the locus defined by $\mathscr{J}(X,\mathfrak{a}_\bullet^m)$, hence by (5.3), $c_X(E)$ is not contained in the locus defined by $\mathscr{J}_+(X,m\cdot \| mD \|)$.

Suppose now that iii) holds and let us deduce ii). Let $A\in \mathrm{CDiv}(X)_{\mathbb{R}}$ be ample such that $(D+A)\in \mathrm{CDiv}(X)_{\mathbb{Q}}$ and let $A'\in \mathrm{CDiv}(X)_{\mathbb{R}}$ be ample such that $A-A'$ is an ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. We choose a positive integer m such that $m(D+A')$ is Cartier. Since $\mathrm{SB}(D+A)=\mathrm{SB}(m(D+A')+m(A-A'))$, by applying Theorem 5.1.3 to the Cartier divisor $m(D+A')$, we see that it is enough to show that $c_X(E)$ is not contained in the locus defined by $\mathscr{J}(X,q\cdot \| m(D+A') \|)$ for any $q\geq 1$. This follows from the inclusion

$$\mathscr{J}_+(X,qm\cdot \| D \|)\subseteq \mathscr{J}(X,q\cdot \| m(D+A') \|)$$

and the assumption in iii).

In order to complete the proof, it is enough to show that if ii) holds, then i) holds too. Let $B\in \mathrm{CDiv}(X)_{\mathbb{R}}$ be ample and such that $(D+B)\in \mathrm{CDiv}(X)_{\mathbb{Q}}$. We choose a positive integer ℓ such that $\ell(D+B)$ is Cartier and apply Theorem 5.1.3 to $\ell(D+B)$. If A is an ample $\mathbb{Q}$-divisor, then $c_X(E)$ is not contained in

$$\mathrm{SB}(\ell(D+B)+A)=\mathrm{SB}(D+\frac{1}{\ell}(A+\ell B))$$

by ii). Therefore $\mathrm{ord}_E(\| \ell(D+B) \|)=0$, hence $\mathrm{ord}_E(\| D+B \|)=0$. Since this holds for all B as above, we conclude that $\sigma_E(D)=0$. □

Corollary 5.1.7. *If X is a normal variety and $D\in \mathrm{CDiv}(X)_{\mathbb{R}}$ is pseudo-effective, then D is nef if and only if $\sigma_E(D)=0$ for all divisors E over X.*

Proof. We have already seen in Proposition 1.7.31ii) that if D is nef, then $\sigma_E(D)=0$ for every divisor E over X. Conversely, if this is the case and $f\colon Y\to X$ is a projective birational morphism, with Y smooth, then Proposition 1.7.35 implies $\sigma_E(f^*(D))=0$ for every divisor E over Y. Since every point on Y is the center of some E, it follows from Corollary 5.1.6 that $f^*(D)+A$ is semiample for every ample $A\in \mathrm{CDiv}(Y)_{\mathbb{R}}$ such that $f^*(D)+A$ is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. Therefore $f^*(D)$ is nef, hence also D is nef. □

Corollary 5.1.8. *If $f\colon Y\to X$ is a birational morphism of smooth projective varieties, then for every $D\in \mathrm{CDiv}(X)_{\mathbb{R}}$ we have $\mathrm{B}_-(f^*(D))=f^{-1}(\mathrm{B}_-(D))$.*

Proof. Since the non-nef locus of a numerical class that is not pseudo-effective is the ambient variety and since D is pseudo-effective if and only if $f^*(D)$ has this property (see Remark 1.4.32), we may assume that D is pseudo-effective. Consider $y\in Y$ and let E be a divisor over Y such that $c_Y(E)=\{y\}$. It follows from Corollary 5.1.6 that $y\in \mathrm{B}_-(f^*(D))$ if and only if $\sigma_E(f^*(D))>0$ and $f(y)\in \mathrm{B}_-(D)$ if and only if $\sigma_E(D)>0$. Since $\sigma_E(D)=\sigma_E(f^*(D))$ by Proposition 1.7.35, we obtain the assertion in the corollary. □

Question 5.1.9. Does the equivalence between i) and iii) in Theorem 5.1.3 hold for arbitrary normal varieties? A positive answer to this question on klt pairs has been recently announced in [CL]. A related question is the following: does the assertion in Corollary 5.1.8 hold if X and Y are only assumed to be normal, instead of smooth (note that a positive answer to the former question implies a positive answer to the latter one, and the converse holds over an uncountable ground field).

5.1.3 Zariski decompositions, revisited

We can use the connection between the asymptotic invariants and the non-nef locus to get a better understanding of Zariski decompositions. We first interpret the movable cone in terms of asymptotic invariants.

Corollary 5.1.10. *If X is a smooth projective variety of dimension $n \geq 2$, then $\alpha \in \mathrm{PEff}(X)$ lies in the closure of the movable cone $\overline{\mathrm{Mov}}^1(X)$ if and only if $\sigma_E(\alpha) = 0$ for every prime divisor E on X.*

Proof. This follows from the definition of $\overline{\mathrm{Mov}}^1(X)$ and Corollary 5.1.6. □

Proposition 5.1.11. *Let X be a smooth projective variety of dimension $n \geq 2$. If $D \in \mathrm{CDiv}(X)_{\mathbb{R}}$ is pseudo-effective and $D = P_\sigma(D) + N_\sigma(D)$ is the divisorial Zariski decomposition of D, then the numerical class of $P_\sigma(D)$ lies in $\overline{\mathrm{Mov}}^1(X)$. Furthermore, if $D = P + N$ is another decomposition with N effective and the numerical class of P lying in $\overline{\mathrm{Mov}}^1(X)$, then $N - N_\sigma(D)$ is effective.*

Proof. It follows from the definition of the divisorial Zariski decomposition and Proposition 1.7.36 that for every prime divisor E on X, we have $\sigma_E(P_\sigma(D)) = \sigma_E(D) - \mathrm{ord}_E(N_\sigma(D)) = 0$. Therefore the first assertion in the proposition follows from Corollary 5.1.10.

Suppose now that $D = P + N$ is a decomposition as in the statement and let E be a prime divisor on X. Using again Proposition 1.7.36 and the convexity of the function σ_E, we obtain

$$\mathrm{ord}_E(N_\sigma(D)) = \sigma_E(D) \leq \sigma_E(P) + \sigma_E(N) = \sigma_E(N) \leq \mathrm{ord}_E(N),$$

where the last inequality follows from Remark 1.7.32. Since this holds for every E, we conclude that $N - N_\sigma(D)$ is effective. □

In particular, we deduce the existence of Zariski decomposition on surfaces.

Proposition 5.1.12. *If X is a smooth projective surface and $D \in \mathrm{CDiv}(X)_{\mathbb{R}}$ is pseudo-effective, then D has a Zariski decomposition, that is, $P_\sigma(D)$ is nef.*

Proof. The assertion follows from Proposition 5.1.11 and the fact that on surfaces, $\overline{\mathrm{Mov}}^1(X)$ coincides with the nef cone. □

Remark 5.1.13. The existence of Zariski decomposition on surfaces has been proved by Zariski for big divisors and by Fujita for pseudo-effective divisors. In fact, what Zariski and Fujita showed is that if X is a smooth projective surface and $D \in \mathrm{PEff}(X)_{\mathbb{R}}$, then one can write $D = P + N$, where

i) P is nef and $N = \sum_{i=1}^{r} a_i E_i$, with all $a_i > 0$.
ii) $(P \cdot E_i) = 0$ for $1 \leq i \leq r$.
iii) The intersection matrix $(E_i \cdot E_j)_{1 \leq i,j \leq r}$ is negative definite.

We refer to [Bǎd01, Chap. 14] for a proof. It is easy to see that given such P and N, we have $P = P_\sigma(D)$ and $N = N_\sigma(D)$. Indeed, it follows from Proposition 5.1.11 that $N = N_\sigma(D) + A$, for some effective divisor A. Since A is supported on $E_1 \cup \ldots \cup E_r$ and

$$(A \cdot E_i) = (P_\sigma(D) \cdot E_i) - (P \cdot E_i) = (P_\sigma(D) \cdot E_i) \geq 0$$

for every i, it follows from iii) that $A = 0$.

This description of the Zaiski decomposition implies that if D is a $\mathbb{Q}$-divisor, then $P_\sigma(D)$ and $N_\sigma(D)$ are $\mathbb{Q}$-divisors, too. Indeed, we have

$$\mathbb{Q} \ni (D \cdot E_j) = \sum_{i=1}^{r} a_i (E_i \cdot E_j)$$

for $1 \leq j \leq r$. It follows from (iii) that we can solve this system of equations to determine the a_i. In particular, these are rational numbers.

Remark 5.1.14. Let X be a smooth projective variety and D a big $\mathbb{R}$-divisor on X. In this case, a Zariski decomposition of D is a decomposition $D = P + N$, where P is nef, N is effective, and for every $m \geq 1$, the natural inclusion

$$H^0(X, \mathscr{O}_X(mP)) \hookrightarrow H^0(X, \mathscr{O}_X(mD)) \tag{5.4}$$

is an isomorphism[3]. Indeed, if $D = P_\sigma(D) + N_\sigma(D)$ gives a Zariski decomposition, then (5.4) is satisfied by Proposition 1.7.36 (for this implication, it is enough to assume that D is pseudo-effective). Conversely, if $D = P + N$ is a decomposition as above, we deduce from the fact that

$$h^0(X, \mathscr{O}_X(\lfloor mP \rfloor)) = h^0(X, \mathscr{O}_X(\lfloor mD \rfloor)) \text{ for all } m \geq 1$$

that P is big (see Proposition 1.4.33). Furthermore, for every prime divisor E on X, we have

$$\mathrm{ord}_E(|\lfloor mD \rfloor|) - \mathrm{ord}_E(|\lfloor mP \rfloor|) = \mathrm{ord}_E(\lfloor mD \rfloor - \lfloor mP \rfloor = \mathrm{ord}_E(\lfloor mD \rfloor) - \mathrm{ord}_E(\lfloor mP \rfloor).$$

Dividing by m and letting m go to infinity, we obtain using Proposition 1.7.26

$$\sigma_E(D) - \sigma_E(P) = \mathrm{ord}_E(D) - \mathrm{ord}_E(P) = \mathrm{ord}_E(N).$$

[3] A decomposition with these properties is also known as a *CKM Zariski decomposition*, where the initials stand for Cutkosky, Kawamata, and Moriwaki.

Since P is nef, we have $\sigma_E(P)=0$, and we deduce that $N=N_\sigma(D)$. Since $D-N_\sigma(D)$ is nef, it follows that D has a Zariski decomposition.

Remark 5.1.15. If $f\colon Y\to X$ is a birational morphism of smooth projective varieties and D is a pseudo-effective $\mathbb{R}$-divisor on X such that $f^*(D)$ has a Zariski decomposition, then $\mathrm{B}_-(D)=f(N_\sigma(f^*(D)))$. Indeed, note first that by Corollary 5.1.8, we have $\mathrm{B}_-(D)=f(\mathrm{B}_-(f^*(D)))$. Therefore it is enough to prove the assertion when $X=Y$ and f is the identity. Let $P=P_\sigma(D)$ and $N=N_\sigma(D)$. Given a point $x\in X$, let F be a divisor over X with center $\{x\}$. Since P is nef, it follows from Proposition 1.7.36 that

$$\sigma_F(D)=\sigma_F(P)+\mathrm{ord}_F(N)=\mathrm{ord}_F(N).$$

Therefore $\sigma_F(D)>0$ if and only if $x\in\mathrm{Supp}(N)$. On the other hand, it follows from Corollary 5.1.6 that $x\in\mathrm{B}_-(D)$ if and only if $\sigma_F(D)>0$. This proves our assertion.

Remark 5.1.16. Let D be a pseudo-effective $\mathbb{R}$-divisor on the smooth projective variety X. It follows from Remark 5.1.15 that if there is a projective, birational morphism $f\colon Y\to X$, with Y smooth, such that $f^*(D)$ has a Zariski decomposition, then $\mathrm{B}_-(D)$ is Zariski closed. Lesieutre [Les] gave an example of such a divisor D in dimension 3 (and a similar example in dimension 4, with D big) such that $\mathrm{B}_-(D)$ is not Zariski closed. In particular, we see that in this example, we cannot have a Zariski decomposition after the pull-back by a birational morphism[4].

5.2 Global generation of adjoint line bundles

5.3 Singularities of theta divisors

5.4 Ladders on Del Pezzo and Mukai varieties

5.5 Skoda-type theorems

[4] As we have mentioned, Nakayama [Nak04] also gave an example with the latter property; however, in his example the non-nef locus is closed.

Chapter 6
Birational rigidity

Apart from Section 10.2, we work over $\mathbb{C}$.

6.1 Factorization of planar Cremona maps

We begin this chapter by reviewing the following celebrated theorem on the structure of the Cremona group of $\mathbb{P}^2$.

Theorem 6.1.1 (Noether–Castelnuovo). *The Cremona group* $\mathrm{Bir}(\mathbb{P}^2)$ *is generated by linear transformations and the standard quadratic transformation*

$$\chi\colon (x:y:z) \dashrightarrow (yz:xz:xy).$$

For clarity of exposition, we shall work without fixing coordinates, but allowing instead to take standard quadratic transformations centered at any triple of distinct non-collinear points of $\mathbb{P}^2$. The freedom in choosing the base points incorporates, implicitly, the role of the linear transformations among the generators of the Cremona group.

Let $\phi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a birational map. This map is defined by a two-dimensional linear system $\mathscr{H} \subset |\mathscr{O}_{\mathbb{P}^2}(r)|$ of curves of degree r with no fixed components. Note that ϕ is an automorphism if and only if $r = 1$.

Suppose that ϕ is not an isomorphism. A minimal sequence of point-blowups

$$f\colon Y = X_{k+1} \xrightarrow{f_k} X_k \to \cdots \to X_1 \xrightarrow{f_1} X_0 = \mathbb{P}^2$$

resolving the indeterminacies of ϕ determines a series of *base points* $p_0, p_1, \dots, p_k$, possibly some infinitely near to others: the centers p_i of the blowups $f_i\colon X_{i+1} \to X_i$. We denote by m_i the multiplicity at the point p_i of the proper transform of $\mathscr{H}$ to X_i. We can assume that the sequence of blowups is ordered such that

$$m_0 \geq m_1 \geq \cdots \geq m_k.$$

Noether's idea to prove the theorem is that taking a standard quadratic tranformation χ centered at points of large multiplicity should lower the degree of the map [Noe70]. The basic computation is the following. Suppose, for example, that the three points p_0, p_1, p_2 are distinct on $\mathbb{P}^2$. It can be shown that $m_1 + m_2 + m_3 > r$, and therefore these points cannot be collinear. Let χ be a standard quadratic transformation centered at these three points. By precomposing ϕ with χ^{-1} (note that this is the same as χ), one obtains a new birational map

$$\phi' = \phi \circ \chi^{-1} \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$$

of degree

$$r' = 2r - m_0 - m_1 - m_2 < r,$$

which means that this operation lowers the degree of the map. One says that χ *untwists* the map ϕ. A recursive application of this process would eventually reduce ϕ to a linear transformation, thus providing the required factorization.

The issue with this approach is that, in general, p_0, p_1, p_2 may fail to be distinct in $\mathbb{P}^2$, and one may not be able to find three distinct points whose multiplicities exceed, together, the degree of the map. As a matter of fact, there may not be three distinct points at all. One is forced to work with infinitely near points. After several attempted proofs, including those of Noether and Clifford which turned out to be fallacious as pointed out by Segre [Seg01], a complete proof of Noether's theorem was finally given by Castelnuovo [Cas01].

Here we present a later proof, due to Alexander [Ale16], which is in some sense closer to the original idea of Noether. We present it here with a small simplification (in the logical structure more than in the computations). We first prove that $\mathrm{Bir}(\mathbb{P}^2)$ is generated by linear transformations, the standard quadratic transformation χ, and the quadratic transformation

$$\omega \colon (x : y : z) \dashrightarrow (x^2 : xy : yz).$$

Theorem 10.1.1 will then follow by observing that ω itself factors as a composition of linear and standard quadratic transformations.

Note that ω has three base points q_1, q_2, q_3, with q_2 infinitely near q_1 and q_3 not lying on the line passing through q_1 with tangent direction q_2. If n_1, n_2, n_3 are the multiplicities of $\mathscr{H}$ at these points, then the map $\phi' = \phi \circ \omega^{-1} \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ has degree $r' = 2r - n_0 - n_1 - n_2$. As we are already doing for χ, we will work without fixing coordinates and allow ω to be centered to any triple of points q_1, q_2, q_3 with the above properties.

Proof of Theorem 10.1.1. Keeping the above notation, let $\phi \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a birational transformation of degree $r > 1$, defined by a linear system $\mathscr{H}$. Let $p_0, \dots, p_k$ the base points of $\mathscr{H}$, and $m_0, \dots, m_k$ be their multiplicities, ordered as above. Let E_i be the exceptional divisor of the blowup $f_i \colon X_{i+1} \to X_i$ centered at p_i, and let F_i be the pullback of E_i to $Y = X_{k+1}$. Finally, let $D \in \mathscr{H}$ be a general member, and let D_Y denote its proper transform on Y. Note that the rational map ϕ lifts, via

$f: Y \to X_0 = \mathbb{P}^2$, to a morphism $g = \phi \circ f: Y \to \mathbb{P}^2$, and D_Y is the pullback, via g, of a general line in $\mathbb{P}^2$.

We set

$$a = a(\phi) := \frac{r - m_0}{2},$$

and define

$$b = b(\phi) := \max\{i \mid m_i > a\}.$$

Lemma 6.1.2. $b \geq 2$.

Proof. Since $f^*D = D_Y + \sum_{i=0}^k m_i F_i$, $F_i \cdot D_Y = m_i$ for every i, and $D_Y^2 = 1$, we have

$$r^2 = D^2 = D \cdot f_* D_Y = f^* D \cdot D_Y = D_Y^2 + \sum_{i=0}^k m_i (F_i \cdot D_Y) = 1 + \sum_{i=0}^k m_i^2.$$

On the other hand, since $K_Y = f^* K_X + \sum_{i=0}^k F_i$ and $K_Y \cdot D_Y = -3$, we have

$$3r = -K_X \cdot D = -K_Y \cdot D_Y + \sum_{i=0}^k (F_i \cdot D_Y) = 3 + \sum_{i=0}^k m_i.$$

Subtracting a times the second identity from the first gives

$$\sum_{i=0}^k m_i (m_i - a) = r^2 - 3ra + 3a - 1.$$

By removing all the negative terms in the left hand side and subtracting $3a - 1$ from the right hand side, we obtain

$$\sum_{i=0}^b m_i (m_i - a) \geq \sum_{i=0}^k m_i (m_i - a) > r(r - 3a) = r(m_0 - a),$$

and hence, subtracting $m_0(m_0 - a)$ from both sides, we get

$$\sum_{i=1}^b m_i (m_i - a) > (r - m_0)(m_0 - a) = 2a(m_0 - a).$$

Notice that $2a \geq m_1$, and hence $2a \geq m_i$ for all $i \geq 1$, since the line through p_0 and p_1 can only meet D in $r - m_0 = 2a$ away from p_0. It follows that

$$\sum_{i=1}^b (m_i - a) > m_0 - a.$$

This implies that $b \geq 2$ because $m_0 \geq m_1$. □

This lemma says that the first three points p_0, p_1, p_2 have multiplicities

$$m_0 \geq m_1 \geq m_2 > a.$$

The proof now goes by induction on the vector $(a,b) \in \frac{1}{2}\mathbb{N} \times \mathbb{N}$ with respect to the lexicographic order. We think of this vector as a measure of the complexity of ϕ. We study two cases, according to the relative position of p_0, p_1, p_2.

Case 1. Suppose that p_0, p_1, p_2 are distinct points in $\mathbb{P}^2$. Note that they cannot be collinear, since $m_0 + m_1 + m_2 > m_0 + 2a = r$. Let $\phi' := \phi \circ \chi^{-1}$ where χ is the standard quadratic transformation centered at these three points, and let $(a', b') := (a(\phi'), b(\phi'))$.

We denote by p'_0, p'_1, p'_2 the base points of χ^{-1}, and let m'_0, m'_1, m'_2 be the multiplicities at these points of the linear system $\mathscr{H}'$ defining ϕ'. Note that $\mathscr{H}'$ is the homaloidal transform of $\mathscr{H}$, it has degree

$$r' = 2r - m_0 - m_1 - m_2,$$

and

$$m'_h = r - m_i - m_j \quad \text{for} \quad \{h, i, j\} = \{0, 1, 2\}.$$

Each point p_i, for $3 \leq i \leq k$ is either mapped to one of p'_0, p'_1, p'_2, or it remains a distinct point of multiplicity m_i of $\mathscr{H}'$. No other base points of $\mathscr{H}'$ are created. The question now is whether $\mathscr{H}'$ achieves its largest multiplicity at p'_0.

If the largest multiplicity of $\mathscr{H}'$ is not achieved at p'_0, then it is larger than m'_0 and we have

$$2a' < r' - m'_0 = r - m_0 = 2a.$$

On the contrary, if m'_0 is the largest multiplicity of $\mathscr{H}'$, then $a' = a$. In this case, however, we get

$$m'_i = r - m_0 - m_j = 2a - m_j < a \quad \text{for} \quad \{i, j\} = \{1, 2\},$$

and therefore $b' < b$. Either way, we have $(a', b') < (a, b)$, and we can apply induction.

Case 2. Suppose now that p_0, p_1, p_2 are not distinct points in $\mathbb{P}^2$. We fix a general point $q \in \mathbb{P}^2$.

If p_1 not is infinitely near p_0, then we let $\phi' := \phi \circ \chi^{-1}$ where χ is the standard quadratic transformation centered at p_0, p_1, q, and denote by p'_0, p'_1, q' the base points of χ^{-1}. If p_1 is infinitely near p_0, then we let $\phi' := \phi \circ \omega^{-1}$ where ω is the quadratic transformation centered at p_0, p_1, q, and denote by p'_0, p'_1, q' the base points of ω^{-1}.

Let $\mathscr{H}'$ denote the linear system defining ϕ', let r' be its degree, and let m'_0, m'_1, n' be the multiplicities of $\mathscr{H}'$ at the points p'_0, p'_1, q'. Note that $r' = 2r - m_0 - m_1$, $m'_i = r - m_i$ for $\{i, j\} = \{1, 2\}$, and

$$n' = r - m_0 - m_1 = 2a - m_1 < a.$$

Furthermore, as in Case 1, ϕ' does not create new base points, and those p_i, for $3 \leq i \leq k$, that are not mapped to any of p'_0, p'_1, q' maintain the same multiplicity m_i in $\mathscr{H}'$.

Let $(a',b') := (a(\phi'),b(\phi'))$. If the largest multiplicity of $\mathscr{H}'$ is larger than m_0, then we get $a' < a$. Otherwise, we have $a' = a$, but then $b' < b$ since $n' < a = a'$. Therefore, $(a',b') < (a,b)$, and induction applied.

To conclude the proof, we are left to verify that ω, given in some fixed coordinates by $(x : y : z) \dashrightarrow (x^2 : xy : yz)$, is the composition of linear transformations and the standard quadratic transformation χ given in the same coordinates by $(x : y : z) \dashrightarrow (yz : xz : xy)$.[1] By precomposing ω with the automorphism α defined by $(x : y : z) \mapsto (x : x+y : z)$, we obtain the transformation

$$\omega \circ \alpha : (x : y : z) \dashrightarrow (x^2 : x(x+y) : (x+y)z).$$

Untwisting this with χ, we get

$$\omega \circ \alpha \circ \chi : (x : y : z) \dashrightarrow (yz : (x+y)z : x(x+y)).$$

This is equal to $\chi \circ \beta$, where β is the linear transformation given by $(x : y : z) \mapsto (x+y : x : z)$. Therefore we have

$$\omega = \chi \circ \beta \circ \chi^{-1} \circ \alpha^{-1} = \chi \circ \beta \circ \chi \circ \alpha^{-1},$$

which gives the required factorization. □

In spite of this important theorem, the Cemona group remains a rather mysterious object of investigation. Mention recent results (classification of finite groups up to conjugation, existence of normal subgroups, topology....)

6.2 Birational rigidity of cubic surfaces of Picard number one

In this section we shall look at smooth cubic surfaces defined over non algebraically closed fields. Let κ be a perfect field, and let $X_\kappa \subset \mathbb{P}^3_\kappa$ be a smooth cubic surface. Since the canonical class of X_κ is defined over κ, the Picard group $\mathrm{Pic}(X_\kappa)$ contains the hyperplane class $\mathscr{O}_{X_\kappa}(1)$. The surface has Picard number one if and only if $\mathrm{Pic}(X_\kappa)$ is generated by $\mathscr{O}_{X_\kappa}(1)$.

Segre proved that if the Picard number is one then X_κ is not rational [Seg51]. His proof was later adjusted by Manin to prove that if two such cubics are birational to each other, then they are projectively equivalent [Man66].[2] These results have been reviewed in the recent treatment [KSC04]. An extension of Manin's proof gives the following more precise theorem [dF].

[1] This is well explained in [KSC04, Page 200], which we followed in our computations. There seems however to be a typo there in the expression of T_2', which should be given by $(x_0^2 : x_0(x_0 + x_1) : (x_0 + x_1)x_2)$.

[2] The hypothesis that κ be perfect is not necessary for these statements.

Theorem 6.2.1. *Let $X_\kappa \subset \mathbb{P}^3_\kappa$ be a smooth cubic surface of Picard number one over a perfect field κ. Suppose that there is a birational map*

$$\phi_\kappa \colon X_\kappa \dashrightarrow X'_\kappa$$

where X'_κ is a smooth projective surface that is either a Del Pezzo surface of Picard number one, or a conic bundle over a curve S'_κ. Then X'_κ is a cubic surface of Picard number one, and there is a birational automorphism $\beta_\kappa \in \mathrm{Bir}(X_\kappa)$ such that $\phi_\kappa \circ \beta_\kappa \colon X_\kappa \to X'_\kappa$ is a projective equivalence. In particular, X_κ is nonrational.

Proof. If X'_κ is a conic bundle over a curve S'_κ then we fix a divisor A'_κ on X'_κ given by the pullback of a very ample divisor on S'_κ. If X'_κ is a Del Pezzo surface of Picard number one, then we set $S'_\kappa = \operatorname{Spec}\kappa$ and $A'_\kappa = 0$. We fix an integer $r' \geq 1$ such that $-r'K_{X'_\kappa} + A'_\kappa$ is very ample. Since the Picard group of X_κ is generated by the class of $-K_{X_\kappa}$, there is a positive integer r such that

$$(\phi_\kappa)_*^{-1}(-r'K_{X'_\kappa} + A'_\kappa) \sim -rK_{X_\kappa}.$$

Let $\overline{\kappa}$ be the algebraic closure of κ, and denote $X = X_{\overline{\kappa}}$, $X' = X'_{\overline{\kappa}}$, $S' = S'_{\overline{\kappa}}$, $A' = A'_{\overline{\kappa}}$ and $\phi = \phi_{\overline{\kappa}}$. Let $D' \in |-r'K_{X'} + A'|$ be a general element, and let

$$D = \phi_*^{-1}D' \in |-rK_X|.$$

We split the proof in two cases.

Case 1. Assume that $\mathrm{mult}_x(D) > r$ for some $x \in X$.

We use the existence of such points of high multiplicity to construct a suitable birational involution of X (defined over κ) that, pre-composed to ϕ, untwists the map. This part of the proof is similar, in spirit, to the proof of Noether's theorem on $\mathrm{Bir}(\mathbb{P}^2)$.

The Galois group of $\overline{\kappa}$ over κ acts on the base points of ϕ and preserves the multiplicities of D at these points. Since D belongs to a linear system with zero-dimensional base locus and $\deg D = 3r$ (as a cycle in $\mathbb{P}^3$), there are at most two points at which D has multiplicity larger than r, and the union of these points is preserved by the Galois action. If there is only one point $x \in X$ (not counting infinitely near ones), then x is defined over κ. Otherwise, we have two distinct points x, y on X whose union $\{x, y\} \subset X$ is defined over κ.

We shall now untwist ϕ by pre-composing with a suitable birational involution $\alpha_1 \in \mathrm{Bir}(X)$, constructed as follows. Let $g \colon \widetilde{X} \to X$ be the blowup of X at the points of multiplicty larger than r. If there is only one such point x, the blowup resolves the indeterminacies of the rational map $X \dashrightarrow \mathbb{P}^2$ given by the linear system $|\mathscr{O}_X(1) \otimes \mathfrak{m}_x|$, which lifts to a double cover $h \colon \widetilde{X} \to \mathbb{P}^2$. The Galois group of this cover is generated by an involution $\widetilde{\alpha}_1$ of $\widetilde{X}$, which descends to a birational involution α_1 of X. If there are two points x, y of multiplicity greater than r, then g resolves the indeterminacies of the rational map $X \dashrightarrow \mathbb{P}^3$ given by the linear system $|\mathscr{O}_X(2) \otimes \mathfrak{m}_x^2 \otimes \mathfrak{m}_y^2|$, which lifts to a double cover $h \colon \widetilde{X} \to Q \subset \mathbb{P}^3$ where Q is a smooth quadric surface. In this case, we denote by $\widetilde{\alpha}_1$ the Galois involution of the cover and by α_1

the birational involution induced on X. In both cases, α_1 is defined over κ. Therefore the composition

$$\phi_1 = \phi \circ \alpha_1^{-1} \colon X \dashrightarrow X'$$

is defined over κ and hence is given by a linear system in $|-r_1K_X|$ for some r_1 (note that $\alpha_1^{-1} = \alpha_1$).

In either case, we have $r_1 < r$. To see this, let E be the exceptional divisor of $g\colon \widetilde{X} \to X$, and let L be the pullback to $\widetilde{X}$ of the hyperplane class of $\mathbb{P}^2$ (resp., of $Q \subset \mathbb{P}^3$) by h. Note that $L \sim g^*(-K_X) - E$ by construction, and $g_*\widetilde{\alpha}_{1*}E \sim -sK_X$ for some s, since this cycle is defined over κ. We observe that there are no lines in X passing through a point of multiplicty larger than r, since D belongs to a movable linear system cut out by forms of degree r. It follows that the involution $\widetilde{\alpha}_1$ does not stabilize the divisor E. This means that $g_*\widetilde{\alpha}_{1*}E$ is supported on a nonempty curve, and therefore $s \geq 1$. If m is the multiplicity of D at x (and hence at y in the second case) and $\widetilde{D}$ is the proper transform of D on $\widetilde{X}$, then $\widetilde{D} + (m-r)E \sim rL$. Applying $(\widetilde{\alpha}_1)_*$ to this divisor and pushing down to X, we obtain $\alpha_{1*}D \sim -r_1K_X$ where $r_1 = r - (m-r)s < r$ since $m > r$. Therefore, this operation lowers the degree of the equations defining the map.

Let $D_1 = \phi_{1*}^{-1}D' \in |-r_1K_X|$. If $\mathrm{mult}_x(D_1) > r_1$ for some $x \in X$, then we proceed as before to construct a new involution α_2, and proceed from there. Since the degree decreases each time, this process stops after finitely many steps. It stops precisely when, letting

$$\phi_i = \phi \circ \alpha_1^{-1} \circ \ldots \circ \alpha_i^{-1} \colon X \dashrightarrow X'$$

and $D_i = \phi_{i*}^{-1}D' \in |-r_iK_X|$, we have $\mathrm{mult}_x(D_i) \leq r_i$ for every $x \in X$. Note that ϕ_i is defined over κ. Then, replacing ϕ by ϕ_i, we reduce to the next case.

Case 2. Assume that $\mathrm{mult}_x(D) \leq r$ for every $x \in X$.

Taking a sequence of blow-ups, we obtain a resolution of indeterminacy

$$\begin{tikzcd} & Y \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X \arrow[rr, dashed, "\phi"] & & X' \end{tikzcd}$$

with Y smooth. Write

$$\begin{aligned} K_Y + \tfrac{1}{r'}D_Y &= p^*(K_X + \tfrac{1}{r'}D) + E' \\ &= q^*(K_{X'} + \tfrac{1}{r'}D') + F' \end{aligned}$$

where E' is p-exceptional, F' is q-exceptional, and $D_Y = p_*^{-1}D = q_*^{-1}D'$. Since X' is smooth and D' is a general hyperplane section, we have $F' \geq 0$ and $\mathrm{Supp}(F') = \mathrm{Ex}(q)$. Note that $K_{X'} + \frac{1}{r'}D'$ is nef. Intersecting with the image in Y of a general complete intersection curve $C \subset X$ we see that $(K_X + \frac{1}{r'}D) \cdot C \geq 0$, and this implies that $r \geq r'$.

Next, we write

$$
\begin{aligned}
K_Y + \tfrac{1}{r} D_Y &= p^*(K_X + \tfrac{1}{r} D) + E \\
&= q^*(K_{X'} + \tfrac{1}{r} D') + F
\end{aligned}
$$

where, again, E is p-exceptional and F is q-exceptional. The fact that $\mathrm{mult}_x(D) \le r$ for all $x \in X$ implies that $E \ge 0$. Intersecting this time with the image in Y of a general complete intersection curve C' in a general fiber of $X' \to S'$, we get $(K_{X'} + \frac{1}{r} D') \cdot C' \ge 0$, and therefore $r = r'$. Note also that $E = E'$ and $F = F'$.

The difference $E - F$ is numerically equivalent to the pullback of A'. In particular, $E - F$ is nef over X and is numerically trivial over X'. Since $p_*(E - F) \le 0$, the Negativity Lemma, applied to p, implies that $E \le F$. Similarly, since $q_*(E - F) \ge 0$, the Negativity Lemma, applied to q, implies that $E \ge F$. Therefore $E = F$. This means that A' is numerically trivial, and hence $S' = \operatorname{Spec} \overline{\kappa}$. Furthermore, we have $\mathrm{Ex}(q) \subset \mathrm{Ex}(p)$, and therefore the inverse map $\sigma = \phi^{-1} \colon X' \dashrightarrow X$ is a morphism.

To conclude, just observe that if $S'_\kappa = \operatorname{Spec} \kappa$ then X'_κ must have Picard number one. But σ, being the inverse of ϕ, is defined over κ. It follows that σ is an isomorphism, as otherwise it would increase the Picard number. Therefore X'_κ is a smooth cubic surface of Picard number one.

Since we can assume without loss of generality to have picked $r' = 1$ to start with, we conclude that, after the reduction step performed in Case 1, ϕ is a projective equivalence defined over κ. The second assertion of the theorem follows by taking $\beta_\kappa = \alpha_1^{-1} \circ \dots \circ \alpha_i^{-1}$, which is defined over κ. $\square$

6.3 The method of maximal singularities

The proof of Theorem 10.2.1 already captures, in the simplest possible setting, the main ideas behind the *method of maximal singularities*, a sophisticated method to study birational links among Fano manifolds of Picard number one and, more generally, among Mori fiber spaces. We recall here the definition of the latter.

Definition 6.3.1. A *Mori fiber space* is a normal projective variety X with $\mathbb{Q}$-factorial terminal singularities, equipped with an extremal Mori contraction $f \colon X \to S$ of fiber type, which means that f is a proper morphism with connected fibers and relative Picard number $\rho(X/S) = 1$, the anticanonical class $-K_X$ is f-ample, and $\dim S < \dim X$.

Mori fiber spaces are the terminal objects produced by the minimal model program within the class of uniruled varieties. In dimension two, they consists of $\mathbb{P}^2$ and ruled surfaces, and any birational equivalence among them factors as a sequence of elementary transformations. In higher dimensions, the factorization process is more complicated, and is studied via the *Sarkisov program*. This consists of a series of elementary links which are used, very much in spirit as in Case 1 of the proof of Theorem 10.2.1, to untwist the map. We shall not discuss the Sarkisov program here. For an introduction to the program, we recommend [**?**].

A new phenomenon occurring in higher dimensions is that some Mori fiber structures are unique in their birational class. This leads to the notions of birational rigidity and superrigidity. Here we focus on the latter.

Definition 6.3.2. A Mori fiber space $f\colon X \to S$ is *birationally superrigid* if every birational map $\phi\colon X \dashrightarrow X'$ from X to another Mori fiber space $f'\colon X' \to S'$ is a fiberwise isomorphism (i.e., ϕ is an isomorphism such that $f' \circ \phi = \psi \circ f$ for some isomorphism $\psi\colon S \to S'$).

The arguments in Case 2 of the proof of Theorem 10.2.1 extend to give sufficient conditions to establish birational superrigidity. The following theorem, which lies at the heat of the menthod of maximal singularities, is proven in [**?**] in the special case where $X = X'$ is a smooth quartic threefold in $\mathbb{P}^4$. The general statement is due to [**?**], whose proof relies of some results from the minimal model program. Here we give a more elementary proof.

Theorem 6.3.3 (Noether–Fano Inequality). *Let $\phi\colon X \dashrightarrow X'$ be a birational map between two Mori fiber spaces $f\colon X \to S$ and $f'\colon X' \to S'$. Fix a sufficiently divisible integer r' and a sufficiently ample divisor on S' such that if A' is the pullback of this divisor to X' then $-r'K_{X'} + A'$ is very ample (if $S' = \operatorname{Spec}\mathbb{C}$ then take $A' = 0$). Let r be the positive rational number such that*

$$\phi_*^{-1}(-r'K_{X'} + A') \sim_{\mathbb{Q}} -rK_X + A$$

where A is the pull-back of a $\mathbb{Q}$-divisor on S. and let $B \subset X$ be the base scheme of the linear system $\phi_*^{-1}|-r'K_{X'} + A'| \subset |-rK_X + A|$. Assume that A is nef and the pair $(X, \frac{1}{r}B)$ is canonical. Then $r = r'$, ϕ is an isomorphism, and there is an isomorphism $\psi\colon S \to S'$ such that $f' \circ \phi = \psi \circ f$.

Proof. Let

$$\begin{tikzcd}
 & Y \arrow[dl, "p"'] \arrow[dr, "q"] & \\
X \arrow[rr, dashed, "\phi"] & & X'
\end{tikzcd}$$

be a resolution of singularities. Note that the exceptional loci $\operatorname{Ex}(p)$ and $\operatorname{Ex}(q)$ have pure codimension 1. Fix a general element $D' \in |-r'K_{X'} + A|$ and let $D_Y = q_*^{-1}D$ (which is the same as q^*D) and $D = p_*D_Y$. Note that $D_Y = p_*^{-1}D$ and $D = \phi_*^{-1}D' \in |-rK_X|$. Write

$$\begin{aligned} K_Y + \tfrac{1}{r'}D_Y &= p^*(K_X + \tfrac{1}{r'}D) + E' \\ &= q^*(K_{X'} + \tfrac{1}{r'}D') + F' \end{aligned}$$

where E' is p-exceptional and F' is q-exceptional. Since X' has terminal singularities and D' is a general hyperplane section, we have $F' \geq 0$ and $\operatorname{Supp}(F') = \operatorname{Ex}(q)$. Since $K_{X'} + \frac{1}{r'}D'$ is numerically equivalent to the pullback of A', which is nef, we have $(K_X + \frac{1}{r'}D) \cdot C \geq 0$ for a general complete intersection curve C in a general fiber of f. This implies that $r \geq r'$.

Next, we write

$$\begin{aligned} K_Y + \tfrac{1}{r} D_Y &= p^*(K_X + \tfrac{1}{r} D) + E \\ &= q^*(K_{X'} + \tfrac{1}{r} D') + F \end{aligned}$$

where E is p-exceptional and F is q-exceptional. Assume that the pair $(X, \frac{1}{r}B)$ is canonical. Since D is defined by a general element of the linear system of divisors cutting out B, and $r \geq 1$, it follows that $(X, \frac{1}{r}D)$ is canonical. This means that $E \geq 0$. Since $K_X + \frac{1}{r}D$ is numerically equivalent to the pullback of A, which is nef by hypothesis, we have $(K_{X'} + \frac{1}{r}D') \cdot C' \geq 0$ for a general complete intersection curve C' in a general fiber of f', and therefore $r = r'$. Note, in particular, that $E = E'$ and $F = F'$, and hence

$$E - F \sim_{\mathbb{Q}} q^*A' - p^*A.$$

Since $E - F$ is p-nef and $p_*(E - F) \leq 0$, we have $E \leq F$ by the Negativity Lemma. Similarly, since $F - E$ is q-nef and $q_*(F - E) \leq 0$, we have $F \leq E$. Therefore $E = F$. This means that $p^*A \sim_{\mathbb{Q}} q^*A'$, and therefore, since A' is the pullback of a very ample divisor on S', there is a (proper) morphism $\psi \colon S \to S'$ fitting in a commutative diagram

$$\begin{tikzcd} & Y \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X \arrow[rr, dashed, "\phi"] \arrow[d, "f"'] & & X' \arrow[d, "f'"] \\ S \arrow[rr, "\psi"] & & S' \end{tikzcd}$$

Computing the Picard number of Y in two ways, we get

$$\begin{aligned} \rho(Y) &= \rho(Y/X) + 1 + \rho(S/S') + \rho(S) \\ &= \rho(Y/X') + 1 + \rho(S). \end{aligned}$$

Note that $\mathrm{Ex}(q) \subset \mathrm{Ex}(p)$ since F contains every q-exceptional divisor in its support, and therefore $\rho(Y/X') \leq \rho(Y/X)$. It follows that $\rho(Y/X') = \rho(Y/X)$ and $\rho(S/S') = 0$. The second identity implies that ψ is an isomorphism, since S' is normal. The first identity implies that $\mathrm{Ex}(p) = \mathrm{Ex}(q)$, and thus the difference $p^*D - q^*D'$ is q-exceptional. Since D is ample, this implies that ϕ is a (proper) morphism. Keeping in mind that X and X' have the same Picard number and X' is normal, it follows that ϕ is an isomorphism too. □

The method of maximal singularities, started in work of Fano [**?**, **?**], was perfected in [**?**] to prove the following result.

Theorem 6.3.4 (Iskovskikh–Manin). *Every smooth quartic threefold $X = X_4 \subset \mathbb{P}^4$ is birationally superrigid. In particular,* $\mathrm{Bir}(X) = \mathrm{Aut}(X)$ *is finite and X is not rational.*

As a matter of fact, in [**?**] there is only mention of the second part of the statement, but the proof itself gives the stronger property that X is birationally superrigid.

This theorem extends to higher dimensions, to the statement that every smooth hypersurface $X \subset \mathbb{P}^N$ of degree N, for $N \geq 4$, is birationally superrigid (see Theorem 10.7.1). We will prove this at the end of the chapter. The proof in higher dimensions requires further techniques, but we shall give a quick proof of the theorem of Iskovskikh and Manin earlier, in Remark 10.4.12.

It is interesting to compare Iskovskikh–Manin's theorem to the following equally influecial theorem, due to [**?**].

Theorem 6.3.5 (Clemens–Griffiths). *Every smooth cubic threefold $X = X_3 \subset \mathbb{P}^4$ is nonrational.*

The two results, which were proved around the same time, gave the first counterexamples to the Lüroth problem. The techniques, though, are very different, and while the first uses the method of maximal singularities, the latter is based on the computation of the intermediate Jacobian. The failure to rationality is, in some sense, of a different nature too: cubic threefolds are not rational, but yet they carry birational structures of del Pezzo fibrations and conic bundles, as well as birational involutions that are not biregular (and in fact their group of birational automorphisms is infinite). These are respectively constructed by taking general linear projections onto one, two, and three dimensional projective spaces.

6.4 Multiplicities and log canonical thresholds

In order to implement the Fano–Noether Inequality to concrete situations (for example, to Fano hypersurfaces in projective spaces, the case of interest in this chapter), one needs to relate conditions on singularities of pairs to other measures of singularities such as multiplicities, which can be controlled in terms of the degrees of the equations. This section is devoted to build such relationship.

6.4.1 Basic properties of multiplicities

The *multiplicity* $e_p(X)$ of a variety X at a point p is defined to be the Hilbert–Samuel multiplicity $e(\mathfrak{m}_p)$ of the maximal ideal $\mathfrak{m}_p$ of the local ring $\mathscr{O}_{X,p}$.

More generally, for any closed subscheme Z of a pure-dimensional scheme X, and an irreducible component T of Z, the *multiplicity of X along Z at T*, denoted by $e_Z(X)_T$ is defined to be the Hilbert–Samuel multiplicity $e(\mathscr{I}_S)$ of the primary ideal $\mathscr{I}_S$ determined by S in the local ring $\mathscr{O}_{X,T}$. If $Z = T$, then we just write $e_T(X)$.

Remark 6.4.1. If D is an effective Cartier divisor on a variety X and $p \in X$ is a regular point, then $e_p(D)$ is simply the multiplicity of a generator of the ideal of D

in the local ring at p [**?**, Example 4.3.9]. If $Z = D_1 \cap \cdots \cap D_n \subset X$ is the complete intersection of n divisors D_i on a variety X, and T is an irreducible component of Z, then $e_Z(X)_T$ is equal to the intersection multiplicity $i(T, D_1 \cdot \ldots \cdot D_n; X)$ [Ful98, Example 7.1.10.(a)].

Proposition 6.4.2. *Let X be a pure-dimensional scheme. For every irreducible closed set $T \subset X$ there is a nonempty open set $T^\circ \subset T$ such that $e_p(X) \geq e_T(X)$ for every point $p \in X$, and equality holds if $p \in T^\circ$.*

Proof. (Give a proof, or quote [**?**, Theorem (4)]) □

If $\alpha = \sum n_i [V_i]$ is a cycle on a variety X, where each V_i is a subvariety, then we define the multiplicity of α along an irreducible subvariety $T \in X$ to be $e_T(\alpha) := \sum n_i e_T(V_i)$, where we set $e_T(V_i) = 0$ if $T \not\subset V_i$.

Remark 6.4.3. If Z is a pure-dimensional closed subscheme of a variety X, and $[Z]$ is the associated fundamental cycle, then $e_p(Z) = e_p([Z])$ for every point $p \in Z$ (cf. [Ful98, Example 4.3.4]).

Proposition 6.4.4. *Let Z be a pure-dimensional closed Cohen-Macaulay subscheme of $\mathbb{P}^m$ of positive dimension.*

i) If H meets properly the embedded tangent cone of Z at a point p, then $e_p(Z \cap H) = e_p(Z)$.
ii) Given a hyperplane in the dual space $\mathscr{H} \subset (\mathbb{P}^m)^\vee$, if $H \in \mathscr{H}$ is general enough, then $e_p(Z \cap H) = e_p(Z)$ for every $p \in Z \cap H$.

Proof. We can assume that $Z \neq \mathbb{P}^m$. Consider any linear subspace $L \subset \mathbb{P}^m$ of dimension $\dim L = m - \dim Z$ that meets properly the embedded tangent cone of Z at p. Then the component of $Z \cap L$ at p is zero-dimensional, and we have $e_p(Z) = l(\mathscr{O}_{Z \cap L, p})$ by [Ful98, Proposition 7.1 and Corollary 12.4]. This implies i).

At any point $p \in Z$, the fiber over p of the conormal variety of Z, viewed as a linear subspace of $(\mathbb{P}^n)^\vee$, contains the dual variety of every component of the embedded projective tangent cone $C_p Z$ of Z at p (e.g., see [**?**, page 219]). It follows then by i) that $e_p(Z \cap H) = e_p(Z)$ as long as H is chosen outside the dual variety $Z_i^\vee$ of each irreducible component Z_i of Z. To conclude, it suffices to observe that $Z_i^\vee$ cannot contain any hyperplane of $(\mathbb{P}^m)^\vee$, since it is irreducible of dimension $\leq m-1$, and $Z_i^{\vee\vee} = Z_i$ is not a point. □

Proposition 6.4.5. *Let Z be a pure-dimensional closed subscheme of $\mathbb{P}^m$. Let $\pi \colon \mathbb{P}^m \smallsetminus \Lambda \to \mathbb{P}^k$ be a linear projection from a center Λ disjoint from Z, and assume that $\pi|_{Z_{\mathrm{red}}}$ is injective over the image of a point $p \in Z$. If $\pi^{-1}(\pi(p))$ meets properly the embedded tangent cone of Z at p, then $e_p(Z) = e_{\pi(p)}(\pi_*[Z])$.*

Proof. By Remark 10.4.3, we can reduce to the case in which Z is a subvariety of $\mathbb{P}^m$. Note that $\pi_*[Z] = [T]$ where $T = \pi(Z)$ is a variety. Let $q = \pi(p) \in T$, let $L \subset \mathbb{P}^k$ be a general line passing through q, and let $\mathfrak{q} \subset \mathscr{O}_{T,q}$ be the ideal generated by the linear forms vanishing along L. Then let $\mathfrak{p} \subset \mathscr{O}_{Z,p}$ be the ideal generated by the linear forms locally vanishing along $\pi^{-1}(L)$. Note that $\mathfrak{p} = \mathfrak{q} \cdot \mathscr{O}_{Z,p}$. Since $\pi^{-1}(q)$ intersects properly the embedded tangent cone C_pZ of Z at p and L is general through q, we may assume that $\pi^{-1}(L)$ intersects properly C_pZ. This implies that the linear forms locally defining $\pi^{-1}(q)$ generate the ideal of the exceptional divisor of the blow up of Z at p, and therefore we have $e(\mathfrak{p}) = e(\mathfrak{m}_p)$ where $\mathfrak{m}_p$ is the maximal ideal of $\mathscr{O}_{Z,p}$. On the other hand, if $\mathfrak{m}_q$ is the maximal ideal of $\mathscr{O}_{T,q}$, then $\mathfrak{q} \subseteq \mathfrak{m}_q$, and hence

$$\mathfrak{p} = \mathfrak{q} \cdot \mathscr{O}_{Z,p} \subseteq \mathfrak{m}_q \cdot \mathscr{O}_{Z,p} \subseteq \mathfrak{m}_p.$$

Therefore $e(\mathfrak{m}_p) = e(\mathfrak{m}_q \cdot \mathscr{O}_{Z,p})$. This implies the proposition, since $e(\mathfrak{m}_p) = e_p(Z)$ by definition and $e(\mathfrak{m}_q \cdot \mathscr{O}_{Z,p}) = e_q(T)$ by [Ful98, Example 4.3.6]. $\square$

6.4.2 Multiplicity bounds

We prove here some inequalities on multiplicities with various geometric flavors. We begin with the following property, due to [**?**].

Proposition 6.4.6. *Let $X \subset \mathbb{P}^N$ be a smooth hypersurface, and let α be an effective cycle on X of pure codimension $k < \frac{1}{2}\dim X$. If $m \in \mathbb{N}$ is such that $\alpha \equiv m \cdot c_1(\mathscr{O}_X(1))^k$, then* $\dim\{x \in \operatorname{Supp}(\alpha) \mid e_x(\alpha) > m\} < k$.

Proof. We need to prove that $e_C(\alpha) \le m$ for every irreducible subvariety C of dimension $\ge k$. First, note that this inequality is trivially satisfied if either $k = 0$ or $C \not\subseteq \operatorname{Supp}(\alpha)$ or $\deg X = 1$. Thus, we may assume that $k \ge 1$, $C \subseteq \operatorname{Supp}(\alpha)$ (that forces $N \ge 4$) and $\deg X \ge 2$. Moreover, it is enough to prove the theorem for the case when $\dim C = k$.

For a point $p = (a_0, \dots, a_N) \in \mathbb{P}^N \smallsetminus X$, let $\pi_p : \mathbb{P}^N \smallsetminus \{p\} \to H_p \cong \mathbb{P}^{N-1}$ be the linear projection from p and set $f_p = \pi_p|_X : X \to H_p$. If $F(x_0, \dots, x_N) = 0$ is the homogeneous equation defining X, then the relative canonical divisor K_{X/H_p} is cut on X by the equation $\sum_{i=0}^N a_i \frac{\partial F}{\partial x_i} = 0$, and moves freely in a base point free linear system, since X is smooth.

For a given subvariety $Y \subset X$, by choosing p general enough we may assume that the general fiber of f_p over $f_p(Y)$ is a reduced set of d points. Then $f_p^{-1} f_p(Y)$ is generically reduced, and we can write

$$\operatorname{Supp}(f_p^{-1} f_p(Y)) = Y \cup R(Y, p),$$

where $R(Y, p)$ is a variety of degree $(d-1)\deg Y$. We say that $R(Y, p)$ is the residual variety of Y under the projection f_p.

We fix k general enough points $p_1, \dots, p_k \in \mathbb{P}^N$, set $R_0 = C$, and define recursively $R_i = R(R_{i-1}, p_i)$ for $i = 1, \dots, k$. We also set $K_0 = X$ and $K_i := K_{X/H_{p_i}}$.

Lemma 6.4.7. *For every $i = 0, \dots, k$,*

i) $\deg R_i = (d-1)^i \deg C$,

ii) $(R_0 \cap \cdots \cap R_i) \supseteq \mathrm{Supp}(K_0 \cap \cdots \cap K_i \cap C)$,
iii) $\dim(R_i \cap \mathrm{Supp}(\alpha)) = k - i$.

Proof. We prove the three assertions by induction on i. For $i = 0$, they follow by hypothesis. So, assume $i \geq 1$ and that i)–iii) are satisfied for $i-1$. Property i) follows from $\deg(R_{i-1} \cup R_i) = d \deg R_{i-1}$. In order to prove ii), it is enough to show that

$$\mathrm{Supp}(R_i \cap R_{i-1}) = \mathrm{Supp}(K_i \cap R_{i-1}). \tag{6.1}$$

We can assume that K_i intersects properly R_{i-1} and each component of the singular locus of R_{i-1}. Since $k < \frac{1}{2}(N-1)$, the secant variety of R_{i-1} has dimension less than N. Thus, for a general p_i, f_{p_i} restricts to a one-to-one morphism on R_{i-1}. Let $U \subset f_{p_i}(R_{i-1})$ be the largest open set such that R_{i-1} restricts to a section of the $\mathbb{A}^1$-bundle $\pi_{p_i}^{-1}(U) \to U$. If p_i is general enough, $R_{i-1} \cap \pi_{p_i}^{-1}(U)$ contains $R_{i-1} \setminus \mathrm{Sing}(R_{i-1})$, hence it intersects each component of $K_i \cap R_{i-1}$. Since

$$X \cap \pi_{p_i}^{-1} f_{p_i}(R_{i-1}) = R_{i-1} \cup R_i$$

is a Cartier divisor on $\pi_{p_i}^{-1} f_{p_i}(R_{i-1})$, we conclude that both R_{i-1} and R_i restrict to Cartier divisors on $\pi_{p_i}^{-1}(U)$. Then for every point $x \in R_{i-1}$ over U, denoting $L = \pi_{p_i}^{-1} f_{p_i}(x)$ ($\cong \mathbb{A}^1$), $R_{i-1}|_L$ and $R_i|_L$ are divisors of L and

$$e_x(X \cap L) = \mathrm{ord}_x(R_{i-1}|_L + R_i|_L).$$

The left hand side of this equation is 1 if and only if $x \notin K_i$, whereas the right hand side is 1 if and only if $x \notin R_i$. This shows that (10.1) holds for the points over U. Suppose now that $x \in R_{i-1}$ is not a point over U. Then $p_i \in T_{R_{i-1},x}$. Since $T_{R_{i-1},x} \subseteq T_{X,x}$, we see that $x \in K_i$. We conclude that $\mathrm{Supp}(R_i \cap R_{i-1})$ is a dense subset of $K_i \cap R_{i-1}$. Since $R_i \cap R_{i-1}$ is closed, equality (10.1) follows. This gives ii).

Before proving iii), we fix the following notation: for two closed subsets $S, T \subseteq \mathbb{P}^N$, let

$$J(S,T) = \{(s,t,p) \in S \times T \times \mathbb{P}^N \mid s \neq t, p \in \overline{st}\}.$$

By counting dimensions, one sees that the map $J(\mathrm{Supp}(\alpha), R_{i-1}) \to \mathbb{P}^N$ is either generically finite or not dominant. Therefore, by choosing p_i general, the intersection of R_i and Z_0 outside $R_{i-1} \cap Z_0$ is zero dimensional or empty. Note that

$$\dim(R_i \cap \mathrm{Supp}(\alpha)) = \max\{\dim(R_i \cap R_{i-1} \cap \mathrm{Supp}(\alpha)), \dim(R_i \cap (X \setminus R_{i-1}) \cap \mathrm{Supp}(\alpha))\}$$

By (10.1), if we pick p_i so that K_i intersects properly $R_{i-1} \cap \mathrm{Supp}(\alpha)$, then we get

$$\dim(R_i \cap R_{i-1} \cap \mathrm{Supp}(\alpha)) = \dim(K_i \cap R_{i-1} \cap \mathrm{Supp}(\alpha)) = \dim(R_{i-1} \cap \mathrm{Supp}(\alpha)) - 1.$$

This gives iii). $\square$

The set $\Sigma := K_1 \cap \cdots \cap K_k \cap C$ contains $(d-1)^k \deg C$ distinct points by Bertini's theorem and Bezout's theorem. By Lemma 10.4.7, $R_k \cap \mathrm{Supp}(\alpha)$ is a zero dimensional set containing Σ. Then, by Bezout's theorem,

$$m(d-1)^k \deg C = \int_X \alpha \cdot [R_k] \geq \sum_{q \in \Sigma} e_q(\alpha) \geq e_C(\alpha)(d-1)^k \deg C.$$

This implies that $e_C(\alpha) \leq m$, so the proof of the Proposition is complete. □

Remark 6.4.8. Because we have assumed $k < \frac{1}{2}\dim X$, the existence of m as in Proposition 10.4.6 follows from Lefschetz Theorem. The proposition holds also if $k = \frac{1}{2}\dim X$ (as long as we assume $\alpha \equiv m(c_1(\mathscr{O}_X(1)))^k$), the same proof extending to this extremal case. The only thing to keep into account is that the equality (10.1) holds only outside a zero dimensional set of $R_i \cap R_{i-1}$. Note also that the statement is trivially true if $k > \frac{1}{2}\dim X$.

The following properties relate multiplicities to discrepancies and log canonical thresholds.

Proposition 6.4.9. *Let A be an effective $\mathbb{Q}$-divisor on a smooth variety X, and suppose that $a_E(X,A) \leq 1$ for some prime divisor E over X. If T is the center of E in X, then $e_T(A) \geq 1$.*

Proof. We can assume that E is an exceptional divisor of a log-resolution $f\colon X' \to X$ of (X,A). Pick a general point $p \in T$, and let $Y \subset X$ be a general complete intersection subvariety of codimension $\operatorname{codim}(Y,X) = \dim T$, passing through p. Then the proper transform Y' of Y meets E transversally, and we have $a_{E'}(Y,A|_Y) \leq 1$ if E' is a component of $E|_{Y'}$. Notice that $\dim Y \geq 2$. If $H \subset Y$ is a general hyperplane section through p, then $(H,A|_H)$ is not klt at p by inversion of adjunction .... Taking a general complete intersection curve $C \subset H$ through p, we see that $(C,A|_C)$ is not klt at p by the same theorem. This is equivalent to $e_p(A|_C) \geq 1$. On the other hand, by taking the hyperplanes cutting out C generally enough, we have $e_p(A|_C) = e_p(A)$. We conclude that $e_T(A) \geq 1$. □

The following theorem relates Hilbert–Samuel multiplicity to log canonical threshold. Consider a local ring $\mathscr{O}_{X,p}$ with maximal ideal $\mathfrak{m}_p$, where p is a regular point of an n-dimensional variety X. If X is 1-dimensional, then an $\mathfrak{m}_p$-primary ideal $\mathfrak{a}$ is locally generated by one equation $h \in \mathscr{O}_{C,p}$, and $e(\mathfrak{a}) = \operatorname{mult}(h) = 1/\operatorname{lct}(h) = 1/\operatorname{lct}(\mathfrak{a})$. In higher dimension there are two natural ways to generalize this relation, by either considering principal ideals or looking at $\mathfrak{m}_p$-primary ideals. In the first case we have

$$n \cdot \operatorname{mult}(h) \geq \frac{n}{\operatorname{lct}(h)} \geq \operatorname{mult}(h)$$

for any $h \in \mathfrak{m}_p$. The $\mathfrak{m}_p$-primary case is treated in the next theorem and its corollary. For an $\mathfrak{m}_p$-primary ideal $\mathfrak{a}$, it establishes the lower bound

$$e(\mathfrak{a}) \geq \left(\frac{n}{\operatorname{lct}(\mathfrak{a})}\right)^n$$

on Hilbert–Samuel multiplicity in terms of the log canonical threshold. Examples show, on the contrary, that there cannot be upper bounds on Hilbert–Samuel multiplicity only in terms of the log canonical threshold if $n \geq 2$ (e.g., take $\mathfrak{a} = (x,y^m) \subset \mathbb{C}[x,y]$ with m arbitrarily large).

Theorem 6.4.10. *Let X be a smooth variety, let $Z \subset X$ be a closed subscheme, and let T be an irreducible component of Z, of codimension n in X. Let $D = \sum_{i=0}^n d_i D_i$ be a $\mathbb{Q}$-divisor with all components passing through T, with simple normal crossings at the generic point of T. Assume that D is either effective (i.e., $d_i \geq 0$ for all i), or irreducible (e.g., $d_i = 0$ for $i \neq 1$). Suppose that, for some $c > 0$, the pair $(X, cZ + D)$ is not klt. Then the length of the local ring $\mathscr{O}_{Z,T}$ satisfies the inequality*

$$l(\mathscr{O}_{Z,T}) \geq \frac{n^n}{n! \cdot c^n} \cdot \prod_{i=1}^n (1 - d_i).$$

Proof. Passing to the completion, we fix an isomorphism $\widehat{\mathscr{O}}_{X,p} \cong k[[x_1, \ldots, x_n]]$ such that each D_i is locally defined by $x_i = 0$, where k is the residue field of $\mathscr{O}_{X,T}$, and restrict to the polynomial ring $R = k[x_1, \ldots, x_n]$. Let $\mathfrak{m} = (x_1, \ldots, x_n)$ denote the maximal ideal at the origin. If $\mathfrak{a} \subset R$ is the ideal determined by the ideal sheaf of Z, then we need to prove that

$$l(R/\mathfrak{a}) \geq \frac{n^n}{n! \cdot c^n} \cdot \prod_{i=1}^n (1 - d_i), \tag{6.2}$$

for any $\mathfrak{m}$-primary ideal $\mathfrak{a}$ of R such that the pair $(R, \mathfrak{a}^c \cdot \prod_{i=1}^n x_i^{d_i})$ is not klt.

We shall start by verifying that (10.2) holds in the special case of monomial ideals. Suppose that $\mathfrak{a}$ is monomial. Let $P(\mathfrak{a}) \subset (\mathbb{R}_{\geq 0})^n$ be the Newton polytope of $\mathfrak{a}$, and let $(u_1, \ldots, u_n)$ be the coordinates in $(\mathbb{R}_{\geq 0})^n$. By the description of multiplier ideals of monomial ideals, the condition that $(R, \mathfrak{a}^c \cdot \prod_{i=1}^n x_i^{d_i})$ is not klt is equivalent to the fact that there is a bounded facet of $P(\mathfrak{a})$ such that, if $\sum_{i=1}^n u_i/a_i = 1$ is the equation of the hyperplane supporting it, then

$$\sum_{i=1}^n \frac{1 - d_i}{a_i} \leq c.$$

Applying the inequality between the arithmetic mean and the geometric mean of the set of numbers $\{(1 - d_i)/a_i\}_{i=1}^n$, we get

$$\left(\prod_{i=1}^n \frac{1 - d_i}{a_i} \right)^{1/n} \leq \frac{1}{n} \cdot \sum_{i=1}^n \frac{1 - d_i}{a_i}.$$

Then (10.2) follows from the fact that, as the length is bounded below by the number of lattice points contained in the area cut out by $\sum_{i=1}^n u_i/a_i \leq 1$ in $(\mathbb{R}_{\geq 0})^n$, we have

$$l(R/\mathfrak{a}) \geq \frac{1}{n!} \cdot \prod_{i=1}^n a_i.$$

The proof of the general case consists in reducing to the monomial case, via a flat degeneration to monomial ideals. To this end, we fix a monomial order. Let $\mathrm{in}(\mathfrak{b})$ denote the monomial initial ideal obtained from an ideal $\mathfrak{b}$. If $d_i \geq 0$ for all i, then the pair $(R, \mathfrak{a}^c \cdot \prod_{i=1}^n x_i^{d_i})$ is effective. Semi-continuity of the log canonical threshold implies that, the pair $(R, \mathrm{in}(\mathfrak{a})^c \cdot \prod_{i=1}^n x_i^{d_i})$ is not klt, and therefore (10.2) follows from the monomial case.

Suppose now that D is irreducible and noneffective. We can assume that $b := -d_1 > 0$ and $d_i = 0$ for $i \neq 1$. By assumption, $(R, \mathfrak{a}^c \cdot x_1^{-b})$ is not klt. The reduction to the monomial setting is more delicate in this case. We first need to reduce to a setting where b is an integer, and then take a suitable monomial order.

Write $b = r/s$ where r, s are positive integers, and let $\widetilde{R} = k[y, x_2, \ldots, x_n]$, with the inclusion $R \subset \widetilde{R}$ given by $x_1 = y^r$. For any ideal $\mathfrak{b} \subset R$ let $\widetilde{\mathfrak{b}} := \mathfrak{b} \cdot \widetilde{R}$. By the ramification formula, $(R, \mathfrak{a}^c \cdot x_1^{-b})$ not being klt implies that $(\widetilde{R}, \widetilde{\mathfrak{a}}^c \cdot y^{-(s+r-1)})$ is not klt. This is equivalent to the condition that

$$y^{s+r-1} \notin \mathscr{J}(\widetilde{\mathfrak{a}}^c).$$

We fix a monomial order in $\widetilde{R}$ such that $y^{s+r-1} < x_2 < \cdots < x_n$, and consider the induced flat deformation to monomial ideals. Then we have

$$y^{s+r-1} \notin \mathrm{in}(\mathscr{J}(\widetilde{\mathfrak{a}}^c)),$$

as otherwise we could find a polynomial $h \in \mathscr{J}(\widetilde{\mathfrak{a}}^c)$ with $\mathrm{in}(h) = y^{s+r-1}$. Because of this particular monomial order we fixed, h must be a polynomial in y of degree $s+r-1$, and since $\mathscr{J}(\widetilde{\mathfrak{a}}^c)$ is $\mathfrak{m}$-primary, it would follow that $y^i \in \mathscr{J}(\widetilde{\mathfrak{a}}^c)$ for some $i \leq s+r-1$, which contradicts our hypothesis.

On the contrary, the restriction theorem for multiplier ideals implies that

$$\mathscr{J}(\mathrm{in}(\widetilde{\mathfrak{a}})^c) \subseteq \mathrm{in}(\mathscr{J}(\widetilde{\mathfrak{a}}^c)),$$

and therefore

$$y^{s+r-1} \notin (\mathscr{J}(\mathrm{in}(\widetilde{\mathfrak{a}})^c)).$$

This means that the pair $(\widetilde{R}, \mathrm{in}(\widetilde{\mathfrak{a}})^c \cdot y^{-(s+r-1)})$ is not klt.

The monomial order of $\widetilde{R}$ induces a monomial order on R, and $\widetilde{\mathrm{in}(\mathfrak{b})} = \mathrm{in}(\widetilde{\mathfrak{b}})$ for any ideal $\mathfrak{b} \subset R$. Applying the ramification formula in the other direction, we conclude that the pair $(R, \mathrm{in}(\mathfrak{a})^c \cdot x_1^{-b})$ is not klt. Since $l(R/\mathrm{in}(\mathfrak{a})) = l(R/\mathfrak{a})$, we have finally reduced this case too to the monomial case. This completes the proof of the theorem. □

Corollary 6.4.11. *With the same assumptions as in Theorem 10.4.10, the multiplicity of X along Z at T satisfies the inequality*

$$e_Z(X)_T \geq \frac{n^n}{c^n} \cdot \prod_{i=1}^n (1 - d_i).$$

Proof. If $\mathscr{I}_Z \subset \mathscr{O}_{X,T}$ is the ideal of Z, then

$$e_Z(X)_T = e(\mathscr{I}_Z) = \lim_{m \to \infty} \frac{n! \cdot l(\mathscr{O}_{X,T}/\mathscr{I}_Z^m)}{m^n}.$$

Therefore the corollary follows by applying Theorem 10.4.10 to the schemes locally defined by the powers $\mathscr{I}_Z^m$. $\square$

Remark 6.4.12. The properties proved thus far are enough to prove Iskovskikh–Manin's theorem (Theorem 10.2). Suppose that $\phi\colon X \dashrightarrow X'$ is a birational map from a smooth quartic threefold $X \subset \mathbb{P}^4$ to a Mori fiber space $X' \to S'$, and assume by way of contradiction that ϕ is not an isomorphism. With the same notation as in Theorem 10.3.3, it follows that $(X, \frac{1}{r}B)$ is not canonical (note that in our setting $A = 0$). Let D be a general member of the linear system $\phi_*^{-1}|-r'K_{X'}+A'| \subset |-rK_X|$ and $Z = D_1 \cap D_2 \subset X$ be the complete intersection of two such divisors. Note that $(X, \frac{1}{r}D)$ and $(X, \frac{1}{r}Z)$ are both not canonical. Proposition 10.4.6 (the easy case $k = 1$) implies that $e_C(D) \le r$ for every curve $C \in X$, and therefore $(X, \frac{1}{r}D)$, and hence $(X, \frac{1}{r}Z)$, are canonical in codimension one, by Proposition 10.4.9. Therefore there is a divisor E over X, with center equal to a point $p \in X$, such that $a_E(X, \frac{1}{r}Z) < 1$. Let $S \subset X$ be the surface cut out by a general hyperplane through p. Note that $a_E(X, S + \frac{1}{r}Z) < 0$. By inversion of adjunction....., there is a divisor F over S, with center p, such that $a_F(S, \frac{1}{r}Z \cap S) < 0$. This means that $\operatorname{lct}_p(S, Z \cap S) < 1/r$. Note that $Z \cap S$ is a zero dimensional scheme. Then, by Remark 10.4.1 and Corollary 10.4.11, we have

$$i(p, (D_1|_S) \cdot (D_2|_S); S) = e_{Z\cap S}(S)_p > 4r^2.$$

On the contrary, the intersection multiplicity in the left hand side is equal to the intersection multiplicity $i(p, \widetilde{D}_1 \cdot \widetilde{D}_2 \cdot X \cdot H; \mathbb{P}^4)$ where $\widetilde{D}_1, \widetilde{D}_2 \subset \mathbb{P}^4$ are hypersurfaces of degree r cutting D_1, D_2 on X, and H is the hyperplane cutting S in X. By Bezout's theorem, this number is bounded above by the product of the degrees of the equations involved, which is equal to $4r^2$. This is in contradiction with the above inequality.

6.5 Log discrepancies via generic projections

In this section we study how log discrepancies behave under generic projections. We will work on possibly singular varieties, and use a variant of the usual notion of log discrepancy called *Mather log discrepancy*. While usual log discrepancies are defined by comparing canonical divisors, Mather log discrepancies are defined (in a more general setting) by comparing sheaves of Kähler differentials.

Definition 6.5.1. Let X be a normal variety of dimension n. Let $f\colon X' \to X$ be a resolution of singularities, and let $\operatorname{jac}_f := \operatorname{Fitt}^0(\Omega_{X'/X}) \subset \mathscr{O}_{X'}$ be the Jacobian ideal of the map. For every prime divisor E on X', we define the *Mather log discrepancy* of a pair (X, Z) along a prime divisor E on X' to be

$$\widehat{a}_E(X, Z) := \operatorname{ord}_E(\operatorname{jac}_f) + 1 - \operatorname{ord}_E(Z).$$

If $Z = 0$ then we drop it from the notation, and write $\widehat{a}_E(X)$.

If X is smooth then $\widehat{a}_E(X,Z) = a_E(X,Z)$. In general, however, the two log discrepancies differ. For instance, if X has locally complete intersection singularities, then it can be shown than $\widehat{a}_E(X,Z) = a_E(X,Z) + \mathrm{ord}_E(\mathrm{jac}_X)$. The next property gives an alternative way of computing Mather discrepancies.

Proposition 6.5.2. *Let E be a prime divisor over a normal affine variety $X \subset \mathbb{A}^N$ of dimension n, and let $\pi\colon \mathbb{A}^N \to U := \mathbb{A}^n$ be a general linear projection. Writing $\mathrm{ord}_E|_{\mathbb{C}(U)} = p \cdot \mathrm{ord}_F$, where F is a prime divisor over U and p is a positive integer, we have $\widehat{a}_E(X) = p \cdot a_F(U)$.*

Proof. We can assume that there is a diagram

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[r, hook] \arrow[d] & \mathbb{A}^N \arrow[d, "\pi"] \\ U' \arrow[r] & U \arrow[r, equal] & \mathbb{A}^n \end{tikzcd}$$

where $X' \to X$ and $U' \to U$ are resolutions such that E is a divisor on X', and F is a divisor on U'. Note that $\mathrm{ord}_E(g^*F) = p$ and $\mathrm{ord}_E(K_{X'/U'}) = p-1$. Denoting by $h\colon X' \to U$ the composition of f with the projection to U, we have $\mathrm{ord}_E(K_{X'/U}) = \mathrm{ord}_E(\mathrm{jac}_h)$. If $x_1,\dots,x_n$ are local parameters in X' centered at a general point of E, then f is locally given by equations $y_i = f_i(x_1,\dots,x_n)$, and jac_f is locally defined by the $n \times n$ minors of the matrix $(\partial f_i/\partial x_j)$. On the other hand, if $\pi\colon \mathbb{A}^N \to U = \mathbb{A}^n$ is a general projection, then jac_h is locally defined by a general linear combination of the $n \times n$ minors of $(\partial f_i/\partial x_j)$, and therefore we have $\widehat{a}_E(X) = \mathrm{ord}_E(K_{X'/U}) + 1$. Then, writing $K_{X'/U} = K_{X'/U'} + K_{U'/U}$, we get

$$\widehat{a}_E(X) = \mathrm{ord}_E(K_{X'/U'}) + \mathrm{ord}_E(g^*K_{U'/U}) + 1 = p \cdot a_F(U).$$

$\square$

Theorem 6.5.3. *Let $X \subset \mathbb{A}^N$ be a normal affine variety of dimension n, and let E be a prime divisor over X. Let $Z \subset X$ be a closed Cohen–Macaulay subscheme of pure codimension k, and let $c \in \mathbb{R}_+$. Then let*

$$\phi : X \to \mathbb{A}^{n-k+1}$$

be the morphism induced by restriction of a general linear projection $\sigma\colon \mathbb{A}^N \to \mathbb{A}^{n-k+1}$. Note that $\phi|_Z$ is a proper finite morphism, and $\phi_*[Z]$ is a cycle of codimension one in $\mathbb{A}^{n-k+1}$; we regard $\phi_*[Z]$ as a Cartier divisor on $\mathbb{A}^{n-k+1}$. Write $\mathrm{ord}_E|_{\mathbb{C}(\mathbb{A}^{n-k+1})} = q \cdot \mathrm{ord}_G$ where G is a prime divisor over $\mathbb{A}^{n-k+1}$ and q is a positive integer. Then

$$q \cdot a_G\left(\mathbb{A}^{n-k+1}, \frac{k!c^k}{k^k} \cdot \phi_*[Z]\right) \le \widehat{a}_E(X, cZ).$$

Proof. We assume that $\mathrm{ord}_E(Z) > 0$ (the case $\mathrm{ord}_E(Z) = 0$ is easier and left to the reader). We factor σ as a composition of two general linear projections $\mathbb{A}^N \to \mathbb{A}^n$

and $\mathbb{A}^n \to \mathbb{A}^{n-k+1}$. For short, we denote $U = \mathbb{A}^n$ and $V = \mathbb{A}^{n-k+1}$. Write $\mathrm{ord}_E|_{\mathbb{C}(U)} = p \cdot \mathrm{ord}_F$ for some prime divisor F over U and some positive integer p. Note that p divides q.

Let $h\colon V' \to V$ be a resolution where the center of ord_F has codimension 1. We can assume that F is a divisor on V'. Let $X' := V' \times_V X$ and $U' := V' \times_V U$, and consider the induced commutative diagram

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d, "\psi'"] \arrow[dd, bend right, "\phi'"'] & X \arrow[d, "\psi"'] \arrow[dd, bend left, "\phi"] \\ U' \arrow[r, "g"] \arrow[d, "\gamma'"] & U \arrow[d, "\gamma"'] \\ V' \arrow[r, "h"] & V \end{tikzcd} .$$

Let $Z' := f^{-1}(Z) \subset X'$ and $Z'' := \psi'(Z') \subset U'$, both defined scheme-theoretically. (In general $Z'' \subset g^{-1}(\psi(Z))$ but the inclusion may be strict.) By base change, the restriction $\phi'|_{Z'}$ is finite, and thus both $\psi'|_{Z'}$ and $\gamma'|_{Z''}$ are finite. Note that

$$p \cdot \mathrm{ord}_F(Z'') = \mathrm{ord}_E((\psi')^{-1}(Z'')) \geq \mathrm{ord}_E(Z') = \mathrm{ord}_E(Z). \tag{6.3}$$

It follows from [**?**, Lemma 1.4] (add explanation here............) that Z' is pure dimensional and $[Z'] = f^*[Z]$. Furthermore, since $\psi'|_{Z'}\colon Z' \to V$ is a finite surjective morphism of schemes, Z'' is also pure dimensional, and $\psi'_*[Z'] \geq [Z'']$. Then, using [**?**, Example 17.4.1] and [**?**, Lemma 3.39] as in the proof of [**?**, Lemma 1.5] (add explanation here............), we get

$$h^*\phi_*[Z] = \phi'_* f^*[Z] = \phi'_*[Z'] \geq \gamma'_*[Z''].$$

The center C of ord_F in U' is contained in V and dominates G. Since G is an irreducible component of $h^*\phi_*[Z]$, it follows that C is an irreducible component of Z'' and the map $\gamma'|_C\colon C \to G$ is finite. Therefore we have

$$\mathrm{ord}_G(\phi_*[Z]) = e_G(h^*\phi_*[Z]) \geq e_G(\gamma'_*[Z'']) \geq e_C([Z'']) = l(\mathscr{O}_{Z'',C}). \tag{6.4}$$

Let $b := \mathrm{ord}_G(K_{V'/V})$ denote the discrepancy of G over V, and let $H := (\gamma')^*F$. Note that H is a smooth divisor at the generic point of C, and $p \cdot \mathrm{ord}_F(H) = q$. Moreover, since $K_{U'/U} = (\gamma')^* K_{V'/V}$, we have $K_{U'/U} = bH + R$ where R does not contain C in its support. Then, by Proposition **??** and equation (10.3), we see that

$$\widehat{a}_E(X, cZ) \geq p \cdot a_F(U', cZ'' - K_{U'/U}) = p \cdot a_F(U', cZ'' - bH).$$

Setting $a := \widehat{a}_E(X, cZ)/q$, we have $a_E(U', cZ'' + (a-b)H) \leq 0$, and this implies that

$$l(\mathscr{O}_{Z'',C}) \geq \frac{(1-a+b)k^k}{k!c^k}. \tag{6.5}$$

by Theorem 10.4.10.

Combining (10.4) and (10.5), we get

$$q \cdot a_G\left(V, \frac{k!c^k}{k^k} \cdot \phi_*[Z]\right) \leq q(b+1-(1-a+b)) = \widehat{a}_E(X, cZ),$$

as stated. $\square$

6.6 Special restriction properties of multiplier ideals

(Maybe move this to the section on inversion of adjunction)

6.7 Birationally rigid Fano hypersurfaces

Theorem 6.7.1. *For any $N \geq 4$, every smooth hypersurface $X = X_N \subset \mathbb{P}^N$ of degree N, is birationally superrigid. In particular,* $\mathrm{Bir}(X) = \mathrm{Aut}(X)$ *is finite and X is not rational.*

Chapter 7
Finite generation of the canonical ring

The goal of this chapter is to present the proof due to Cascini and Lazić [CL12] for the finite generation of the canonical ring. We then explain, following [CL13], how this result in suitable generality implies the known results in the Minimal Model Program from [BCHM10].

Chapter 8
Extension theorems and applications

Chapter 9

The canonical bundle formula and subadjunction

Chapter 10
Arc spaces

10.1 Jet schemes

In this section we introduce the jet schemes and prove some of their basic properties. We will mostly use the definition for varieties over a field, but it is sometimes convenient to also have available a relative version of this notion and this requires no extra effort. We thus start by working with schemes of finite type over a fixed Noetherian ring R, all morphisms being morphisms of schemes over R. If X is such a scheme, $m \in \mathrm{Z}_{\geq 0}$, and A is an R-algebra, an *A-valued m-jet* on X (or simply an *m-jet* if $A = R$) is a morphism $\operatorname{Spec} A[t]/(t^{m+1}) \to X$. We first show that these objects are parametrized by a scheme of finite type over R.

Proposition 10.1.1. *Given a scheme X of finite type over R and a non-negative integer m, there exists a scheme $J_m(X)$ of finite type over R such that for every R-algebra A, we have a functorial isomorphism*

$$\operatorname{Hom}(\operatorname{Spec} A, J_m(X)) \simeq \operatorname{Hom}(\operatorname{Spec} A[t]/(t^{m+1}), X).$$

In other words, the scheme $J_m(X)$ represents the functor that takes an R-algebra A to the set of A-valued m-jets of X. It follows that if the scheme exists, then it is unique; it is called the m^{th} *jet scheme* of X. Whenever we need to specify the ground ring, we write $J_m(X/R)$ instead of $J_m(X)$. Other common notation in the literature for $J_m(X)$ is X_m and $\mathscr{L}_m(X)$. We will use the notation X_m for the set of k-valued points of $J_m(X)$ when $R = k$ is a field.

Before giving the proof of the proposition, we need some preparations. We first note that if $J_m(X)$ exists for some X, then we get a canonical morphism $\pi_m^X \colon J_m(X) \to X$. Indeed, for every R-algebra A, let us denote by j_m^A the closed immersion corresponding to the projection $A[t]/(t^{m+1}) \to A$, that maps t to 0. The morphism π_m^X corresponds to the natural transformation of functors

$$\operatorname{Hom}(\operatorname{Spec} A[t]/(t^{m+1}), X) \to \operatorname{Hom}(\operatorname{Spec} A, X),\ \gamma \to \gamma \circ j_m^A.$$

If X is understood from the context, we write π_m instead of π_m^X.

Lemma 10.1.2. *If $J_m(X)$ exists for a scheme X and U is an open subset of X, then $J_m(U)$ exists and $J_m(U) \simeq (\pi_m^X)^{-1}(U)$.*

Proof. Note that for every R-algebra A, a morphism $\gamma\colon \operatorname{Spec} A[t]/(t^{m+1}) \to X$ factors through U if and only if $\gamma \circ j_m^A$ factors through U (factoring through U is a set theoretic statement). With this observation, the fact that $(\pi_m^X)^{-1}(U)$ satisfies the definition of $J_m(U)$ follows from the definition of $J_m(X)$. $\square$

We can now prove the existence of jet schemes.

Proof of Proposition 9.1.1. We first prove the assertion when X is affine. Let us choose a closed embedding $X \hookrightarrow \mathbb{A}_R^N$ and let $g_1, \ldots, g_r \in R[x_1, \ldots, x_N]$ be generators for the ideal of X. For every R-algebra A, giving a morphism $\gamma\colon \operatorname{Spec} A[t]/(t^{m+1}) \to X$ is equivalent to giving a morphism of R-algebras

$$\phi\colon R[x_1, \ldots, x_N]/(g_1, \ldots, g_r) \to A[t]/(t^{m+1}),$$

hence to giving

$$\phi(x_i) = \sum_{j=0}^{m} a_i^{(j)} t^j \text{ for } 1 \leq i \leq N$$

such that $g_\ell(\phi(x_1), \ldots, \phi(x_N)) = 0$ in $A[t]/(t^{m+1})$ for all ℓ. For every ℓ, there are polynomials $G_\ell^{(0)}, \ldots, G_\ell^{(m)}$ in the variables $x_i^{(j)}$, with $1 \leq i \leq N$ and $0 \leq j \leq m$, such that

$$g_\ell\left(\sum_{j=0}^{m} a_1^{(j)} t^j, \ldots, \sum_{j=0}^{m} a_N^{(j)} t^j\right) = \sum_{j=0}^{m} G_\ell^{(j)}(\underline{a}) t^j,$$

where $\underline{a} = (a_i^{(j)})_{i,j}$. We hence conclude that

$$J_m(X) \simeq \operatorname{Spec}(R[x_i^{(j)} \mid 1 \leq i \leq N, 0 \leq j \leq m]/(G_\ell^{(j)} \mid 1 \leq \ell \leq r, 0 \leq j \leq m)).$$

We now consider the case of an arbitrary scheme X of finite type over R. Let $X = \bigcup_i U_i$ be a finite affine open cover of X. By the case we have already proved, for every i we have the jet scheme $J_m(U_i)$. Lemma 9.1.2 implies that for every i and j, we have a canonical isomorphism $(\pi_m^{U_i})^{-1}(U_i \cap U_j) \to (\pi_m^{U_j})^{-1}(U_i \cap U_j)$, both schemes being isomorphic to $J_m(U_i \cap U_j)$. Furthermore, these isomorphisms satisfy the cocycle condition and therefore we can glue the schemes $J_m(U_i)$ to obtain a scheme $J_m(X)$, together with a morphism $\pi_m\colon J_m(X) \to X$. It is now straightforward to check that $J_m(X)$ satisfies the desired universal property. The key observation is that given a morphism $\gamma\colon \operatorname{Spec} A[t]/(t^{m+1}) \to X$ and $f \in A$, then the corresponding morphism $\gamma_f\colon \operatorname{Spec} A_f[t]/(t^{m+1}) \to X$ factors through some U_i if and only if $\gamma_f \circ j_m^{A_f}$ factors through U_i. This completes the proof of the proposition. $\square$

If $f\colon X \to Y$ is a morphism of schemes as above, then we obtain a morphism $f_m\colon J_m(X) \to J_m(Y)$ that corresponds to the natural map

$$\operatorname{Hom}(\operatorname{Spec} A[t]/(t^{m+1}), X) \to \operatorname{Hom}(\operatorname{Spec} A[t]/(t^{m+1}), Y),\ \gamma \to f \circ \gamma.$$

It is clear that in this way we obtain a functor J_m from the category of schemes of finite type over R to itself.

For every scheme X as above and every $p > q$, truncation of jets induces a morphism $\pi^X_{p,q}\colon J_p(X) \to J_q(X)$. Indeed, for every R-algebra A, we have a map

$$\mathrm{Hom}(\mathrm{Spec}\,A[t]/(t^{p+1}), X) \to \mathrm{Hom}(\mathrm{Spec}\,A[t]/(t^{q+1}), X),$$

given by the composition with the closed immersion corresponding to the quotient homomorphism $A[t]/(t^{p+1}) \to A[t]/(t^{q+1})$. Note that we obtain in this way a transformation of functors $J_p \to J_q$. We write $\pi_{p,q}$ instead of $\pi^X_{p,q}$ whenever the scheme is understood from the context. It is clear that if $p > q > r$, then $\pi^X_{q,r} \circ \pi^X_{p,q} = \pi^X_{p,r}$ and $\pi^X_{m,0} = \pi^X_m$. We also note that by the proof of Proposition 9.1.1, all morphisms $\pi^X_{p,q}$ are affine.

Example 10.1.3. It follows from the proof of Proposition 9.1.1 that $J_m(\mathbb{A}^N_R) \simeq \mathbb{A}^{(m+1)N}_R$. Furthermore, if $p > q$, then via these isomorphisms, the projection $\pi_{p,q}\colon J_p(\mathbb{A}^N_R) \to J_q(\mathbb{A}^N_R)$ gets identified to the projection onto the first $(q+1)N$ coordinates.

Example 10.1.4. It is clear from definition that $J_0(X) \simeq X$. The first jet scheme $J_1(X)$ is isomorphic to the total tangent space $\mathscr{S}pec(\mathrm{Sym}^\bullet(\Omega_{X/R}))$. Clearly, it is enough to give a canonical isomorphism when $X = \mathrm{Spec}(S)$ is an affine scheme over R. In this case, giving a morphism $\mathrm{Spec}\,A \to \mathrm{Spec}(\mathrm{Sym}^\bullet(\Omega_{S/R}))$ is equivalent to giving a homomorphism of R-algebras $\phi\colon S \to A$ and a morphism of S-modules $\Omega_{S/R} \to A$, that is, an R-derivation $D\colon S \to A$ (where A is an S-module via ϕ). Giving such a pair (ϕ, D) is equivalent to giving a morphism of R-algebras $S \to A[t]/(t^2)$, mapping $s \in S$ to $\phi(s) + D(s)t$. Therefore $\mathrm{Spec}(\mathrm{Sym}^\bullet(\Omega_{X/R}))$ satisfies the universal property of $J_1(X)$.

Example 10.1.5. Let us see in a concrete case how to write down explicit equations for jet schemes. Suppose that $X \hookrightarrow Y = \mathbb{A}^2_R$ is the cuspidal curve defined by $(x^2 + y^3)$ and let us compute $J_2(X) \subseteq J_2(Y) \simeq \mathrm{Spec}\,R[x, x', x'', y, y', y'']$. Since we have

$$(x + x't + x''t^2)^2 + (y + y't + y''t^2)^3$$
$$= (x^2 + y^3) + (2xx' + 3y^2y')t + (2xx'' + (x')^2 + 3yy'' + 3y(y')^2)t^2 \bmod (t^3),$$

it follows that $J_2(X)$ is defined by the ideal

$$(x^2 - y^3, 2xx' + 3y^2y', 2xx'' + (x')^2 + 3yy'' + 3y(y')^2).$$

Remark 10.1.6. The functor J_m is the right adjoint of the functor

$$X \rightsquigarrow X \times_{\mathrm{Spec}\,R} \mathrm{Spec}\,R[t]/(t^{m+1}).$$

In other words, for every schemes X and Y of finite type over R, we have a functorial isomorphism

$$\alpha^m_{Y,X}\colon \mathrm{Hom}(Y, J_m(X)) \simeq \mathrm{Hom}(Y \times_{\mathrm{Spec}\,R} \mathrm{Spec}\,R[t]/(t^{m+1}), X).$$

Indeed, when Y is affine, this follows from the definition of $J_m(X)$, and the extension to arbitrary Y is standard. As in the case of an arbitrary adjoint pair of functors, we can express the above bijection in terms of a "universal object". More precisely, by taking $Y = J_m(X)$, we obtain the "universal family of jets"

$$\tau_X^m = \alpha^m_{J_m(X),X}(\mathrm{Id}_{J_m(X)})\colon J_m(X) \times_{\operatorname{Spec} R} \operatorname{Spec} R[t]/(t^{m+1}) \to X$$

such that for every $\gamma\colon Y \to J_m(X)$, we have

$$\alpha^m_{Y,X}(\gamma) = \tau_X^m \circ (\gamma \times \mathrm{Id}_{\operatorname{Spec} R[t]/(t^{m+1})}).$$

Like every right adjoint functor, the functor J_m commutes with fibered products: given any two morphisms $X \to S$ and $Y \to S$, we have a canonical isomorphism

$$J_m(X \times_S Y) \simeq J_m(X) \times_{J_m(S)} J_m(Y).$$

Example 10.1.7. If G is an algebraic group over R, then by applying the functor J_m to the multiplication map $G \times_{\operatorname{Spec} R} G \to G$, $(x,y) \to xy$ and to the inverse map $G \to G$, $x \to x^{-1}$, we see that $J_m(G)$ is an algebraic group over R. Furthermore, if $p > q$, then the projection $\pi^G_{p,q}\colon J_p(G) \to J_q(G)$ is a morphism of algebraic groups over R. If G acts algebraically on a scheme X over R, then by applying J_m to the map $G \times X \to X$, $(g,x) \to gx$ we deduce that $J_m(G)$ has an induced action on $J_m(X)$.

In addition to the projection $\pi^X_m\colon J_m(X) \to X$, we also have a canonical section $\sigma^X_m\colon X \to J_m(X)$ of π^X_m. At the level of A-valued points, this maps a morphism $\phi\colon \operatorname{Spec} A \to X$ to $\phi \circ p$, where p is the morphism of schemes corresponding to the inclusion $A \hookrightarrow A[t]/(t^{m+1})$. It is clear that we have $\pi^X_m \circ \sigma^X_m = \mathrm{Id}_X$ for every m (in particular, π^X_m is surjective). More generally, for every $p > q$, we have $\pi^X_{p,q} \circ \sigma^X_p = \sigma^X_q$.

Remark 10.1.8. If X is a scheme of finite type over R, $R \to S$ is a homomorphism of Noetherian rings, and $X_S = X \times_{\operatorname{Spec} R} \operatorname{Spec} S$, then there is a canonical isomorphism

$$J_m(X_S/S) \simeq J_m(X/R) \times_{\operatorname{Spec} R} \operatorname{Spec} S.$$

This follows immediately by considering the A-valued points for both sides.

Remark 10.1.9. If $f\colon X \to Y$ is a closed immersion, then $f_m\colon J_m(X) \to J_m(Y)$ is a closed immersion, too. Indeed, this assertion is local over Y, hence it is enough to prove it when Y (hence also X) is affine. In this case, the assertion follows from the description of jet schemes by equations given in the proof of Proposition 9.1.1.

Remark 10.1.10. If $f\colon X \to Y$ is a morphism and $Z \hookrightarrow Y$ is a closed subscheme, then $J_m(f^{-1}(Z)) \simeq f_m^{-1}(J_m(Z))$. Indeed, this is a special case of the fact that the functor J_m commutes with fiber products.

Remark 10.1.11. If S is any Noetherian scheme and $f\colon X \to S$ is a scheme of finite type over S, then we can define $J_m(X/S)$ as in the case when S is affine. Existence follows by gluing the schemes $J_m(f^{-1}(U_i)/\mathscr{O}(U_i))$, where $S = \bigcup_i U_i$ is a finite affine open cover of S. However, since we will not make use of this more general setting, we do not pursue it any further.

We now extend the assertion in Lemma 9.1.2 from open immersions to étale morphisms.

Lemma 10.1.12. *If $f\colon X \to Y$ is étale , then the following diagram is Cartezian:*

$$\begin{tikzcd} J_m(X) \arrow[r, "f_m"] \arrow[d, "\pi_m^X"'] & J_m(Y) \arrow[d, "\pi_m^Y"] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

In particular, f_m is étale.

Proof. For every R-algebra A and every commutative diagram

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r] \arrow[d, "j"'] & X \arrow[d] \\ \operatorname{Spec} A[t]/(t^{m+1}) \arrow[r] & Y, \end{tikzcd}$$

there is a unique morphism $\operatorname{Spec} A[t]/(t^{m+1}) \to X$ that makes the resulting triangles in the above diagram commutative. This is a consequence of the fact that f is formally étale and j is a closed immersion, defined by a nilpotent ideal. The assertion in the lemma now follows from the definition of jet schemes. □

Given a scheme F, a morphism of schemes $f\colon X \to Y$ is *locally trivial*, *with fiber F* if there is a cover $Y = \bigcup_i U_i$ of Y by open subsets such that each $f^{-1}(U_i)$ is isomorphic over U_i with $U_i \times_{\operatorname{Spec} R} F$.

Corollary 10.1.13. *If X is a smooth scheme over R of relative dimension n, then $J_m(X)$ is smooth over R, of relative dimension $(m+1)n$. Moreover, for every $p > q$, the morphism $\pi_{p,q}^X\colon J_p(X) \to J_q(X)$ is locally trivial, with fiber $\mathbb{A}_R^{(p-q)n}$.*

Proof. Since X is smooth over R, of relative dimension n, it follows that X can be covered by open subsets U on which we have coordinates $x_1,\ldots,x_n$ (that is, $dx_1,\ldots,dx_n$ trivialize $\Omega_{X/R}$ on U). In this case $(x_1,\ldots,x_n)$ defines an étale morphism $U \to \mathbb{A}_R^n$, and we obtain $U_m \simeq U \times_{\operatorname{Spec} R} \mathbb{A}_R^{mn}$ over U by Lemma 9.1.12 and Example 9.1.3. The last assertion in the corollary follows from this. Since $\pi_m^X\colon J_m(X) \to X$ is locally trivial, with fiber $\mathbb{A}_R^{mn}$, it follows that X_m is smooth over R. □

Remark 10.1.14. Arguing as in the proof of Corollary 9.1.13, we see that if $f\colon X \to Y$ is a smooth morphism of relative dimension n, then $f_m\colon J_m(X) \to J_m(Y)$ is smooth, of relative dimension $(m+1)n$. Indeed, X is covered by open subsets U with the property that there are $x_1,\ldots,x_n \in \mathscr{O}(U)$ such that the map they define $U \to Y \times_{\operatorname{Spec} R} \mathbb{A}_R^n$ is étale. By Lemma 9.1.12, the corresponding morphism

$J_m(U) \to J_m(Y) \times_{\operatorname{Spec} R} \mathbb{A}_R^{(m+1)n}$ is étale, which implies that f_m is smooth, of relative dimension $(m+1)n$.

Note that if we assume in addition that f is surjective, then f_m is surjective, too. Indeed, given a point $\gamma \in J_m(Y)$, consider $y = \pi_m^Y(\gamma)$ and let $k(y) \hookrightarrow k(\gamma)$ be the corresponding extension of residue fields. We can find $x \in X$ such that $f(x) = y$ and let us choose a field L containing both $k(\gamma)$ and the residue field $k(x)$ of x. Therefore we obtain a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} L \arrow[r, "\widetilde{x}"] \arrow[d, "j"'] & X \arrow[d, "f"] \\ \operatorname{Spec} L[t]/(t^{m+1}) \arrow[r, "\widetilde{\gamma}"] & Y, \end{tikzcd}$$

where $\widetilde{x}$ and $\widetilde{\gamma}$ correspond to x and γ, respectively. Since f is formally smooth and j is a closed embedding, defined by a nilpotent ideal, there is a morphism $\operatorname{Spec} L[t]/(t^{m+1}) \to X$ such that the resulting triangles in the above diagram are commutative. This corresponds to an L-valued point of X_m whose corresponding point $\delta \in J_m(X)$ has the property that $f_m(\delta) = \gamma$.

In general, properties of a scheme do not carry over to properties of its jet schemes. Smoothness is an exception, as we saw in Corollary 9.1.13, and we will see in Remark 9.1.16 below that connectedness is also preserved. On the other hand, the next example shows that irreducibility or reduceness are not preserved. We will discuss later a condition on singularities that guarantees that the jet schemes of an algebraic variety are reduced and irreducible.

Example 10.1.15. Let X be an (irreducible) singular curve defined over an algebraically closed field. If $x_0 \in X$ is a singular (closed) point, then $\dim \pi_1^{-1}(x_0) \geq 2$. Since $\dim(\pi_1^{-1}(X_{\mathrm{sm}})) = 2$, it follows that $\pi_1^{-1}(x_0)$ gives an irreducible component of $J_1(X)$ different from the closure of $\pi_1^{-1}(X_{\mathrm{sm}})$.

Suppose now that $Y \subset \mathbb{A}^2 = \operatorname{Spec} k[x,y]$ is defined by the ideal (xy). In this case $J_1(Y) \subset \operatorname{Spec} k[x,x',y,y']$ is defined by $I = (xy, xy' + x'y)$. Since $x^2y' = x(xy' + x'y) - x'(xy)$, it follows that $xy' \in \operatorname{Rad}(I)$, but it is clear that $xy' \notin I$.

Another piece of structure that the jet schemes have is an action of the multiplicative group over R, namely of $\mathbb{G}_{m,R} = \operatorname{Spec} R[y, y^{-1}]$. This is given by reparametrization of t. In fact, we have a morphism

$$\Phi_m = \Phi_m^X \colon \mathbb{A}_R^1 \times_{\operatorname{Spec} R} J_m(X) \to J_m(X),$$

which at the level of A-valued points is taking a pair (a, γ), with $a \in A$ and γ: $\operatorname{Spec} A[t]/(t^{m+1}) \to X$ to $\gamma \circ \phi_a$, where ϕ_a corresponds to the morphism of A-algebras $A[t]/(t^{m+1}) \to A[t]/(t^{m+1})$ that takes t to at. Note that the morphisms Φ_m are compatible with the projections $\pi_{p,q}$ in the sense that we have commutative diagrams

$$\begin{tikzcd} \mathbb{A}^1_R \times_{\operatorname{Spec} R} J_p(X) \arrow[r, "\Phi_p"] \arrow[d, "(\mathrm{Id},\pi_{p,q})"'] & J_p(X) \arrow[d, "\pi_{p,q}"] \\ \mathbb{A}^1_R \times_{\operatorname{Spec} R} J_q(X) \arrow[r, "\Phi_q"] & J_q(X) \end{tikzcd}$$

for every $p > q$. It is clear that Φ^X_m is functorial in X and it restricts to an action of $\mathbb{G}_{m,R}$ on $J_m(X)$. We also see that the restriction of Φ_m to $J_m(X) \overset{(0,\mathrm{Id})}{\hookrightarrow} \mathbb{A}^1_R \times_{\operatorname{Spec} R} J_m(X)$ is equal to $\sigma_m \circ \pi_m$.

If Z is an irreducible component of $J_m(X)$, then $\Phi_m(\mathbb{A}^1_R \times_{\operatorname{Spec} R} Z)$ is irreducible and contains Z, hence it is equal to Z. We deduce that Φ_m induces a morphism $\mathbb{A}^1_R \times_{\operatorname{Spec} R} Z \to Z$ (where we consider on Z the reduced scheme structure). In particular, this gives $\sigma_m \circ \pi_m(Z) \subseteq Z$. This, in turn, implies the set-theoretic equality $\pi_m(Z) = \sigma_m^{-1}(Z)$, hence $\pi_m(Z)$ is a closed subset of X.

Remark 10.1.16. If X is a connected scheme, then $J_m(X)$ is connected for every $m \geq 0$. Indeed, suppose that we can write $J_m(X) = Z \cup Z'$, where Z and Z' are disjoint closed subsets. Since both Z and Z' are unions of irreducible components of X, it follows from the above discussion that $\pi_m(Z)$ and $\pi_m(Z')$ are both closed subsets of X. Furthermore, they are disjoint (since $x \in \pi_m(Z) \cap \pi_m(Z')$ implies $\sigma_m(x) \in Z \cap Z'$) and $X = \pi_m(Z) \cup \pi_m(Z')$ (since $X = \pi_m(J_m(X))$). This contradicts the fact that X is connected.

We end this section with two remarks, showing that in the geometric setting we can recover the smoothness of a scheme and the order of a hypersurface from the information given by the jet schemes. We now assume that the ground ring is an algebraically closed field.

Remark 10.1.17. Let X be a smooth n-dimensional variety and $H \subset X$ an effective Cartier divisor. If $p \in H$ is a closed point and $d = \operatorname{ord}_p(H)$, then

$$(\pi^H_m)^{-1}(p) = (\pi^X_m)^{-1}(p) \simeq \mathbb{A}^{mn} \text{ for } m < d, \tag{10.1}$$

while $(\pi^H_d)^{-1}(p) \neq (\pi^X_d)^{-1}(p)$. In fact, $(\pi^H_d)^{-1}(p) \simeq C_pH \times \mathbb{A}^{(d-1)n}$, where C_pH is the tangent cone of H at p, and the canonical morphism $(\pi^H_d)^{-1}(p) \to (\pi^H_1)^{-1}(p) \simeq T_pH$ corresponds to the projection to the first component, followed by the canonical inclusion $C_pH \hookrightarrow T_pH$.

In order to check these assertions, after restricting to a suitable affine open neighborhood $U \subseteq X$ of p, we may assume that X is affine and that we have $u_1, \ldots, u_n \in \mathscr{O}(X)$ giving a system of coordinates. Let $\mathfrak{m}_p$ denote the ideal defining p in X. The equality in (9.1) is clear: for every k-algebra A, if γ: $\operatorname{Spec} A[t]/(t^{m+1}) \to X$ lies in $(\pi^X_m)^{-1}(p)$, then $\gamma^{-1}(\mathfrak{m}_p) \subseteq (t)$. Since $f \in \mathfrak{m}_p^d$, it follows that $\gamma^{-1}(f) = (0)$ whenever $m \leq d-1$.

In order to check the other assertions, let us consider the homogeneous polynomial $g \in k[x_1, \ldots, x_n]$ of degree d such that $f - g(u_1, \ldots, u_n) \in \mathfrak{m}_p^{d+1}$. If $D \subset X$ is the hypersurface defined by $g(u_1, \ldots, u_n)$, then $(\pi^H_d)^{-1}(p) = (\pi^D_d)^{-1}(p)$. Moreover, in this case g defines $C_pH \subseteq T_pX \simeq \mathbb{A}^n$. By Lemma 9.1.12 applied to the étale morphism $X \to \mathbb{A}^n$ defined by $(u_1, \dots, u_n)$, it is enough to prove the remaining assertions when $X = \mathbb{A}^n$, $p = 0$, and H is defined by g. For any k-algebra A, an A-valued point of $(\pi_d^H)^{-1}(p)$ is determined by those $(a_{i,j}) \in A^{dn}$ such that

$$g\left(\sum_{j=1}^{d} a_{1,j}t^j, \dots, \sum_{j=1}^{d} a_{n,j}t^j\right) \equiv 0 \ (\operatorname{mod} t^{d+1}).$$

Since the left-hand side of the above expression is equal to $t^d g(a_{1,1}, \dots, a_{n,1})$ mod (t^{d+1}), we obtain the isomorphism $(\pi_d^H)^{-1}(0) \simeq H \times \mathbb{A}^{(d-1)n}$, as well as the last assertion.

Remark 10.1.18. It follows from Corollary 9.1.13 that if X is smooth, then all maps $\pi_{p,q}^X$, with $p > q$, are surjective. The converse also holds: in fact, if $x \in X$ is a singular closed point, then $(\pi_m^X)^{-1}(x) \to (\pi_1^X)^{-1}(x) \simeq T_xX$ is not surjective for $m \gg 0$. Indeed, it is enough to show that for every $x \in X$, the image of $(\pi_m^X)^{-1}(x) \to (\pi_1^X)^{-1}(x)$, for $m \gg 0$, is contained in the tangent cone C_xX of X at x. In order to show this, we may assume that X is a closed subscheme of some $\mathbb{A}^n$. Since C_xX is the scheme-theoretic intersection of finitely many cones of the form C_xH, for suitable hypersurfaces $H \subset \mathbb{A}^n$ containing X, the assertion follows from Remark 9.1.17.

10.2 Arc schemes

We work in the same setting as in the previous section, with schemes of finite type over a Noetherian ring R. If X is such a scheme, then we have the inverse system of schemes $(J_m(X))_{m \geq 0}$, with the transition morphisms given by $\pi_{p,q}^X \colon J_p(X) \to J_q(X)$ for $p > q$. Since these morphisms are affine, the inverse limit of this system exists. It is denoted by $J_\infty(X)$ and it is called the *arc scheme* of X. When we need to emphasize the ground ring, we write $J_\infty(X/R)$. This scheme is denoted in the literature also by X_∞ or $\mathscr{L}(X)$. In the case when $R = k$ is a field, we will denote by X_∞ the set of k-valued points of $J_\infty(X)$.

We recall how the inverse limit is constructed. If $U \subseteq X$ is an affine open subset, then we consider $\operatorname{Spec}(\varinjlim_m \mathscr{O}(J_m(U)))$. Since the direct limit of rings commutes with localization, it is straightforward to check that these schemes glue together to a scheme $J_\infty(X)$. Moreover, the natural maps $\operatorname{Spec}(\varinjlim_m \mathscr{O}(J_m(U))) \to J_m(U)$ glue to give $\pi_{\infty,m}^X \colon J_\infty(X) \to J_m(X)$ such that $\pi_{p,q}^X \circ \pi_{\infty,p}^X = \pi_{\infty,q}^X$ for $p > q$. We also write π_∞^X for $\pi_{\infty,0}^X$ and we drop the upper index if the scheme X is clear from the context. It is easy to see that $J_\infty(X)$, together with these morphisms, is the inverse limit of the inverse system $(J_m(X))_{m \geq 0}$, that is

$$\mathrm{Hom}(Y, J_\infty(X)) \simeq \varprojlim_m \mathrm{Hom}(Y, J_m(X)) \tag{10.2}$$

for every scheme Y over R.

Example 10.2.1. If $X = \mathrm{Spec}\, R$, then $J_\infty(X) = \mathrm{Spec}\, R$. If $X = \mathbb{A}^n_R$, with $n \geq 1$, then it follows from Example 9.1.3 that $J_\infty(X)$ is isomorphic to an infinite-dimensional affine space over R, that is, to $\mathbb{A}^{\mathbb{N}}_R := \mathrm{Spec}\, R[x_n; n \geq 0]$. Moreover, each morphism $\pi_{\infty,m} \colon J_\infty(X) \to J_m(X)$ is given by the projection onto the first $(m+1)n$ components. In particular, this shows that $J_\infty(X)$ is in general not of finite type over R, and in fact, it is not Noetherian.

Lemma 10.2.2. *For every R-algebra A, there is a functorial map*

$$\mathrm{Hom}(\mathrm{Spec}\, A[\![t]\!], X) \to \mathrm{Hom}(\mathrm{Spec}\, A, J_\infty(X)). \tag{10.3}$$

This is a bijection if either X is affine or A is a local ring.

Proof. Using the fact that $J_\infty(X) = \varprojlim_m J_m(X)$ and the definition of jet schemes, we see that

$$\mathrm{Hom}(\mathrm{Spec}\, A, J_\infty(X)) \simeq \varprojlim_m \mathrm{Hom}(\mathrm{Spec}\, A[t]/(t^{m+1}), X).$$

The morphism in (9.3) is then obtained by composing with the compatible morphisms $\mathrm{Spec}\, A[t]/(t^{m+1}) \to \mathrm{Spec}\, A[\![t]\!]$ induced by the obvious projections. The fact that (9.3) is a bijection when X is affine is clear, since $A[\![t]\!] \simeq \varprojlim_m A[t]/(t^{m+1})$.

When A is a local ring, the fact that (9.3) is a bijection can be reduced the the case when X is affine, as follows. If B is any local ring, then $\mathrm{Hom}(\mathrm{Spec}\, B, X) = \bigcup_U \mathrm{Hom}(\mathrm{Spec}\, B, U)$, where the union is over all affine open subsets U of X (this is due to the fact that a morphism $\phi\colon \mathrm{Spec}\, B \to X$ factors through $U \subseteq X$ if and only if ϕ maps the unique closed point of $\mathrm{Spec}\, B$ to U). Since $(A, \mathfrak{m})$ is a local ring, both $A[\![t]\!]$ and $A[t]/(t^{m+1})$ are local rings, with the maximal ideal generated by $\mathfrak{m}$ and t. We note that a morphism $\mathrm{Spec}\, A[t]/(t^{m+1}) \to X$ factors through an open subset U if and only if its restriction to $\mathrm{Spec}\, A$ factors through U. We thus conclude that

$$\varprojlim_m \mathrm{Hom}(\mathrm{Spec}\, A[t]/(t^{m+1}), X) = \bigcup_U \varprojlim_m \mathrm{Hom}(\mathrm{Spec}\, A[t]/(t^{m+1}), U)$$

and

$$\mathrm{Hom}(\mathrm{Spec}\, A[\![t]\!], X) = \bigcup_U \mathrm{Hom}(\mathrm{Spec}\, A[\![t]\!], U),$$

where U varies over the affine open subsets of X. Since (9.3) is a bijection when X is affine, we conclude that it is a bijection also when A is a local ring. $\square$

We will use the above lemma especially when $A = K$ is a field. In general, a morphism $\mathrm{Spec}\, A[\![t]\!] \to X$ is called an A-valued arc on X. The above lemma says that when X is affine or A is a local ring, we have a bijection between the A-valued arcs on X and the A-valued points of X_∞. When $A = K$ is a field, we will denote by 0 the closed point of $\operatorname{Spec} K[\![t]\!]$ and by η its generic point.

If $f\colon X \to Y$ is a morphism of schemes of finite type over R, then the morphisms $f_m\colon J_m(X) \to J_m(Y)$ induce a morphism $f_\infty\colon J_\infty(X) \to J_\infty(Y)$ such that we have commutative diagrams

$$\begin{tikzcd} J_\infty(X) \arrow[r, "f_\infty"] \arrow[d, "\pi^X_{\infty,m}"'] & J_\infty(Y) \arrow[d, "\pi^Y_{\infty,m}"] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

In this way we obtain a functor J_∞ from schemes of finite type over R to schemes over R and each $\pi_{\infty,m}$ gives a natural transformation.

Remark 10.2.3. The general properties of the functor J_∞ can be deduced by "passing to limit" from the corresponding properties of the functors J_m, that we discussed in the previous section. For example, we have the following:

1) If $f\colon X \to Y$ is an étale morphism, then we have a Cartezian diagram

$$\begin{tikzcd} J_\infty(X) \arrow[r, "f_\infty"] \arrow[d, "\pi^X_{\infty}"'] & J_\infty(Y) \arrow[d, "\pi^Y_{\infty}"] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

This follows from Lemma 9.1.12 and the fact that inverse limits commute with fiber products.
2) If X is a smooth scheme over R, of relative dimension n, then each $J_\infty(X) \to J_m(X)$ is locally trivial, with fiber $\mathbb{A}^{\mathbb{N}}_R$ (if $n > 0$) or $\operatorname{Spec} R$ (if $n = 0$).
3) If $f\colon X \to Y$ is smooth and surjective, then $f_\infty\colon J_\infty(X) \to J_\infty(Y)$ is surjective. Furthermore, if $R = k$ is an algebraically closed field, then we also have surjectivity for the map between the corresponding sets of k-valued arcs. Both assertions follow using the argument in Remark 9.1.14.
4) If $f\colon X \to Y$ is a closed immersion, then $f_\infty\colon J_\infty(X) \to J_\infty(Y)$ is a closed immersion. This follows from Remark 9.1.9 and the fact that an inductive limit of surjective ring homomorphisms is again surjective.
5) J_∞ commutes with fibered products, that is, for every two morphisms $f\colon X \to S$ and $g\colon Y \to S$, we have a canonical isomorphism

$$J_\infty(X \times_S Y) \simeq J_\infty(X) \times_{J_\infty(S)} J_\infty(Y).$$

This follows from Remark 9.1.8 and the fact that inverse limits commute with fiber products. In particular, we see that if $f\colon X \to Y$ is a morphism and $Z \hookrightarrow Y$ is a closed subscheme, then

$$J_\infty(f^{-1}(Z)) = f_\infty^{-1}(J_\infty(Z)).$$

6) If G is an algebraic group over R, then $J_\infty(G)$ is a group scheme over R. Furthermore, if G acts algebraically on a scheme over R, then $J_\infty(G)$ has an algebraic action on $J_\infty(X)$. Both assertions follow from Example 9.1.7, by taking the inverse limit.

Remark 10.2.4. It follows from the definition of $J_\infty(X)$ that a basis of open sets for its Zariski topology is given by the subsets of the form $\pi_{\infty,m}^{-1}(U)$, where m varies over the non-negative integers and U varies over the open subsets of $J_m(X)$.

For every scheme X, the system of sections $(\sigma_m)_{m\geq 1}$ define by (9.2) a morphism $\sigma_\infty = \sigma_\infty^X\colon X \to J_\infty(X)$ such that $\pi_{\infty,m}\circ\sigma_\infty = \sigma_m$. In particular, we have $\pi_\infty\circ\sigma_\infty = \mathrm{Id}_X$.

Recall that for a scheme X we also have the morphism $\Phi_m\colon \mathbb{A}^1_R \times_{\operatorname{Spec} R} J_m(X) \to J_m(X)$ for every $m\geq 0$. Using (9.2), we obtain a morphism

$$\Phi_\infty = \Phi_\infty^X\colon \mathbb{A}^1_R \times_{\operatorname{Spec} R} J_\infty(X) \to J_\infty(X)$$

such that for every $m \geq 0$, we have a commutative diagram

$$\begin{tikzcd}
\mathbb{A}^1_{\operatorname{Spec} R} \times_{\operatorname{Spec} R} J_\infty(X) \arrow[r, "\Phi_\infty"] \arrow[d, "(\mathrm{Id},\pi_{\infty,m})"'] & J_\infty(X) \arrow[d, "\pi_{\infty,m}"] \\
\mathbb{A}^1_{\operatorname{Spec} R} \times_{\operatorname{Spec} R} J_m(X) \arrow[r, "\Phi_m"] & J_m(X).
\end{tikzcd}$$

The morphism Φ_∞ restricts to an action of $\mathbb{G}_{m,R}$ on $J_\infty(X)$ and the restriction of Φ_∞ to $J_\infty(X) \overset{(0,\mathrm{Id})}{\hookrightarrow} \mathbb{A}^1_R \times_{\operatorname{Spec} R} J_\infty(X)$ is equal to $\sigma_\infty\circ\pi_\infty$.

Remark 10.2.5. If R has equicharacteristic 0 (that is, if $\mathbb{Q}\subseteq R$), there is an easy way to write down explicitly the equations of jet schemes and arc schemes for affine schemes, by "formally differentiating" the original equations. Let us start with the case $S = R[x_1,\ldots,x_N]$. We consider the polynomial rings

$$S_m = R[x_i^{(j)} \mid 1\leq i\leq N; 0\leq j\leq m] \text{ and } S_\infty = R[x_i^{(j)} \mid 1\leq i\leq N; j\geq 0]$$

(we make the convention $x_i^{(0)} = x_i$ and sometimes write $x_i' = x_i^{(1)}$, $x_i'' = x_i^{(2)}$). Note that we have

$$S = S_0 \subseteq S_1 \subseteq S_2 \subseteq \ldots \subseteq S_\infty = \bigcup_{m\geq 0} S_m.$$

On S_∞ we consider the unique R-derivation D given by $D(x_i^{(j)}) = x_i^{(j+1)}$ for all i and j. For every $f\in S_\infty$, we define $f^{(j)}$ recursively by putting $f^{(0)} = f$ and $f^{(j)} = D(f^{(j-1)})$ for $j\geq 1$. Note that if $f\in S$, then $f^{(j)}\in S_m$ for all $j\leq m$.

For an R-algebra A, we parametrize the morphisms $S\to A[t]/(t^{m+1})$ in a slightly different way than in the proof of Proposition 9.1.1: a morphism $\phi\colon S\to A[t]/(t^{m+1})$ is determined by

$$\phi(x_i) = \sum_{j=0}^{m} \frac{a_i^{(j)}}{j!} t^j.$$

For every $f \in S$, we have

$$\phi(f) = \sum_{j=0}^{m} \frac{f^{(j)}(a, a', \ldots, a^{(m)})}{j!} t^j \text{ in } A[t]/(t^{m+1}).$$

Indeed, in order to check this, it is enough to note that both sides are additive and multiplicative in f and the equality trivially holds when $f = x_i$. This implies that if $X \hookrightarrow \mathbb{A}^N$ is the closed subscheme defined by the ideal $I = (f_1, \ldots, f_r) \subseteq S$, the jet scheme $J_m(X)$ is defined in $\operatorname{Spec} S_m$ by $(f_i^{(j)} \mid 1 \leq i \leq r, 0 \leq j \leq m)$. We deduce from this that $J_\infty(X)$ is defined in $\operatorname{Spec} S_\infty$ by $(f_i^{(j)} \mid 1 \leq i \leq r, j \geq 0)$.

Remark 10.2.6. Suppose that we are still in the equicharacteristic 0 case. It follows from Remark 9.2.5 that if X is affine, then there is an R-derivation δ of $\mathscr{O}(J_\infty(X))$ that satisfies the following universal property: if $g\colon \mathscr{O}(X) \to T$ is a morphism of R-algebras such that T has an R-derivation δ_T, then there is a unique morphism of R-algebras $h\colon \mathscr{O}(J_\infty(X)) \to T$ such that the composition $\mathscr{O}(X) \to \mathscr{O}(J_\infty(X)) \xrightarrow{h} T$ is equal to g and $\delta_T \circ h = h \circ \delta$. In order to see this, let us write $X = \operatorname{Spec} S/I$, for a polynomial ring S. It follows from Remark 9.2.5 that the derivation D induces an R-derivation δ on $\mathscr{O}(J_\infty(X))$ and it is straightforward to see that this satisfies the universal property. In the case when $R = k$ is a field of characteristic zero, this is the starting point for the connection between arc schemes and differential algebra, see [Bui94].

We now turn to some properties that hold for arc schemes, in spite of the fact that they do not hold for jet schemes.

Lemma 10.2.7. *For every scheme X, the closed immersion $J_\infty(X_{\mathrm{red}}) \hookrightarrow J_\infty(X)$ is a homeomorphism of topological spaces. Moreover, if $X_1, \ldots, X_r$ are the irreducible components of X, then we have an equality of sets $J_\infty(X) = \cup_{i=1}^r J_\infty(X_i)$.*

Proof. For the first assertion, it is enough to show that for every R-algebra K, which is a field, the two schemes have the same K-valued points. By Lemma 9.2.2, this is equivalent to the fact that the injective map

$$\operatorname{Hom}(\operatorname{Spec} K[\![t]\!], X_{\mathrm{red}}) \to \operatorname{Hom}(\operatorname{Spec} K[\![t]\!], X)$$

is a bijection. This is clear, since $K[\![t]\!]$ is reduced.

For the second assertion, it is enough to prove that for every K as above, every K-valued point of $J_\infty(X)$ is a K-valued point of some $J_\infty(X_i)$. By Lemma 9.2.2, we need to show that every morphism $\operatorname{Spec} K[\![t]\!] \to X$ factors through some X_i. This is a consequence of the fact that $K[\![t]\!]$ is a domain. □

The next proposition is the first indication of the connection between arc schemes and birational geometry.

Proposition 10.2.8. *Let $f\colon X \to Y$ be a proper scheme morphism and assume that $Z \subset Y$ is a closed subset such that f induces an isomorphism $X \smallsetminus f^{-1}(Z) \simeq Y \smallsetminus Z$. In this case f_∞ induces a map*

$$\widetilde{f}\colon J_\infty(X) \smallsetminus J_\infty(f^{-1}(Z)) \to J_\infty(Y) \smallsetminus J_\infty(Z)$$

which induces bijections on K-valued points for every R-algebra K that is a field. In particular, $\widetilde{f}$ is a bijection inducing isomorphisms of residue fields between the corresponding points.

Proof. Note first that the conclusion is independent of the scheme structures we consider on Z and $f^{-1}(Z)$, by Lemma 9.2.7. Let K be an R-algebra that is a field. By Lemma 9.2.2, showing that the induced map

$$\mathrm{Hom}(\mathrm{Spec}\, K, J_\infty(X) \smallsetminus J_\infty(f^{-1}(Z))) \to \mathrm{Hom}(\mathrm{Spec}\, K, J_\infty(Y) \smallsetminus J_\infty(Z))$$

is a bijection, is equivalent to showing that the following map

$$\{\gamma\colon \mathrm{Spec}\, K[\![t]\!] \to X \mid \gamma(\eta) \in X \smallsetminus f^{-1}(Z)\} \to \{\delta\colon \mathrm{Spec}\, K[\![t]\!] \to Y \mid \delta(\eta) \in Y \smallsetminus Z\}$$

is bijective. If $\delta\colon \mathrm{Spec}\, K[\![t]\!] \to Y$ is such that $\delta(\eta) \in Y \smallsetminus Z$, then δ induces $\widetilde{\delta}\colon \mathrm{Spec}\, K(\!(t)\!) \to Y \smallsetminus Z \simeq X \smallsetminus f^{-1}(Z) \subseteq X$. It follows from the valuative criterion for properness that given the commutative diagram

$$\begin{tikzcd} \mathrm{Spec}\, K(\!(t)\!) \arrow[r, "\widetilde{\delta}"] \arrow[d] & X \arrow[d, "f"] \\ \mathrm{Spec}\, K[\![t]\!] \arrow[r, "\delta"] & Y, \end{tikzcd}$$

there is a unique morphism $\gamma\colon \mathrm{Spec}\, K[\![t]\!] \to X$ such that the resulting two triangles in the above diagram are commutative. This completes the proof of the proposition. $\square$

Remark 10.2.9. With the notation in Proposition 9.2.8, if f is birational, but not proper, the same argument implies that the map $J_\infty(X) \smallsetminus J_\infty(f^{-1}(Z)) \to J_\infty(Y) \smallsetminus J_\infty(Z)$ induces *injections* between the K-valued points for every R-algebra K that is a field. Indeed, we simply use the valuative criterion for separatedness (recall that all schemes are assumed separated).

We point out that while the morphism $\widetilde{f}$ in Proposition 9.2.8 is bijective, it is very far from being a homeomorphism. In fact, one of the key results of the theory, the birational transformation rule that will be discussed in Sections 9.3 and 9.7 below, shows in particular how the codimensions of certain subsets change under this morphism. This is one of the peculiar phenomena when working with arc schemes, which are not Noetherian.

We use Proposition 9.2.8 to prove the following result due to Kolchin [Kol73].

Theorem 10.2.10. *If X is an irreducible scheme of finite type over a field k of characteristic* 0*, then $J_\infty(X)$ is irreducible.*

Proof. We first consider the case when X is smooth and irreducible. We have seen that in this case each $J_m(X)$ is smooth and connected, hence irreducible. Since a basis of open subsets of $J_\infty(X)$ is given by the subsets of the form $\pi_{\infty,m}^{-1}(U)$, for various $m \geq 0$ and various open subsets of $J_m(X)$, it follows that every two nonempty open subsets of $J_\infty(X)$ have nonempty intersection. Therefore $J_\infty(X)$ is irreducible in this case.

Suppose now that X is arbitrary. We argue by induction on $n = \dim(X)$, the case $n = 0$ being a consequence of the smooth case. Let us assume that we know the theorem in dimension less than n. By Lemma 9.2.7, we may assume that X is also reduced. Since we are over a field of characteristic 0, there is a smooth, irreducible scheme Y and a proper birational morphism $f\colon Y \to X$. Since $J_\infty(Y)$ is irreducible, in order to prove that $J_\infty(X)$ is irreducible, it is enough to show that $J_\infty(X)$ is contained in the closure of $f_\infty(J_\infty(Y))$. Let Z be a closed subset of X such that f induces an isomorphism $Y \smallsetminus f^{-1}(Z) \to X \smallsetminus Z$. It follows from Proposition 9.2.8 that $J_\infty(X) \smallsetminus J_\infty(Z)$ is contained in the image of f_∞. Therefore it is enough to show that $J_\infty(Z)$ is contained in the closure of $f_\infty(J_\infty(Y))$.

Let $Z_1, \ldots, Z_r$ be the irreducible components of Z. By induction, we know that each $J_\infty(Z_i)$ is irreducible. Furthermore, Lemma 9.2.7 implies the equality of sets $J_\infty(Z) = \cup_{i=1}^r J_\infty(Z_i)$. Therefore it is enough to find in each $J_\infty(Z_i)$ a nonempty open subset that is contained in the image of f_∞. Let W_i be an irreducible component of $f^{-1}(Z_i)$ that dominates Z_i. Since we are in characteristic 0, it follows from the generic smoothness theorem that there are open subsets $U_i \subseteq Z_i$ and $V_i \subseteq W_i$ such that f induces a smooth surjective morphism $V_i \to U_i$. It follows from the property 3) in Remark 9.2.3 that $J_\infty(U_i)$ is contained in the image of f_∞. Since $J_\infty(U_i)$ is a nonempty open subset of $J_\infty(Z_i)$, this completes the proof of the theorem. □

In fact, Kolchin's theorem holds in a more general setting, in which the ground field is endowed with a derivation, see [Kol73] and also [Gil02] for a scheme-theoretic approach. For a different proof of the above version of Kolchin's theorem, without using resolution of singularities, see [IK03] and [NS05].

Example 10.2.11. The above result fails in positive characteristic. Consider the following example from [NS05]. Suppose that $X \hookrightarrow \mathbb{A}^3 = \operatorname{Spec} k[x,y,z]$ is the hypersurface defined by $(x^2 - y^2 z)$, where k is a field of characteristic 2. We have a corresponding embedding

$$J_\infty(X) \hookrightarrow J_\infty(\mathbb{A}^3) = \operatorname{Spec} k[x_j, y_j, z_j; j \geq 0],$$

defined by the ideal generated by the coefficients of

$$\left(\sum_{j\geq 0} y_j^2 t^{2j}\right) \cdot \left(\sum_{j\geq 0} z_j t^j\right) - \sum_{j\geq 0} x_j^2 t^{2j}$$

(as usual, we write $x_0 = x$, $y_0 = y$, and $z_0 = z$). We have two nonempty open subsets of $J_\infty(X)$ with empty intersection, namely $U = J_\infty(X_{\mathrm{sm}})$ and $V = (z_1 \neq 0)$. Note that $X_{\mathrm{sm}} = X \cap (y \neq 0)$. We can check that U and V are disjoint by showing that they have no common K-valued points, where K is a field containing k. Therefore we need to check that if we have the following equality in $K[\![t]\!]$

$$\left(\sum_{j\geq 0} b_j^2 t^{2j}\right)\cdot\left(\sum_{j\geq 0} c_j t^j\right) = \sum_{j\geq 0} a_j^2 t^{2j}, \tag{10.4}$$

with $b_0 \neq 0$, then $c_1 = 0$. Indeed, differentiating (9.4) with respect to t gives

$$\left(\sum_{j\geq 0} b_j^2 t^{2j}\right)\cdot\left(\sum_{j\geq 1} jc_j t^{j-1}\right) = 0.$$

Since $b_0 \neq 0$, we conclude that $c_j = 0$ for all j odd. In particular, $c_1 = 0$. It is clear that $J_\infty(X_{\mathrm{sm}})$ is nonempty. In order to see that V is nonempty, it is enough to consider its intersection with the closed subscheme of $J_\infty(\mathbb{A}^3)$ defined by $(x_j, y_j; j \geq 0)$, which is contained in $J_\infty(X)$. We conclude that in this example $J_\infty(X)$ is not irreducible.

Corollary 10.2.12. *If X is a scheme of finite type over a field k of characteristic zero and $X_1, \ldots, X_r$ are the irreducible components of X, then $J_\infty(X_1), \ldots, J_\infty(X_r)$ are the irreducible components of $J_\infty(X)$. In particular, $J_\infty(X)$ has finitely many irreducible components.*

Proof. It follows from Theorem 9.2.10 that each $J_\infty(X_i)$ is irreducible, while Lemma 9.2.7 gives the set-theoretic decomposition $J_\infty(X) = \cup_{i=1}^r J_\infty(X_i)$. Since $\pi_\infty(J_\infty(X_i)) = X_i$ for every i, it follows that $J_\infty(X_i) \not\subseteq J_\infty(X_j)$ whenever $i \neq j$. We therefore obtain the assertion in the corollary. $\square$

Corollary 10.2.13. *If X is a connected scheme of finite type over a field k of characteristic zero, then $J_\infty(X)$ is connected.*

Proof. If $X_1, \ldots, X_r$ are the irreducible components of X, then the $J_\infty(X_i)$ are the irreducible components of $J_\infty(X)$ by Corollary 9.2.12. Moreover, we have $J_\infty(X_i) \cap J_\infty(X_j) = J_\infty(X_i \cap X_j)$, hence this intersection is empty if and only if $X_i \cap X_j = \emptyset$. Since X is connected, we conclude that $J_\infty(X)$ is connected. $\square$

We have seen in Corollary 9.2.12 that at least over a field of characteristic zero, the arc schemes have finitely many irreducible components. Using an argument similar to the one in the proof of Theorem 9.2.10, we extend this assertion to certain subsets of arc schemes that will be our main focus in the following sections. The general subsets of $J_\infty(X)$ tend to be rather badly behaved, due to the fact that $J_\infty(X)$ is not Noetherian. However, we will be interested in subsets that come "from a finite level" which, as we will see, tend to be much better behaved.

Definition 10.2.14. Let X be a scheme of finite type over R. A *cylinder* in $J_\infty(X)$ is a subset of the form $C = \pi_{\infty,m}^{-1}(S)$, for some m and some constructible subset $S \subseteq J_m(X)$. We recall that a subset of a Noetherian scheme is constructible if it can be written as a finite union of locally closed subsets. Note that given a cylinder C, we may write it as $\pi_{\infty,m}^{-1}(S)$, with m as large as we want. It is then clear that cylinders form an algebra of sets, that is, the union and the intersection of finitely many cylinders, as well as the difference of two cylinders, are again cylinders.

Proposition 10.2.15. *If X is a scheme of finite type over a field k of characteristic* 0 *and $C = \pi_{\infty,m}^{-1}(S) \subseteq J_\infty(X)$ is a cylinder, then the following hold:*

i) C has finitely many irreducible components.
ii) The set of points $\gamma \in C$ with the property that the residue field $k(\gamma)$ is a finite extension of k is dense in C.

Proof. Note that in order to prove the first assertion, it is enough to write C as the union of finitely many subsets, all of them irreducible with respect to the induced topology. For the assertion in ii), recall that a basis of open subsets of $J_\infty(X)$ is given by the subsets of the form $U = \pi_{\infty,q}^{-1}(V)$, for various q and various open subsets $V \subseteq J_q(X)$. Since for every such U, the intersection $U \cap C$ is again a cylinder, we see that it is enough to show that every nonempty cylinder C contains a point γ whose residue field is finite over k.

We first prove i) and ii) when X is smooth and irreducible. Note that S is a finite union of irreducible locally closed subsets of $J_m(X)$, hence we may assume that S is locally closed in $J_m(X)$ and irreducible. Since each $\pi_{p,m}$ is locally trivial, with fiber an affine space, it follows that each $\pi_{p,m}^{-1}(S)$ is irreducible. Arguing as in the proof of Theorem 9.2.10 (in the smooth case), we deduce that C is irreducible. If S is nonempty, we can find a closed point $\gamma_m \in S$, hence the reside field K of γ_m is finite over k. Since $\pi_{\infty,m}^{-1}(\gamma_m)$ is isomorphic to either $\operatorname{Spec} K$ (if $\dim(X) = 0$) or to $\mathbb{A}_K^\infty$ (if $\dim(X) \geq 1$), it follows that there is $\gamma \in \pi_{\infty,m}^{-1}(\gamma_m) \subseteq C$ with residue field K. This completes the proof when X is smooth and irreducible.

We prove the general case by induction on $n = \dim(X)$, the case $n = 0$ being clear. Note that if $X_1, \ldots, X_r$ are the irreducible components of X (say, with reduced scheme structures), then Lemma 9.2.7 gives an equality of sets $J_\infty(X) = J_\infty(X_1) \cup \ldots \cup J_\infty(X_r)$. Since each $C \cap J_\infty(X_i)$ is a cylinder in $J_\infty(X_i)$, we see that it is enough to prove the proposition when X is an integral scheme.

In this case, since the ground field has characteristic 0, there is a resolution of singularities $f \colon Y \to X$. Let Z be a proper closed subset of X such that f induces an isomorphism $Y \smallsetminus f^{-1}(Z) \to X \smallsetminus Z$. In this case, it follows from Proposition 9.2.8 that $J_\infty(X) \smallsetminus J_\infty(Z) \subseteq f_\infty(J_\infty(Y))$. We thus have the decomposition

$$C = f_\infty(f_\infty^{-1}(C)) \cup (C \cap J_\infty(Z)).$$

Note that $C \cap J_\infty(Z) = (\pi_{\infty,m}^Z)^{-1}(J_m(Z) \cap S)$ is a cylinder in $J_\infty(Z)$, while $f_\infty^{-1}(C) = (\pi_m^Y)^{-1}(f_m^{-1}(S))$ is a cylinder in $J_\infty(Y)$. The smooth case thus implies that $f_\infty^{-1}(C)$ is a finite union of irreducible subsets and if nonempty, then it contains a point with residue field finite over k. Therefore $f_\infty(f_\infty^{-1}(C))$ has the same properties. On the other hand, the induction hypothesis implies that $C \cap J_\infty(Z)$ is a finite union of irreducible subsets and if nonempty, then it contains a point with residue field finite over k. We thus conclude that C has the same properties. This completes the proof of the proposition. $\square$

We end this section with two interesting examples of spaces of arcs.

Example 10.2.16. Suppose that X is a toric variety (for simplicity, we work over an algebraically closed field k and only consider k-valued points). In this case, X contains an open subset T, which is a torus, and whose natural action on itself extends to an action on X. We thus obtain an action of T_∞ on X_∞ and the orbits of this action have been described by Ishii in [Ish04], as follows. For the basic facts about toric varieties that we will use, we refer to [Ful93]. Let Δ be the fan of X and $N \simeq \mathbb{Z}^n$ the corresponding lattice. If $D^{(1)}, \ldots, D^{(r)}$ are the the invariant prime divisors on X, then each $D^{(i)}$ is itself a toric variety, with corresponding torus a quotient of T. Arguing, for example, by induction on $\dim(X)$, we see that it is enough to describe the T_∞-orbits that are contained in

$$X_\infty^\circ := X_\infty \smallsetminus (D^{(1)} \cup \ldots \cup D^{(r)})_\infty = X_\infty \smallsetminus \cup_{i=1}^r D_\infty^{(i)}.$$

We will show that these orbits are parametrized by the lattice points in the support $|\Delta|$ of Δ.

Given $\gamma \in X_\infty$, there is a cone $\sigma \in \Delta$ such that the image of γ in X lies in U_σ. In this case $\gamma \in (U_\sigma)_\infty$, that is, it corresponds to a ring homomorphism $\gamma^* \colon k[\sigma^\vee \cap M] \to k[\![t]\!]$, where M is the dual lattice of N. Note that $\gamma \notin D_\infty^{(i)}$ for every i if and only if $\gamma^*(\chi^u) \neq 0$ for every $u \in \sigma^\vee \cap M$. We assume that this is the case, and consider the map

$$\sigma^\vee \cap M \ni u \to \mathrm{ord}_t(\gamma^*(\chi^u)) \in \mathbb{Z}_{\geq 0}.$$

Since this is clearly additive, it follows that there is a unique $v \in \sigma \cap N$ such that the map is given by $\langle -, v \rangle$. Note that $v = 0$ if and only if $\gamma \in T_\infty$. Given any $v \in \sigma \cap N$, we get an arc $\gamma_v \in (U_\sigma)_\infty$ corresponding to the ring homomorphism

$$k[\sigma^\vee \cap M] \to k[\![t]\!],\ \chi^u \to t^{\langle u,v \rangle}.$$

It is easy to see that as an arc in X_∞, this is independent of the choice of σ. We note that if τ is the the unique face of σ containing v, then the image of γ_v in X lies in the orbit O_τ of X corresponding to τ.

Given an arbitrary arc γ as above, if v is the corresponding element in $\sigma \cap N$, we see that there is a unique $\delta \in T_\infty$ such that $\gamma = \delta \cdot \gamma_v$; indeed, we have $\delta^*(\chi^u) = \gamma^*(\chi^u) \cdot t^{-\langle u,v \rangle}$. We thus obtain a bijection between the T_∞-orbits in $(U_\sigma)_\infty$ that are not contained in any $D_\infty^{(i)}$ and $\sigma \cap N$. By varying σ, we obtain a bijection between the T_∞ orbits in X_∞° and $|\Delta| \cap N$. We point out that it is not known how to give a similar description for the T_m-orbits of X_m, when X is a toric variety.

Note that if $f \colon Y \to X$ is an equivariant morphism of toric varieties, the morphism $f_\infty \colon Y_\infty \to X_\infty$ induces a morphism $f_\infty^\circ \colon Y_\infty^\circ \to X_\infty^\circ$. It is clear that if ϕ is the corresponding lattice homomorphism, then $f_\infty(\gamma_v)=\gamma_{\phi(v)}$. In particular, if f is proper and birational, then we have a bijection between the orbits in Y_∞° and X_∞° and f_∞ induces a bijection between the corresponding orbits. Note that if Y is smooth, then Y_∞ is irreducible, and therefore also Y_∞° is irreducible. For an arbitrary X, by taking a toric resolution of singularities $f\colon Y\to X$, we obtain that X_∞° is irreducible in arbitrary characteristic.

Suppose now that X is a toric variety with fan Δ. Once we have an orbit decomposition as above, the next question is to describe the orbit closures. Given $v,w\in|\Delta|\cap N$, we claim that $T_\infty\cdot\gamma_v$ is contained in the closure of $T_\infty\cdot\gamma_w$ if and only if there is a cone $\sigma\in\Delta$ (equivalently, for every $\sigma\ni v$) such that $w\in\sigma$ and $v-w\in\sigma$).

Suppose first that $\gamma_v\in\overline{T_\infty\cdot\gamma_w}$. If $\sigma\in\Delta$ is such that $v\in\sigma$, then $\gamma_v\in(U_\sigma)_\infty^\circ$. By assumption, this implies $\gamma_w\in(U_\sigma)_\infty^\circ$, hence $w\in\sigma\cap N$. Moreover, since γ_v lies in the closure of γ_w, it follows that for every $u\in\sigma^\vee\cap M$, we have

$$\langle u,v\rangle=\mathrm{ord}_t(\gamma_v^*(\chi^u))\geq\mathrm{ord}_t(\gamma_w^*(\chi^u))=\langle u,w\rangle.$$

This implies that $v-w\in\sigma$.

Conversely, suppose that there is a cone $\sigma\in\Delta$ such that $w,v-w\in\sigma$ (hence also $v\in\sigma$). Recall that we have a morphism $U_\sigma\times U_\sigma\to U_\sigma$ induced by

$$k[\sigma^\vee\cap M]\to k[\sigma^\vee\cap M]\otimes k[\sigma^\vee\cap M],\ \chi^u\to\chi^u\otimes\chi^u$$

(this extends the T-action). This induces a morphism $(U_\sigma)_\infty\times(U_\sigma)_\infty\to(U_\sigma)_\infty$ which restricts to $\alpha\colon(U_\sigma)_\infty^\circ\times(U_\sigma)_\infty^\circ\to(U_\sigma)_\infty^\circ$. It is clear that $\alpha(\gamma_{v-w},\gamma_w)=\gamma_v$, hence $\gamma_v\in\alpha((U_\sigma)_\infty^\circ\times\{\gamma_w\})\subseteq\overline{T_\infty\cdot\gamma_w}$, where the inclusion follows from the irreducibility of $(U_\sigma)_\infty^\circ$.

We end this example by noting that if Z is a closed subset of X_∞° which is preserved by the T_∞-action, then Z has finitely many irreducible components, each of these being the closure of some orbit $T_\infty\cdot\gamma_v$. In order to see this, let $\Lambda=\{v\in|\Delta|\cap N\mid\gamma_v\in Z\}$, hence $Z=\sqcup_{v\in\Lambda}T_\infty\cdot\gamma_v$. On Λ we consider the order given by $v\geq w$ precisely when $T_\infty\cdot\gamma_v\subseteq\overline{T_\infty\cdot\gamma_w}$. We claim that the set S of minimal elements in Λ is finite; in this case, it is clear that that the irreducible components of Z are given by the orbit closures $\overline{T_\infty\cdot\gamma_v}$, for $v\in S$. In order to check the claim, note that it is enough to show that for every $\sigma\in\Delta$, the set $S(\sigma)$ of minimal elements in $\Lambda\cap\sigma$ is finite. If we choose a system of nonzero generators $v_1,\ldots,v_r$ for $\sigma\cap N$, we have a surjective semigroup homomorphism $\phi:\mathbb{Z}^r_{\geq 0}\to\Lambda\cap\sigma$, given by $\phi(m_1,\ldots,m_r)=\sum_{i=1}^r m_iv_i$. If on $\mathbb{Z}^r_{\geq 0}$ we consider the order given by $(m_1,\ldots,m_r)\leq(m_1',\ldots,m_r')$ precisely when $m_i\leq m_i'$ for all i, then we see that $\phi^{-1}(S(\sigma))$ is contained in the set of minimal elements of $\phi^{-1}(\Lambda\cap\sigma)$. Since every subset of $\mathbb{Z}^r_{\geq 0}$ has finitely many minimal elements (this follows easily by induction on r), we conclude that $S(\sigma)$ is finite.

Example 10.2.17. Let $M=M_{m,n}(k)$ be the affine space of $m\times n$ matrices over an algebraically closed field k, with $n\geq m\geq 1$. For every r with $1\leq r\leq m$, let $D_r(M)\hookrightarrow M$ be the *generic determinantal variety* defined by the ideal generated by all $r\times r$ minors. As a set, $D_r(M)$ consists of all matrices of rank $\leq r-1$. Note that the group $G = GL_m(k) \times GL_n(k)$ acts on M by $(A,B) \cdot T = ATB^{-1}$. The orbits of this action consist precisely of the matrices of the same rank. We use this action in order to describe (set-theoretically) $D_r(M)_q$ and $D_r(M)_\infty$, following [Doc13].

Note that $M_\infty = M_{m,n}(k[[t]])$. It follows from the structure theorem for modules over a principal ideal domain that the orbits of G_∞ on M_∞ are parametrized by m-tuples $\underline{d} = (d_1, d_2, \ldots, d_m)$, with $0 \le d_1 \le d_2 \le \ldots \le d_m \le \infty$ and $d_i \in \mathbb{Z} \cup \{\infty\}$. An element in the orbit corresponding to $\underline{d} = (d_1, \ldots, d_m)$ is the matrix $A(\underline{d}) = \mathrm{diag}(t^{d_1}, \ldots, t^{d_m})$, with the convention $t^\infty = 0$. It is clear that $D_r(M)_\infty$ is the union of the orbits corresponding to those $\underline{d}$ with $d_r = \infty$.

Moreover, we can also describe the inverse image of $D_r(M)_q \subseteq M_q$ in M_∞. Indeed, this is the union of those $G_\infty \cdot A(\underline{d})$ with $\sum_{i=1}^m d_i \ge q+1$.

10.3 The birational transformation rule I

From now on, unless explicitly mentioned otherwise, we work over a fixed algebraically closed field k. While the main result in this section also has a version in positive characteristic (see [EM09]), for the sake of simplicity we prefer to state and prove it when $\mathrm{char}(k) = 0$, which is the case that we will need for applications. Recall that if X is a scheme of finite type over k, we denote by X_m and X_∞ the sets of k-valued points of $J_m(X)$ and $J_\infty(X)$, respectively. We keep the same notation for the different maps between these spaces. Note that X_∞ is only considered as a topological space, with the topology induced by the Zariski topology on $J_\infty(X)$ (we now refer to it as the *space of arcs* of X). It is clear that we have a homeomorphism of topological spaces $X_\infty \simeq \varprojlim_m X_m$. Since $J_m(X)$ is a scheme of finite type over k, there is no loss of information in only considering its k-valued points. Furthermore, since we will mostly be interested in cylinders in $J_\infty(X)$, it follows from Proposition 9.2.15 that we may, indeed, restrict to the k-valued points.

As in Section 9.2, we define a *cylinder* in X_∞ to be a subset of the form $\pi_{\infty,m}^{-1}(S)$, where $S \subseteq X_m$ is a constructible subset. It is clear that the set of cylinders in X_∞ form an algebra of subsets.

The main examples of cylinders arise as follows. Suppose that $Z \hookrightarrow X$ is a closed subscheme of X, defined by the ideal $\mathscr{I}_Z$. The ring $k[[t]]$ is a DVR, with the discrete valuation denoted by ord_t. We associate to Z a function $\mathrm{ord}_Z \colon X_\infty \to \mathrm{Z}_{\ge 0} \cup \{\infty\}$ that measures the order of vanishing of an arc along Z. More precisely, for $\gamma\colon \operatorname{Spec} k[[t]] \to X$, we consider the ideal $\gamma^{-1}(\mathscr{I}_Z)$ in $k[[t]]$. This ideal is generated by $t^{\mathrm{ord}_Z(\gamma)}$, with the convention that $\mathrm{ord}_Z(\gamma) = \infty$ when the ideal is zero. With this notation, for every $m \in \mathrm{Z}_{\ge 0}$, we have the following *contact loci*

$$\mathrm{Cont}^{\ge m}(Z) = \{\gamma \in X_\infty \mid \mathrm{ord}_Z(\gamma) \ge m\},$$

$$\mathrm{Cont}^{m}(Z) = \{\gamma \in X_\infty \mid \mathrm{ord}_Z(\gamma) = m\}.$$

It is clear that both sets are cylinders. Indeed, we have $\mathrm{Cont}^{\geq m}(Z) = \pi_{\infty,m-1}^{-1}(Z_{m-1})$ (with the convention that the right-hand side is equal to X_∞ when $m = 0$) and $\mathrm{Cont}^m(Z) = \mathrm{Cont}^{\geq m}(Z) \smallsetminus \mathrm{Cont}^{\geq (m+1)}(Z)$. Note that $\mathrm{Cont}^{\geq m}(Z)$ is closed, while $\mathrm{Cont}^m(Z)$ is locally closed. We may define in the same way the sets

$$\mathrm{Cont}^{\geq m}(Z)_p, \mathrm{Cont}^m(Z)_p \subseteq J_p(X)$$

whenever $p \leq m$. When $\mathfrak{a}$ is the ideal defining Z, we also write $\mathrm{Cont}^m(\mathfrak{a})$ and $\mathrm{Cont}^{\geq m}(\mathfrak{a})$ instead of $\mathrm{Cont}^m(Z)$ and $\mathrm{Cont}^{\geq m}(Z)$, respectively.

10.3.1 Cylinders in the space of arcs of a smooth variety

In this section we concentrate on cylinders in spaces of arcs of smooth varieties, which are much easier to study, due to the fact that the morphisms $\pi_{\infty,m}$ are locally trivial. We will turn to the more delicate study of cylinders in spaces of arcs of singular varieties in Section 9.7.

Lemma 10.3.1. *If C is a cylinder in X_∞, where X is a smooth variety, the subset $\pi_{\infty,m}(C)$ of X_m is constructible for every $m \geq 0$.*

Proof. Suppose that $C = \pi_{\infty,p}^{-1}(S)$, for some $p \geq 0$ and some constructible subset S of X_p. If $p \leq m$, then $\pi_{\infty,m}(C) = \pi_{m,p}^{-1}(S)$ by the subjectivity of $\pi_{\infty,m}$, hence $\pi_{\infty,m}(C)$ is clearly constructible. On the other hand, if $p > m$, then the subjectivity of $\pi_{\infty,p}$ implies $\pi_{\infty,m}(C) = \pi_{p,m}(S)$, and this is constructible by Chevalley's theorem. $\square$

Lemma 10.3.2. *If $C = \pi_{\infty,m}^{-1}(S)$ is a cylinder in X_∞, where X is a smooth variety, then*

i) The closure $\overline{C}$ of C is a cylinder. In fact, $\overline{C} = \pi_{\infty,m}^{-1}(\overline{S})$.
ii) C is closed, open, or locally closed if and only if S has the same property.
iii) If S is locally closed and $S_1, \ldots, S_r$ are the irreducible components of S, then $\pi_{\infty,m}^{-1}(S_1), \ldots, \pi_{\infty,m}^{-1}(S_m)$ are the irreducible components of C.

Proof. We note that since X is smooth, the morphism $\pi_{\infty,m}$ is surjective. Furthermore, each $\pi_{p,m}$ is locally trivial; in particular, it is flat, hence open. Since the topology on X_∞ is the inverse limit topology, it follows that $\pi_{\infty,m}$ is open.

We first prove i). The inclusion $\overline{C} \subseteq \pi_{\infty,m}^{-1}(\overline{S})$ follows since the right-hand side is closed. Therefore we only need to prove the reverse inclusion. If $\gamma \in X_\infty \smallsetminus \overline{C}$, there is an open subset W of X_∞ such that $\gamma \in W$ and $W \cap C = \emptyset$. Therefore $\pi_{\infty,m}(\gamma)$ lies in $\pi_{\infty,m}(W)$, which is open and does not intersect S. Therefore $\gamma \notin \pi_{\infty,m}^{-1}(\overline{S})$.

For ii), we only need to prove that if C is open, closed, or locally closed, then S has the same property. If C is open, then $S = \pi_{\infty,m}(C)$ is open since $\pi_{\infty,m}$ is open. By considering the complement of S, we also obtain that if C is closed, then S is closed. Suppose now that C is locally closed. In this case, C is open in $\overline{C}$, which is equal to $\pi_{\infty,m}^{-1}(\overline{S})$, by i). If we write $C = \pi_{\infty,m}^{-1}(\overline{S}) \cap U$, for some open subset U of X_∞, then $S = \pi_{\infty,m}(C) = \overline{S} \cap \pi_{\infty,m}(U)$. Since $\pi_{\infty,m}(U)$ is open in X_m, we deduce that S is locally closed in X_m.

For iii), note first that each $\pi_{\infty,m}^{-1}(S_i)$ is closed in C. Furthermore, we have seen in the proof of Proposition 9.2.15 that since X is smooth and S_i is irreducible, $\pi_{\infty,m}^{-1}(S_i)$ is irreducible as well. If $i \neq j$, then S_i is not contained in S_j, and since $\pi_{\infty,m}$ is surjective, we conclude that $\pi_{\infty,m}^{-1}(S_i)$ is not contained in $\pi_{\infty,m}^{-1}(S_j)$. We thus obtain the assertion in iii). □

If $C = \pi_{\infty,m}^{-1}(S)$ is a cylinder in X_∞, where X is a smooth variety of dimension n, we put

$$\operatorname{codim}(C) := \operatorname{codim}(S, X_m) = (m+1)n - \dim(S).$$

Note that for $p > m$, the morphism $\pi_{p,m}\colon X_p \to X_m$ is locally trivial, with fiber $\mathbb{A}^{(p-m)n}$. This implies that $\dim(\pi_{p,m}^{-1}(S)) = \dim(S) + (p-m)n$, hence $\operatorname{codim}(C, X_\infty)$ is well-defined. It is clear from definition that if $C_1 \subseteq C_2$ are cylinders, then $\operatorname{codim}(C_1) \geq \operatorname{codim}(C_2)$. We also have $\operatorname{codim}(C) = \operatorname{codim}(\overline{C})$, since the same property holds for constructible subsets in X_m. The following result is very useful.

Proposition 10.3.3. *If Y is a proper closed subscheme of the smooth variety X, then*

$$\lim_{m \to \infty} \operatorname{codim}(\operatorname{Cont}^{\geq m}(Y)) = \infty.$$

Proof. Let $C_m = \operatorname{Cont}^{\geq m}(Y)$. Since $C_m \supseteq C_{m+1}$ for every m, it follows that the sequence $\{\operatorname{codim}(C_m)\}_{m \geq 1}$ is non-decreasing. We conclude that if it does not go to infinity, then there are N and m_0 such that $\operatorname{codim}(C_m) = N$ for $m \geq m_0$. In this case, there is a common irreducible component of all C_m, with $m \geq m_0$. Indeed, for every $m > m_0$, if C is an irreducible component of C_m with $\dim(C) = \dim(C_m)$, then for every p with $m_0 \leq p < m$, we see that C is also an irreducible component of C_p. Since C_{m_0} has only finitely many irreducible components, one of these has to be an irreducible component for all C_m, with $m \geq m_0$. We thus have a cylinder that is contained in Y_∞, which contradicts the lemma below. □

Lemma 10.3.4. *If X is a smooth variety and Y is a proper subscheme of X, then for every cylinder $C \subseteq X_\infty$, we have $C \not\subseteq Y_\infty$.*

Proof. It is enough to show that if $\gamma \in X_\infty$, then $\pi_{\infty,m}^{-1}(\pi_{\infty,m}(\gamma_m)) \not\subseteq Y_\infty$ for every $m \geq 0$. Let $p = \pi_m(\gamma_m) \in X$. If $x_1, \ldots, x_n$ are local coordinates on X in a neighborhood of p, such that $x_i(p) = 0$, we have an isomorphism $\widehat{\mathscr{O}_{X,p}} \simeq k[[y_1, \ldots, y_n]]$ that maps each x_i to y_i. Let $f \in k[[y_1, \ldots, y_n]]$ be the formal power series that corresponds to the image in $\widehat{\mathscr{O}_{X,p}}$ of a nonzero element in the ideal defining Y. Recall that we have a bijection $\pi_\infty^{-1}(p) \simeq (tk[[t]])^n$ such that $\gamma\colon \mathscr{O}_{X,p} \to k[[t]]$ corresponds to $u = (\gamma(x_1), \ldots, \gamma(x_n))$. Note that every such γ induces a unique homomorphism $\widehat{\mathscr{O}_{X,p}} \to k[[t]]$.

Since $\pi_{\infty,m}^{-1}(\pi_{\infty,m}(\gamma_m)) \subseteq Y_\infty$, we deduce that for every $w \in (t^{m+1}k[[t]])^n$, we have $f(u+w) = 0$. It is clear that there is $g \in k[[t, y_1, \ldots, y_n]]$ such that $f(u + t^m v) = g(t, v_1, \ldots, v_n)$ for every $v = (v_1, \ldots, v_n) \in (tk[[t]])^n$ and that g is nonzero since f is nonzero. Therefore in order to get a contradiction it is enough to show that if g has the property that $g(t,w_1,\ldots,w_n)=0$ for every $w\in(tk[\![t]\!])^n$, then $g=0$.

The key case is when $n=1$. Let us write $g=\sum_{i\geq 0}g_iy^i$, where $g_i\in k[\![t]\!]$. Suppose that g is nonzero and let $r=\min\{i\mid g_i\neq 0\}$. Since $g(t,w)=0$ for every $w\in tk[\![t]\!]$, it follows that $\sum_{i\geq r}g_i(t)w^{i-r}=0$ for every $w\in tk[\![t]\!]\smallsetminus\{0\}$. On the other hand, it is clear that we can write $\sum_{i\geq r}g_i(t)w^{i-r}=\sum_{j\geq 0}P_jt^j$, where each P_j is a polynomial in the coefficients of w. Since the P_j vanish when $w\neq 0$, and since the ground field is infinite, it follows that the P_j vanish for every w. In particular, by taking $w=0$, we obtain $g_r=0$, a contradiction.

The general case now follows by induction on n. Indeed, if $n\geq 2$, let us write $g=\sum_{i\geq 0}g_i(t,y_1,\ldots,y_{n-1})y_n^i$. The case $n=1$ implies that $g_i(t,w_1,\ldots,w_{n-1})=0$ for all $i\geq 0$ and all $(w_1,\ldots,w_{n-1})\in(tk[\![t]\!])^{n-1}$. We conclude that $g_i=0$ for all i by induction, hence $g=0$. □

Corollary 10.3.5. *If $f\colon Y\to X$ is a proper birational morphism between smooth varieties, then each $f_m\colon Y_m\to X_m$ is surjective.*

Proof. Let $\gamma_m\in X_m$. If Z is a proper closed subset of X such that f is an isomorphism over $X\smallsetminus Z$, then $X_\infty\smallsetminus Z_\infty$ is contained in the image of f_∞ by Proposition 9.2.8. On the other hand, Lemma 9.3.4 implies $(\pi^X_{\infty,m})^{-1}(\gamma_m)\not\subseteq Z_\infty$. By combining these assertions, we deduce $\gamma_m\in\mathrm{Im}(\pi^X_{\infty,m}\circ f_\infty)\subseteq\mathrm{Im}(f_m)$. □

The next result shows that for closed irreducible cylinders in the space of arcs of a smooth variety, the notion of codimension that we defined agrees with the codimension from the point of view of the Zariski topology.

Proposition 10.3.6. *If X is a smooth variety and C is a closed cylinder in X_∞, then*

$$\mathrm{codim}(C)=\max\{r\geq 0\mid \exists C\subsetneq W_1\subsetneq\ldots\subsetneq W_r\subseteq X_\infty\mid W_i\,\textit{closed, irreducible}\}.$$

Proof. The assertion follows from our definition of codimension if we show that all irreducible closed subsets W of X containing C must be cylinders. This is the content of the next lemma. □

Lemma 10.3.7. *Let X be a smooth variety and C a cylinder in X_∞. If W is a closed, irreducible subset of X_∞ containing C, then W is a cylinder.*

Proof. Since W is closed in X_∞, it follows from the definition of the topology on the space of arcs that there are closed subsets $Z_m\subseteq X_m$ such that $W=\cap_{m\geq 0}C_m$, where $C_m=\pi^{-1}_{\infty,m}(Z_m)$. Since $\overline{\pi_{\infty,m}(W)}\subseteq Z_m$ for every m, we may replace each Z_m by $\overline{\pi_{\infty,m}(W)}$ and thus assume that each Z_m is irreducible and that $C_{m+1}\subseteq C_m$ for every m. In particular, we have $\mathrm{codim}(C_{m+1})\geq\mathrm{codim}(C_m)$, with equality if and only if $C_m=C_{m+1}$. Since $C\subseteq C_m$, we must have $\mathrm{codim}(C_m)\leq\mathrm{codim}(C)$ for every m. iI follows that the sequence $(\mathrm{codim}(C_m))_{m\geq 1}$ is eventually constant, hence $(C_m)_{m\geq 1}$ is eventually constant. We conclude that $C=C_m$ for $m\gg 0$ and therefore C is a cylinder. □

While we will not make use of the following two results, they can simplify certain arguments when working over an uncountable field.

Proposition 10.3.8. *If X is a smooth variety and the ground field is uncountable, then for every descending sequence of nonempty cylinders $C_1 \supseteq C_2 \supseteq \ldots$, we have $\cap_{i\geq 1} C_i \neq \emptyset$.*

Proof. We give the proof following [Bat98]. Since we work over an uncountable field, every descending sequence of nonempty constructible subsets of a scheme of finite type has nonempty intersection (see Proposition E.0.1). Consider the descending sequence of subsets of X_0

$$\pi_{\infty,0}(C_1) \supseteq \pi_{\infty,0}(C_2) \supseteq \ldots,$$

which are all constructible by Lemma 9.3.1, and clearly nonempty. Let γ_0 be an element in the intersection. By choice of γ_0, it follows that the descending sequence

$$\pi_{\infty,1}(C_1) \cap \pi_{1,0}^{-1}(\gamma_0) \supseteq \pi_{\infty,1}(C_2) \cap \pi_{1,0}^{-1}(\gamma_0) \supseteq \ldots$$

consists of nonempty subsets of X_1, which are constructible by Lemma 9.3.1. We may thus choose γ_1 in the intersection of these sets. Repeating this, we obtain a sequence $(\gamma_i)_{i\geq 0}$ such that $\gamma_{i+1} \in \pi_{\infty,i+1}(C_m) \cap \pi_{i+1,i}^{-1}(\gamma_i)$ for every $i \geq 0$ and $m \geq 1$. Therefore the sequence $(\gamma_i)_{i\geq 0}$ defines an element $\gamma \in X_\infty$. For every m, we have $\pi_{\infty,i}(\gamma) = \gamma_i \in \pi_{\infty,i}(C_m)$, and since C_m is a cylinder, we see by taking $i \gg 0$ that $\gamma \in C_m$. Therefore $\gamma \in \cap_{m\geq 1} C_m$. □

Corollary 10.3.9. *If $f: Y \to X$ is a proper, birational morphism of smooth varieties over an uncountable ground field, then $f_\infty: Y_\infty \to X_\infty$ is surjective.*

Proof. Given $\gamma \in X_\infty$, for every $m \geq 1$ we consider $\gamma_m = \pi_{\infty,m}^X(\gamma)$ and the cylinder $C_m = (\pi_{\infty,m}^Y)^{-1}(f_m^{-1}(\gamma_m))$. This is a descending sequence of cylinders, which are all nonempty by Corollary 9.3.5. It follows from Proposition 9.3.8 that there is $\delta \in \cap_{m\geq 1} C_m$. Therefore $\pi_{\infty,m}^X(\gamma) = \pi_{\infty,m}^X(f_\infty(\delta))$ for every $m \geq 1$, hence $\gamma = f_\infty(\delta)$. □

10.3.2 The key result

In this section, unless explicitly mentioned otherwise, we assume that the ground field has characteristic 0. In order to be able to state the birational transformation formula, we now introduce the notion of piecewise trivial fibration. Let F be a reduced scheme. Given a morphism $f: W' \to W$ of schemes of finite type over k and constructible subsets $A \subseteq W$ and $A' \subseteq W'$ such that f induces a map $g: A' \to A$, we say that g is *piecewise trivial*, with fiber F, if there is a decomposition $A = \cup_{j=1}^r A_j$, with each A_j locally closed in W and such that $g^{-1}(A_j)$ is locally closed in W' and it is isomorphic over A_j to $A_j \times F$ (where we consider on both A_j and $g^{-1}(A_j)$ the reduced structures). Of course, if this is the case, then $A = f(A')$. Note that in the definition, we may always assume that the A_j are mutually disjoint. If g is piecewise trivial, with fiber $\operatorname{Spec}(k)$, then we say that g is a *piecewise isomorphism*.

Lemma 10.3.10. *Let $f\colon X \to Y$ be a morphism of schemes of finite type over k, F a reduced scheme, and $B \subseteq X$ a constructible subset.*

i) Suppose that $B = \bigcup_{i=1}^r B_i$, with each B_i constructible and $f^{-i}(f(B_i)) \cap B = B_i$ (that is, B_i is a union of fibers of $B \to Y$). In this case, $B \to f(B)$ is piecewise trivial with fiber F if and only if each $B_i \to f(B_i)$ has the same property.

ii) If every $y \in f(B)$ has an open neighborhood U_y in Y such that $B \cap f^{-1}(U_y) \to f(B) \cap U_y$ is piecewise trivial with fiber F, then $B \to f(B)$ is piecewise trivial with fiber F.

iii) If every $x \in B$ has an open neighborhood V_x in X such that $B \cap V_x$ is a union of fibers of $B \to Y$ and $B \cap V_x \to f(B \cap V_x)$ is piecewise trivial, with fiber F, then $B \to f(B)$ is piecewise trivial, with fiber F.

Proof. The equivalence in i) follows from definition. The assertions in ii) and iii) follow arguing by Noetherian induction on Y, respectively X. □

Lemma 10.3.11. *If $f\colon W' \to W$ is a morphism of schemes of finite type over k and f induces a map $g\colon A' \to A$, where A' and A are constructible subsets of W' and W, respectively, then g is a piecewise isomorphism if and only if it is bijective.*

Proof. It is clear that if g is a piecewise isomorphism, then it is bijective, hence we only need to prove the converse. Suppose that g is bijective. Since A' is constructible, we can write it as a disjoint union $A' = \sqcup_{i=1}^r A'_i$, with each A'_i locally closed in W'. Since g is bijective, we obtain a corresponding decomposition $A = \sqcup_{i=1}^r g(A'_i)$, with each $g(A'_i)$ constructible by Chevalley's theorem. Clearly, it is enough to show that each $A'_i \to g(A'_i)$ is a piecewise isomorphism (note that $A'_i = g^{-1}(g(A'_i))$ by the injectivity of g). Therefore we may assume that A' is locally closed in W'.

We now consider a decomposition $A = \sqcup_{j=1}^s A_j$, with each A_j locally closed in W. Since $g^{-1}(A_j) = A' \cap f^{-1}(A_j)$ is locally closed in W', and since it is clearly enough to show that each $g^{-1}(A_j) \to A_j$ is a piecewise isomorphism, we may assume that A' and A are locally closed in W' and W, respectively. Hence after replacing W' by A' and W by A, we may assume that $A' = W'$ and $A = W$. Furthermore, we may replace W and W' by the corresponding reduced schemes and thus assume that both W and W' are reduced.

Arguing by Noetherian induction with respect to W, we see that if U is an open subset of W, it is enough to prove that $f^{-1}(U) \to U$ is a piecewise isomorphism. In particular, if W is reducible and $W_1, \ldots, W_r$ are its irreducible components, we may replace W by $W_1 \smallsetminus \cup_{i \neq 1} W_i$. Therefore we may assume that W is irreducible. Since f is surjective, there is an irreducible component W'' of W' that dominates W. In this case, we can find an open subset V of W that is contained in $f(W'')$. After replacing $W' \to W$ by $f^{-1}(V) \to V$, we may assume that both W' and W are irreducible and reduced. It is clear that $\dim(W') = \dim(W)$. Since we are in characteristic 0, we can find open subsets U' of W' and U of W such that f induces a morphism $U' \to U$ that is finite and smooth (hence étale). After replacing U by $f(U')$, we see that we have a finite bijective étale morphism $U' \to U$. This must be an isomorphism. Furthermore, since f is bijective, we have $U' = f^{-1}(U)$. This completes the proof of the lemma. $\square$

Corollary 10.3.12. *Let $f: X \to Y$ and $g: Y \to Z$ be morphisms of schemes of finite type and F a reduced scheme. Suppose that $A \subseteq X$, $B \subseteq Y$, and $C \subseteq Z$ are constructible subsets such that we get induced maps $A \to B$ which is piecewise trivial with fiber F, and $B \to C$ which is a piecewise isomorphism. The composition $A \to B \to C$ is then piecewise trivial, with fiber F.*

Proof. By assumption, we can write C as a disjoint union of locally closed subsets $C_1, \ldots, C_s$ in Z such that each $g^{-1}(C_j)$ is locally closed in Y and $g^{-1}(C_j) \to C_j$ is an isomorphism. We may similarly write B as a disjoint union of subsets $B_1, \ldots, B_r$ that are locally closed in Y, such that each $f^{-1}(B_i)$ is locally closed in A and it is isomorphic to $B_i \times F$ over B_i. It is then clear that each $B_i \cap g^{-1}(C_j)$ is locally closed in Y, hence $g(B_i) \cap C_j$ is locally closed in C_j, and thus in Z, and $(g \circ f)^{-1}(g(B_i) \cap C_j)$ is isomorphic to $(g(B_i) \cap C_j) \times F$. $\square$

Example 10.3.13. Note that we can have morphisms of schemes of finite type $f: X \to Y$ and $g: Y \to Z$ such that g is (piecewise) trivial, f is a piecewise isomorphism, but $g \circ f$ is not piecewise trivial. Indeed, suppose that $f: X = (\mathbb{A}^2 \smallsetminus \{0\}) \sqcup \{0\} \to Y = \mathbb{A}^2$ is the obvious morphism and $g: \mathbb{A}^2 \to \mathbb{A}^1$ is the projection onto the first component. It is then clear that for every locally closed subset W of Z containing the origin, we have $(g \circ f)^{-1}(W) \not\simeq W \times \mathbb{A}^1$.

The result that is responsible for most of the applications of the spaces of arcs is the birational transformation formula. This describes the behavior of f_∞ when $f: Y \to X$ is a proper, birational morphism, with Y smooth. In this section we only consider the easier case when X is smooth, too, when the result is due to Kontsevich [Kon]. For the more general version, see Section 9.7. Recall that if $f: Y \to X$ is a proper, birational morphism between two smooth varieties, then the morphism of line bundles $f^*\omega_X \to \omega_Y$ corresponds to a section of $\omega_Y \otimes f^*(\omega_X^{-1})$ defining an effective divisor $K_{Y/X}$. This has the property that $\dim f^{-1}(f(y)) \geq 1$ for every $y \in \mathrm{Supp}(K_{Y/X})$ and if $U = X \smallsetminus f(\mathrm{Supp}(K_{Y/X}))$, then $f^{-1}(U) \to U$ is an isomorphism (see Lemma B.2.3).

Theorem 10.3.14. *If $f: Y \to X$ is a birational morphism between smooth varieties over k, then for every $e \in \mathbb{Z}_{\geq 0}$ and every $m \geq 2e$, the contact locus $\mathrm{Cont}^e(K_{Y/X})_m$ has the following properties:*

i) If $\gamma_m, \gamma'_m \in Y_m$ are such that $f_m(\gamma_m) = f_m(\gamma'_m)$ and $\gamma_m \in \mathrm{Cont}^e(K_{Y/X})_m$, then

$$\pi^Y_{m,m-e}(\gamma_m) = \pi^Y_{m,m-e}(\gamma'_m).$$

In particular, we also have $\gamma'_m \in \mathrm{Cont}^e(K_{Y/X})_m$.

ii) The map

$$\mathrm{Cont}^e(K_{Y/X})_m \to f_m(\mathrm{Cont}^e(K_{Y/X})_m)$$

is piecewise trivial, with fiber $\mathbb{A}^e$.

Before proving the general case, we illustrate the theorem in an important special case.

Example 10.3.15. Suppose that X is a smooth variety and $f\colon Y \to X$ is the blow-up along a smooth subvariety $Z \hookrightarrow X$ of codimension $r \geq 2$. In this case we allow the ground field to have arbitrary characteristic. If E is the exceptional divisor, then $K_{Y/X} = (r-1)E$ (see Lemma **??**). In particular, we see that the contact locus in the theorem is empty, unless $a := \frac{e}{r-1} \in \mathbb{Z}$, which we henceforth assume. Note that both assertions in Theorem 9.3.14 are local over X. Since locally on X we can find an étale morphism to some $\mathbb{A}^n$ such that Z is the pull-back of the linear subspace defined by $(x_1, \ldots, x_r)$, it is easy to see, using Lemma 9.1.12, that we may assume that $X = \mathbb{A}^n$ and Z is defined by $(x_1, \ldots, x_r)$.

In particular, we have $X_m = (k[t]/(t^{m+1}))^{\oplus n}$. For $1 \leq i \leq r$, let us consider the chart $U \subset Y$ with coordinates $y_1, \ldots, y_n$, such that $y_j = x_j$ for $j > r$, $y_i = x_i$, and $y_j = x_i x_j$ for $j \leq r$, $j \neq i$. In this case the morphism $U_m \to X_m$ gets identified to

$$\phi_i\colon (k[t]/(t^{m+1}))^{\oplus n} \to (k[t]/(t^{m+1}))^{\oplus n},$$

$$(u_1, \ldots, u_n) \to (u_1 u_i, \ldots, u_i, \ldots, u_r u_i, u_{r+1}, \ldots, u_n).$$

It is clear that $f_m(\mathrm{Cont}^e(K_{Y/X})_m \cap U_m)$ is equal to

$$\{w = (w_1, \ldots, w_n) \in \mathbb{A}^n_m, \mathrm{ord}_t(w_i) = a \leq \mathrm{ord}_t(w_j) \text{ for } 1 \leq j \leq r\} \tag{10.5}$$

and moreover, it is easy to check that the inverse image of (9.5) in Y_m is contained in U_m. Note that given any $u_i, g \in k[t]/(t^{m+1})$ with $\mathrm{ord}_t(g) \geq a = \mathrm{ord}(u_i)$, there is $u_j \in k[t]/(t^{m+1})$ such that $u_i u_j = g$; furthermore, this only depends on the class of u_j in $k[t]/(t^{m+1-a})$, which is uniquely determined. Since $e \geq a$, we obtain assertion i) in the theorem in this case. Moreover, it is clear that after identifying

$$U_m \simeq (k[t]/(t^{m+1}))^{\oplus(n-r+1)} \times (k[t]/(t^{m-a+1}))^{\oplus(r-1)} \times k^{\oplus(r-1)a},$$

the morphism ϕ_i gets identified to the projection on the product of the first two components. Since $(r-1)a = e$, this shows that assertion ii) in the theorem holds in this case.

The proof that we give for Theorem 9.3.14 follows [Loo02]. The key ingredient is a functorial description for the fibers of certain projections $\pi_{p,m}\colon X_p \to X_m$. This, in turn, is a consequence of the following easy lemma.

Lemma 10.3.16. *Let A and R be k-algebras. Given $m, p \in \mathbb{Z}_{\geq 0}$ with $m \leq p \leq 2m+1$, consider the map induced by truncation*

$$\theta_{p,m}\colon \mathrm{Hom}_{k-\mathrm{alg}}(R, A[t]/(t^{p+1})) \to \mathrm{Hom}_{k-\mathrm{alg}}(R, A[t]/(t^{m+1})).$$

For every morphism of k-algebras $\overline{\alpha}\colon R \to A[t]/(t^{m+1})$, there is a natural action of $\mathrm{Der}_k(R, t^{m+1}A[t]/t^{p+1}A[t])$ *on $\theta_{p,m}^{-1}(\overline{\alpha})$ that makes this fiber a principal homogeneous space whenever it is nonempty.*

Proof. Since $m \leq p \leq 2m+1$, the $A[t]/(t^{p+1})$-module $t^{m+1}A[t]/t^{p+1}A[t]$ is in fact an $A[t]/(t^{m+1})$-module, hence an R-module via $\overline{\alpha}$. It is cleat that if $\alpha\colon R \to A[t]/(t^{p+1})$ is a lifting of $\overline{\alpha}$, then any other k-linear lift α' of $\overline{\alpha}$ can be uniquely written as $\alpha' = \alpha + D$, for some k-linear map $D\colon R \to \frac{t^{m+1}A[t]}{t^{p+1}A[t]}$. Using again the hypothesis that $p \leq 2m+1$, we see that for every $u, v \in R$, we have

$$\alpha'(u)\alpha'(v) = \alpha(u)\alpha(v) + \overline{\alpha}(u)D(v) + \overline{\alpha}(v)D(u).$$

In other words, α' is a k-algebra homomorphism if and only if D is a derivation. This gives the assertion in the lemma. $\square$

Corollary 10.3.17. *Let X be a scheme of finite type over k. For every $m, p \in \mathbb{Z}_{\geq 0}$ such that $m \leq p \leq 2m+1$, if $\gamma_p \in X_p$ and $\gamma_m = \pi_{p,m}(\gamma_p)$, then we have a scheme-theoretic isomorphism*

$$\pi_{p,m}^{-1}(\gamma_m) \simeq \mathrm{Hom}_{k[t]/(t^{m+1})}(\gamma_m^*\Omega_X, (t^{m+1})/(t^{p+1})). \tag{10.6}$$

Proof. Note that the right-hand side of (9.6) is a finite-dimensional k-vector space V. As an algebraic variety, this is $\mathrm{Spec}(\mathrm{Sym}(V^*))$, such that for a k-algebra A, its A-valued points are in natural bijection to $\mathrm{Hom}_k(V^*, A) \simeq V \otimes_k A$.

In order to prove (9.6), we may replace X by an affine open neighborhood of the image of γ_p in X and thus assume $X = \mathrm{Spec}(R)$. For every k-algebra A, the set of A-valued points of $\pi_{p,m}^{-1}(\gamma_m)$ consists of the k-algebra homomorphisms $\delta_p\colon R \to A[t]/(t^{p+1})$ such that the following diagram is commutative:

$$\begin{tikzcd}
R \arrow[r, "\delta_p"] \arrow[d, "\gamma_m"'] & A[t]/(t^{p+1}) \arrow[d, "q"] \\
k[t]/(t^{m+1}) \arrow[r, "j"] & A[t]/(t^{m+1}),
\end{tikzcd}$$

where q and j are the natural projection, respectively, inclusion. It follows from the lemma that the set of such δ_p is in natural bijection to

$$\mathrm{Der}_k(R, t^{m+1}A[t]/t^{p+1}A[t]) \simeq \mathrm{Hom}_R(\Omega_R, t^{m+1}A[t]/t^{p+1}A[t])$$

$$\simeq \mathrm{Hom}_{k[t]/(t^{m+1})}(\gamma_m^*\Omega_R, (t^{m+1})/(t^{p+1}) \otimes_k A) \simeq V \otimes_k A,$$

where the last isomorphism follows from the fact that $\gamma_m^*\Omega_R$ is a finitely generated $k[t]/(t^{m+1})$-module. $\square$

Remark 10.3.18. The isomorphism in Corollary 9.3.17 is natural in the pair (X, γ_p) in an obvious sense.

Remark 10.3.19. With the notation in Corollary 9.3.17, if $p > m$ and $\gamma_{p-1} \in X_{p-1}$ is the image of γ_p, then the natural projection $\pi_{p,m}^{-1}(\gamma_m) \to \pi_{p-1,m}^{-1}(\gamma_m)$ corresponds via the isomorphisms given using the corollary to the map

$$\mathrm{Hom}_{k[t]/(t^{m+1})}(\gamma_m^*\Omega_X, (t^{m+1})/(t^{p+1})) \to \mathrm{Hom}_{k[t]/(t^{m+1})}(\gamma_m^*\Omega_X, (t^{m+1})/(t^p))$$

induced by the natural projection $(t^{m+1})/(t^{p+1}) \to (t^{m+1})/(t^p)$.

Remark 10.3.20. If X is a smooth n-dimensional variety, the assertion in Corollary 9.3.17 globalizes as follows. In this case, for every m and p with $m \le p \le 2m+1$, we have a geometric vector bundle $E_{p,m}$ over X_m, whose geometric fiber over $\gamma_m \in X_m$ is

$$\mathrm{Hom}_{k[t]/(t^{m+1})}(\gamma_m^*\Omega_X, (t^{m+1})/(t^{p+1})) \simeq \left((t^{m+1})/(t^{p+1})\right)^{\oplus n} \simeq k^{(p-m)n}.$$

By using Lemma 9.3.16 one can check that $E_{p,m}$ has a natural action on X_p over X_m. Furthermore, if $h\colon Z \to X_m$ is a scheme morphism such that there is $\widetilde{h}\colon Z \to X_p$ with $\pi_{p.m} \circ \widetilde{h} = h$, then $\widetilde{h}$ induces a morphism $h^*(E_{p,m}) \to Z \times_{X_m} X_p$ over Z which is an isomorphism.

Proof of Theorem 9.3.14. The proof of part i) is the most involved. We proceed in several steps. Let $Z = f(\mathrm{Supp}(K_{Y/X}))$, hence f is an isomorphism over $Y \smallsetminus X$. Since Y is smooth, we may choose $\gamma, \gamma' \in Y_\infty$ that map to $\gamma_m, \gamma'_m \in Y_m$, respectively. We denote by γ_q and γ'_q the images of, respectively, γ and γ' in Y_q, for every q.

Step 1. It is enough to show that there is $\delta \in Y_\infty$ such that

1) $\pi^Y_{\infty,m-e}(\delta) = \pi^Y_{\infty,m-e}(\gamma)$ and
2) $f_\infty(\delta) = f_\infty(\gamma')$.

Indeed, since $\gamma_m \in \mathrm{Cont}^e(K_{Y/X})_m$ and $M \ge 2e$, condition 1) implies that $\delta \in \mathrm{Cont}^e(K_{Y/X})$. In particular, $\delta \notin f_\infty^{-1}(Z_\infty)$. In this case, condition 2) together with Remark 9.2.9 implies $\delta = \gamma'$, and using one more time condition 1) we conclude that $\pi^Y_{m,m-e}(\gamma_m) = \pi^Y_{m,m-e}(\gamma'_m)$.

Step 2. In order to find $\delta \in Y_\infty$ that satisfies 1) and 2) above, it is enough to construct $\delta_p \in Y_p$ for every $p \ge m$ such that the following hold:

a) $\delta_m = \gamma_m$.
b) $\pi^Y_{p-1,p-e-1}(\delta_{p-1}) = \pi^Y_{p,p-e-1}(\delta_p)$ for every $p \ge m+1$.
c) $f_p(\delta_p) = f_p(\gamma'_p)$ for every $p \ge m$.

Indeed, in this case, it follows from condition b) that there is a unique $\delta \in Y_\infty$ such that $\pi^Y_{\infty,p-e}(\delta) = \pi^Y_{p,p-e}(\delta_p)$ for every $p \ge m$. In particular, this condition for $p = m$, together with a) imply $\pi^Y_{\infty,m-e}(\delta) = \pi^Y_{\infty,m-e}(\gamma)$. On the other hand, condition c) implies $f_\infty(\delta) = f_\infty(\gamma')$.

Step 3. We construct the δ_p as in Step 2 by induction on p. For $p = m$, we take $\delta_m = \gamma_m$ and condition c) is satisfied since by assumption $f_m(\gamma_m) = f_m(\gamma'_m)$. Suppose now that δ_p is constructed for some $p \ge m$ and let us construct δ_{p+1}. Let $\alpha_{p+1} \in Y_{p+1}$ be an arbitrary lift of δ_p. Once we make this choice, the set of elements $\delta_{p+1} \in Y_{p+1}$ such that $\beta_{p-e} := \pi^Y_{p,p-e}(\delta_p) = \pi^Y_{p+1,p-e}(\delta_{p+1})$ is in bijection, by Corollary 9.3.17, with

$$\mathrm{Hom}_{k[t]/(t^{p-e+1})}(\beta_{p-e}^{*}\Omega_Y, (t^{p-e+1})/(t^{p+2}))$$

(note that the corollary can be applied since $p \geq m \geq 2e$ implies $p+1 \leq 2(p-e+1)$). We need to show that we can choose such δ_{p+1} such that $f_{p+1}(\delta_{p+1}) = f_{p+1}(\gamma'_{p+1})$.

Step 4. Note now that the lift $f_{p+1}(\alpha_{p+1}) \in X_{p+1}$ of $f_{p-e}(\gamma'_{p-e})$ induces by Corollary 9.3.17 a bijection between the set of such lifts and

$$\mathrm{Hom}_{k[t]/(t^{p-e+1})}(\beta_{p-e}^{*}(f^{*}\Omega_X), (t^{p-e+1})/(t^{p+2})).$$

Another such lift is provided by $f_{p+1}(\gamma'_{p+1})$; if via the above bijection this lift corresponds to D, we see that in order to construct δ_{p+1} as desired, we need to show that D lies in the image of the canonical map

$$\begin{aligned}\tau_{p+1} \colon \mathrm{Hom}_{k[t]/(t^{p-e+1})}&(\beta_{p-e}^{*}\Omega_Y, (t^{p-e+1})/(t^{p+2})) \\ &\to \mathrm{Hom}_{k[t]/(t^{p-e+1})}(\beta_{p-e}^{*}(f^{*}\Omega_X), (t^{p-e+1})/(t^{p+2})).\end{aligned}$$

Step 5. On the other hand, since by assumption $f_{p+1}(\alpha_{p+1})$ and $f_{p+1}(\gamma'_{p+1})$ map to the same element in X_p, it follows that the composition $\overline{D}$ of D with the projection $(t^{p-e+1})/(t^{p+2}) \to (t^{p-e+1})/(t^{p+1})$ lies in the image of

$$\begin{aligned}\tau_{p} \colon \mathrm{Hom}_{k[t]/(t^{p-e+1})}&(\beta_{p-e}^{*}\Omega_Y, (t^{p-e+1})/(t^{p+1})) \\ &\to \mathrm{Hom}_{k[t]/(t^{p-e+1})}(\beta_{p-e}^{*}(f^{*}\Omega_X), (t^{p-e+1})/(t^{p+1})).\end{aligned}$$

Therefore in order to complete the proof of the first part of the theorem, it is enough to show that the natural morphism $\mathrm{Coker}(\tau_{p+1}) \to \mathrm{Coker}(\tau_p)$ is an isomorphism.

Step 6. Since $\gamma_m \in \mathrm{Cont}^e(K_{Y/X})_m$ and $m \geq 2e$, we have $\beta_{p-e} \in \mathrm{Cont}^e(K_{Y/X})_{p-e}$. This implies using the structure theorem for modules over a principal ideal domain that if we consider the morphism of free $k[t]/(t^{p-e+1})$-modules of rank n

$$\beta_{p-e}^{*}(f^{*}\Omega_X) \to \beta_{p-e}^{*}\Omega_Y,$$

then we can choose bases such that the morphism is represented by a diagonal matrix with the entries $(t^{a_1}, \ldots, t^{a_n})$, for some nonnegative integers $a_1, \ldots, a_n$ such that $\sum_{i=1}^{n} a_i = e$. We thus conclude that

$$\mathrm{Coker}(\tau_{p+1}) \simeq \mathrm{Coker}(\tau_p) \simeq \oplus_{i=1}^{n}(t^{p-e+1})/(t^{p-e+1+a_i})$$

(the key point is that $p-e+1+a_i \leq p+1$ for all i). This completes the proof of the first part in the theorem.

In order to prove the second assertion in the theorem, we may cover Y by affine open subsets U such that $\Omega_Y|_U \simeq \mathscr{O}_U^n$. It is a consequence of part i) that $U_m \cap \mathrm{Cont}^e(K_{Y/X})_m$ is a union of fibers of f_m. By Lemma 9.3.10 it is thus enough to show that for every such U, the map induced by f_m from $U_m \cap \mathrm{Cont}^e(K_{Y/X})_m$ to its image is a piecewise trivial vibration with fiber $\mathbb{A}^e$.

Following Remark 9.3.20, we have a geometric vector bundle E on Y_{m-e} whose geometric fiber over a jet $\alpha \in Y_{m-e}$ is

$$\mathrm{Hom}_{k[t]/(t^{m-e+1})}(\alpha^*\Omega_Y, (t^{m+1})/(t^{m-e+1})).$$

By the assumption on U, there is a section $\sigma\colon U_{m-e} \to U_m$ of $\pi^U_{m,m-e}$ (we also see that E is trivial on U_{m-e}, but this will not be important). Similarly, we have a vector bundle F on X_{m-e} whose geometric fiber over a jet $\beta \in X_{m-e}$ is

$$\mathrm{Hom}_{k[t]/(t^{m-e+1})}(\beta^*\Omega_X, (t^{m+1})/(t^{m-e+1})).$$

Note that we have a morphism of algebraic varieties $\psi\colon E \to f^*_{m-e}(F)$ that is linear on the fibers over Y_{m-e}, induced by $f^*(\Omega_X) \to \Omega_Y$. We claim that if $\alpha \in \mathrm{Cont}^e(K_{Y/X})_{m-e}$, then the induced linear map between the corresponding fibers $E_{(\alpha)} \to F_{(f_{m-e}(\alpha))}$ has a rank e kernel. Indeed, the map gets identified to the linear map

$$\begin{aligned}\mathrm{Hom}_{k[t]/(t^{m-e+1})}&(\alpha^*\Omega_Y, (t^{m+1})/(t^{m-e+1}))\\ &\to \mathrm{Hom}_{k[t]/(t^{m-e+1})}(\alpha^*(f^*\Omega_X), (t^{m+1})/(t^{m-e+1})).\end{aligned} \tag{10.7}$$

By the structure theorem for modules over a principal ideal domain and the assumption on α, we may choose bases such that the map $\alpha^*(f^*(\Omega_X)) \to \alpha^*(\Omega_Y)$ is given by a diagonal matrix, with entries $t^{a_1}, \ldots, t^{a_n}$, with $\sum_{i=1}^n a_i = e$. In this case, the kernel of the map in (9.7) is given by

$$\oplus_{i=1}^n (t^{m-a_i+1})/(t^{m-e+1}) \simeq \oplus k^{a_i} \simeq k^e.$$

Using the section σ, we define a morphism $E|_{U_{m-e}} \to Y_m$, which is an isomorphism onto U_m. Moreover, using $f_m \circ \sigma$, we define an isomorphism $f^*_{m-e}(F)|_{U_{m-e}} \to U_{m-e} \times_{X_{m-e}} X_m$ such that the diagram

$$\begin{array}{ccc} E|_{U_{m-e}} & \xrightarrow{\psi|_{U_{m-e}}} & f^*_{m-e}(F)|_{U_{m-e}} \\ \downarrow & & \downarrow \\ U_m & \xrightarrow{f_m} & X_m, \end{array}$$

is commutative. It follows from the first assertion in the theorem that if $\gamma_m, \gamma'_m \in \mathrm{Cont}^e(K_{Y/X})_m$ lie in the same fiber of f_m, then they lie in the same fiber of $\pi^Y_{m,m-e}$. Let $W = U_{m-e} \cap \mathrm{Cont}^e(K_{Y/X})_{m-e}$. We this see that after restricting to W, the right vertical map in the above diagram is injective, hence a piecewise isomorphism onto its image by Lemma 9.3.11. As we have seen above, the morphism $\psi|_W$ is a morphism of vector bundles, with kernel having constant rank e, hence it is piecewise trivial onto its image, with fiber $\mathbb{A}^e$. We conclude using Corollary 9.3.12 that the morphism $E|_W \to X_m$ is piecewise trivial, onto its image, with fiber $\mathbb{A}^e$. Since the left vertical map in the above diagram is an isomorphism, it follows that the morphism $W \to X_m$ is piecewise trivial onto its image, with fiber $\mathbb{A}^e$. This completes the proof of the theorem. □

Corollary 10.3.21. *Let $F: Y \to X$ be a proper, birational morphism between smooth varieties and e, m two integers, with $m \geq e$. If $S \subseteq \mathrm{Cont}^e(K_{Y/X})_m$ is a constructible subset and $C = (\pi^Y_{\infty,m})^{-1}(S)$, then $f_\infty(C)$ is a cylinder in X_∞ and $\mathrm{codim}(f_\infty(C)) = \mathrm{codim}(C) + e$. Moreover, if S is a union of fibers of f_m, then $f_\infty(C) = (\pi^X_{\infty,m})^{-1}(f_m(S))$.*

Proof. Let $Z = f(\mathrm{Supp}(K_{Y/X}))$, with the reduced scheme structure. Since $f^{-1}(Z)$ has the same support as $K_{Y/X}$, there is $r \geq 1$ such that $\mathscr{O}_Y(-rK_{Y/X})$ is contained in the ideal defining $f^{-1}(Z)$. Note that for every $p > m$, we may replace S by $(\pi^Y_{p,m})^{-1}(S)$ and therefore we may assume that $m \gg 0$. In particular, we may assume that $m \geq re$. Furthermore, after possibly replacing m by $m+e$, we may assume that $m \geq 2e$ and $S = (\pi^Y_{m,m-e})^{-1}(\pi^Y_{m,m-e}(S))$. In this case S is a union of fibers of f_m. Indeed, if $\alpha \in S$ and $\beta \in Y_m$ are such that $f_m(\alpha) = f_m(\beta)$, then it follows from Theorem 9.3.14 that $\pi^Y_{m,m-e}(\alpha) = \pi^Y_{m,m-e}(\beta)$. Therefore $\beta \in S$.

By Chevalley's theorem, $T := f_m(S) \subseteq X_m$ is constructible. We claim that $f_\infty(C) = (\pi^X_{\infty,m})^{-1}(T)$. The inclusion "$\subseteq$" is trivial. For the reverse one, suppose $\gamma \in (\pi^X_{\infty,m})^{-1}(T)$. We claim that $\gamma \notin Z_\infty$. Indeed, otherwise $\gamma_m := \pi^X_{\infty,m}(\gamma) \in Z_m$. By assumption, we have $\gamma_m = f_m(\delta_m)$ for some $\delta_m \in \mathrm{Cont}^e(K_{Y/X})_m$. Therefore $\delta_m \in f_m^{-1}(Z_m) = (f^{-1}(Z))_m$, which contradicts the fact that $\mathrm{ord}_{f^{-1}(Z)}(\delta_m) \leq \mathrm{ord}_{rK_{Y/X}}(\delta_m) = re$. We conclude that $\gamma \notin Z_\infty$, hence by Proposition 9.2.8 there is $\delta \in Y_\infty$ such that $f_\infty(\delta) = \gamma$. Since S is a union of fibers of f_m and $\pi^X_{\infty,m}(\gamma) \in f_m(S)$, it follows that $\delta \in C$, hence $\gamma \in f_\infty(C)$. This concludes the proof of the equality $f_\infty(C) = (\pi^Y_{\infty,m})^{-1}(T)$, which implies, in particular, that $f_\infty(C)$ is a cylinder. Furthermore, the assertion about codimensions follows from the fact that by Theorem 9.3.14, $S \to f_m(S)$ is piecewise trivial, with fiber $\mathbb{A}^e$. □

Remark 10.3.22. With the notation in the above proof, if the ground field is uncountable, then the equality $f_\infty(C) = (\psi^X_{\infty,m})^{-1}(T)$ follows easily since f_∞ is surjective by Corollary 9.3.9.

Example 10.3.23. Let $X = X(\Delta)$ be a smooth toric variety, where Δ is a fan in $N_{\mathbb{R}}$, for a lattice $N \simeq \mathbb{Z}^n$. We will use the description of the orbits in the arc space of X from Example 9.2.16. Suppose that $v \in |\Delta| \cap N$. We claim that the orbit $T_\infty \cdot \gamma_v$ is a cylinder in X_∞. Indeed, let $\sigma \in \Delta$ be a cone containing v. Since σ is a nonsingular cone, if $\dim(\sigma) = r$, then there is a basis $e_1, \ldots, e_n$ of N, such that $e_1, \ldots, e_r$ are the primitive generators of the rays of σ. If $e_1^*, \ldots, e_n^*$ is the dual basis of the dual lattice and $x_i = \chi^{e_i^*}$, then $U_\sigma \simeq k[x_1, \ldots, x_r, x_{r+1}^{\pm 1}, \ldots, x_n^{\pm 1}]$. If $v = \sum_{i=1}^r a_i v_i$, we see that

$$T_\infty \cdot \gamma_v = \{\gamma \in (U_\sigma)_\infty \mid \mathrm{ord}_t(\gamma^*(x_i)) = a_i \text{ for } 1 \leq i \leq r\}.$$

Therefore $T_\infty \cdot \gamma_v$ is a cylinder in X_∞ of codimension $\sum_{i=1}^r a_i$. Recall that on X we have the *canonical divisor* $K_X = -\sum_i D_i$, where the D_i are the invariant prime divisors on X (for this and the other facts about toric varieties, we refer to [Ful93]). This corresponds to a piecewise linear function ϕ_{K_X} on $|\Delta|$ that takes value 1 on each primitive ray generator. We thus see that

$$\operatorname{codim}(T_\infty \cdot \gamma_v) = \phi_{K_X}(v).$$

Suppose now that $f\colon X' \to X$ is a birational toric morphism corresponding to the identity on the lattice N. Given a lattice point v in the support of the fan Δ' of X', the morphism $f_\infty\colon X'_\infty \to X_\infty$ induces a bijection between the orbits $O' = T_\infty \cdot \gamma_v \subseteq X'_\infty$ and $O = T_\infty \cdot \gamma_v \subseteq X_\infty$. It follows from the previous discussion that

$$\operatorname{codim}(O) - \operatorname{codim}(O') = \phi_{K_X}(v) - \phi_{K_{X'}}(v) = -\phi_{K_{X'/X}}(v),$$

where $\phi_{K_{X'/X}}$ is the piecewise linear function on $|\Delta'|$ corresponding to the divisor $K_{X'/X} = K_{X'} - f^*(K_X)$. Note that this is compatible with Corollary 9.3.21, since for every $\gamma \in O'$, we have $\operatorname{ord}_{K_{X'/X}}(\gamma) = \operatorname{ord}_{K_{X'/X}}(\gamma_v) = -\phi_{K_{X'/X}}(v)$.

Corollary 10.3.24. *If $f\colon Y \to X$ is a proper, birational morphism between smooth varieties, then for every cylinder $C \subseteq Y_\infty$, the closure $C' := \overline{f_\infty(C)}$ is a cylinder. Moreover, if C is irreducible, then*

$$\operatorname{codim}(C') = \operatorname{codim}(C) + \min\{m \mid C \cap \operatorname{Cont}^m(K_{Y/X}) \neq \emptyset\}$$

and $f_\infty(C)$ contains a nonempty open subcylinder of $\overline{f_\infty(C)}$.

Proof. By Proposition 9.3.2, C has finitely many irreducible components, say $C_1, \ldots, C_r$, and each of these is a cylinder. Since $\overline{f_\infty(C)} = \cup_{i=1}^r \overline{f_\infty(C_i)}$, we see that it is enough to prove the corollary when C is irreducible.

By Lemma 9.3.4, we have $e := \min\{m \mid C \cap \operatorname{Cont}^m(K_{Y/X}) \neq \emptyset\} < \infty$. Note that $C_0 := C \cap \operatorname{Cont}^e(K_{Y/X})$ is an open dense subcylinder of C, hence $\overline{f_\infty(C)} = \overline{f_\infty(C_0)}$. On the other hand, $f_\infty(C_0)$ is a cylinder by Corollary 9.3.21 and its codimension is $\operatorname{codim}(C) + e$. It follows from Proposition 9.3.21 that $\overline{f_\infty(C_0)}$ is a cylinder, as well, of the same codimension with $f_\infty(C_0)$. For the last assertion, note that if we write $C_0 = (\pi^Y_{\infty,m})^{-1}(S)$, with $m \gg 0$, then it follows from Corollary 9.3.21 that $f_\infty(C_0) = (\pi^X_{\infty,m})^{-1}(f_m(S))$. Since $f_m(S)$ contains an open subset of $\overline{f_m(S)}$, we deduce that $f_\infty(C)$ contains an open subcylinder of $\overline{f_\infty(C)}$. $\square$

Proposition 10.3.25. *If $f\colon Y \to X$ is a proper, birational morphism between smooth varieties, then for every irreducible closed cylinder $C \subseteq X_\infty$, there is a unique irreducible closed cylinder $C_Y \subseteq Y_\infty$ such that $C = \overline{f(C_Y)}$.*

Proof. Note that $T := f_\infty^{-1}(C)$ is a closed cylinder in Y_∞. Furthermore, if Z is a proper closed subset of X such that f is an isomorphism over $X \smallsetminus Z$, then by Proposition 9.2.8, f_∞ is bijective over $X_\infty \smallsetminus Z_\infty$. If $T_1, \ldots, T_r$ are the irreducible components of T, then

$$C = \overline{C \smallsetminus Z_\infty} = \overline{f(T_1)} \cup \ldots \cup \overline{f(T_r)}.$$

Since C is irreducible, it follows that there is i such that $\overline{f(T_i)} = C$. We may and do take $C_Y = T_i$.

Suppose now that $C'_Y \neq C_Y$ is another irreducible closed cylinder in $Y\infty$ such that $\overline{f_\infty(C'_Y)} = C$. We assume, for example, that $C_Y \not\subseteq C'_Y$. Applying the last assertion in Corollary 9.3.24 for $C_Y \smallsetminus C'_Y$ and C'_Y, we deduce that there are nonempty open subcylinders V_1 and V_2 in C that are contained in $f_\infty(C_Y \smallsetminus C'_Y)$ and $f_\infty(C'_Y)$, respectively. Furthermore, we may assume that $V_1, V_2 \subseteq X_\infty \smallsetminus Z_\infty$, hence $V_1 \cap V_2 = \emptyset$. This contradicts the fact that V_1 and V_2 are open in C and C is irreducible.

□

10.4 First applications: classical and stringy E-functions

Our first goal is to explain Kontsevich's result saying that any two birational Calabi-Yau varieties have the same Hodge numbers. More generally, any two K-equivalent smooth projective varieties have the same Hodge numbers. The proof is an easy consequence of the formalism of Hodge-Deligne polynomials and of the birational transformation formula proved in the previous section. Later in this section we define following [Bat98] the stringy Hodge-Deligne polynomial of a klt pair (X,D) in terms of a log resolution. Another application of the birational transformation formula gives the independence of the chosen log resolution. In this section we work over the field $\mathbb{C}$ of complex numbers.

10.4.1 The Hodge-Deligne polynomial

We start with a review of the Hodge-Deligne polynomial. If X is a smooth, projective, complex algebraic variety, its Hodge polynomial is given by

$$E(X) = E(X;u,v) := \sum_{p,q=0}^{\dim(X)} (-1)^{p+q} h^{p,q}(X) u^p v^q \in \mathbb{Z}[u,v],$$

with $h^{p,q}(X) = h^q(X, \Omega_X^p)$. It is a consequence of Hodge theory (see Corollary 2.1.15) that $\dim_{\mathbb{C}} H^i(X^{\mathrm{an}}, \mathbb{C}) = \sum_{p+q=i} h^{p,q}(X)$. This implies that $E(X;t,t)$ is the Poincaré polynomial:

$$E(X;t,t) = \sum_{i=0}^{2\dim(X)} (-1)^i \dim_C H^i(X^{\mathrm{an}}, \mathbb{C}) t^i.$$

In particular, we have

$$E(X;1,1) = \chi^{\mathrm{top}}(X^{\mathrm{an}}) := \sum_{i\geq 0} (-1)^i \dim_{\mathbb{C}} H^i(X^{\mathrm{an}}, \mathbb{C}).$$

The Hodge polynomial can be extended to arbitrary schemes of finite type over $\mathbb{C}$. More precisely, to every such scheme X one can associate a polynomial $E(X) = E(X;u,v) \in \mathbb{Z}[u,v]$ such that

1) $E(X) = E(Y)$ if X and Y are isomorphic.
2) $E(X) = E(X_{\mathrm{red}})$.
3) If Y is a closed subscheme of X, then
$$E(X) = E(Y) + E(X \smallsetminus Y).$$
4) If X is a smooth, projective variety, then $E(X)$ is the Hodge polynomial of X.

This invariant is called the *Hodge-Deligne polynomial*.

Remark 10.4.1. There is at most one invariant that satisfies conditions 1)-4) above. Indeed, we argue by induction on $n = \dim(X)$. By 2), it is enough to only consider reduced schemes. If $n = 0$, then X is a disjoint union of points, and 3) and 4) above imply $E(X)$ is the number of points of X. Suppose now that $E(Y;u,v)$ is determined when $\dim(Y) \leq n-1$. If $X_1,\ldots,X_r$ are the irreducible components of X and $Z = X_2 \cup \ldots \cup X_r$, then $E(X) = E(Z) + E(X \smallsetminus Z)$. Arguing by induction on the number of irreducible components, we see that it is enough to consider the case when X is irreducible. Furthermore, we may assume that X is affine: indeed, if U is an affine open subset of X, then $E(X) = E(U) + E(X \smallsetminus U)$ and $\dim(X \smallsetminus U) < n$.

Suppose now that X is an irreducible affine variety. We can embed X as a dense open subset of a projective variety W. Since $E(X) = E(W) - E(W \smallsetminus X)$ and $\dim(W \smallsetminus X) < n$, it follows that it is enough to determine $E(W)$. Let us consider now a projective morphism $f\colon W' \to W$ that gives a resolution of singularities of W. Therefore W' is smooth and irreducible, hence $E(W')$ is the Hodge polynomial of W'. On the other hand, if Y is a proper closed subset of W such that f is an isomorphism over $W \smallsetminus Y$, then

$$E(W') - E(f^{-1}(Y)) = E(f^{-1}(W \smallsetminus Y)) = E(W \smallsetminus Y) = E(W) - E(Y). \tag{10.8}$$

Since $\dim(Y) < n$ and $\dim(f^{-1}(Y)) < n$, it follows that $E(W)$ is determined by (9.8).

Remark 10.4.2. It is clear that if X is a smooth n-dimensional projective variety, then $h^{p,q}(X) = 0$, unless $p,q \leq n$. In particular, $E(X;u,v)$ has degree $\leq n$ with respect to each of u and v. Since $h^{n,n}(X) = h^n(X,\omega_X) = h^0(X,\mathscr{O}_X) = 1$, it follows that the total degree of the Hodge polynomial of X is $2n$ and the only term of total degree $2n$ is $(uv)^n$. By running the argument in Remark 9.4.1, we see that for every scheme X of finite type over $\mathbb{C}$, the polynomial $E(X;u,v)$ has degree d with respect to each of u and v, where $d = \dim(X)$. Moreover, its total degree is $2d$ and the only term of total degree $2d$ is $a(X)(uv)^d$, where $a(X)$ is the number of irreducible components of X of maximal dimension.

Remark 10.4.3. It is a consequence of Hodge theory that if X is a smooth, projective variety, then $h^{p,q}(X) = h^{q,p}(X)$ for all p and q. The argument in Remark 9.4.1 then implies that for every reduced scheme X of finite type over $\mathbb{C}$, the Hodge-Deligne polynomial satisfies $E(X;u,v) = E(X;v,u)$.

Remark 10.4.4. For every scheme X of finite type over $\mathbb{C}$, we have $E(X;1,1) = \chi^{\mathrm{top}}(X)$. We have seen that this is a consequence of the Hodge decomposition when X is smooth and projective. For the general case, a key point is that for algebraic varieties, the Euler-Poincaré characteristic is equal to the Euler-Poincaré characteristic with compact support:

$$\chi^{\mathrm{top}}(X) = \chi_c^{\mathrm{top}}(X) := \sum_{i=0}^{2\dim(X)} (-1)^i \dim_{\mathbb{C}} H_c^i(X^{\mathrm{an}}, \mathbb{C})$$

(see [Ful93, p. 141-142]). If Y is a closed subset of X, then we have the long exact sequence for cohomology with compact support

$$\ldots \to H_c^i(X \smallsetminus Y, \mathbb{C}) \to H_c^i(X, \mathbb{C}) \to H_c^i(Y, \mathbb{C}) \to H_c^{i+1}(X \smallsetminus Y, \mathbb{C}) \to \ldots,$$

hence we obtain $\chi_c^{\mathrm{top}}(X) = \chi_c^{\mathrm{top}}(Y) + \chi_c^{\mathrm{top}}(X \smallsetminus Y)$. Since we also have $E(X;1,1) = E(Y;1,1) + E(X \smallsetminus Y;1,1)$ and $\chi_c^{\mathrm{top}}(-)$ agrees with $E(-;1,1)$ on smooth projective varieties, the argument in Remark 9.4.1 implies that the two invariants agree for all X.

Remark 10.4.5. The existence of the Hodge-Deligne polynomial follows from the existence of a mixed Hodge structure on the cohomology with compact support of a scheme of finite type[1] over $\mathbb{C}$ (see [Del74]). More precisely, for such X, the $\mathbb{Q}$-vector space $H_c^i(X, \mathbb{Q})$ carries a finite increasing filtration $W_\bullet$ and $H_c^i(X, \mathbb{C})$ carries a finite decreasing filtration $F^\bullet$ such that for every i, the induced F-filtration on W_i/W_{i-1} makes it a pure Hodge structure of weight i. In this case, if $d = \dim(X)$, one puts

$$E(X;u,v) := \sum_{p,q=0}^{d} \left(\sum_{i=0}^{2d} (-1)^i \dim_{\mathbb{C}} \mathrm{Gr}_F^p \mathrm{Gr}_{p+q}^W H_c^i(X^{\mathrm{an}}, \mathbb{C}) \right) u^p v^q.$$

Note that when X is a smooth projective variety, then $\mathrm{Gr}_m^W H^i(X^{\mathrm{an}}, \mathbb{C}) = 0$ unless $m = i$ and $\dim_{\mathbb{C}} \mathrm{Gr}_F^p H^i(X^{\mathrm{an}}, \mathbb{C}) = h^{p,i-p}(X)$. Therefore in this case the above definition recovers the Hodge polynomial of X. In order to check the additivity property, one uses the fact that if Y is a closed subscheme of X and $U = X \smallsetminus Y$, then we have a long exact sequence

$$\ldots \to H_c^i(U^{\mathrm{an}}, \mathbb{Q}) \to H_c^i(X^{\mathrm{an}}, \mathbb{Q}) \to H_c^i(Y^{\mathrm{an}}, \mathbb{Q}) \to H_c^{i+1}(U^{\mathrm{an}}, \mathbb{Q}) \to \ldots.$$

Furthermore, this satisfies a suitable strictness property with respect to the two filtrations, which implies that for every p and q one gets a long exact sequence

$$\ldots \to \mathrm{Gr}_F^p \mathrm{Gr}_{p+q}^W H_c^i(U^{\mathrm{an}}; \mathbb{C}) \to \mathrm{Gr}_F^p \mathrm{Gr}_{p+q}^W H_c^i(X^{\mathrm{an}}; \mathbb{C}) \to \mathrm{Gr}_F^p \mathrm{Gr}_{p+q}^W H_c^i(Y^{\mathrm{an}}; \mathbb{C}) \to \ldots.$$

[1] Note that the cohomology groups only depend on the reduced scheme structure on X.

This immediately implies $E(X) = E(Y) + E(U)$.

There is another approach to proving the existence of the Hodge-Deligne polynomial. This makes use of a result of Bittner, whose proof relies on the weak factorization theorem (see Remark 9.5.13 below). The advantage of that approach is that it applies directly to any algebraically closed field of characteristic 0.

Proposition 10.4.6. *The Hodge-Deligne polynomial is multiplicative in the following sense: if X and Y are two schemes of finite type over $\mathbb{C}$, then $E(X \times Y) = E(X) \cdot E(Y)$.*

Proof. Indeed, arguing as in the proof of Remark 9.4.1, we see that it is enough to check the assertion when X and Y are smooth projective varieties[2]. In this case, if $\pi_1 \colon X \times Y \to X$ and $\pi_2 \colon X \times Y \to Y$ are the projections, then $\Omega_{X\times Y} \simeq \pi_1^*(\Omega_X) \oplus \pi_2^*(\Omega_Y)$. Therefore $\Omega^p_{X\times Y} \simeq \oplus_{i+j=p}(\pi_1^*(\Omega^i_X) \otimes \pi_2^*(\Omega^j_Y))$ and the Künneth formula implies

$$h^{p,q}(X \times Y) = \sum_{i+j=p} h^q(X \times Y, \pi_1^*(\Omega^i_X) \otimes \pi_2^*(\Omega^j_Y))$$

$$= \sum_{i+j=p} \sum_{a+b=q} \dim_C H^a(X, \Omega^i_X) \otimes H^b(Y, \Omega^j_Y) = \sum_{i+j=p} \sum_{a+b=q} h^{i,a}(X) \cdot h^{j,b}(Y).$$

This gives $E(X \times Y) = E(X) \cdot E(Y)$. □

Example 10.4.7. For every smooth projective curve C of genus g, the Hodge polynomial of C is given by $E(C) = 1 + g(u+v) + uv$. In particular, $E(\mathbb{P}^1) = 1 + uv$. Since the Hodge polynomial of a point is $E(\operatorname{Spec}\mathbb{C}) = 1$, we conclude that $E(\mathbb{A}^1) = uv$. It follows from Remark 9.4.6 that $E(\mathbb{A}^n) = (uv)^n$, and using the decomposition $\mathbb{P}^n = \mathbb{P}^{n-1} \sqcup \mathbb{A}^n$, we see by induction on n that

$$E(\mathbb{P}^n) = 1 + uv + \ldots + (uv)^n.$$

Let X be a scheme of finite type over $\mathbb{C}$ and A a constructible subset of X. We can write $A = A_1 \sqcup \ldots \sqcup A_r$ as a disjoint union of locally closed subsets. We define $E(A;u,v) := \sum_{i=1}^r E(A_i;u,v)$. One can check that this is independent of the choice of decomposition. Furthermore, if $W_1, \ldots, W_m$ are constructible subsets of X that are mutually disjoint, then $E(W_1 \cup \ldots \cup W_m) = \sum_{i=1}^m E(W_i)$. We leave these assertions as an exercise for the reader. We will prove a more general statement in Proposition 9.5.5 below.

Example 10.4.8. Suppose that $f \colon X \to Y$ is a morphism of schemes of finite type over $\mathbb{C}$ inducing a map $g \colon A \to B$, where $A \subseteq X$ and $B \subseteq Y$ are constructible. If g is piecewise trivial, with fiber F, then $E(A) = E(B) \cdot E(F)$. Indeed, it follows from definition that we can write $B = B_1 \sqcup \ldots \sqcup B_r$, with B_i locally closed in Y and $g^{-1}(B_i)$ locally closed in X, such that $g^{-1}(B_i) \simeq B_i \times F$. In this case, using Remark 9.4.6, we obtain

[2] For a more formal argument, see Remark 9.5.10 below

$$E(A) = \sum_{i=1}^{r} E(g^{-1}(B_i)) = \sum_{i=1}^{r} E(B_i) \cdot E(F) = E(B) \cdot E(F).$$

Example 10.4.9. If $f\colon V \to X$ is a geometric vector bundle of rank r, then $E(V) = E(X) \cdot (uv)^r$. If $\mathbb{P}(V) \to X$ is the corresponding projective bundle, then $E(\mathbb{P}(V)) = E(X) \cdot (1 + uv + \ldots + (uv)^{r-1})$.

10.4.2 Hodge numbers of K-equivalent varieties

Our goal in this section is to prove the following result of Kontsevich [Kon]. Recall that a *Calabi-Yau variety* is a smooth projective variety X such that $\omega_X \simeq \mathscr{O}_X$ (one sometimes adds other conditions, such as simply-connectedness or the vanishing of certain Hodge numbers, but we we will not need these conditions).

Theorem 10.4.10. *If X and Y are birational complex Calabi-Yau varieties, then $h^{p,q}(X) = h^{p,q}(Y)$ for every p and q.*

In fact, the theorem has a more precise form, involving K-equivalent varieties. Suppose that X and Y are two complete, birational $\mathbb{Q}$-Gorenstein varieties over an algebraically closed field k of characteristic 0. Since X and Y are birational, we can find a smooth variety W, having proper, birational morphisms $f\colon W \to X$ and $g\colon W \to Y$. Indeed, by assumption, we have a rational map $\phi\colon X \dashrightarrow Y$ and we can find a birational morphism $W' \to X$ such that the composition $W' \to X \dashrightarrow Y$ is a morphism. It is then enough to take W to be a resolution of singularities of W'. Given such W, one says that X and Y are *K-equivalent* if $K_{W/X} = K_{W/Y}$. Note that the definition is independent of the choice of W. Indeed, given another smooth variety W_1 with proper birational morphisms $f_1\colon W_1 \to X$ and $g_1\colon W_1 \to Y$, we can find a smooth variety Z, with proper, birational morphisms $p\colon Z \to W$ and $p_1\colon Z \to W_1$ such that $f \circ p = f_1 \circ p_1$ and $g \circ p = g_1 \circ p_1$ (one can simply run the previous argument for the birational map $W \dashrightarrow W_1$). By symmetry, it is enough to compare the condition in terms of Z with the condition in terms of W. By Remark 3.1.8, we have

$$K_{Z/X} = K_{Z/W} + p^*(K_{W/X}) \text{ and } K_{Z/Y} = K_{Z/W} + p^*(K_{W/Y}).$$

We see that $K_{Z/X} = K_{Z/Y}$ if and only if $K_{W/X} = K_{W/Y}$ (note that if D_1 and D_2 are divisors on W such that $p^*(D_1) = p^*(D_2)$, then $D_1 = p_*(p^*(D_1)) = p_*(p^*(D_2)) = D_2$).

Lemma 10.4.11. *Suppose that X and Y are complete normal varieties, with canonical singularities. If $f\colon W \to X$ and $g\colon W \to Y$ are proper, birational morphisms, with W smooth, then in order for X and Y to be K-equivalent, it is enough to have $K_{W/X}$ and $K_{W/Y}$ linearly equivalent.*

Proof. Let L be the common field of rational functions of X, Y, and W. By assumption, there is a nonzero $\phi \in L$ and a positive integer m such that $\mathrm{div}_W(\phi) = m(K_{W/X} - K_{W/Y})$. Since $K_{W/Y}$ is g-exceptional, it follows that

$$\mathrm{div}_Y(\phi) = g_*(\mathrm{div}_W(\phi)) = g_*(mK_{W/X} - mK_{W/Y}) = g_*(mK_{W/X}),$$

hence this is effective. Therefore $\phi \in \mathscr{O}_Y(Y) = \mathscr{O}_W(W)$, which implies that $K_{W/X} - K_{W/Y}$ is effective. Applying f_*, we obtain that $K_{W/Y} - K_{W/X}$ is effective, and by putting these together, we obtain $K_{W/X} = K_{W/Y}$. □

Remark 10.4.12. In fact, with the notation in the lemma, one can easily check using Corollary 1.6.36 that in order to obtain the K-equivalence of X and Y, it is enough to assume that $K_{W/X}$ and $K_{W/Y}$ are numerically equivalent.

Corollary 10.4.13. *Any two birational Calabi-Yau varieties are K-equivalent.*

Proof. Let W be a smooth variety, having proper, birational morphisms $f\colon W \to X$ and $g\colon W \to Y$. In this case we have $f^*(\omega_X) \simeq \mathscr{O}_W \simeq g^*(\omega_Y)$. Since $\mathscr{O}_W(K_{W/X}) \simeq \omega_W \otimes f^*(\omega_X^{-1})$ and $\mathscr{O}_W(K_{W/Y}) \simeq \omega_W \otimes g^*(\omega_Y^{-1})$, we conclude that $K_{W/X}$ and $K_{W/Y}$ are linearly equivalent. Lemma 9.4.11 implies that X and Y are K-equivalent. □

It follows from Corollary 9.4.13 that Theorem 9.4.10 is a special case of the following more general version, also due to Kontsevich.

Theorem 10.4.14. *If X and Y are smooth, complete, K-equivalent complex varieties, then $h^{p,q}(X) = h^{p,q}(Y)$ for every p and q.*

Before giving the proof of this theorem, we need some preparations, extending the definition of the Hodge-Deligne polynomial to cylinders in the arc space of a smooth variety. Let X be a smooth, n-dimensional, complex algebraic variety and let $C \subseteq X_\infty$ be a cylinder. If $C = \pi_{\infty,m}^{-1}(S)$, for a constructible subset $S \subseteq X_m$, we put

$$E(C;u,v) = E(S;u,v)\cdot(uv)^{-mn} \in \mathbb{Z}[u,v,u^{-1},v^{-1}].$$

Note that this is independent of the representation of C: if $p > m$ and we write $C = \pi_{\infty,p}^{-1}(T)$, then $T = \pi_{p,m}^{-1}(S) \to S$ is piecewise trivial with fiber $\mathbb{A}^n$, hence $E(T) = E(S)\cdot(uv)^{(p-m)n}$ by Example 9.4.8. It is clear from definition that if $S \subseteq X$, then $E(\pi_\infty^{-1}(S)) = E(S)$. In particular, $E(X_\infty) = E(X)$.

Lemma 10.4.15. *With the above notation, the following hold:*

i) If $C_1,\ldots,C_r$ are mutually disjoint cylinders in X_∞, then

$$E(C_1 \cup \ldots \cup C_r) = \sum_{i=1}^{r} E(C_i).$$

ii) For every cylinder C, every monomial u^iv^j that appears in $E(C)$ with nonzero coefficient satisfies $i,j \le n - \mathrm{codim}(C)$.

Proof. For i), note that we can find m such that $C_i = \pi_{\infty,m}^{-1}(S_i)$ for $1 \le i \le r$, hence the assertion follows from the additivity of the Hodge-Deligne polynomial on the constructible subsets of X_m. The upper bounds in ii) follow from the definition of $E(C)$ and Remark 9.4.2. □

The new phenomenon in the setting of cylinders is that we might have a cylinder C and a countable family of pairwise disjoint subcylinders $C_m \subseteq C$ such that $C \smallsetminus \bigcup_m C_m$ is "small", in a suitable sense. We want to assert that in this case $E(C) = \sum_m E(C_m)$. In order to make sense of this, we need to work in a completion of $\mathbb{Z}[u^{\pm 1}, v^{\pm 1}]$. The easiest approach is to consider

$$\widetilde{T} := \mathbb{Z}[\![u^{-1}, v^{-1}]\!][u,v].$$

We consider on $\widetilde{T}$ the linear topology in which a basis of open neighborhoods of 0 is given by the subgroups $\{(uv)^N \mathbb{Z}[u^{-1}, v^{-1}] \mid N \in \mathbb{Z}\}$. Therefore a sequence $(a_m)_{m\geq 1}$ in $\widetilde{T}$ has the property that $\lim_{m\to\infty} a_m = a$, for some $a \in \widetilde{T}$, if and only if for every $M > 0$, there is m_0 such that for all $m \geq m_0$, we have $a_m - a = \sum_{p,q \leq -M} \alpha_{p,q} u^p v^q$. It is clear that $\widetilde{T}$ is complete. Another fact that we will use, which holds in every abelian group endowed with a linear topology, is that the convergence of a series $\sum_{m\geq 1} a_m$ and its sum, assuming convergence, are independent of the order. We may thus consider series in $\widetilde{T}$ indexed by arbitrary countable sets.

Example 10.4.16. Suppose that $(C_m)_{m\geq 1}$ is a sequence of cylinders in X_∞. We have seen in Lemma 9.4.15 that all monomials $u^i v^j$ that appear in $E(C_m)$ with nonzero coefficient satisfy $i, j \leq n - \mathrm{codim}(C_m)$ and $(uv)^{n-\mathrm{codim}(C_m)}$ is one such monomial. It this follows from definition that $\lim_{m\to\infty} E(C_m) = 0$ in $\widetilde{T}$ if and only if $\lim_{m\to\infty} \mathrm{codim}(C_m) = \infty$.

Lemma 10.4.17. *Let X be a smooth variety and C a cylinder in X_∞. If $(C_m)_{m\geq 1}$ is a sequence of pairwise disjoint subcylinders of C, such that*

$$\lim_{m\to\infty} \mathrm{codim}(C \smallsetminus (C_1 \cup \ldots \cup C_m)) = \infty,$$

then $E(C) = \sum_{m\geq 1} E(C_m)$.

Proof. Since the C_i are pairwise disjoint cylinders contained in C, it is clear that

$$E(C) - \sum_{\ell=1}^{m} E(C_\ell) = E(C \smallsetminus (C_1 \cup \ldots \cup C_m)).$$

It follows from our assumption and Example 9.4.16 that

$$\lim_{m\to\infty} \left(E(C) - \sum_{\ell=1}^{m} E(C_\ell) \right) = 0,$$

which implies the assertion in the lemma. □

Corollary 10.4.18. *Let X be a smooth variety and C a cylinder in X_∞. If $(C_m)_{m\geq 1}$ is a sequence of pairwise disjoint subcylinders of C, such that there is a proper closed subscheme Y of X and a function $\nu \colon \mathbb{Z}_{>0} \to \mathbb{Z}_{\geq 0}$ with $\lim_{m\to\infty} \nu(m) = \infty$ such that*

$$C \smallsetminus (C_1 \cup \ldots \cup C_m) \subseteq \mathrm{Cont}^{\geq \nu(m)}(Y),$$

then $E(C) = \sum_{m \geq 1} E(C_m)$.

Proof. The assertion follows by combining Lemma 9.4.17 and Proposition 9.3.3. □

Corollary 10.4.19. *Let* $f\colon W \to X$ *be a proper, birational morphism between smooth varieties. If* $R \subseteq X_\infty$ *is a cylinder and* $C = f_\infty^{-1}(R)$*, then*

$$E(R) = \sum_{e \geq 0} E(C \cap \mathrm{Cont}^e(K_{W/X})) \cdot (uv)^{-e}.$$

Proof. It follows from Corollary 9.3.21 that each $f_\infty(\mathrm{Cont}^e(K_{W/X}))$ is a cylinder in X_∞. Note also that these cylinders are pairwise disjoint, since f_∞ is injective on $W_\infty \smallsetminus (K_{W/X})_\infty$ by Proposition 9.2.8. In order to apply Corollary 9.4.18, let $Z = f(\mathrm{Supp}(K_{W/X}))$, with the reduced scheme structure. Since $f^{-1}(Z)$ and $K_{W/X}$ have the same support, there is $\ell > 0$ such that the ℓ^{th} power of the ideal defining $f^{-1}(Z)$ is contained in $\mathscr{O}_W(-K_{W/X})$. We have

$$X_\infty \smallsetminus \bigcup_{e=0}^{j} f_\infty(\mathrm{Cont}^e(K_{W/X})) \subseteq Z_\infty \cup f_\infty(\mathrm{Cont}^{\geq (j+1)}(K_{W/X})) \subseteq \mathrm{Cont}^{\geq \lceil (j+1)/\ell \rceil}(Z).$$

We can thus apply Corollary 9.4.18 to conclude that

$$E(R) = \sum_{e \geq 0} E(f_\infty(\mathrm{Cont}^e(K_{W/X})) \cap R). \tag{10.9}$$

On the other hand, we have by Corollary 9.3.21

$$E(f_\infty(\mathrm{Cont}^e(K_{W/X})) \cap R) = E(f_\infty(C \cap \mathrm{Cont}^e(K_{W/X})) = E(C \cap \mathrm{Cont}^e(K_{W/X})) \cdot (uv)^{-e}. \tag{10.10}$$

The assertion in the corollary follows by combining (9.9) and (9.10). □

Kontsevich's theorem is an easy consequence of Corollary 9.4.19.

Proof of Theorem 9.4.14. Let W be a smooth variety, with proper, birational morphisms $f\colon W \to X$ and $g\colon W \to Y$. By applying Corollary 9.4.19 with $R = X_\infty$, we obtain

$$E(X) = \sum_{e \geq 0} E(\mathrm{Cont}^e(K_{W/X})) \cdot (uv)^{-e}. \tag{10.11}$$

Applying the same argument for $g\colon W \to Y$, we obtain

$$E(Y) = \sum_{e \geq 0} E(\mathrm{Cont}^e(K_{W/Y})) \cdot (uv)^{-e}.$$

Since $K_{W/X} = K_{W/Y}$, by assumption, we conclude that X and Y have the same Hodge polynomials. □

10.4.3 Stringy E-functions

In this section we introduce following [Bat98] a variant of the Hodge-Deligne polynomial for certain singular varieties, that behaves well with respect to birational morphisms. However, in general this is not a polynomial. We define it as a formal power series, and then show that it is a rational function. In the process of doing this, we define the (Hodge realizations of) motivic integerals of certain functions defined on the space of arcs.

Let us first consider this in a simple case. For the various notions of singularities of pairs that we use in this section, see Section 3.1. Suppose that Y is a variety that has canonical singularities and is 1-Gorenstein, that is, K_Y is a Cartier divisor. If $f\colon X \to Y$ is a resolution of singularities, then by assumption $K_{X/Y}$ is an effective divisor. Motivated by formula (9.11) in the proof of Theorem 9.4.14, we put

$$E_{\rm st}(Y;u,v) := \sum_{e\geq 0} E(\mathrm{Cont}^e(K_{X/Y});u,v)\cdot(uv)^{-e} \in \widetilde{T}. \tag{10.12}$$

Since $E(\mathrm{Cont}^e(K_{X/Y});u,v)\cdot(uv)^{-e}$ has degree in each of u and v bounded above by $n-\mathrm{codim}(\mathrm{Cont}^e(K_{X/Y}))-e$, it follows from Proposition 9.3.3 that $E_{\rm st}(Y;u,v)$ is well-defined. Of course, one needs to show that the definition is independent of resolution, but we will do this in a more general setting later.

We generalize this in two ways. First, it is convenient to drop the assumption that X is 1-Gorenstein and only assume that K_Y is $\mathbb{Q}$-Cartier. Furthermore, it is natural to work with pairs (Y,D), where D is a $\mathbb{Q}$-divisor on Y such that K_Y+D is $\mathbb{Q}$-Cartier. Instead of requiring canonical singularities, it will turn out to be enough to require that the pair (Y,D) is klt. However, in this case we need to treat contact loci of possibly non-effective, which are not cylinders. In order to treat these sets, we may the following rather ad-hoc definition.

Definition 10.4.20. A subset $C\subseteq X_\infty$ is a *limit of cylinders* if there is a sequence of pairwise disjoint cylinders $(C_m)_{m\geq 1}$, a proper closed subscheme Y of X, and a function $\nu\colon \mathbb{Z}_{>0}\to\mathbb{Z}_{\geq 0}$ such that

i) $C=\bigsqcup_{m\geq 1} C_m$,
ii) $\bigcup_{i\geq m} C_i \subseteq \mathrm{Cont}^{\geq \nu(m)}(Y)$ for all m, and
iii) $\lim_{m\to\infty}\nu(m)=\infty$.

Given C and a sequence of cylinders $(C_m)_{m\geq 1}$ as above, we see that if $\nu(m)\geq N$ for all $m\geq m_0$, then $C_{m+1}\cup\ldots\cup C_{m+p}\subseteq \mathrm{Cont}^{\geq N}(Y)$ for all $m\geq m_0$ and $p\geq 1$. Since $\lim_{m\to\infty}\nu(m)=\infty$, it follows from Lemma 9.3.3 that the series $\sum_{m\geq 1}E(C_m)$ is Cauchy, hence convergent in $\widetilde{T}$. We denote its sum by $E(C)$.

Remark 10.4.21. Note that if $C\subseteq X_\infty$ and $(C_m)_{m\geq 1}$ is a sequence of pairwise disjoint cylinders whose union is equal to C, then for a proper closed subscheme Y of X there is a function ν that satisfies ii) and iii) above if and only if for every N, the contact locus $\mathrm{Cont}^{\geq N}(Y)$ contains all but finitely many C_m.

Lemma 10.4.22. *If $C \subseteq X_\infty$ is a limit of cylinders, then $E(C)$ is independent on the choice of the sequence $(C_m)_{m\geq 1}$.*

Proof. Suppose that $(C'_m)_{m\geq 1}$ is another sequence of pairwise disjoint cylinders such that $C = \bigsqcup_{m\geq 1} C'_m$ and that Z and $\mu\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{\geq 0}$ satisfy conditions ii) and iii) in Definition 9.4.20. In this case, for every m we can apply Corollary 9.4.18 to the cylinder C_m and the sequence of subcylinders $(C_m \cap C'_i)_{i\geq 1}$. We thus obtain $E(C_m) = \sum_{i\geq 1} E(C_m \cap C'_i)$ and summing over m, we deduce

$$\sum_{m\geq 1} E(C_m) = \sum_{i,m\geq 1} E(C_m \cap C'_i).$$

Reversing the roles of the two sequences, we also obtain

$$\sum_{i\geq 1} E(C'_i) = \sum_{i,m\geq 1} E(C_m \cap C'_i).$$

Therefore $\sum_{m\geq 1} E(C_m) = \sum_{i\geq 1} E(C'_i)$. □

Remark 10.4.23. Of course, if C is a cylinder, then it is a limit of cylinders, and the new definition of $E(C)$ agrees with the old one.

In the setting that we are interested in, the divisors have rational coefficients, hence we will need to work in a suitable extension of $\widetilde{T}$. Given a positive integer ℓ, let us consider the extension

$$\widetilde{T}^{(\ell)} := \mathbb{Z}[\![u^{-1/\ell}, v^{-1/\ell}]\!][u^{1/\ell}, v^{1/\ell}] \simeq \widetilde{T}[y]/(y^\ell - uv)$$

of $\widetilde{T}$. Note that $\widetilde{T}^{(\ell)}$ and $\widetilde{T}$ are abstractly isomorphic, and we put the topology on $\widetilde{T}^{(\ell)}$ that makes them homeomorphic. With respect to this topology, $\widetilde{T}$ is a closed subspace of $\widetilde{T}^{(\ell)}$.

Suppose now that $\phi\colon X_\infty \to \frac{1}{\ell}\mathbb{Z} \cup \{\infty\}$ is a function such that $\phi^{-1}(\alpha)$ is a limit of cylinders for every $\alpha \in \frac{1}{\ell}\mathbb{Z}$. We attach to ϕ the following integral-like invariant[3]

$$\int_{X_\infty} (uv)^\phi := \sum_{\alpha\in\frac{1}{\ell}\mathbb{Z}} E(\phi^{-1}(\alpha)) \cdot (uv)^\alpha \in \widetilde{T}^{(\ell)},$$

if the series in convergent.

Example 10.4.24. Suppose that $Y^{(1)}, \ldots, Y^{(r)}$ are proper closed subschemes of X and $a_1, \ldots, a_r$ are rational numbers. Let ℓ be a positive integer such that $\ell a_i \in \mathbb{Z}$ for every i. We consider the function $\phi\colon X_\infty \to \frac{1}{\ell}\mathbb{Z} \cup \{\infty\}$ given by

$$\phi(\gamma) = \sum_{i=1}^r a_i \cdot \mathrm{ord}_{Y^{(i)}}(\gamma)$$

[3] This is the "Hodge realization" of the motivic integral that will be introduced in the next section.

(with the convention that $\phi(\gamma) = \infty$ if some $\mathrm{ord}_{Y^{(i)}}(\gamma) = \infty$). We claim that $\phi^{-1}(q)$ is a limit of cylinders for every $q \in \frac{1}{\ell}\mathbb{Z}$.

For every $\nu = (\nu_1, \ldots, \nu_r) \in \mathbb{Z}^r_{\geq 0}$, consider the cylinder $C_\nu = \bigcap_{i=1}^r \mathrm{Cont}^{\nu_i}(Y^{(i)})$. With this notation, we have $\phi^{-1}(q) = \bigsqcup_{a_1\nu_1+\ldots+a_r\nu_r=q} C_\nu$. If Y is the closed subscheme defined by the product of the ideals defining the Y_i, then for every N, all but finitely many of the C_ν are contained in $\mathrm{Cont}^{\geq N}(Y)$. This implies that $\phi^{-1}(q)$ is a limit of cylinders and

$$E(\phi^{-1}(q)) = \sum_{a_1\nu_1+\ldots+a_r\nu_r=q} E(C_\nu).$$

We conclude that

$$\int_{X_\infty} (uv)^\phi = \sum_{\nu \in \mathbb{Z}^r_{\geq 0}} E(C_\nu) \cdot (uv)^{\sum_i a_i \nu_i}, \tag{10.13}$$

in the sense that one side is convergent if and only if the other one is, and if this is the case, then we have equality.

If we consider one more proper closed subscheme $Y^{(r+1)}$ and consider the function $\phi' = \phi + 0 \cdot \mathrm{ord}_{Y^{(r+1)}}$, then following our convention $\phi = \phi'$ on $X_\infty \smallsetminus Y^{(r+1)}_\infty$, but the two functions might differ on $Y^{(r+1)}_\infty$. However, we have $\int_{X_\infty} (uv)^\phi = \int_{X_\infty} (uv)^{\phi'}$. Indeed, this is a consequence of formula (9.13) and of the fact that for every ν, we have

$$E(C_\nu) = \sum_{m \geq 0} E(C_\nu \cap \mathrm{Cont}^m(Y^{(r+1)}))$$

by Corollary 9.4.18. Similarly, if $Y' = \sum_{i=1}^r b_i \cdot \mathrm{ord}_{Y^{(i)}}$ and ϕ' is the corresponding function, then we may consider $\phi + \phi'$, with the convention that $\phi(\gamma) + \phi'(\gamma) = \infty$ if either $\phi(\gamma) = \infty$ or $\phi'(\gamma) = \infty$. It is not necessarily true that $\phi_1 + \phi_2$ is equal to $\psi := \sum_i (a_i + b_i) \cdot \mathrm{ord}_{Y_i}$ everywhere (the two functions might disagree on $Y^{(i)}_\infty$ in case $a_i = -b_i$). However, we have

$$\int_{X_\infty} (uv)^{\phi+\phi'} = \int_{X_\infty} (uv)^\psi.$$

Example 10.4.25. Let us specialize to the case of divisors. If X is a smooth variety and F is a $\mathbb{Q}$-divisor on X, we write $F = \sum_{i=1}^r a_i F_i$, with the a_i nonzero and the F_i distinct prime divisors. We put $\mathrm{ord}_F := \sum_{i=1}^r a_i \cdot \mathrm{ord}_{F_i}$, as in Example 9.4.24. Note that if we allow some coefficients to be zero, then we get a different function. However, as we have seen, the corresponding integrals are the same. Similarly, if F' is another $\mathbb{Q}$-divisor, then the functions $\mathrm{ord}_F + \mathrm{ord}_{F'}$ and $\mathrm{ord}_{F+F'}$ might not agree everywhere (in case some prime divisor appears with opposite coefficients in F and F'). However, the two functions have the same integral.

We now turn to the definition of the stringy E-function. Let (Y, D) be a pair with Y normal and $K_Y + D$ being $\mathbb{Q}$-Cartier. For a resolution of singularities $f \colon X \to Y$ of Y, we write $K_X + D_X = f^*(K_Y + D)$, as in Section 3.1.3. We fix a positive integer ℓ such that $\ell(K_Y + D)$ is Cartier, hence ℓD_X has integer coefficients. We consider the function $\mathrm{ord}_{D_X}\colon X_\infty \to \frac{1}{\ell}\mathbb{Z}\cup\{\infty\}$. The *stringy E-function* of the pair (Y,D) is

$$E_{\mathrm{st}}(Y,D) = E_{\mathrm{st}}(Y,D;u^{1/\ell},v^{1/\ell}) := \int_{X_\infty} (uv)^{\mathrm{ord}_{D_X}} \in \widetilde{T}^{(\ell)},$$

assuming that this is defined. If $D=0$, then we simply write $E_{\mathrm{st}}(Y)$. Note that if Y is 1-Gorenstein and has canonical singularities, we recover our previous definition (note that when $D=0$, we have $D_X = -K_{X/Y}$).

By Example 9.4.24, $\mathrm{ord}_{D_X}^{-1}(q)$ is a limit of cylinders for every $q\in\frac{1}{\ell}\mathbb{Z}$. Moreover, if we write $D_X = \sum_{i=1}^r a_iF_i$, with the F_i distinct prime divisors, and put $C_\nu = \bigcap_{i=1}^r \mathrm{Cont}^{\nu_i}(F_i)$ for every $\nu\in\mathbb{Z}_{\geq 0}^r$, then

$$E_{\mathrm{st}}(Y,D) = \sum_{\nu\in\mathbb{Z}_{\geq 0}^r} E(C_\nu)\cdot(uv)^{\sum_i a_i\nu_i}.$$

We first show that the definition is independent of the chosen resolution and that it satisfies a "change of variable" formula under proper, birational morphisms.

Proposition 10.4.26. *If (Y,D) is a pair as above, then the definition of $E_{\mathrm{st}}(Y,D)$ (in particular, the convergence of the corresponding series) is independent of the choice of resolution of singularities.*

Before proving this, we give the following "change of variable" formula.

Proposition 10.4.27. *Let $g\colon W\to X$ be a proper birational morphism between two smooth varieties. If $Y^{(1)},\ldots,Y^{(r)}$ are proper closed subschemes of X and $a_1,\ldots,a_r$ are rational numbers, then for the functions $\phi=\sum_{i=1}^r a_i\cdot\mathrm{ord}_{Y^{(i)}}$ and $\psi=\phi\circ g_\infty - \mathrm{ord}_{K_{W/X}}$ (with the convention that $\psi(\gamma)=\infty$ if either $\phi(g_\infty(\gamma))=\infty$ or $\mathrm{ord}_{K_{W/X}}(\gamma)=\infty$), the following holds:*

$$\int_{X_\infty}(uv)^{\phi} = \int_{W_\infty}(uv)^{\psi},$$

in the sense that one integral exists if and only if the other one does, and if this is the case, then they are equal.

Proof. For every $\nu=(\nu_1,\ldots,\nu_r)\in\mathbb{Z}_{\geq 0}^r$, we put $C_\nu=\bigcap_{i=1}^r\mathrm{Cont}^{\nu_i}(Y_i)$. It follows from Example 9.4.24 that $\phi^{-1}(q)$ is a limit of cylinders and

$$E(\phi^{-1}(q)) = \sum_{a_1\nu_1+\ldots+a_r\nu_r=q} E(C_\nu).$$

By definition, we have $\psi = -\mathrm{ord}_{K_{W/X}} + \sum_{i=1}^r a_i\cdot\mathrm{ord}_{g^{-1}(Y_i)}$. Therefore $\psi^{-1}(q')$ is a limit of cylinders for every q' and we have

$$E(\psi^{-1}(q')) = \sum_{a_1\nu_1+\ldots+a_r\nu_r-e=q'} E(g_\infty^{-1}(C_\nu\cap\mathrm{Cont}^e(K_{W/X})).$$

Furthermore, applying Corollary 9.4.19 to each C_ν gives

$$E(C_\nu) = \sum_{e\geq 0} E(g_\infty^{-1}(C_\nu)) \cap \mathrm{Cont}^e(K_{W/X})) \cdot (uv)^{-e}.$$

By putting all these together, we obtain

$$\int_{X_\infty} (uv)^\phi = \sum_q E(\phi^{-1}(q)) \cdot (uv)^q = \sum_{\nu \in \mathbb{Z}^r_{\geq 0}} E(C_\nu) \cdot (uv)^{\sum_i a_i \nu_i}$$

$$= \sum_{\nu \in \mathbb{Z}^r_{\geq 0}, e\geq 0} E(g_\infty^{-1}(C_\nu) \cap \mathrm{Cont}^e(K_{W/X})) \cdot (uv)^{-e+\sum_i a_i \nu_i} = \int_{Y_\infty} (uv)^\psi.$$

This completes the proof of the proposition. □

Proof of Proposition 9.4.26. By dominating any two resolutions by a third one, we see that it is enough to consider two proper birational morphisms $f\colon X \to Y$ and $g\colon W \to X$ and show that

$$\int_{X_\infty} (uv)^{\mathrm{ord}_{D_X}} = \int_{W_\infty} (uv)^{\mathrm{ord}_{D_W}}.$$

Note that $D_W = (D_X)_W = g^*(D_X) - K_{W/X}$, hence using Proposition 9.4.27, we obtain

$$\int_{X_\infty} (uv)^{\mathrm{ord}_{D_X}} = \int_{W_\infty} (uv)^{\mathrm{ord}_{D_X} \circ g_\infty - \mathrm{ord}_{K_{W/X}}} = \int_{W_\infty} (uv)^{\mathrm{ord}_{D_W}}.$$

□

Once we know that the stringy E-function does not depend on the choice of resolution, it follows from definition that it satisfies the following "birational transformation formula".

Proposition 10.4.28. *Let (Y,D) a pair as above and $g\colon Z \to Y$ a proper birational morphism, with Z normal. If we write, as usual $K_Z + D_Z = g^*(K_Y + D)$, then*

$$E_{\mathrm{st}}(Z, D_Z) = E_{\mathrm{st}}(Y, D),$$

in the sense that one side exists if and only if the other one does, and if this is the case, then they are equal.

Proof. Let $f\colon X \to Z$ be a resolution of singularities. We use f to compute $E_{\mathrm{st}}(Z, D_Z; u, v)$ and $g \circ f$ to compute $E(Y, D; u, v)$. Since $D_X = (D_Z)_X$, the equality in the proposition is clear. □

The following case is particularly important. We first recall that if Y is a $\mathbb{Q}$-Gorenstein variety, then a resolution of singularities $f\colon X \to Y$ is *crepant* if $K_{X/Y} = 0$.

Corollary 10.4.29. *If Y is a complete variety that has a crepant resolution $f\colon X \to Y$, with X projective, then $E_{\mathrm{st}}(X)$ is equal to the Hodge polynomial of X.*

Proof. The assertion follows from the fact that if ϕ is the zero function on X_∞, then $\int_{X_\infty}(uv)^\phi = E(X_\infty) = E(X)$. $\square$

Finally, we turn to the criterion for the existence of $E_{\rm st}(Y,D)$ and to its explicit computation in terms of a log resolution of (Y,D).

Proposition 10.4.30. *If Y is an n-dimensional normal variety and D is a $\mathbb{Q}$-divisor on Y such that $K_Y + D$ is $\mathbb{Q}$-Cartier, then $E_{\rm st}(Y,D)$ is defined if and only if (Y,D) is klt. If this is the case and $f\colon X \to Y$ is a resolution of singularities, with $D_X = \sum_{i=1}^r a_i F_i$ having simple normal crossings, then*

$$E_{\rm st}(Y,D) = \sum_{J\subseteq\{1,\ldots,r\}} E(F_J^\circ)\cdot\prod_{j\in J}\frac{uv-1}{(uv)^{1-a_i}-1},$$

where for every $J \subseteq \{1,\ldots,r\}$, we put $F_J^\circ = (\cap_{j\in J}F_j) \smallsetminus \big(\cup_{j\not\in J}F_j\big)$, with the convention that $F_\emptyset^\circ = X \smallsetminus (F_1 \cup \ldots \cup F_r)$ and the corresponding product is equal to 1. In particular, we see that $E_{\rm st}(Y,D)$ is a rational function.

Proof. Let $f\colon X \to Y$ be a resolution as in the proposition (for example, a log resolution of (Y,D)). For every $\nu = (\nu_1,\ldots,\nu_r) \in \mathbb{Z}_{\geq 0}^r$, we put $C_\nu = \bigcap_{i=1}^r \mathrm{Cont}^{\nu_i}(F_i)$. The key computation is that of $E(C_\nu)$.

Let us fix ν and put $J = \{i \mid \nu_i \geq 1\}$. It is clear that if $\gamma \in C_\nu$, then $\pi_\infty(\gamma) \in F_J^\circ$. In particular, if $F_J^\circ = \emptyset$, then $C_\nu = \emptyset$. Suppose now that $F_J^\circ \neq \emptyset$. Let m be an integer such that $m \geq \nu_i$ for every i. In this case we have $C_\nu = \pi_{\infty,m}^{-1}(S)$, where $S = \cap_i \mathrm{Cont}^{\nu_i}(F_i)_m$.
Claim. The projection $S \to F_J^\circ$ is locally trivial, with fiber

$$\mathbb{A}^{mn-\sum_i \nu_i} \times (\mathbb{A}^1 \smallsetminus \{0\})^{|J|}.$$

Indeed, this assertion is local on X, hence we may assume that we have a system of coordinates $x_1,\ldots,x_n$ on X such that each F_i is defined by one of these coordinates. We may assume that $J = \{1,\ldots,s\}$. In this case, by Corollary 9.1.13 and its proof, we have an isomorphism over X

$$J_m(X) \simeq X \times (tk[t]/(t^{m+1}))^{\oplus n}$$

that maps an m-jet γ with $\pi_m(\gamma) = p$ to

$$(p, \gamma^*(x_1 - x_1(p)),\ldots,\gamma^*(x_n - x_n(p))).$$

Via this isomorphism S corresponds to

$$F_J^\circ \times \{(u_1,\ldots,u_n) \in (tk[t]/(t^{m+1}))^{\oplus n} \mid \mathrm{ord}(u_i) = \nu_i \,\text{for}\, 1 \leq i \leq s\}$$

$$\simeq F_J^\circ \times \left(\prod_{i=1}^{|J|}\mathbb{A}^{m-\nu_i}\right) \times (\mathbb{A}^1 \smallsetminus \{0\})^{|J|} \times \mathbb{A}^{m(n-|J|)}.$$

This immediately implies the assertion in the claim.

Using the claim, we obtain

$$E(C_\nu) = E(S)\cdot(uv)^{-mn} = E(F_J^\circ)(uv-1)^{|J|}\cdot(uv)^{-\sum_i \nu_i}.$$

By definition, we have

$$E(Y,D) = \sum_{\nu\in\mathbb{Z}^r_{\geq 0}} E(C_\nu)(uv)^{\sum_i a_i\nu_i}.$$

The sum is considered in $\widetilde{T}^{(\ell)}$, where ℓ is a positive integer such that $\ell a_i\in\mathbb{Z}$ for all i. The sum of the terms corresponding to those ν with $\{i \mid \nu_i\geq 1\} = J$ is

$$S_J = E(F_J^\circ)(uv-1)^{|J|}\cdot \sum_{(\nu_i)\in\mathbb{Z}^{|J|}_{\geq 1}} (uv)^{\sum_{i\in J}(a_i-1)\nu_i}.$$

It follows from the topology on $\widetilde{T}^{(\ell)}$ that all S_J are convergent if and only if $a_i<1$ for all $i\in J$. By definition, this means precisely that (Y,D) is klt.

Suppose now that $a_i<1$ for all i. We can compute S_J using the formula for the geometric series, and we obtain

$$S_J = E(F_J^\circ)\cdot(uv-1)^{|J|}\cdot\prod_{i\in J}\frac{(uv)^{a_i-1}}{1-(uv)^{a_i-1}} = E(F_J^\circ)\cdot\prod_{j\in J}\frac{uv-1}{(uv)^{1-a_i}-1}.$$

Summing over all subsets J of $\{1,\dots,r\}$ gives the formula in the proposition. $\square$

Remark 10.4.31. With the notation in Proposition 9.4.30, suppose that ℓ is a positive integer such that $\ell(K_Y+D)$ is Cartier. If (Y,D) is kit, it follows from the formula for $E_{\mathrm{st}}(Y,D)$ that this rational function can be evaluated at $(u^{1/\ell},v^{1/\ell})=(1,1)$. We then obtain the *stringy Euler-Poincaré characteristic*

$$\chi_{\mathrm{st}}(Y,D) := E_{\mathrm{st}}(Y,D;1,1) = \sum_{J\subseteq\{1,\dots,r\}}\chi^{\mathrm{top}}(F_J^\circ)\cdot\prod_{j\in J}\frac{1}{1-a_i}.$$

Example 10.4.32. Let us compute, following [Bat98], the stringy E-function of toric pairs. For the basic facts on toric varieties, we refer to [Ful93]. Suppose that $Y=Y(\Delta)$ is a toric variety with fan Δ in $N_{\mathbb{R}}\simeq\mathbb{R}^n$. Let $D_1,\dots,D_d$ be the prime invariant divisors on Y and let $D=\sum_{i=1}^d a_iD_i$ be a toric divisor such that K_Y+D is $\mathbb{Q}$-Cartier. Recall that on a toric variety we may take $K_Y=-\sum_{i=1}^d D_i$ (see [Ful93, Chapter 4.3]). Furthermore, K_Y+D is $\mathbb{Q}$-Cartier if and only if there is a function $\psi=\psi_{K_Y+\Delta}$ on $|\Delta|$ that is linear on each cone in Δ and such that $\psi(v_i)=1-a_i$ for $1\leq i\leq r$, where v_i is the primitive generator of the ray corresponding to the divisor D_i (see[Ful93, Chapter 3.3]). Let $f\colon X\to Y$ be a toric resolution of singularities corresponding to a fan Δ_X refining Δ (see [Ful93, Chapter 2.6]). If we write as usual $K_X+D_X=f^*(K_Y+D)$, then $\psi_{K_X+D_X}=\psi$. Let $F_1,\dots,F_r$ be the prime invariant divisors on X, corresponding to the primitive ray generators $w_1,\dots,w_r$. With this notation, we have $D_X=\sum_{j=1}^r(1-\psi(w_j))F_j$. Note that since X is smooth, $\sum_{j=1}^r F_j$

has simple normal crossings. Therefore (Y,D) is klt if and only if $\psi(w_j) > 0$ for all j (this is equivalent to $\psi > 0$ on $|\Delta_X| \smallsetminus \{0\} = |\Delta| \smallsetminus \{0\}$ and it is further equivalent to $a_i < 1$ for all i). Let us assume that this is indeed the case

Note that if $J \subseteq \{1,\ldots,r\}$, then F_J° is nonempty if and only if the rays in Σ corresponding to the elements of J span a cone of Δ_X. Furthermore, if this is the case and σ is this cone, then $F_J^\circ \simeq (\mathbb{A}^1 \smallsetminus \{0\})^{\dim(\sigma)}$, hence $E(F_J^\circ) = (uv-1)^{n-\dim(\sigma)}$. It thus follows from Proposition 9.4.30 that

$$E_{\mathrm{st}}(Y,D) = (uv-1)^n \cdot \sum_{\sigma \in \Delta_X} \prod_{w_j \in \sigma} \frac{1}{(uv)^{\psi(w_j)-1}}.$$

We can interpret this expression directly on Δ, as follows. The formula for the geometric series implies

$$\frac{1}{(uv)^{\psi(w_j)-1}} = \frac{(uv)^{-\psi(w_j)}}{1-(uv)^{-\psi(w)}} = \sum_{i \geq 1} (uv)^{-\psi(iw_j)}.$$

We thus obtain

$$E_{\mathrm{st}}(Y,D) = (uv-1)^n \cdot \sum_{\sigma \in \Delta_X} \sum_{w \in \mathrm{Int}(\sigma) \cap N} (uv)^{-\psi(w)} = (uv-1)^n \cdot \sum_{w \in |\Delta| \cap N} (uv)^{-\psi(w)}.$$

Remark 10.4.33. One can develop the framework of (Hodge realizations of) motivic integrals in a more formal way, making it more similar to usual integration theories. In particular, one can define measurable sets and measurable functions and treat more general integrals, not just those of functions of the form $\sum_{i=1}^r a_i \cdot \mathrm{ord}_{Y_i}$, as we did in this section. This is done in detail in [Bat98]. A somewhat different approach, using semialgebraic subsets in the space of arcs is pursued in [DL99]. On the other hand, since for the applications that we have in mind we only need to deal with the rather special subsets and functions that we considered, we preferred to take this more hands-on approach.

10.4.4 Historical comments

This story started when Batyrev proved in [Bat99a] that K-equivalent smooth projective varieties have the same Betti numbers (while the paper only appeared in 1999, it had been available for a few years before that). Batyrev's argument used p-adic integration to show that general reductions to positive characteristic of the two varieties have the same zeta function, hence the two varieties have the same Betti numbers via the Weil conjectures. Motivated by this, Kontsevich introduced in his talk [Kon] at Orsay in 1995 motivic integration in order to prove that K-equivalent smooth projective varieties have in fact the same Hodge numbers. This appeared in [Bat98], together with the definition of the stringy E-function, in the context of Hodge realizations of motivic integrals that we discussed in this section. Motivic integration was then extended to singular varieties [DL99], to formal schemes [Seb04], and to an arithmetic setting [DL01]. For nice introductions to the circle of ideas around geometric motivic integrations, see [BL04], [Cra04], and [Vey06]. On the other hand, a version of Hodge invariants had been introduced in the context of quotients of smooth varieties by finite groups by Batyrev and Dais [BD96]. This was the *orbifold Hodge polynomial* $E_{\mathrm{orb}}(X;u,v) \in \mathbb{Z}[u^{1/\ell}, v^{1/\ell}]$, inspired by string theory and whose definition involved in an essential way the group action. The fact that the orbifold Hodge polynomial agrees with the stringy E-function in the case of orbifolds is one aspect of the *McKay correspondence*, proved for global quotients in [Bat99b] and [DL02] in the case of quotients $\mathbb{A}^n/G$, of an affine space by a linear action of a finite group, and in the general case of varieties with quotient singularities in [Yas04]. An interesting aspect is that while motivic integration became an important construction, with many applications, in the end it was not really necessary for the proof of Kontsevich's theorem. It was independently observed by Ito [Ito03] and [Wan98] that once we know that two K-equivalent varieties have general reductions to positive characteristic having the same zeta functions, then standard arguments in p-adic Hodge theory imply that the two varieties have the same Hodge numbers.

10.5 Introduction to motivic integration

As the reader has probably noticed, the constructions in the previous section have only made use of the additivity and multiplicativity of the Hodge-Deligne polynomial. One can thus redo those arguments and constructions by working with the universal invariant that has these two properties, namely the class in the Grothendieck ring of varieties. In this section we introduce this formalism and explain the changes that have to be made in this setting. As an application of this formalism, we introduce an important invariant of singularities of hypersurfaces, Denef and Loeser's motivic zeta function.

10.5.1 The Grothendieck group of varieties

We now introduce the ring in which the universal Euler-Poincaré characteristic lives, the Grothendieck group of varieties. The definition can be given in a very general setting. Suppose that S is a Noetherian scheme. The Grothendieck group $K_0(\mathrm{Var}/S)$ is the quotient of the free abelian group on the set of symbols $[X/S]$, where X is a scheme of finite type over S, by the subgroup generated by the following relations:

i) $[X/S] = [Y/S]$ if X and Y are isomorphic as schemes over S.
ii) $[X/S] = [X_{\mathrm{red}}/S]$ for every X.
iii) If Z is a closed subscheme of X and $U = X \smallsetminus Z$, then

$$[X/S] = [Z/S] + [U/S].$$

Due to property ii) above, if Z is a locally closed subset of X, the element $[Z] \in K_0(\mathrm{Var}/S)$ is well-defined, independent on the scheme structure we consider on Z. We also note that relation iii) above for $Z = X$ implies $[\emptyset] = 0$. When S is understood from the context, then we simply write $[X]$ instead of $[X/S]$ and when $S = \mathrm{Spec}(R)$, for a ring R, we write $K_0(\mathrm{Var}/R)$ and $[X/R]$ for the corresponding objects.

In fact, $K_0(\mathrm{Var}/S)$ becomes a commutative ring, with multiplication given by

$$[X] \cdot [Y] = [X \times_S Y].$$

Note that the unit element is $[S]$.

If $f \colon T \to S$ is a morphism of Noetherian schemes, then we have an induced morphism of Grothendieck rings

$$f^* \colon K_0(\mathrm{Var}/S) \to K_0(\mathrm{Var}/T),\ [X/S] \to [X \times_S T/T].$$

If f is of finite type, then we also have a *group* homomorphism

$$f_* \colon K_0(\mathrm{Var}/T) \to K_0(\mathrm{Var}/S),\ [Y/T] \to [Y/S].$$

Note that these maps satisfy the *projection formula*

$$f_*(f^*(\alpha) \cdot \beta) = \alpha \cdot f_*(\beta) \text{ for every } \alpha \in K_0(\mathrm{Var}/S), \beta \in K_0(\mathrm{Var}/T).$$

Indeed, it is enough to check this when $\alpha = [X/S]$ and $\beta = [Y/T]$, when the assertion follows from the following isomorphism of schemes over S

$$Y \times_T (T \times_S X) \simeq Y \times_S X.$$

The class of the affine line $\mathbb{A}^1_S$ in $K_0(\mathrm{Var}/S)$ is denoted by $\mathbb{L}$ (or $\mathbb{L}_S$ if S is not understood from the context). Therefore we have $[\mathbb{A}^n_S] = \mathbb{L}^n$. Moreover, the decomposition $\mathbb{P}^n_S = \mathbb{P}^{n-1}_S \sqcup \mathbb{A}^n_S$ implies by induction on n that

$$[\mathbb{P}^n_S/S] = \sum_{i=0}^{n} \mathbb{L}^i.$$

Let S be a Noetherian scheme and A an abelian group. An *Euler-Poincaré characteristic with values in* A on schemes over S is a map α that associates to a scheme X of finite type over S an element $\alpha(X) \in A$ such that

i) $\alpha(X) = \alpha(Y)$ if X and Y are isomorphic as schemes over S.
ii) $\alpha(X) = \alpha(X_{\mathrm{red}})$.
iii) If Y is a closed subscheme of X, then $\alpha(X) = \alpha(Y) + \alpha(X \smallsetminus Y)$.

In other words, α induces a group homomorphism $K_0(\mathrm{Var}/S) \to A$. In what follows we will identify α with this homomorphism. Note that the map $X/S \to [X/S] \in K_0(\mathrm{Var}/S)$ is the universal Euler-Poincaré characteristic. If A is a ring and α is an Euler-Poincaré characteristic with values in A, then we say that α is *multiplicative* if the induced map $K_0(\mathrm{Var}/S) \to A$ is a ring homomorphism.

Example 10.5.1. If $S = \mathrm{Spec}(k)$ is a finite field, then for every finite field extension K/k, we obtain a multiplicative Euler-Poincaré characteristic $K_0(\mathrm{Var}/S) \to \mathbb{Z}$ mapping $[X]$ to the number of elements of $X(K)$.

Example 10.5.2. If $S = \mathrm{Spec}(\mathbb{C})$, then the topological Euler-Poincaré characteristic $\chi^{\mathrm{top}}(X)$ gives a multiplicative Euler-Poincaré characteristic on $K_0(\mathrm{Var}/\mathbb{C})$. Indeed, we have seen in Remark 9.4.4 that this gives an Euler-Poincaré characteristic and the fact that it is multiplicative is an immediate consequence of Künneth's formula.

Example 10.5.3. As we have discussed in the previous section, the above example can be refined by the Hodge-Deligne polynomial. More precisely, still assuming that $S = \mathrm{Spec}(\mathbb{C})$, the Hodge-Deligne polynomial gives a multiplicative Euler-Poincaré characteristic $K_0(\mathrm{Var}/\mathbb{C}) \to \mathbb{Z}[u, v]$.

Our next goal is to show that given a constructible subset in a scheme of finite type over S, we can define its class in $K_0(\mathrm{Var}/S)$. In order to do this, we will need the following lemma, which extends condition iii) in the definition of the Grothendieck group of varieties.

Proposition 10.5.4. *Suppose that S is a Noetherian scheme and X is a scheme of finite type over S. If we have a decomposition $X = Y_1 \sqcup \ldots \sqcup Y_r$, where all Y_i are locally closed subsets of X, then $[X] = [Y_1] + \ldots + [Y_r]$ in $K_0(\mathrm{Var}/S)$.*

Proof. We argue by Noetherian induction, hence we may assume that this property holds for all proper closed subschemes of X. Let Z be an irreducible component of X and η_Z its generic point. If i is such that $\eta_Z \in Y_i$, then $Z \subseteq \overline{Y_i}$, and since Y_i is open in $\overline{Y_i}$, it follows that there is a nonempty open subset U of X contained in Y_i (for example, we may take U to consist of the points in $Y_i \cap Z$ that do not lie on any irreducible component of X different from Z). By definition, we have

$$[Y_i] = [U] + [Y_i \smallsetminus U] \text{ and } [X] = [U] + [X \smallsetminus U]. \tag{10.14}$$

Applying the induction hypothesis for $X \smallsetminus U$ and the decomposition $X \smallsetminus U = (Y_i \smallsetminus U) \sqcup \bigsqcup_{j \neq i} Y_j$, we have

$$[X \smallsetminus U] = [Y_i \smallsetminus U] + \sum_{j \neq i} [Y_j]. \tag{10.15}$$

By combining (9.14) and (9.15), we get the formula in the proposition. $\square$

Suppose now that X is a scheme of finite type over a Noetherian scheme S and W is a constructible subset of X. Consider a disjoint decomposition $W = W_1 \sqcup \ldots \sqcup W_r$, with each W_i locally closed in X. We put $[W] := \sum_{i=1}^{r} [W_i] \in K_0(\mathrm{Var}/S)$.

Proposition 10.5.5. *With the above notation, the following hold:*

i) The definition of $[W]$, for W constructible in X, is independent of the disjoint decomposition.
ii) If $W_1, \ldots, W_s$ are pairwise disjoint constructible subsets of X, and $W = \bigcup_{i=1}^{r} W_i$, then $[W] = \sum_{i=1}^{s} [W_i]$.

Proof. Suppose that we have two decompositions into locally closed subsets

$$W = W_1 \sqcup \ldots \sqcup W_r \text{ and } W = W' \sqcup \ldots \sqcup W'_s.$$

Let us also consider the decomposition $W = \bigsqcup_{i,j}(W_i \cap W'_j)$. It follows from Proposition 9.5.4 that $[W_i] = \sum_{j=1}^{s}[W_i \cap W'_j]$ for every i and $[W'_j] = \sum_{i=1}^{r}[W_i \cap W'_j]$ for every j. Therefore

$$\sum_{i=1}^{r}[W_i] = \sum_{i=1}^{r}\sum_{j=1}^{s}[W_i \cap W'_j] = \sum_{j=1}^{s}\sum_{i=1}^{r}[W_i \cap W'_j] = \sum_{j=1}^{s}[W'_j].$$

This proves i). The assertion in ii) follows from i): if we consider disjoint unions $W_i = W_{i,1} \sqcup \ldots \sqcup W_{i,m_i}$ for every i, with each $W_{i,j}$ locally closed in X, then $W = \bigsqcup_{i,j} W_{i,j}$, and

$$[W] = \sum_{i,j}[W_{i,j}] = \sum_{i}[W_i].$$

$\square$

Remark 10.5.6. It follows from Proposition 9.5.5 that if $\alpha\colon K_0(\mathrm{Var}/S) \to A$ is an Euler-Poincaré characteristic and $W \subseteq X$ is a constructible subset of a scheme X of finite type over S, then we can define $\alpha(W)$ by writing $W = \sqcup_{i=1}^{r} W_i$, with W_i locally closed subsets of X and putting $\alpha(W) = \sum_{i=1}^{r}\alpha(W_i)$. It is clear that the resulting map is additive on disjoint constructible subsets of X.

Let F be a scheme of finite type over S. We define piecewise trivial fibrations in this more general setting in the same way as before. More precisely, given a morphism $f\colon X \to Y$ of schemes of finite type over S and constructible subsets $A \subseteq X$ and $B \subseteq Y$ such that f induces a map $g\colon A \to B$, we say that g is *piecewise trivial with fiber F* if we can write $B = \bigcup_i B_i$, with B_i locally closed in Y and $g^{-1}(B_i)$ locally closed in X for every i, such that $g^{-1}(B_i)_{\mathrm{red}}$ is isomorphic over B_i to $(B_i \times F)_{\mathrm{red}}$.

Corollary 10.5.7. *If $f\colon X \to Y$ is a morphism of schemes of finite type over S inducing a piecewise trivial map $g\colon A \to B$ with fiber F, where $A \subseteq X$ and $B \subseteq Y$ are constructible, then $[A/S] = [B/S] \cdot [F/S]$ in $K_0(\mathrm{Var}/S)$.*

Proof. It is enough to consider a cover as in the definition of piecewise trivial fibrations consisting of pairwise disjoint subsets. $\square$

Example 10.5.8. If $E \to X$ is a rank r vector bundle, it follows that $[E] = [X] \cdot \mathbb{L}^r$. Similarly, if $\mathbb{P}(E) \to X$ is the corresponding projectivized vector bundle, we have $[\mathbb{P}(E)] = [X] \cdot (1 + \mathbb{L} + \ldots + \mathbb{L}^{r-1})$.

We end this subsection with a discussion of the Grothendieck group $K_0(\mathrm{Var}/S)$, when S is a scheme of finite type over an algebraically closed field k of characteristic 0. We keep this assumption for the rest of this subsection.

Proposition 10.5.9. *For every S, the group $K_0(\mathrm{Var}/S)$ is generated by the classes $[X]$, with X a smooth variety, with a projective morphism $X \to S$.*

Proof. The argument follows the one in Remark 9.4.1, hence we omit it. □

Remark 10.5.10. Let $\alpha\colon K_0(\mathrm{Var}/S) \to A$ be an Euler-Poincaré characteristic, where A is a ring. It follows from Proposition 9.5.9 that in order to check that α is multiplicative, it is enough to check that $\alpha(X \times_S Y) = \alpha(X) \cdot \alpha(Y)$, whenever X and Y are smooth varieties, with projective morphisms to S.

While Proposition 9.5.9 gives generators for $K_0(\mathrm{Var}/S)$, it is natural to ask about the relations between these generators. This is answered by the following result of Bittner [Bit04] which says that the relations are generated by those corresponding to smooth blow-ups.

Theorem 10.5.11. *For every S, the kernel of the natural morphism from the free abelian group on isomorphism classes of smooth varieties, projective over S, to $K_0(\mathrm{Var}/S)$ is generated by the following elements:*

i) $[\emptyset]$
ii) $([\mathrm{Bl}_Y X] - [D]) - ([X] - [Y])$,

where X and Y are varieties as above, with Y a subvariety of X, and where $\mathrm{Bl}_Y X$ is the blow-up of X along Y, with exceptional divisor D.

We do not give a proof of this theorem, since we will not use it. We only mention that the main ingredient in its proof is the following weak factorization theorem of Abramovich, Karu, Matsuki, and Włodarczyk.

Theorem 10.5.12. (*[AKMW02]*) *If S is a scheme of finite type over an algebraically closed field field of characteristic zero, then every birational map between two smooth varieties, projective over S, can be realized as a composition of blow-ups and blow-downs of smooth irreducible centers on smooth projective varieties.*

Remark 10.5.13. The presentation of the Grothendieck group in Theorem 9.5.11 gives an easy way to construct Euler-Poincaré characteristics. Note that the Hodge polynomial of a smooth projective variety makes sense over any field k. If k is algebraically closed, of characteristic zero, this can be extended to an Euler-Poincaré characteristic, the Hodge-Deligne polynomial $E\colon K_0(\mathrm{Var}/k) \to \mathbb{Z}[u,v]$. By Theorem 9.5.11, in order to prove this it is enough to show that if X is a smooth projective variety and Y is a smooth subvariety, then $E(X) - E(Y) = E(\mathrm{Bl}_Y X) - E(D)$. This is elementary to check. Since the Hodge-Deligne polynomial is available over any algebraically closed field of characteristic 0, we see that all results in Section 9.4 extend to this setting.

We now explain how Bittner's result implies a theorem of Larsen and Lunts, relating the Grothendieck group of varieties with stable birational geometry. We keep the assumption that k is an algebraically closed field of characteristic zero. Recall that two varieties X and Y are *stably birational* if $X \times \mathbb{P}^m$ and $Y \times \mathbb{P}^n$ are birational for some $m, n \geq 0$.

Let SB/k denote the set of stably birational equivalence classes of varieties over k. We denote the class of X in SB/k by $\langle X\rangle$. Note that SB/k is a commutative semigroup, with multiplication induced by $\langle X\rangle\cdot\langle Y\rangle=\langle X\times Y\rangle$. Of course, the identity element is $\mathrm{Spec}\,k$. Let us consider the semigroup algebra $\mathbb{Z}[\mathrm{SB}/k]$ associated to the semigroup SB/k.

Proposition 10.5.14. *There is a unique ring homomorphism $\Phi\colon K_0(\mathrm{Var}/k)\to\mathbb{Z}[\mathrm{SB}/k]$ such that $\Phi([X])=\langle X\rangle$ for every smooth projective variety X over k.*

Proof. Uniqueness is a consequence of Proposition 9.5.9. In order to prove the existence of a group homomorphism Φ as in the proposition, we apply Theorem 9.5.11. This shows that it is enough to check that whenever X and Y are smooth projective varieties, with Y a closed subvariety of X, we have

$$\langle \mathrm{Bl}_Y(X)\rangle-\langle E\rangle=\langle X\rangle-\langle Y\rangle,$$

where $\mathrm{Bl}_Y X$ is the blow-up of X along Y, and E is the exceptional divisor. In fact, we have $\langle X\rangle=\langle \mathrm{Bl}_Y(X)\rangle$ since X and $\mathrm{Bl}_Y(X)$ are birational, and $\langle Y\rangle=\langle E\rangle$, since E is birational to $Y\times\mathbb{P}^{r-1}$, where $r=\mathrm{codim}_X(Y)$.

In order to check that Φ is a ring homomorphism, it is enough to show that $\Phi(uv)=\Phi(u)\Phi(v)$ when u and v vary over a system of group generators of $K_0(\mathrm{Var}/k)$. By Proposition 9.5.9, we may take this system to consist of classes of smooth projective varieties, in which case the assertion is clear. □

Since $\langle\mathbb{P}^1\rangle=\langle\mathrm{Spec}\,k\rangle$, it follows that $\Phi(\mathbb{L})=0$, hence Φ induces a ring homomorphism

$$\overline{\Phi}\colon K_0(\mathrm{Var}/k)/(\mathbb{L})\to \mathrm{Z}[\mathrm{SB}/k].$$

Theorem 10.5.15. ([LL03]) *The above ring homomorphism $\overline{\Phi}$ is an isomorphism.*

Proof. The key point is to show that we can define a map

$$\mathrm{SB}/k\to K_0(\mathrm{Var}/k)/(\mathbb{L})$$

such that whenever X is a smooth projective variety, $\langle X\rangle$ is mapped to $[X]\,\mathrm{mod}\,(\mathbb{L})$. Note first that by Hironaka's theorem on resolution of singularities, for every variety Y over k, there is a smooth projective variety X that is birational to Y. In particular, $\langle X\rangle=\langle Y\rangle$. We claim that if X_1 and X_2 are stably birational smooth projective varieties, then $[X]-[Y]\in(\mathbb{L})$.

Suppose that $X_1\times\mathbb{P}^m$ and $X_2\times\mathbb{P}^n$ are birational. It follows from Theorem 9.5.12 that $X_1\times\mathbb{P}^m$ and $X_2\times\mathbb{P}^n$ are connected by a chain of blow-ups and blow-downs with smooth centers. Note that

$$[X_1]-[X_1\times\mathbb{P}^m]=-[X_1]\cdot\mathbb{L}(1+\mathbb{L}+\ldots+\mathbb{L}^{m-1})\in(\mathbb{L}).$$

Similarly, we have $[X_2]-[X_2\times\mathbb{P}^n]\in(\mathbb{L})$. Therefore in order to prove our claim, it is enough to show the following: if Z and W are smooth projective varieties, with Z a closed subvariety of W, then $[\mathrm{Bl}_Z W]-[W]\in(\mathbb{L})$, where $\mathrm{Bl}_Z(W)$ is the blow-up of

W along Z. Let $r=\operatorname{codim}_W(Z)$, and let E be the exceptional divisor, so $E\simeq\mathbb{P}_Z(N)$, where N is the normal bundle of Z in W. Our claim follows from

$$[\mathrm{Bl}_Z(W)]-[W]=[E]-[Z]=[Z]\cdot[\mathbb{P}^{r-1}]-[Z]=[Z]\cdot\mathbb{L}(1+\mathbb{L}+\ldots+\mathbb{L}^{r-2}).$$

We thus get a group homomorphism $\Psi\colon\mathbb{Z}[\mathrm{SB}/k]\to K_0(\mathrm{Var}/k)/(\mathbb{L})$ such that $\Psi(\langle X\rangle)=[X]\bmod(\mathbb{L})$ for every smooth projective variety X. It is clear that $\overline{\Phi}$ and Ψ are inverse maps, which proves the theorem. □

Remark 10.5.16. It was shown in [Poo02] that for every field k of characteristic 0, the Grothendieck group $K_0(\mathrm{Var}/k)$ is not a domain. The idea of the proof is the following. One shows that there are abelian varieties A and B over k such that $A\times A\simeq B\times B$, but such that $A\times_{\mathrm{Spec}(k)}\mathrm{Spec}(\overline{k})\not\simeq B\times_{\mathrm{Spec}(k)}\mathrm{Spec}(\overline{k})$. Since

$$([A/k]+[B/k])\cdot([A/k]-[B/k])=[A/k]^2-[B/k]^2=0 \text{ in } K_0(\mathrm{Var}/k),$$

it is enough to show that both $[A/k]-[B/k]$ and $[A/k]+[B/k]$ are nonzero in $K_0(\mathrm{Var}/k)$.

One now observes that if $\mathrm{AB}/\overline{k}$ is the semigroup of isomorphism classes of abelian varieties over $\overline{k}$, then there is a semigroup homomorphism $\tau\colon\mathrm{SB}/\overline{k}\to\mathrm{AB}/\overline{k}$ that for an abelian variety V, maps $\langle V\rangle$ to the isomorphism class of V. Recall that for a smooth projective variety X over $\overline{k}$, there is a morphism $f\colon X\to\mathrm{Alb}(X)$ to an abelian variety (the *Albanese variety of* X) that has the following universal property: for every morphism $g\colon X\to V$ to an abelian variety, there is a unique morphism $h\colon\mathrm{Alb}(X)\to V$ such that $h\circ f=g$. We simply define the value of τ on $\langle X\rangle$ to be the isomorphism class of $\mathrm{Alb}(X)$. In order to show that this is well-defined, one proceeds as in the proof of Theorem 9.5.15, and one reduces to showing that $\mathrm{Alb}(X\times\mathbb{P}^n)\simeq\mathrm{Alb}(X)$ and $\mathrm{Alb}(\mathrm{Bl}_Y(X))\simeq\mathrm{Alb}(X)$ whenever X is a smooth projective variety and Y is a smooth closed subvariety. Both assertions follow from the universal property of the Albanese variety and the fact that any rational map from a projective space to an abelian variety is constant. Furthermore, one sees from the universal property that $\mathrm{Alb}(X\times Y)\simeq\mathrm{Alb}(X)\times\mathrm{Alb}(Y)$, hence τ is a semigroup homomorphism.

We thus have a sequence of ring homomorphisms

$$K_0(\mathrm{Var}/k)\to K_0(\mathrm{Var}/\overline{k})\xrightarrow{\Phi}\mathbb{Z}[\mathrm{SB}/\overline{k}]\to\mathbb{Z}[\mathrm{AV}/\overline{k}],$$

where the first one is the pull-back via $\mathrm{Spec}(\overline{k})\to\mathrm{Spec}(k)$ and the third one is induced by τ. Since the images of both $[A/k]-[B/k]$ and $[A/k]+[B/k]$ by the composition of the above homomorphisms are clearly nonzero, we conclude that $K_0(\mathrm{Var}/k)$ is not a domain.

We note that it is an open question whether the localization $K_0(\mathrm{Var}/k)[\mathbb{L}^{-1}]$ is a domain.

Remark 10.5.17. Recall that two varieties X and Y over k are piecewise isomorphic if there are decompositions $X=\sqcup_{i=1}^r X_i$ and $Y=\sqcup_{i=1}^r Y_i$, with X_i and Y_i locally closed in X and Y, respectively, such that $X_i \simeq Y_i$ for every i. It follows from Lemma 9.5.4 that if X and Y are piecewise isomorphic, then $[X] = [Y]$ in $K_0(\mathrm{Var}/k)$. It is an open question (raised by Larsen and Lunts) whether the converse holds. For some results in small dimension, see [LS10].

10.5.2 Motivic integration

We now explain how the results in Sections 9.4.2 and 9.4.3 can be lifted to the level of the Grothendieck ring of varieties. The idea is simply to replace the Hodge-Deligne polynomial by the universal Euler-Poincaré characteristic. In order to do this, there is one more step needed: as in the case of the Hodge-Deligne polynomial, we need to suitably complete the ring where our Euler-Poincaré characteristic takes values.

Let k be an algebraically closed field of characteristic 0. We consider the localization of $K_0(\mathrm{Var}/k)$ obtained by inverting $\mathbb{L}$:

$$\mathscr{M}_k := K_0(\mathrm{Var}/k)[\mathbb{L}^{-1}].$$

For every $m \in \mathbb{Z}$, let $F^m\mathscr{M}_k$ be the subgroup of $\mathscr{M}_k$ generated by

$$\{[Y] \cdot \mathbb{L}^{-N} \mid Y \text{ scheme of finite type over } k,\ \dim(Y) - N \leq -m\}.$$

Note that $F^{m+1}\mathscr{M}_k \subseteq F^m\mathscr{M}_k$ for every m and we consider on $\mathscr{M}_k$ the linear topology induced by this family of subgroups. It is clear from definition that $F^{m_1}\mathscr{M}_k \cdot F^{m_2}\mathscr{M}_k \subseteq F^{m_1+m_2}\mathscr{M}_k$. It is well-known and easy to check that in this case $\mathscr{M}_k$ is a topological ring. Therefore its completion

$$\widehat{\mathscr{M}_k} := \varprojlim_m \mathscr{M}_k/F^m\mathscr{M}_k$$

is a ring, called the *completed Grothendieck ring of varieties over* k, and the canonical morphism $\psi \colon \mathscr{M}_k \to \widehat{\mathscr{M}_k}$ is a ring homomorphism.

Remark 10.5.18. It is not known whether ψ is injective. This leads to several delicate issues, coming from the fact that by going to the completed Grothendieck ring, we might lose some information.

Remark 10.5.19. The Euler-Poincaré characteristic $E\colon K_0(\mathrm{Var}/k) \to \mathbb{Z}[u,v]$ given by the Hodge-Deligne polynomial induces a ring homomorphism $\mathscr{M}_k \to \mathbb{Z}[u^{\pm 1}, v^{\pm 1}]$ (recall that $E(\mathbb{L}) = uv$). It follows from the universal property of the completion that this induces a continuous ring homomorphism

$$\widehat{E} \colon \widehat{\mathscr{M}_k} \to \mathbb{Z}[\![u^{-1}, v^{-1}]\!][u,v].$$

We now define the *motivic measure* of cylinders in the space of arcs of a smooth variety, and more generally, the measure of a limit of cylinders. Let X be a smooth

n-dimensional variety over k. If $C \subseteq X_\infty$ is a cylinder, we write $C = \pi_{\infty,m}^{-1}(S)$ and put

$$[C] := [S] \cdot \mathbb{L}^{-mn} \in \mathscr{M}_k.$$

Since each projection map $X_p \to X_m$, with $p > m$, is locally trivial, with fiber $\mathbb{A}^{(p-m)n}$, we see that $[C]$ is well-defined. It follows from Proposition 9.5.5 that if $C_1, \ldots, C_r$ are pairwise disjoint cylinders in X_∞, then

$$[C_1 \cup \ldots \cup C_r] = \sum_{i=1}^{r} [C_i].$$

With a slight abuse of notation, we denote by $[C]$ also the image of this element in $\widehat{\mathscr{M}_k}$. This should not cause any confusion, since we will always specify the ring we consider. We omit the proofs of the following results, which follow verbatim the proofs of Corollaries 9.4.18 and 9.4.19.

Proposition 10.5.20. *Let X be a smooth variety and C a cylinder in X_∞. If $(C_m)_{m\geq 1}$ is a sequence of pairwise disjoint subcylinders of C, such that there is a proper closed subscheme Y of X and a function $\nu\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{\geq 0}$ with $\lim_{m\to\infty} \nu(m) = \infty$ such that*

$$C \smallsetminus (C_1 \cup \ldots \cup C_m) \subseteq \mathrm{Cont}^{\geq \nu(m)}(Y),$$

then $[C] = \sum_{m\geq 1} [C_m]$ in $\widehat{\mathscr{M}_k}$.

Proposition 10.5.21. *Let $f\colon W \to X$ be a proper, birational morphism between smooth varieties. If $R \subseteq X_\infty$ is a cylinder and $C = f_\infty^{-1}(R)$, then*

$$[R] = \sum_{e\geq 0} [C \cap \mathrm{Cont}^e(K_{W/X})] \cdot \mathbb{L}^{-e} \text{ in } \widehat{\mathscr{M}_k}.$$

As a corollary, one obtains the following version of Kontsevich's theorem.

Corollary 10.5.22. *If X and Y are smooth, projective varieties that are K-equivalent, then $[X]$ and $[Y]$ have the same image in $\widehat{\mathscr{M}_k}$.*

Remark 10.5.23. Since the kernel of the composition $K_0(\mathrm{Var}/k) \to \mathscr{M}_k \to \widehat{\mathscr{M}_k}$ is not understood, we can not conclude from Corollary 9.5.22 that $[X] = [Y]$ in $K_0(\mathrm{Var}/k)$. In fact, it is an open question whether this holds.

We can proceed as in Section 9.4.3 in order to define a motivic version of the stringy E-function of a pair. We omit the proofs, which follow verbatim the ones for the Hodge realizations. If X is a smooth variety and $C \subseteq X_\infty$ is a limit of cylinders and $(C_m)_{m\geq 1}$ is a sequence of cylinders as in Definition 9.4.20, then we put

$$[C] := \sum_{m\geq 1} [C_m] \in \widehat{\mathscr{M}_k}.$$

We see as in the case of $E(C)$ that $[C]$ is well-defined and is independent of the sequence $(C_m)_{m\geq 1}$.

For every positive integer ℓ, we consider the ring

$$\widehat{\mathscr{M}_k}[\mathbb{L}^{1/\ell}] \simeq \widehat{\mathscr{M}_k}[y]/(y^\ell - \mathbb{L}).$$

Note that $\widehat{\mathscr{M}_k}[\mathbb{L}^{1/\ell}]$ is isomorphic as a group with ℓ copies on $\widehat{\mathscr{M}_k}$. This isomorphism induces a topology on $\widehat{\mathscr{M}_k}[\mathbb{L}^{1/\ell}]$ which makes it a topological ring. Note that the inclusion $\widehat{\mathscr{M}_k} \hookrightarrow \widehat{\mathscr{M}_k}[\mathbb{L}^{1/\ell}]$ is a homeomorphism onto image.

Suppose now that $\phi\colon X_\infty \to \frac{1}{\ell}\mathbb{Z} \cup \{\infty\}$ is a function such that $\phi^{-1}(\alpha)$ is a limit of cylinders for every $\alpha \in \frac{1}{\ell}\mathbb{Z}$. The *motivic integral* $\int_{X_\infty} \mathbb{L}^\phi$ is defined by

$$\int_{X_\infty} \mathbb{L}^\phi = \sum_{\alpha \in \frac{1}{\ell}\mathbb{Z}} [\phi^{-1}(\alpha)] \cdot \mathbb{L}^\alpha \in \widehat{\mathscr{M}_k}[\mathbb{L}^{1/\ell}],$$

if the series in convergent.

The following analogue of Proposition 9.4.27 gives a change of variable formula for motivic integrals.

Proposition 10.5.24. *Let $g\colon W \to X$ be a proper birational morphism between two smooth varieties. If $Y^{(1)}, \ldots, Y^{(r)}$ are proper closed subschemes of X and $a_1, \ldots, a_r$ are rational numbers, then for the functions $\phi = \sum_{i=1}^r a_i \cdot \mathrm{ord}_{Y^{(i)}}$ and $\psi = \phi \circ g_\infty - \mathrm{ord}_{K_{W/X}}$ (with the convention that $\psi(\gamma) = \infty$ if either $\phi(g_\infty(\gamma)) = \infty$ or $\mathrm{ord}_{K_{W/X}}(\gamma) = \infty$), the following holds:*

$$\int_{X_\infty} \mathbb{L}^\phi = \int_{W_\infty} \mathbb{L}^\psi,$$

in the sense that one integral exists if and only if the other one does, and if this is the case, then they are equal.

We can now define the motivic version of the stringy E-function. Let (Y,D) be a pair with Y normal and $K_Y + D$ being $\mathbb{Q}$-Cartier. Let ℓ be a positive integer such that $\ell(K_Y + D)$ is a Cartier divisor. For a resolution of singularities $f\colon X \to Y$ of Y, we write as usual $K_X + D_X = f^*(K_Y + D)$. We consider the function $\mathrm{ord}_{D_X}\colon X_\infty \to \frac{1}{\ell}\mathbb{Z} \cup \{\infty\}$ and the *motivic stringy E-function* of the pair (Y,D) is

$$E_{\mathrm{st}}^{\mathrm{mot}}(Y,D) := \int_{X_\infty} \mathbb{L}^{\mathrm{ord}_{D_X}} \in \widehat{\mathscr{M}_k}[\mathbb{L}^{1/\ell}],$$

assuming that this is defined.

Remark 10.5.25. Recall that by Remark 9.5.19, we have a continuous ring homomorphism $\widehat{\mathscr{M}_k} \to \mathbb{Z}[[u^{-1}, v^{-1}]][u,v]$. For every positive integer ℓ, this induces a continuous ring homomorphism $\widehat{\mathscr{M}_k}[\mathbb{L}^{1/\ell}] \to \mathbb{Z}[[u^{-1/\ell}, v^{-1/\ell}]][u^{1/\ell}, v^{1/\ell}]$ that maps $\mathbb{L}^{1/\ell}$ to $(uv)^{1/\ell}$. It follows from definition that if $E_{\mathrm{st}}^{\mathrm{mot}}(Y,D)$ is defined, then it is mapped by this morphism to $E_{\mathrm{st}}(Y,D)$.

Proposition 10.5.26. *If (Y,D) is a pair as above, then the definition of $E_{\mathrm{st}}^{\mathrm{mot}}(Y,D)$ (in particular, the convergence of the corresponding series) is independent of the choice of resolution of singularities.*

Proposition 10.5.27. *Let (Y,D) be a pair as above and $g\colon Z \to Y$ a proper birational morphism, with Z normal. If we write, as usual $K_Z + D_Z = g^*(K_Y + D)$, then*

$$E_{\mathrm{st}}^{\mathrm{mot}}(Z, D_Z) = E_{\mathrm{st}}^{\mathrm{mot}}(Y, D),$$

in the sense that one side exists if and only if the other one does, and if this is the case, then they are equal.

Proposition 10.5.28. *For a pair (Y,D), the motivic stringy E-function is defined if and only if (Y,D) is klt. If this is the case and $f\colon X \to Y$ is a resolution of singularities of Y, with $D_X = \sum_{i=1}^r a_i F_i$ a simple normal crossing divisor, then*

$$E_{\mathrm{st}}^{\mathrm{mot}}(Y,D) = \sum_{J \subseteq \{1,\ldots,r\}} [F_J^\circ] \cdot \prod_{j \in J} \frac{\mathbb{L}-1}{\mathbb{L}^{1-a_i}-1},$$

where for every $J \subseteq \{1,\ldots,r\}$, we put $F_J^\circ = (\cap_{j\in J} F_j) \smallsetminus \big(\cup_{j \notin J} F_j\big)$, with the convention that $F_\emptyset^\circ = X \smallsetminus (F_1 \cup \ldots \cup F_r)$ and the corresponding product is equal to 1.

Proof. If $E_{\mathrm{st}}^{\mathrm{mot}}(Y,D)$ is well-defined, then it follows from Remark 9.5.25 that $E_{\mathrm{st}}(Y,D)$ is well-defined, hence (Y,D) is klt by Proposition 9.4.30. The converse follows if we prove the explicit formula in the proposition, and its proof follows verbatim the proof of Proposition 9.4.30, using the universal Euler-Poincaré characteristic instead of the Hodge-Deligne polynomial. □

Example 10.5.29. Suppose that $Y \subset \mathbb{A}^n$ is the cone over a smooth, projective hypersurface $Z \subset \mathbb{P}^{n-1}$ of degree d, where $n \geq 3$. We have seen in Example 3.1.16 that Y has klt singularities if and only if $d < n$. Moreover, the blow-up $\pi\colon X \to Y$ of 0 gives a log resolution of Y and if F is the exceptional divisor, then $F \simeq Z$ and $K_{X/Y} = (n-1-d)F$. On the other hand, we have $X \smallsetminus F \simeq Y \smallsetminus \{0\}$, and we have a morphism $Y \smallsetminus \{0\} \to Z$ that is locally trivial, with fiber $\mathbb{A}^1 \smallsetminus \{0\}$. Therefore $[X \smallsetminus F] = [Z] \cdot (\mathbb{L}-1)$. It follows from Proposition 9.4.30 that if $d < n$, then

$$E_{\mathrm{st}}^{\mathrm{mot}}(Y) = [F] \cdot \frac{\mathbb{L}-1}{\mathbb{L}^{n-d}-1} + [X \smallsetminus F]$$

$$= [Z] \cdot \frac{\mathbb{L}-1}{\mathbb{L}^{n-d}-1} + [Z](\mathbb{L}-1) = \frac{[Z](\mathbb{L}-1)\mathbb{L}^{n-d}}{\mathbb{L}^{n-d}-1}.$$

Example 10.5.30. We also have a formula for the motivic stringy E-function of toric pairs. With the notation in Example 9.4.32, we have

$$E_{\mathrm{st}}^{\mathrm{mot}}(Y,D) = (\mathbb{L}-1)^n \cdot \sum_{\sigma \in \Delta_X} \prod_{w_j \in \sigma} \frac{1}{\mathbb{L}^{\psi(w_j)-1}}.$$

In terms of the fan of Y, this can be written as

$$E_{\mathrm{st}}^{\mathrm{mot}}(Y,D) = (\mathbb{L}-1)^n \cdot \sum_{w \in |\Delta| \cap N} \mathbb{L}^{-\psi(w)}.$$

Remark 10.5.31. Using deep model-theoretic tools, Cluckers and Loeser gave in [CL08] a much more general and refined construction of motivic integrals. In particular, their work implies that the stringy invariants $E_{\mathrm{st}}^{\mathrm{mot}}(X,D)$ can be defined in the localization of $K_0(\mathrm{Var}/k)$ at the set $\{\mathbb{L}\} \cup \{[\mathbb{P}^N; N \geq 1\}$, and not just in the image of this ring in $\mathscr{M}_k$, as follows from the discussion in this section. In particular, this implies that if X and Y are K-equivalent smooth projective varieties, then $[X] = [Y]$ in this localization of $K_0(\mathrm{Var}/k)$.

10.5.3 The motivic zeta function

Suppose now that X is a smooth variety and D is an effective divisor on X. Instead of computing $E_{\mathrm{st}}^{\mathrm{mot}}(X,D)$, one can extract more information about the pair (X,D) by recording the measures of the contact loci $\mathrm{Cont}^m(D)$ in a generating function. This is the motivic zeta function of Denef and Loeser [DL98] that we now introduce. In fact, we will work with general closed subschemes of X.

Let $n = \dim(X)$. In order to keep more information, we work in the localization of the Grothendieck group of varieties over X, namely $\mathscr{M}_X := K_0(\mathrm{Var}/X)[\mathbb{L}_X^{-1}]$. As before, if $C = \pi_{\infty,m}^{-1}(S)$ is a cylinder in X_∞, we put

$$[C/X] := [S/X] \cdot \mathbb{L}_X^{-mn} \in \mathscr{M}_X,$$

and this is well-defined. Recall that if $a_X : X \to \mathrm{Spec}(k)$ is the structural morphism, then we have an induced group homomorphism $(a_X)_* : K_0(\mathrm{Var}/X) \to K_0(\mathrm{Var}/k)$, further inducing $(a_X)_* : \mathscr{M}_X \to \mathscr{M}_k$. The projection formula gives

$$(a_X)_*([V/X] \cdot \mathbb{L}_X^m) = (a_X)_*([V]) \cdot \mathbb{L}^m.$$

This implies that if $C \subseteq X_\infty$ is a cylinder, then $(a_X)_*([C/S])$ is equal to our old $[C] \in \mathscr{M}_k$.

For a proper closed subscheme Y of X, the *motivic zeta function* of Y is the following generating series

$$Z_Y^{\mathrm{mot}}(T) := \sum_{m=0}^{\infty} [\mathrm{Cont}^m(Y)/X] \cdot T^m \in \mathscr{M}_X[\![T]\!].$$

Remark 10.5.32. The original definition of the motivic zeta function [DL99] had coefficients in $\mathscr{M}_k$. In other words, one considered

$$Z_{Y,X}^{\mathrm{mot}}(T) := \sum_{m=0}^{\infty} [\mathrm{Cont}^m(Y)] \cdot T^m \in \mathscr{M}_k[\![T]\!].$$

Note that $(a_X)_*$ induces a group homomorphism (that we denote in the same way) $\mathscr{M}_X[\![T]\!] \to \mathscr{M}_k[\![T]\!]$ and we have $Z_{Y,X}^{\mathrm{mot}} = (a_X)_*(Z_Y^{\mathrm{mot}})$. The definition of the motivic integrals using the Grothendieck group of varieties over X is due to Looijenga

[Loo02]. We also mention that the definition is usually given for hypersurfaces in the affine space, but working in our more general setting does not cause any additional difficulties.

Remark 10.5.33. One can also consider the following generating function:

$$\widetilde{Z}_{Y,X}^{\mathrm{mot}}(T) := \sum_{m=0}^{\infty} [\mathrm{Cont}^{\geq m}(Y)] \cdot T^m \in \mathscr{M}_k[[T]]$$

(and, of course, one can also define a corresponding lifting in $\mathscr{M}_X[[T]]$). Since $\mathrm{Cont}^{\geq (m+1)}(Y) = (\pi_{\infty,m}^X)^{-1}(Y_m)$, we have

$$\widetilde{Z}_{Y,X}^{\mathrm{mot}}(T) = [X] + \sum_{m \geq 0} [J_m(Y)] \cdot \mathbb{L}^{-mn} T^{m+1}.$$

On the other hand, since

$$[\mathrm{Cont}^m(Y)] = [\mathrm{Cont}^{\geq m}(Y)] \smallsetminus [\mathrm{Cont}^{\geq (m+1)}(Y)],$$

an easy computation implies that $\widetilde{Z}_{Y,X}^{\mathrm{mot}}(T)$ and $Z_{Y,X}^{\mathrm{mot}}(T)$ are related by

$$\widetilde{Z}_{Y,X}^{\mathrm{mot}}(T) \cdot (T-1) + [X] = T \cdot Z_{Y,X}^{\mathrm{mot}}(T).$$

One advantage of working in the Grothendieck ring of varieties over X is that we easily obtain local versions by specialization: given a closed point $x \in X$, the *local motivic zeta function* of Y at x is

$$Z_{Y,x}^{\mathrm{mot}}(T) := \sum_{m=0}^{\infty} [\mathrm{Cont}^m(Y) \cap \pi_\infty^{-1}(x)] \cdot T^m \in \mathscr{M}_k[[T]].$$

It is clear that this is equal to $i_x^*(Z_Y^{\mathrm{mot}}(T))$, where $i_x \colon \mathrm{Spec}(k) \to X$ corresponds to x and $i_x^* \colon \mathscr{M}_X[[T]] \to \mathscr{M}_k[[T]]$ is induced by the ring homomorphism $i_x^* \colon K_0(\mathrm{Var}/X) \to K_0(\mathrm{Var}/k)$.

Remark 10.5.34. In fact, Denef and Loeser define the motivic zeta function to have coefficients in a certain Grothendieck group of varieties over k with group action. More precisely, suppose that D is a divisor in X defined by $f \in \mathscr{O}(X)$. In this case, there is a morphism $\mathrm{Cont}^m(D) \to \mathbb{A}^1 \smallsetminus \{0\}$ that takes γ to the coefficient of t^m in $\gamma^*(f) \in k[[t]]$. Note that the fiber $\mathrm{Cont}^m(D)^\circ$ of this map over 1 is a cylinder and $[\mathrm{Cont}^m(D)] = [\mathrm{Cont}^m(D)^\circ] \cdot (\mathbb{L}-1)$. Indeed, let us consider the corresponding map $\mathrm{Cont}^m(D)_p \to \mathbb{A}^1 \smallsetminus \{0\}$ at a finite level $p \geq m$ and the fiber $\mathrm{Cont}^m(D)_p^\circ$ over 1. Recall that we have an action of $\mathbb{G}_m$ on $\mathrm{Cont}^m(D)_p$ induced by $t \to \lambda t$, and the morphism to $\mathbb{A}^1 \smallsetminus \{0\}$ is compatible with this action. This easily implies that $\mathrm{Cont}^m(D)_p \simeq \mathrm{Cont}^m(D)_p^\circ \times (\mathbb{A}^1 \smallsetminus \{0\})$, hence $[\mathrm{Cont}^m(D)] = [\mathrm{Cont}^m(D)^\circ] \cdot (\mathbb{L}-1)$. We can thus rewrite

$$Z_{D,X}^{\mathrm{mot}}(T) = \mathbb{L}^{-1} \cdot \sum_{m \geq 0} [\mathrm{Cont}^m(D)^\circ] \cdot T^m. \tag{10.16}$$

On the other hand, each $\mathrm{Cont}^m(D)_p^\circ$ still carries an action of the group μ_m of m^{th} roots of 1 in k. If $\widehat{\mu} = \varprojlim_m \mu_m$ (where for $d|m$, the morphism $\mu_m \to \mu_d$ is given by $\lambda \to \lambda^{m/d}$), then one considers the category of schemes of finite type over k with an algebraic action of $\widehat{\mu}$, which factors through some μ_m. One defines a Grothendieck group of such schemes $K_0^\mu(\mathrm{Var}/k)$ and one defines a lift of the series in (9.16) to a formal power series with coefficients in $K_0^\mu(\mathrm{Var}/k)[\mathbb{L}^{-1}]$. There are some subtleties in the definition of this more refined Grothendieck ring and in its connection to $\mathscr{M}_k$, for which we refer to [DL98] and [Loo02].

Denef and Loeser define in [DL98], using the motivic zeta function, a "motivic incarnation" for the nearby cycles of D. It is then useful to work in the Grothendieck ring of varieties with $\widehat{\mu}$-action, in order to also recover the monodromy action on the nearby cycles. However, we will not pursue this further in what follows.

Theorem 10.5.35. *Let X be an n-dimensional smooth variety and Y a proper closed subscheme on X. If $f\colon W \to X$ is a log resolution of (X,Y) which is an isomorphism over $X \smallsetminus Y$ and if $f^{-1}(Y) = \sum_{i=1}^r a_iF_i$ and $K_{W/X} = \sum_{i=1}^r k_iF_i$, where the F_i are distinct prime divisors, then*

$$Z_Y^{\mathrm{mot}}(T) = \sum_{J\subseteq\{1,\ldots,r\}} [F_J^\circ/X] \cdot \prod_{j\in J} \frac{(\mathbb{L}-1)T^{a_j}}{\mathbb{L}^{k_j+1} - T^{a_j}},$$

where $F_J^\circ = \bigcap_{j\in J} F_j \smallsetminus \bigcup_{i\notin J} F_i$. In particular, $Z_Y^{\mathrm{mot}}(T)$ is a rational function.

Proof. We may assume that $a_i \geq 1$ for all i (note that by assumption f is an isomorphism over $X \smallsetminus Y$, hence $K_{W/X}$ is supported on $f^{-1}(Y)$). For every $\nu \in \mathbb{Z}_{\geq 0}^r$, we put

$$C_\nu = \bigcap_{i=1}^r \mathrm{Cont}^{\nu_i}(F_i) \subseteq W_\infty.$$

It is clear that

$$f_\infty^{-1}(\mathrm{Cont}^m(Y)) = \mathrm{Cont}^m(f^{-1}(Y)) = \bigsqcup_{\sum_i a_i\nu_i = m} C_\nu.$$

Note that this is a finite union, since all a_i are positive. Furthermore, it follows from Proposition 9.2.8 that f_∞ is bijective over $\mathrm{Cont}^m(Y)$, hence

$$\mathrm{Cont}^m(Y) = \bigsqcup_{\sum_i a_i\nu_i = m} f_\infty(C_\nu).$$

On the other hand, it follows from Corollary 9.3.21 and its proof that each $f_\infty(C_\nu)$ is a cylinder in X_∞ and

$$[f_\infty(C_\nu)/X] = [C_\nu/X] \cdot \mathbb{L}^{-\sum_i k_i\nu_i}.$$

Moreover, it follows from the proof of Proposition 9.4.30 that if $J \subseteq \{1,\ldots,r\}$ is such that $\nu_i \geq 1$ precisely when $i \in J$, then

$$[C_\nu/X] = [F_J^\circ/X]\cdot(\mathbb{L}-1)^{|J|}\mathbb{L}^{-\sum_i \nu_i}.$$

By putting these together, we conclude that

$$Z_Y^{\mathrm{mot}}(T) = \sum_{\nu\in\mathbb{Z}_{\geq 0}^r} [C_\nu/X]\cdot \mathbb{L}^{-\sum_i k_i\nu_i} T^{\sum_i a_i\nu_i}$$

$$= \sum_{J\subseteq\{1,\ldots,r\}} \sum_{\nu\in\mathbb{Z}_{>0}^{|J|}} [F_J^\circ/X]\cdot(\mathbb{L}-1)^{|J|}\mathbb{L}^{-\sum_i(k_i+1)\nu_i} T^{\sum_i a_i\nu_i}$$

$$= \sum_{J\subseteq\{1,\ldots,r\}} [F_J^\circ/X]\cdot \prod_{j\in J}\frac{(\mathbb{L}-1)T^{a_j}}{\mathbb{L}^{k_j+1}-T^{a_j}}.$$

□

Remark 10.5.36. The formula in Theorem 9.5.35 induces an obvious formula for $Z_{Y,X}^{\mathrm{mot}}$. Moreover, by restricting over a closed point $x\in X$, we obtain

$$Z_{Y,x}^{\mathrm{mot}} = \sum_{J\subseteq\{1,\ldots,r\}} [F_J^\circ\cap f^{-1}(x)]\cdot \prod_{j\in J}\frac{(\mathbb{L}-1)T^{a_j}}{\mathbb{L}^{k_j+1}-T^{a_j}},$$

Example 10.5.37. Suppose that Y is a smooth subvariety of the smooth variety X, of codimension $r\geq 1$. In this case one can compute the motivic zeta function directly or one can use the formula in Theorem 9.5.35 for the blow-up of X along Y to conclude

$$Z_Y^{\mathrm{mot}}(T) = [(X\smallsetminus Y)/X] + [Y/X]\cdot\frac{(\mathbb{L}-1)T}{\mathbb{L}^r - T}.$$

Example 10.5.38. Suppose that X is a smooth surface and $C\subset X$ is a curve having a unique singular point x, which is a node. As we have seen in Example 3.1.17, the blow-up $f\colon W\to X$ of x, with exceptional divisor E, gives a log resolution of (X,C). Moreover, we have $K_{W/X}=E$ and $f^*(C)=\widetilde{C}+2E$, where $\widetilde{C}$ is the proper transform of C. Note that $E\simeq\mathbb{P}^1$ and E intersects $\widetilde{C}$ in two points. Since $\widetilde{C}\smallsetminus E\simeq C\smallsetminus\{x\}$, it follows from Theorem 9.5.35 that

$$Z_{C,X}^{\mathrm{mot}}(T) = [X\smallsetminus C] + [C\smallsetminus\{x\}]\cdot\frac{(\mathbb{L}-1)T}{\mathbb{L}-T} + (\mathbb{L}-1)\cdot\frac{(\mathbb{L}-1)T^2}{\mathbb{L}^2-T^2} + 2\frac{(\mathbb{L}-1)^2T^3}{(\mathbb{L}^2-T^2)(\mathbb{L}-T)}$$

$$= [X\smallsetminus C] + [C\smallsetminus\{x\}]\cdot\frac{(\mathbb{L}-1)T}{\mathbb{L}-T} + \frac{(\mathbb{L}-1)^2T^2}{(\mathbb{L}-T)^2}.$$

Example 10.5.39. It is sometimes easier to compute directly $Z_{D,X}^{\mathrm{mot}}$, rather than use resolution of singularities. Suppose, for example, that $X=\operatorname{Spec} k[x,y]$ and D is the prime divisor defined by (x^a-y^b), where a and b are relatively prime positive integers. Let us compute $\widetilde{Z}_{D,X}^{\mathrm{mot}}(T)$. Note that for every m, we have a decomposition

$$\mathrm{Cont}^{\geq m}(D) = C_{m,1}\cup C_{m,2},$$

where

$$C_{m,1} = \{(u,v) \in (k[\![t]\!])^2 \mid \mathrm{ord}(u) \geq m/a, \mathrm{ord}(v) \geq m/b\} \text{ and}$$
$$C_{m,2} = \{(u,v) \in (k[\![t]\!])^2 \mid \mathrm{ord}(u^a) = \mathrm{ord}(v^b) < m\}.$$

We thus obtain $\widetilde{Z}^{\mathrm{mot}}_{D,X}(T) = S_1 + S_2$, where

$$S_1 = \sum_{m\geq 0} [C_{m,1}]T^m \text{ and } S_2 = \sum_{m\geq 0} [C_{m,2}]T^m.$$

Note that if $(u,v) \in C_{m,2}$, then there is p with $m > pab$ such that $u = t^{pb}u'$, $v = t^{pa}v'$, and $(u',v') \in \mathrm{Cont}^{\geq(m-pab+1)}(D) \cap \pi_\infty^{-1}(X \smallsetminus \{0\})$. For every $p < m/ab$, these conditions define a subcylinder $C^{(p)}_{m,2}$ of $C_{m,2}$. Since $D \smallsetminus \{0\}$ is smooth, it is easy to deduce that

$$[C^{(p)}_{m,2}] = [D \smallsetminus \{0\}] \cdot \mathbb{L}^{-(pa+pb+m-pab)}.$$

Therefore

$$S_2 = [D \smallsetminus \{0\}] \cdot \sum_{p;m\geq pab} \mathbb{L}^{pab-pa-pb}(\mathbb{L}^{-1}T)^m = [D \smallsetminus \{0\}] \cdot \sum_{p\geq 0} \frac{\mathbb{L}^{-(pa+pb)}T^{pab}}{1-\mathbb{L}^{-1}T}$$
$$= \frac{[D \smallsetminus \{0\}]}{(1-\mathbb{L}^{-1}T)(1-\mathbb{L}^{-(a+b)}T^{ab})}.$$

On the other hand, it is clear that

$$[C_{m,1}] = \mathbb{L}^{-\lceil m/a\rceil - \lceil m/b\rceil}.$$

We deduce that

$$S_1 = \left(\sum_{m=0}^{ab-1} \mathbb{L}^{-\lceil m/a\rceil - \lceil m/b\rceil} T^m\right) \cdot \left(\sum_{\ell\geq 0} \mathbb{L}^{-\ell(a+b)}T^{\ell ab}\right) = \frac{\sum_{m=0}^{ab-1} \mathbb{L}^{-\lceil m/a\rceil - \lceil m/b\rceil} T^m}{1-\mathbb{L}^{-(a+b)}T^{ab}},$$

hence

$$\widetilde{Z}^{\mathrm{mot}}_{D,X}(T) = \frac{\sum_{m=0}^{ab-1} \mathbb{L}^{-\lceil m/a\rceil - \lceil m/b\rceil} T^m}{1-\mathbb{L}^{-(a+b)}T^{ab}} + \frac{[D \smallsetminus \{0\}]}{(1-\mathbb{L}^{-1}T)(1-\mathbb{L}^{-(a+b)}T^{ab})}.$$

Example 10.5.40. Suppose that $X = \mathbb{A}^n$, with $n \geq 2$, and $H \subset X$ is the cone over a smooth, projective, degree d hypersurface Z in $\mathbb{P}^{n-1}$. If $f\colon W \to X$ is the blow-up of 0, with exceptional divisor E, then f gives a log resolution of (X,H), see Example 3.1.16. If $\widetilde{H}$ is the proper transform of H, then $f^*(H) = \widetilde{H} + dE$ and $K_{W/X} = (n-1)E$. Moreover, we have $E \simeq \mathbb{P}^{n-1}$ and $\widetilde{H} \cap E \simeq Z$. Note also that $\widetilde{H} \smallsetminus E \simeq H \smallsetminus \{0\}$ and $H \smallsetminus \{0\}$ is locally trivial over Z, with fiber $\mathbb{A}^1 \smallsetminus \{0\}$. Therefore $[\widetilde{H} \smallsetminus E] = [Z] \cdot (\mathbb{L}-1)$. Similarly, we have $[W \smallsetminus (E \cup \widetilde{H})] = [\mathbb{P}^{n-1} \smallsetminus Z] \cdot (\mathbb{L}-1)$. We deduce from Theorem 9.5.35 that

$$Z^{\mathrm{mot}}_{H,X}(T) = [\mathbb{P}^{n-1} \smallsetminus Z] \cdot (\mathbb{L}-1) + [Z] \cdot \frac{(\mathbb{L}-1)^2 T}{\mathbb{L}-T} + [\mathbb{P}^{n-1} \smallsetminus Z] \cdot \frac{(\mathbb{L}-1)T^d}{\mathbb{L}^n - T^d}$$

$$+[Z] \cdot \frac{(\mathbb{L}-1)^2 T^{d+1}}{(\mathbb{L}-T)(\mathbb{L}^n - T^d)} = [\mathbb{P}^{n-1} \smallsetminus Z] \cdot \frac{(\mathbb{L}-1)\mathbb{L}^n}{\mathbb{L}^n - T^d} + [Z] \cdot \frac{(\mathbb{L}-1)^2 \mathbb{L}^n T}{(\mathbb{L}-T)(\mathbb{L}^n - T^d)}.$$

We now introduce an important specialization of the motivic zeta function. Let us assume that we work over $k = \mathbb{C}$. Motivated by the analogy with the p-adic zeta function (see Section 9.5.4 below), it is natural to try to evaluate the motivic zeta function at $T = \mathbb{L}^{-s}$. In order to make sense of this, let us assume that s is a nonnegative integer. In this case

$$Z^{\mathrm{mot}}_{Y,X}(\mathbb{L}^{-s}) = \sum_{m \geq 0} [\mathrm{Cont}^m(Y)] \cdot \mathbb{L}^{-sm} = \int_{X_\infty} \mathbb{L}^{-s \cdot \mathrm{ord}_Y}.$$

Note that this is well-defined in $\widehat{\mathscr{M}_k}$: this is an immediate consequence of Proposition 9.3.3. We compute it using Theorem 9.5.35. With the notation in the theorem, we obtain

$$Z^{\mathrm{mot}}_{Y,X}(\mathbb{L}^{-s}) = \sum_{J \subseteq \{1,\ldots,r\}} [F_J^\circ] \cdot \prod_{j \in J} \frac{(\mathbb{L}-1)}{\mathbb{L}^{sa_j + k_j + 1} - 1}.$$

Recall now that we have a ring homomorphism $\widehat{E} \colon \widehat{\mathscr{M}_k} \to \mathbb{Z}[[u^{-1}, v^{-1}]][u,v]$ (see Remark 9.5.19). By applying this to $Z^{\mathrm{mot}}_{Y,X}(\mathbb{L}^{-s})$, we obtain

$$\widehat{E}(Z^{\mathrm{mot}}_{Y,X}(\mathbb{L}^{-s})) = \sum_{J \subseteq \{1,\ldots,r\}} E(F_J^\circ) \cdot \prod_{j \in J} \frac{1}{\sum_{\ell=0}^{sa_j + k_j} (uv)^\ell}.$$

We can further evaluate this rational function at $u = v = 1$ to obtain the rational number

$$\sum_{J \subseteq \{1,\ldots,r\}} \chi^{\mathrm{top}}(F_J^\circ) \cdot \prod_{j \in J} \frac{1}{sa_j + k_j + 1}. \qquad (10.17)$$

Definition 10.5.41. Given a smooth complex variety X and a proper closed subscheme Y of X, the *topological zeta function* of Y is the rational function that in terms of a log resolution as above, is given by

$$Z^{\mathrm{top}}_Y = \sum_{J \subseteq \{1,\ldots,r\}} \chi^{\mathrm{top}}(F_J^\circ) \cdot \prod_{j \in J} \frac{1}{sa_j + k_j + 1}. \qquad (10.18)$$

Of course, one does not need the motivic zeta function in order to make this definition. The issue, however, is independence of the log resolution. The above computation shows that for every nonnegative integer s, the value $Z^{\mathrm{top}}_Y(s)$ is equal to the expression in (9.17), obtained by the above specialization procedure from the motivic zeta function of Y. Since a rational function is uniquely determined by its values on an infinite set, we obtain the independence on the choice of log resolution.

The topological zeta function was introduced by Denef and Loeser in [DL92] and its independence of log resolution was proved using p-adic integration. The above argument using the motivic zeta function was given in [DL98].

The main open question concerning this circle of ideas is the so-called monodromy conjecture. This was first made by Igusa in the setting of p-adic zeta functions. It admits analogues in the setting of motivic zeta functions or topological zeta functions, due to Denef and Loeser. Since both the topological zeta function and the p-adic zeta functions can be obtained by specialization from the motivic one[4], the strongest statement is the one involving the motivic zeta function. In fact, in each case there are two statements, a weaker one in terms of the monodromy action on the cohomology of the Milnor fiber and a stronger one in terms of the roots of the Bernstein polynomial (the fact that the second formulation implies the first one is a consequence of Malgrange's Theorem **??**).

Conjecture 10.5.42 (Monodromy conjecture for the motivic zeta function). If X is a smooth complex variety and D is a divisor on X, then $Z^{\mathrm{mot}}_{D,X}(T)$ lies in the subring of $\mathscr{M}_k[[T]]$ generated by $\mathscr{M}_k$ and $\frac{(\mathbb{L}-1)T^N}{\mathbb{L}^\nu - T^N}$, where ν and N vary over the positive integers such that

i) $\exp\left(-2\pi i\frac{\nu}{N}\right)$ is an eigenvalue for the monodromy action on the Milnor fiber of D at some point $x \in D$ (weak version), or
ii) $-\frac{\nu}{N}$ is a root of the Bernstein-Sato polynomial attached to D (strong version).

A positive answer to this conjecture would imply a positive answer to the next one.

Conjecture 10.5.43 (Monodromy conjecture for the topological zeta function). If X is a smooth complex variety and D a divisor on X, then for every pole s of $Z^{\mathrm{top}}_D(T)$, the following holds:

i) $\exp(2\pi i \operatorname{Re}(s))$ is an eigenvalue for the monodromy action on the Milnor fiber of D at some point $x \in D$ (weak version), or
ii) $\operatorname{Re}(s)$ is a root of the Bernstein-Sato polynomial attached to D (strong version).

The above conjectures, as well as the corresponding one in the p-adic setting have generated a lot of interest and many special cases are known. The strong version has been checked when $X = \mathbb{A}^2$ (in the p-adic context) in [**?**]. More is known about the weak version: this holds, for example, for the motivic zeta function of a hypersurface in $\mathbb{A}^3$ that is non-degenerate with respect to the Newton polyhedron [BV], for non-degenerate hypersurfaces in arbitrary dimension, under some restrictive conditions (this was shown in the p-adic setting in [Loe90]), for quasi-ordinary hypersurfaces [ABCNLMH05], and for hyperplane arrangements [BMT11]. In general, the weak version seems more amenable, since A'Campo formula (see Theorem **??**) describes the zeta function of the monodromy action in terms of a log resolution. In order to prove, for example, the weak version of Conjecture 9.5.43 it is enough to find a log

[4] For the precise statement in the latter case, see Section 9.5.4 below.

resolution and show that certain candidate poles for the topological zeta function as described in (9.18) are not really poles and that the remaining ones appear in the monodromy zeta function.

Remark 10.5.44. One can formulate the monodromy conjecture also for general closed subschemes. The weak version is in terms of *Verdier monodromy* (see for example [VPV10] where this is checked for the topological zeta function of a subscheme of $\mathbb{A}^2$). The strong version is in terms of the Bernstein-Sato polynomial of a subscheme, in the sense of [BMS06]. It is checked, for example, in the case of motivic zeta functions of monomial subschemes of $\mathbb{A}^n$ in [HMY07].

10.5.4 A brief summary of Archimedean and p-adic zeta functions

We first discuss the Archimedean side of the story. Suppose that $f \in \mathbb{C}[x_1, \ldots, x_n]$ is a non-constant polynomial. If $\phi \in \mathscr{C}_0^\infty(\mathbb{C}^n)$ is a $\mathscr{C}^\infty$ function on $\mathbb{C}^n$, with compact support, then it is easy to see that for every $s \in \mathbb{C}$, with $\mathrm{Re}(s) > 0$, the following integral is well-defined:

$$Z_{f,\phi}(s) := \int_{\mathbb{C}^n} |f(z)|^{2s} \phi(z) dz d\bar{z}.$$

Moreover, this is a holomorphic function[5] on $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > 0\}$. The story started in 1954, with the following problem of I. Gel'fand: show that $Z_{f,\phi}$ admits a meromorphic continuation to $\mathbb{C}$. There is also a real version of the problem, in which f has real coefficients. In this case, it is more natural to put

$$Z_{f,\phi}(s) := \int_{\mathbb{R}^n} |f(x)|^{s} \phi(x) dx.$$

One case of the problem that is easy to handle is that when $f = x_1^{a_1} \cdots x_n^{a_n}$ is a monomial. In this case one can use, for example, integration by parts to show that $Z_{f,\phi}$ admits a meromorphic continuation, with all poles of the form $-\frac{j}{a_i}$ for some i and some positive integer j. The general case of the problem has been solved in two ways, and both solutions turned out to be very influential. The first argument was given independently by Atiyah [Ati70] and Bernstein and S. Gel'fand [BG69], using Hironaka's theorem on resolution of singularities. The idea is that given a log resolution of singularities $\pi\colon Y \to \mathbb{A}^n$ of $(\mathbb{A}^n, V(f))$, one can use the change of variable formula to compute $Z_{f,\phi}(s)$ as an integral on $Y(\mathbb{C})$ (or $Y(\mathbb{R})$, depending on the context). In this case, one is reduced essentially to the monomial case. Note that this argument is very close to the one that we gave for the analytic interpretation of multiplier ideals and log canonical threshold in Chapter 4.6. An upshot of

[5] In fact, it is natural to also let ϕ vary. In this way one obtains a holomorphic function on $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > 0\}$ with values in distributions. The value at s is denoted by $|f|^{2s}$, the *complex power* of f at s.

this method is that given such a resolution, if we write $\pi^*(V(f)) = \sum_{i=1}^N a_i E_i$ and $K_{Y/\mathbb{A}^n} = \sum_{i=1}^N k_i E_i$, then the poles of $Z_{f,\phi}$ are among the rational numbers $-\frac{k_i+j}{a_i}$, with $1 \leq i \leq N$ and $j \in \mathbb{Z}_{>0}$. In particular, we see that $Z_{f,\phi}$ is holomorphic in the half-plane $\{s \mid \mathrm{Re}(s) > -\mathrm{lct}(f)\}$.

The second proof of Gel'fand's problem was obtained only a couple of years later by Bernstein [Ber72]. In order to achieve this, Bernstein developed the theory of D-modules on the affine space, and in particular, he proved the existence of what is nowadays called the Bernstein-Sato polynomial (see Chapter 4.7 for a discussion of this invariant). The main point is that the functional equation

$$b_f(s) f^s = P(s, x, \partial_x) \bullet f^s$$

allows applying integration by parts directly for f, without making use of resolution of singularities. As a consequence of this method, one obtains that the poles of $Z_{f,\phi}$ are of the form $\lambda - m$, where λ is a root of b_f and m is a nonnegative integer. It is instructive to compare the two estimates for the poles obtained via the two methods, keeping in mind that in the presence of a log resolution as above, by Lichtin's Theorem **??**, every root λ of b_f is of the form $\lambda = -\frac{k_i+j}{a_i}$, for some i and some $j \in \mathbb{Z}_{>0}$.

Let us discuss now the p-adic side of the story. Suppose that p is a positive prime integer and K is a finite extension of the field $\mathbb{Q}_p$ of p-adic rational numbers (for example, one can simply take $K = \mathbb{Q}_p$). The integral closure of the ring $\mathbb{Z}_p$ of p-adic integers in K is a complete DVR denoted by O_K. Let π be a generator of the maximal ideal of O_K and $q = p^r$ the number of elements in the residue field of O_K. Igusa introduced in [Igu74], [Igu75] the following p-adic analogue of the complex powers. If $f \in K[x_1, \ldots, x_n]$ is a non-constant polynomial, then the *local zeta function* (or *p-adic zeta function*) of f is given by

$$Z_{f,K}(s) = \int_{O_K^n} |f(x)|_p^s d\mu$$

where $s \in \mathbb{C}$ (note that since O_K^n is compact, in this case one does not have to use the auxiliary function ϕ). In the above integral, the absolute value is the p-adic one, given by $|u|_p = \left(\frac{1}{q}\right)^{\mathrm{ord}(u)}$, where $\mathrm{ord}(-)$ is the discrete valuation on O_K. The measure is the product measure on K^n of the Haar measure on K. Explicitly, the measure on K is characterized by the fact that it is invariant under translations and $\mu(O_K) = 1$. These conditions imply that

$$\mu\left(a + \prod_{i=1}^n (\pi^{m_i} O_K)\right) = \left(\frac{1}{q}\right)^{\sum_i m_i}$$

for every $a \in K^n$ and every $m_1, \ldots, m_n \in \mathbb{Z}$. It is again easy to check that $Z_{f,K}(s)$ is well-defined when $\mathrm{Re}(s) > 0$ and it gives a holomorphic function in this half-plane.

In fact, $Z_{f,K}$ has a very down-to-earth interpretation. After possibly multiplying f by a power of π, we may assume that $f \in O_K[x_1, \ldots, x_n]$. It then follows from the definition of the integral that

$$Z_{f,K}(s) = \sum_{m\in\mathbb{Z}_{\geq 0}} \mu(\{u \in O_K^n \mid \mathrm{ord}(f(u)) = m\}) \cdot \left(\frac{1}{q}\right)^{ms}. \tag{10.19}$$

Moreover, note that the set $\{u \in O_K^n \mid \mathrm{ord}(f(u)) \geq m\}$ is the disjoint union of a_m translates of $\prod_{i=1}^n (\pi^m O_K)$, where

$$a_m = \#\{\overline{u} \in (O_K/\pi^m O_K)^n \mid f(\overline{u}) = 0\},$$

with the convention $a_0 = 1$. Therefore

$$\mu(\{u \in O_K^n \mid \mathrm{ord}(f(u)) = m\}) = \frac{a_m}{q^{mn}} - \frac{a_{m+1}}{q^{(m+1)n}}. \tag{10.20}$$

If one considers the *Poincaré power series* of f

$$P_{f,K}(T) := \sum_{m\geq 0} \frac{a_m}{q^{mn}} T^m \in \mathbb{Q}[\![T]\!],$$

then an easy computation using (9.19) and (9.20) shows that $P_{f,K}$ is related to $Z_f(s)$ by

$$1 - tZ_{f,K}(s) = (1-t)P_{f,K}(s),$$

for $\mathrm{Re}(s) > 0$, where $t = q^{-s}$.

Using the above explicit description of Igusa's zeta function, it is easy to compute $Z_f(s)$ when $f = x_1^{a_1} \cdots x_n^{a_n}$ is a monomial. In this case one deduces that $Z_{f,K}(s)$ is a rational function of q^{-s}, with the denominator $\prod_{i=1}^n (1 - q^{-(a_i s+1)})$. A fundamental result of Igusa is that for *every* f, the local zeta function $Z_{f,K}(s)$ is a rational function of q^{-s}. In particular, it admits a meromorphic continuation to $\mathbb{C}$. Moreover, in light of the above relation with the Poincaré power series, this implies that $P_{f,K}$ is a rational function, a statement that had been conjectured by Borevich and Shafarevich.

The idea is to use a log resolution (over K) and the change of variable formula for p-adic integrals. In this case, one can again reduce to a monomial computation, though in this case the argument is considerably more involved. Given a resolution $\pi\colon Y \to \mathbb{A}_K^n$, if we write $\pi^*(V(f)) = \sum_{i=1}^N a_i E_i$ and $K_{Y/X} = \sum_{i=1}^N k_i E_i$, then Igusa showed that

$$Z_{f,K}(s) = \sum_J \frac{h_J(q^{-s})}{\prod_{i\in J}(1 - q^{-(a_i s + k_i + 1)})},$$

where the sum is over those $J \subseteq \{1, \ldots, N\}$ such that $\cap_{i\in J} E_i \neq \emptyset$, and where each h_J is a polynomial. In particular, we see that if s is a pole of $Z_{f,K}$, then there is i such that $\mathrm{Re}(s) = -\frac{k_i+1}{a_i}$. It is interesting to compare this result with the corresponding estimate for the poles in the Archimedean setting.

Motivated by many examples, Igusa made his monodromy conjecture concerning the poles of the local zeta functions. If $f \in L[x_1, \ldots, x_n]$, where L is a number field, then the conjecture says that for almost all p-adic completions K of L, if s is a pole of $Z_{f,K}$, then $\exp(2\pi i \mathrm{Re}(s))$ is an eigenvalue for the monodromy action on the cohomology of the Milnor fiber of f. The stronger version of the conjecture predicts that in fact, in this case $\mathrm{Re}(s)$ is a root of the Bernstein-Sato polynomial b_f of f. We stress that in the setting of p-adic integrals there is no integration by parts, which makes this conjectural relation to the Bernstein-Sato polynomial very striking. For an introduction to both Archimedean and p-adic zeta functions, we refer the reader to Igusa's book [Igu00]. For a detailed discussion of the monodromy conjecture for the p-adic zeta functions, see Denef's Bourbaki talk [Den91].

The analogy between Igusa's zeta function and the motivic zeta function is pretty transparent: one replaces O_K by another type of complete DVR, the formal power series ring $\mathbb{C}[[t]]$, and the role of $q = \#\mathbb{A}^1(\mathbf{F}_q)$ is played by $\mathbb{L}$. In fact, one can prove a precise connection between the motivic zeta function and the p-adic one, under the assumption that one has a log resolution of f that has good reduction (that is, the resolution is defined over O_K and it induces a log resolution also when taking the fiber over the closed point of $\mathrm{Spec}(O_K)$). Note that when we start with a polynomial over a number field L, this will be the case for almost all of the p-adic completions of L. For the precise formula relating the motivic and the p-adic zeta functions in the good reduction case, see [DL98].

10.6 Applications to singularities

In this section we give some applications of the birational transformation formula to singularities of pairs for which the ambient variety is smooth. The key point is that one can set a dictionary between divisorial valuations and valuations associated to cylinders such that the log discrepancy corresponds to the codimension of the cylinder. This allows the description of invariants like the log canonical threshold and minimal log discrepancy in terms of codimensions of contact loci and allows proving some properties of these invariants by elementary geometric arguments.

10.6.1 Divisorial valuations and cylinders

We give a description of divisorial valuations in terms of cylinders in the space of arcs. Let X be a fixed smooth variety over an algebraically closed field k of characteristic 0.

We first show that if C is an irreducible, closed cylinder, then we can associate to C a valuation ord_C of the function field of X. Let ξ be the generic point of $\overline{\pi_\infty(C)}$ and suppose that $f \in \mathscr{O}_{X,\xi}$ is nonzero. If U is an open neighborhood of ξ such that $f \in \mathscr{O}_X(U)$, then we put

$$\mathrm{ord}_C(f) := \min\{\mathrm{ord}_{V(f)}(\gamma) \mid \gamma \in C_U := C \cap \pi_\infty^{-1}(U)\}.$$

Note that since $C_U \not\subseteq V(f)_\infty$ by Lemma 9.3.7, we have $\mathrm{ord}_C(f) \in \mathbb{Z}_{\geq 0}$. Moreover, we have $\mathrm{ord}_C(f) = \mathrm{ord}_{V(f)}(\gamma)$ for all γ is a suitable open subcylinder of C_U. Since C is irreducible, every open subcylinder of C is dense; in particular, the definition of $\mathrm{ord}_C(f)$ is independent of the choice of U. As usual, we put $\mathrm{ord}_C(0) = \infty$. Given two nonzero functions $f_1, f_2 \in \mathscr{O}_{X,\xi}$, we can choose U as above such that $f_1, f_2 \in \mathscr{O}_X(U)$. Since C is irreducible and since $\mathrm{ord}_C(f_i)$ is achieved on an open subcylinder of C_i, it is clear that we have

i) $\mathrm{ord}_C(f_1 + f_2) \geq \min\{\mathrm{ord}_C(f_1), \mathrm{ord}_C(f_2)\}$ and
ii) $\mathrm{ord}_C(f_1 f_2) = \mathrm{ord}_C(f_1) + \mathrm{ord}_C(f_2)$.

This implies that ord_C can be extended to a valuation of the fraction field of X with values in $\mathbb{Z}$. It follows from definition that given a nonzero f defined on an open neighborhood of ξ, we have $\mathrm{ord}_C(f) = 0$ if and only if $\pi_\infty(C) \not\subseteq V(f)$. Indeed, if f is defined on U and $\gamma \in U_\infty$, then $\mathrm{ord}_t(\gamma^*(f)) = 0$ if and only if $\pi_\infty(\gamma) \in V(f)$. We say that C is *non-dominating* if $\overline{\pi_\infty(C)} \neq X$. We see that this is the case if and only if ord_C is not the trivial valuation (recall that the trivial valuation is the one identically equal to 0 on all nonzero elements).

We can also define $\mathrm{ord}_C(\mathfrak{a})$ when $\mathfrak{a}$ is an ideal sheaf in X, as follows. If U is an affine open subset of X intersecting $\pi_\infty(C)$, then

$$\mathrm{ord}_C(\mathfrak{a}) := \min\{\mathrm{ord}_C(f) \mid f \in \mathfrak{a}(U)\}.$$

It is clear that the definition is independent of U and that $\mathrm{ord}_C(\mathfrak{a}) \geq m$ if and only if $C \subseteq \mathrm{Cont}^{\geq m}(\mathfrak{a})$.

Remark 10.6.1. Suppose that $\phi \colon Y \to X$ is a proper, birational morphism of smooth varieties. If C is an irreducible closed cylinder in Y_∞, then $C_X := \overline{\phi_\infty(C)}$ is an irreducible, closed cylinder in X_∞. Indeed, Corollary 9.3.21 implies that this is a cylinder and the other properties are obvious. For every $\delta \in Y_\infty$, if $\gamma = \phi_\infty(\delta)$, then for every $f \in \mathscr{O}_X(U)$, where U is an open neighborhood of $\pi^X_\infty(\gamma)$, we have $\mathrm{ord}_t(\delta^*(f \circ \phi) = \mathrm{ord}_t(\gamma^*(f))$. Therefore it follows from definition that $\mathrm{ord}_C = \mathrm{ord}_{C_X}$. Note that C is non-dominating if and only if C_X has the same property.

A valuation v of the function field of X with values in $\mathbb{Z}$ is *divisorial* if it is of the form $q \cdot \mathrm{ord}_E$, for a divisor E over X and a positive integer q. Of course, in this case both q and E are uniquely determined (note that the image of v is equal to $q\mathbb{Z}$). The following is the main result of this section, setting up a dictionary between divisorial valuations and cylinders in the space of arcs.

Theorem 10.6.2. *If X is a smooth variety, then the following hold:*

i) If C is a non-dominating irreducible, closed cylinder in X_∞, then ord_C *is a divisorial valuation.*
ii) For every divisor E over X and every positive integer q, there is a unique maximal cylinder $C_q(E)$ which is non-dominating, irreducible, and closed, such that $\mathrm{ord}_C = q \cdot \mathrm{ord}_E$. *Moreover, we have*

$$\mathrm{codim}(C_q(E)) = q(\mathrm{ord}_E(K_{-/X}) + 1).$$

The theorem was first proved in [ELM04], but we follow here the approach in [Zhu].

Proof of Theorem 9.6.2. We first construct the cylinders $C_q(E)$ (which we also denote by $C_q^X(E)$, when the variety X is not understood from the context). Let E be a divisor over X and q a positive integer. Suppose that $f\colon Y \to X$ is a proper, birational morphism, with Y a smooth variety, and such that E is a smooth prime divisor on Y. Consider the closed cylinder $C_q^Y(E) = \mathrm{Cont}^{\geq q}(E)$. Since

$$C_q^Y(E) = (\pi_{q-1}^Y)^{-1}(E_{q-1})$$

and E is smooth, it follows that $C_q^Y(E)$ is an irreducible cylinder of codimension q. Moreover, it is non-dominating since $\pi_\infty^Y(C_q^Y(E)) = E$. We claim that if $v = \mathrm{ord}_{C_q^Y(E)}$, then $v = q \cdot \mathrm{ord}_E$. In order to check this, we may restrict to any affine open subset that intersects E, hence we may assume that Y is affine and E is defined by (y). Since there is an arc on Y with order q along E, it follows that $v(y) = q$. If $g \in \mathscr{O}(Y)$ is written $g = y^m h$, where $h \not\in (y)$, then $v(g) = m \cdot v(y) + v(h) = mq = \mathrm{ord}_E(g)$ (since h does not vanish on $E = \pi_\infty(C_q^Y(E))$, it follows that $v(h) = 0$).

If $k = \mathrm{ord}_E(K_{Y/X})$ and we write $K_{Y/X} = kE + D$, for some effective divisor D, it is clear that for every $\gamma \in \mathrm{Cont}^q(E) \cap (\pi_\infty^Y)^{-1}(Y \smallsetminus D)$, we have $\mathrm{ord}_{K_{Y/X}}(\gamma) = kq$. We thus conclude from Corollary 9.3.24 that $C_q(E) := \overline{f_\infty(C_q^Y(E))}$ is an irreducible closed cylinder in X_∞, with $\mathrm{codim}(C_q(E)) = kq + q = q(\mathrm{ord}_E(K_{-/X}) + 1)$. By Remark 9.6.1, it follows that $C_q(E)$ is non-dominating and the corresponding valuation is equal to $q \cdot \mathrm{ord}_E$.

Suppose now that T is an arbitrary non-dominating, irreducible closed cylinder in X_∞. Since ord_T is a nontrivial valuation with values in $\mathbb{Z}$, there is a unique positive integer q such that the image of ord_T is $q\mathbb{Z}$. In order to finish the proof of the theorem, it is enough to show that there is a divisor E over X such that $T \subseteq C_q(E)$ and the two cylinders induce the same valuation. Let $Z := \overline{\pi_\infty(T)}$, which by assumption is an irreducible proper closed subset of X. In order to prove our assertion, we may replace X by any open subset intersecting Z and T by $T \cap \pi_\infty^{-1}(U)$.

Let us consider first the case when Z is a prime divisor. We show that in this case $T \subseteq C_q(Z)$ and the two cylinders define the same valuation. After replacing X by an open subset, we may assume that X is affine, Z is smooth and defined by (z). Since $\mathrm{ord}_T(h) = 0$ whenever $h \not\in (z)$, it follows that $\mathrm{ord}_T = \mathrm{ord}_T(z) \cdot \mathrm{ord}_Z$, and by definition of q we must have $q = \mathrm{ord}_T(z)$. The inclusion $T \subseteq C_q(Z) = \mathrm{Cont}^{\geq q}(Z)$ is clear, and this completes the proof in this case.

We now consider the case when $\mathrm{codim}(Z) \geq 2$. After replacing X by an open subset, we may assume that Z is smooth. Let $\phi\colon W \to X$ be the blow-up along Z, with exceptional divisor F. It follows from Proposition 9.3.25 that there is an irreducible closed cylinder T_W in W_∞ such that $T = \overline{\phi_\infty(T_W)}$. Since T is non-dominating it follows that T_W is non-dominating and it follows from Remark 9.6.1 that $\mathrm{ord}_T = \mathrm{ord}_{T_W}$. If we know the assertion for T_W, then there is a divisor E over W such that $T_W \subseteq C_q^W(E)$ and $\mathrm{ord}_{T_W} = q \cdot \mathrm{ord}_E$. In this case $T = \overline{\phi_\infty(T_W)} \subseteq \overline{\phi_\infty(C_q^W(E))} = C_q(E)$ and $\mathrm{ord}_T = q \cdot \mathrm{ord}_E$, hence we obtain the assertion for T.

On the other hand, since $\pi_\infty^X(T) \subseteq Z$, we have $T_W \subseteq \mathrm{Cont}^{\geq 1}(Z)$. Since $\mathrm{ord}_F(K_{W/X}) \geq 1$, this implies $e' := \min\{m \mid T_W \cap \mathrm{Cont}^m(K_{W/X}) \neq \emptyset\} \geq 1$, and the formula in Corollary 9.3.24 gives $\mathrm{codim}(T_W) = \mathrm{codim}(T) - e' < \mathrm{codim}(T)$. Since the codimension is always a nonnegative integer, we may argue by induction on $\mathrm{codim}(T)$ and thus assume that we know the assertion for T_W. As we have seen, this implies the assertion for T, and thus completes the proof of the theorem. □

Corollary 10.6.3. *If C is a non-dominating irreducible closed cylinder in X_∞ such that $\mathrm{ord}_C = q \cdot \mathrm{ord}_E$ for some divisor E over X and some positive integer q, then*

$$\mathrm{codim}(C) \geq q(\mathrm{ord}_E(K_{-/X}) + 1).$$

Proof. It follows from the theorem that $C \subseteq C_q(E)$, hence

$$\mathrm{codim}(C) \geq \mathrm{codim}(C_q(E)) = q(\mathrm{ord}_E(K_{-/X}) + 1).$$

□

Remark 10.6.4. Let C is a non-dominating, irreducible closed cylinder in X_∞ and $Z = \overline{\pi_\infty(C)} \subseteq X$. If E is a divisor over X and q is a positive integer such that $\mathrm{ord}_C = q \cdot \mathrm{ord}_E$, then $Z = c_X(E)$. Indeed, this follows from the fact that given a function $f \in \mathscr{O}_X(U)$, where U is an open subset that intersects Z, we have $\mathrm{ord}_C(f) > 0$ if and only if f vanishes on Z. We also note that if $C = C_q(E)$, then in fact $\pi_\infty(C)$ is closed in X. Indeed, it follows from the definition of $C_q(E)$ that this is preserved by the morphism $\Phi_\infty \colon \mathbb{A}^1 \times X_\infty \to X_\infty$, hence $\pi_\infty(C) = \sigma_\infty^{-1}(C)$, where $\sigma_\infty \colon X \to X_\infty$ is the canonical section of π_∞.

Our next result identifies the cylinders of the form $C_q(E)$ as the irreducible components of contact loci.

Proposition 10.6.5. *Let X be a smooth variety and C an irreducible closed cylinder in X. The following are equivalent:*

i) There is a divisor E over X and a positive integer q such that $C = C_q(E)$.
ii) There is a proper closed subscheme Y of X and a positive integer m such that C is an irreducible component of $\mathrm{Cont}^{\geq m}(Y)$.

Proof. Suppose first that C is a non-dominating, irreducible closed cylinder in X_∞ and let ξ be the generic point of $\overline{\pi_\infty(C)}$. We define a graded sequence of ideals $\mathfrak{a}_\bullet$, by putting

$$\mathfrak{a}_m = \{f \mid \mathrm{ord}_C(f) \geq m\}.$$

Since X is not necessarily affine, let us give a few details. If U is an affine open subset with $\xi \in U$, then we put $\mathfrak{a}_m(U) := \{f \in \mathscr{O}_X(U) \mid \mathrm{ord}_C(f) \geq m\}$, while if $\xi \notin U$, then we put $\mathfrak{a}_m(U) = \mathscr{O}_X(U)$. It is an easy exercise to see that these glue and give a coherent ideal sheaf. Furthermore, it is clear from definition that all $\mathfrak{a}_m$ are nonzero and $\mathfrak{a}_\bullet = (\mathfrak{a}_m)_{m \geq 1}$ is a graded sequence of ideals.

Consider now $C_m = \mathrm{Cont}^{\geq m}(\mathfrak{a}_m)$. It is clear that $C \subseteq C_m$ for all m. Moreover, if $m = pq$, then $\mathfrak{a}_p^q \subseteq \mathfrak{a}_m$, hence $C_m \subseteq \mathrm{Cont}^{\geq m}(\mathfrak{a}_p^q) = C_p$. It follows that if we put $C'_m = C_{m!}$, then we have

$$C \subseteq \ldots \subseteq C'_{m+1} \subseteq C'_m \subseteq \ldots \subseteq C'_1.$$

We claim that we can find irreducible components C''_m of C'_m such that

$$C \subseteq \ldots \subseteq C''_{m+1} \subseteq C''_m \subseteq \ldots \subseteq C''_1.$$

Indeed, if A_m is the set of irreducible components of C'_m that contain C, since C is irreducible, it follows that each A_m is nonempty. Furthermore, we can define maps $\alpha_m \colon A_{m+1} \to A_m$ such that for every $Z \in A_{m+1}$, we have $Z \subseteq \alpha_m(Z)$. Since the A_m are nonempty finite sets, it follows that $\varprojlim_m A_m$ is nonempty. An element of $\varprojlim_m A_m$ corresponds precisely to a sequence of irreducible components $(C''_m)_{m \geq 1}$, as required.

Since $\mathrm{codim}(C''_m) \leq \mathrm{codim}(C''_{m+1}) \leq \mathrm{codim}(C)$ for every m, it follows that there is m_0 such that $\mathrm{codim}(C''_m) = \mathrm{codim}(C''_{m_0})$ for every $m \geq m_0$. Since the C''_m are irreducible, we deduce that $C''_m = C''_{m_0} =: B$ for all $m \geq m_0$. For every m, we have $B \subseteq C_m$; indeed, if $N \geq \max\{m, m_0\}$, then $B = C''_N \subseteq C_{N!} \subseteq C_m$. We claim that $\mathrm{ord}_C = \mathrm{ord}_B$. After replacing X by an affine open neighborhood of ξ, we may assume that X is affine. Since $C \subseteq B$, it is clear that $\mathrm{ord}_C \geq \mathrm{ord}_B$ on $\mathscr{O}(X)$. On the other hand, let $f \in \mathscr{O}(X)$ and suppose that $\mathrm{ord}_C(f) = r$. In this case $f \in \mathfrak{a}_r$ and since $B \subseteq C_r$, it follows that $\mathrm{ord}_B(f) \geq \mathrm{ord}_{C_r}(f) \geq r$. This shows that indeed $\mathrm{ord}_C = \mathrm{ord}_B$.

Suppose now that $C = C_q(E)$. It follows from Theorem 9.6.2 that C is the unique maximal irreducible closed cylinder inducing a given valuation, hence $C = B$. In particular, C is an irreducible component of a contact locus. This completes the proof of i)⇒ii).

Conversely, suppose that C is an irreducible component of $\mathrm{Cont}^{\geq m}(Y)$, where Y is a proper closed subscheme of X and $m \geq 1$. We argue as in the proof of Theorem 9.6.2. Let $Z = \overline{\pi_\infty(C)}$. After replacing X by an affine open subset intersecting Z, we may assume that X is affine and Z is smooth. Suppose first that Z is a prime divisor. We may assume that Z is defined by a principal ideal (y). Let us write the ideal of Y as $I_Y = (y^r) \cdot \mathfrak{b}$, where $\mathfrak{b}$ is an ideal not contained in (y). We may replace X by $X \smallsetminus V(\mathfrak{b})$ and thus assume that $I_Y = (y^r)$. In this case, it is clear that $C = C_q(Z)$, where $q = \lceil m/r \rceil$.

Let us consider now the case when $\mathrm{codim}_X(Z) \geq 2$. Let $\phi \colon W \to X$ be the blow-up along Z, with exceptional divisor F. We apply Proposition 9.3.25 and consider the unique irreducible closed cylinder C_W in W_∞ such that $\overline{\phi_\infty(C_W)} = C$. Note that $C_W \subseteq \phi_\infty^{-1}(C) \subseteq \mathrm{Cont}^{\geq m}(\phi^{-1}(Y))$. Since C_W is irreducible, it follows that there is an irreducible component C' of $\mathrm{Cont}^{\geq m}(\phi^{-1}(Y))$ containing C_W. We then have $C = \overline{\phi_\infty(C_W)} \subseteq \overline{\phi_\infty(C')} \subseteq \mathrm{Cont}^{\geq m}(Y)$. Since C is an irreducible component of $\mathrm{Cont}^{\geq m}(Y)$, we deduce that $C = \overline{\phi_\infty(C')}$, and the uniqueness in Proposition 9.3.25 implies $C_W = C'$. Therefore C_W is an irreducible component of $\mathrm{Cont}^{\geq m}(\phi^{-1}(Y))$. We now argue as in the proof of Theorem 9.6.2, by induction on $\mathrm{codim}(C)$. Since $\mathrm{codim}(C_W) < \mathrm{codim}(C)$, it follows by induction that $C_W = C_q^W(E)$ for some divisor E over W and some positive integer q. In this case $C = \overline{\phi_\infty(C_W)} = C_q^X(E)$. This completes the proof of the proposition. $\square$

As we will see in the next section, Theorem 9.6.2 allows translating the description of classes of singularities of pairs and of invariants of such pairs in terms of codimensions of certain contact loci. However, it is sometimes useful to also have an explicit description of the codimensions of the contact loci along a subscheme and of the irreducible components of minimal codimension of these loci in terms of a log resolution. We give this in the next proposition. Note that this is very close to the computations that we did in Chapters 9.4 and 9.5 (cf., for example, the proof of Theorem 9.5.35).

Proposition 10.6.6. *Let X be a smooth variety and Z a proper closed subscheme of X. Let $f\colon Y \to X$ be a log resolution of (X,Z) that is an isomorphism over $X \smallsetminus Y$ and let us write*

$$f^{-1}(Z) = \sum_{i=1}^N a_i E_i \text{ and } K_{Y/X} = \sum_{i=1}^N k_i E_i.$$

For every nonnegative integer m, we have

$$\operatorname{codim}(\operatorname{Cont}^m(Z)) = \min\left\{ \sum_{i=1}^N (k_i+1)\nu_i \mid \sum_{i=1}^N a_i\nu_i = m, \bigcap_{\nu_i \geq 1} E_i \neq \emptyset \right\}$$

and the number of irreducible components of $\operatorname{codim}(\operatorname{Cont}^{\geq m}(Z))$ *of minimal codimension is equal to*

$$\sum_{J \subseteq \{1,\dots,N\}} |\{\nu \in \mathbb{Z}^J_{>0} \mid \sum_{i\in J} a_i\nu_i = m, \sum_{i\in J}(k_i+1)\nu_i = \operatorname{codim}(\operatorname{Cont}^{\geq m}(Z))\} \cdot \beta_J,$$

where β_J is the number of connected components of $\cap_{i\in J} E_i$.

Proof. Since f is an isomorphism over $X \smallsetminus Y$, we may assume that $a_i \geq 1$ for all i. For every $\nu \in \mathbb{Z}^N_{\geq 0}$, let $C_\nu = \cap_{i=1}^N \operatorname{Cont}^{\nu_i}(E_i) \subseteq Y_\infty$. It is clear that C_ν is nonempty if and only if $\cap_{i\in J(\nu)} E_i \neq \emptyset$, where $J(\nu) = \{i \mid \nu_i \geq 1\}$. Furthermore, if this is the case, then C_ν has $\beta_{J(\nu)}$ disjoint irreducible components, all of them of codimension $\sum_{i=1}^N \nu_i$. Since $C_\nu \subseteq \operatorname{Cont}^e(K_{Y/X})$, where $e = \sum_{i=1}^N k_i\nu_i$, it follows from Proposition 9.2.8 that the $f_\infty(C_\nu)$ are mutually disjoint. Furthermore, using also Corollary 9.3.21, we see that each $f_\infty(C_\nu)$ is a disjoint union of $\beta_{J(\nu)}$ irreducible subcylinders, all of them of codimension $\sum_{i=1}^N (k_i+1)\nu_i$. Since $\operatorname{Cont}^m(Y) = \sqcup_\nu f_\infty(C_\nu)$, where the union is over those ν such that $\sum_i a_i\nu_i = m$ (note that this is a finite set since all a_i are positive), both assertions in the proposition are clear. $\square$

Corollary 10.6.7. *With the notation in Proposition 9.6.6, we have for every $m \in \mathbb{Z}_{\geq 0}$*

$$\operatorname{codim}(\operatorname{Cont}^{\geq m}(Z)) = \min\left\{ \sum_{i=1}^N (k_i+1)\nu_i \mid \sum_{i=1}^N a_i\nu_i \geq m, \bigcap_{\nu_i \geq 1} E_i \neq \emptyset \right\}$$

and the number of irreducible components of $\mathrm{Cont}^{\geq m}(Z)$ *of minimal codimension is equal to*

$$\sum_{J\subseteq\{1,\ldots,N\}} |\{\nu\in\mathbb{Z}^J_{>0} \mid \sum_{i\in J} a_i\nu_i\geq m, \sum_{i\in J}(k_i+1)\nu_i=\mathrm{codim}(\mathrm{Cont}^{\geq m}(Z))\}\cdot\beta_J.$$

Proof. Let us demote by $\alpha(C)$ the number of irreducible components of minimal codimension of a cylinder C. For every $j\geq 0$, we have a disjoint decomposition

$$\mathrm{Cont}^{\geq m}(Z)=\bigsqcup_{i=m}^{m+j}\mathrm{Cont}^i(Z)\bigsqcup\mathrm{Cont}^{\geq(m+j+1)}(Z).$$

By Proposition 9.3.3, for $j\gg 0$, we have $\mathrm{codim}(\mathrm{Cont}^{\geq(m+j+1)}(Z))>\mathrm{codim}(\mathrm{Cont}^{\geq m}(Z))$, hence

$$\mathrm{codim}(\mathrm{Cont}^{\geq m}(Z))=\min\{\mathrm{codim}(\mathrm{Cont}^i(Z))\mid m\leq i\leq m+j\}$$

and $\alpha(\mathrm{Cont}^{\geq m}(Z))=\sum_i\alpha(\mathrm{Cont}^i(Z))$, where the sum is over those i with $m\leq i\leq m+j$ such that $\mathrm{codim}(\mathrm{Cont}^i(Z))=\mathrm{codim}(\mathrm{Cont}^{\geq m}(Z))$. The assertions in the statement now follow from the ones in Proposition 9.6.6. □

Remark 10.6.8. One can use the description of the contact loci in terms of a log resolution in Proposition 9.6.6 and Corollary 9.6.7 in order to show that if C is an irreducible component of some $\mathrm{Cont}^{\geq m}(Z)$, with $m\geq 1$, then ord_C is a divisorial valuation. Arguing as in the proof of Proposition 9.6.5 one then sees that given any non-dominating irreducible closed cylinder in X_∞, there is some $C'\supseteq C$, with C' an irreducible component of some contact locus, such that $\mathrm{ord}_C=\mathrm{ord}_{C'}$. This implies that ord_C is a divisorial valuation, giving another proof for this assertion from Theorem 9.6.2; see [ELM04] for details. However, the proof that we presented has the advantage that does not make use of resolution of singularities. In fact, it only uses the birational transformation formula for smooth blow-ups, which as we have seen, is an easy exercise. While we have implicitly used the general case of this formula to show that $C_q(E)$ is a cylinder, and to compute its codimension, this can also be done by only considering smooth blow-ups: it is known that in arbitrary characteristic, one can realize a divisor over X as lying on a composition of blow-ups with smooth centers, after possibly restricting to suitable open subsets after each step (see [KM98, Lemma 2.45]). For the details on how to carry this out in arbitrary characteristic, see [Zhu].

Remark 10.6.9. It follows from the formula in Corollary 9.6.7 that if Z is a proper closed subscheme in a smooth variety, then for every positive integers m and p, we have

$$\mathrm{codim}(\mathrm{Cont}^{\geq mp}(Z))\leq p\cdot\mathrm{codim}(\mathrm{Cont}^{\geq m}(Z)).$$

It would be interesting to find a direct geometric argument for this, which does not rely on log resolutions (and would thus also hold in positive characteristic).

10.6.2 Applications to log canonical thresholds

The results in the previous section allow the description of log canonical and klt pairs in terms of cylinders in the space of arcs. We first give a statement for higher-codimension pairs in the sense of Chapter 3.1. Suppose that $(X,\mathscr{Z})$ is such a pair, that is, $\mathscr{Z}=\sum_{i=1}^r q_iZ_i$, where the Z_i are proper closed subschemes of X and $q_i\in\mathbb{R}$. We assume that X is smooth. If C is a non-dominating, irreducible closed cylinder in X_∞, we put $\mathrm{ord}_C(\mathscr{Z}):=\sum_{i=1}^r q_i\cdot\mathrm{ord}_C(\mathfrak{a}_i)$, where $\mathfrak{a}_i$ is the ideal defining Z_i.

Proposition 10.6.10. *Let $(X,\mathscr{Z})$ be a pair as above, with X smooth. The pair $(X,\mathscr{Z})$ is log canonical (klt) if and only if for every non-dominating, irreducible closed cylinder $C\subseteq X_\infty$, we have $\mathrm{codim}(C)\geq\mathrm{ord}_C(\mathscr{Z})$ (respectively, $\mathrm{codim}(C)>\mathrm{ord}_C(\mathscr{Z})$).*

Proof. Let us prove the description of log canonical pairs: the argument for klt pairs is entirely analogous. Suppose first that $(X,\mathscr{Z})$ satisfies the condition on cylinders. If E is a divisor over X, applying this condition for $C_1(E)$, we obtain using Theorem 9.6.2

$$\mathrm{codim}(C_1(E))=1+\mathrm{ord}_E(K_{Y/X})\geq\mathrm{ord}_{C_1(E)}(\mathscr{Z})=\mathrm{ord}_E(\mathscr{Z}).$$

Since this holds for every E, it follows that $(X,\mathscr{Z})$ is log canonical. Conversely, suppose that that the pair $(X,\mathscr{Z})$ is log canonical. Given any irreducible, closed, non-dominating cylinder $C\subseteq X_\infty$, it follows from Theorem 9.6.2 that there is a divisor E over X and a positive integer q such that $C\subseteq C_q(E)$ and $\mathrm{ord}_C=\mathrm{ord}_{C_q(E)}=q\cdot\mathrm{ord}_E$. Using the fact that $(X,\mathscr{Z})$ is log canonical, we conclude that

$$\mathrm{codim}(C)\geq\mathrm{codim}(C_q(E))=q(\mathrm{ord}_E(K_{-/X})+1)\geq q\cdot\mathrm{ord}_E(\mathscr{Z})=\mathrm{ord}_C(\mathscr{Z}).$$

This completes the proof of the proposition. □

This proposition implies the following formula for the log canonical threshold of a closed subscheme.

Corollary 10.6.11. *If X is an n-dimensional smooth variety and Z is a closed subscheme of X defined by the nonzero ideal $\mathfrak{a}$, then*

$$\mathrm{lct}(\mathfrak{a})=\min_C\frac{\mathrm{codim}(C)}{\mathrm{ord}_C(\mathfrak{a})}=\min_{m\geq 1}\frac{\mathrm{codim}(\mathrm{Cont}^{\geq m}(Z))}{m}=n-\max_{m\geq 0}\frac{\dim(Z_m)}{m+1},\tag{10.21}$$

where the first minimum is over all irreducible, closed cylinders $C\subseteq X_\infty$ such that $\mathrm{ord}_C(\mathfrak{a})>0$. Moreover, this minimum is achieved if and only if $C=C_q(E)$ for some positive integer q and some divisor E over X that computes $\mathrm{lct}(\mathfrak{a})$.

Proof. It follows from Proposition 9.6.10 and the definition of the log canonical threshold that $\mathrm{lct}(\mathfrak{a})$ is the largest $t\in\mathbb{Q}_{>0}$ such that $\mathrm{codim}(C)\geq t\cdot\mathrm{ord}_C(\mathfrak{a})$ for every non-dominating irreducible closed cylinder $C\subseteq X_\infty$. Therefore

$$\mathrm{lct}(\mathfrak{a}) = \inf_C \frac{\mathrm{codim}(C)}{\mathrm{ord}_C(\mathfrak{a})}, \tag{10.22}$$

where C varies over the irreducible closed cylinders such that $\mathrm{ord}_C(\mathfrak{a}) > 0$.

Suppose now that C is an irreducible, closed cylinder with $\mathrm{ord}_C(\mathfrak{a}) > 0$. It follows from Theorem 9.6.2 that there is a divisor E over X and a positive integer q such that $C \subseteq C_q(E)$ and $\mathrm{ord}_C = q \cdot \mathrm{ord}_E$. Therefore we have

$$\mathrm{codim}(C) \geq \mathrm{codim}(C_q(E)) = q(1+\mathrm{ord}_E(K_{-/X})) \geq q \cdot \mathrm{lct}(\mathfrak{a}) \cdot \mathrm{ord}_E(\mathfrak{a}) = \mathrm{lct}(\mathfrak{a}) \cdot \mathrm{ord}_C(\mathfrak{a}).$$

We thus see that C achieves the infimum in (9.22) if and only if $C = C_q(E)$ and E computes $\mathrm{lct}(\mathfrak{a})$. In particular, this shows that the infimum in (9.22) is a minimum.

Suppose now that C is an irreducible closed cylinder with $\mathrm{ord}_C(\mathfrak{a}) = m > 0$. Note that $C \subseteq \mathrm{Cont}^{\geq m}(Z)$ and if C' is an irreducible component of $\mathrm{Cont}^{\geq m}(Z)$ that contains C, then $\mathrm{codim}(C) \geq \mathrm{codim}(C')$ and $\mathrm{ord}_{C'}(\mathfrak{a}) = m$. We thus deduce from (9.22) that

$$\mathrm{lct}(\mathfrak{a}) = \min_{m \geq 1} \frac{\mathrm{codim}(\mathrm{Cont}^{\geq m}(Z))}{m}.$$

Since $\mathrm{Cont}^{\geq m}(Z)) = (\pi^X_{\infty,m-1})^{-1}(Z_{m-1})$, it follows that $\mathrm{codim}(\mathrm{Cont}^{\geq m}(Z)) = mn - \dim(Z_{m-1})$, and we obtain the last equality in (9.21). $\square$

Remark 10.6.12. If instead of the log canonical threshold $\mathrm{lct}(\mathfrak{a})$ one is interested in $\mathrm{lct}_W(\mathfrak{a})$, where W is a closed subset of X, then in the proofs of Proposition 9.6.10 and Corollary 9.6.11 we only consider the divisors E over X with $c_X(E) \cap W \neq \emptyset$. In light of Remark 9.6.4, when we consider cylinders C over X_∞, the condition translates to $\overline{\pi_\infty(C)} \cap W \neq \emptyset$. Moreover, note that when C is a component of $\mathrm{Cont}^{\geq m}(\mathfrak{a})$, then $\pi_\infty(C)$ is closed. We thus obtain

$$\mathrm{lct}_W(\mathfrak{a}) = \min_{m \geq 1} \frac{\mathrm{codim}^W(\mathrm{Cont}^{\geq m}(\mathfrak{a}))}{m},$$

where $\mathrm{codim}^W(\mathrm{Cont}^{\geq m}(\mathfrak{a}))$ is the smallest codimension of an irreducible component of $\mathrm{Cont}^{\geq m}(\mathfrak{a})$ whose image in X intersects W.

Remark 10.6.13. It follows from Remark 9.6.4 and the proof of Corollary 9.6.11 that if $c = \mathrm{lct}(\mathfrak{a})$, then the non-klt centers of $(X, \mathfrak{a}^c)$ are the sets of the form $\pi_m(T)$, where m is a nonnegative integer and T is an irreducible component of $V(\mathfrak{a})_m$, with $\dim(T) = (n - \mathrm{lct}(\mathfrak{a})) \cdot (m+1)$.

Remark 10.6.14. Let X be a smooth variety and $\mathfrak{a}$ a nonzero ideal on X defining a subscheme Z. Let $\alpha(m)$ denote the number of irreducible components of $\mathrm{Cont}^m(\mathfrak{a})$ of codimension $\mathrm{lct}(\mathfrak{a}) \cdot m$. We can estimate $\alpha(m)$ as follows. Consider a log resolution $f \colon Y \to X$ of (X, Z) that is an isomorphism over $X \smallsetminus Z$ and let us write

$$f^{-1}(Z) = \sum_{i=1}^N a_i E_i \text{ and } K_{Y/X} = \sum_{i=1}^N k_i E_i.$$

Let $J_0 = \{i \in \{1,\dots,N\} \mid k_i + 1 = \mathrm{lct}(\mathfrak{a}) \cdot a_i\}$ and $\Lambda = \{J \subseteq J_0 \mid \cap_{i \in J} E_i \neq \emptyset\}$. It follows easily from Proposition 9.6.6 that

$$\alpha(m) = \#\{\nu \in \mathbb{Z}_{\geq 0}^N \mid \sum_i a_i \nu_i = m, \{i \mid \nu_i \geq 1\} \in \Lambda\}.$$

We deduce that if $d \geq 1$ is the largest number of elements of a set in Λ, then $\limsup_{m\to\infty} \frac{\alpha_m}{m^{d-1}} \in (0,\infty)$.

As an application of Corollary 9.6.11, we give another proof for the inversion of adjunction formula for log canonical thresholds in the case of smooth ambient varieties.

Corollary 10.6.15. *Let X be a smooth variety and $H \subset X$ a smooth subvariety of codimension* 1*. If $\mathfrak{a}$ is an ideal on X such that $\mathfrak{a} \cdot \mathscr{O}_H$ is nonzero, then* $\mathrm{lct}_H(\mathfrak{a}) \geq \mathrm{lct}(\mathfrak{a} \cdot \mathscr{O}_H)$.

Proof. Let Z be the subscheme defined by $\mathfrak{a}$. It is enough to show that if $\mathrm{lct}_H(\mathfrak{a}) < \tau$, then $\mathrm{lct}(\mathfrak{a} \cdot \mathscr{O}_H) < \tau$. It follows from Corollary 9.6.11 (see also Remark 9.6.12) that since $\mathrm{lct}_H(\mathfrak{a}) < \tau$, there is $m \geq 0$ and an irreducible component W of Z_m such that $\pi_m(W) \cap H \neq 0$ and

$$\dim(W) > (m+1)(n-\tau).$$

Let us consider $W \cap H_m \subseteq (Z \cap H)_m$. Note that $W \cap H_m$ is nonempty: if $x \in \pi_m(W) \cap H$, then the constant m-jet $\sigma_m(x)$ lies in W, hence in $W \cap H_m$. On the other hand, since H is locally defined in X by one equation, H_m is locally defined in X_m by $(m+1)$ equations. We deduce that if W_H is an irreducible component of $W \cap H_m$, then

$$\dim(W_H) \geq \dim(W) - (m+1) > (m+1)(n-1-\tau).$$

Another application of Corollary 9.6.11 gives $\mathrm{lct}(\mathfrak{a} \cdot \mathscr{O}_H) < \tau$ and this completes the proof. $\square$

10.6.3 Applications to minimal log discrepancies: semicontinuity

10.6.4 Characterization of locally complete intersection rational singularities

10.7 The birational transformation rule II: the general case

10.7.1 Spaces of arcs of singular varieties

10.7.2 The general birational transformation formula

10.8 Inversion of adjunction for locally complete intersection varieties

10.9 The formal arc theorem and the curve selection lemma

10.9.1 Complete rings and the Weierstratrass preparation theorem

In this section we review some facts about rings with linear topologies and their completions. Since we deal with more general rings than usual (for example, we need to handle completions of certain non-Noetherian rings) we develop carefully what we need. In particular, we give the proof of the Weierstrass preparation theorem in the form that we will need for treating rings of formal power series in infinitely many variables.

Recall that if R is a ring, a linear topology on R is defined by a weakly decreasing sequence[6] of ideals $(I_j)_{j\geq 1}$. In this case, a basis of open sets of some $a \in R$ is given by $\{a+I_j \mid j \geq 1\}$ and with this topology R becomes a topological ring. For example, if $\mathfrak{a}$ is an ideal in R, then the sequence of ideals $(\mathfrak{a}^j)_{j\geq 1}$ defines the $\mathfrak{a}$-adic topology on R. Note that a topology on R defined by a sequence $(I_j)_{j\geq 1}$ is coarser than the $\mathfrak{a}$-adic topology if and only if for every j, we have $\mathfrak{a}^N \subseteq I_j$ for $N \gg 0$. We always make the assumption that there are open sets in R different from R and the empty set; equivalently, I_j is a proper ideal of R for $j \gg 0$. All topologies we will consider will be linear topologies.

Suppose that R is a ring with a linear topology given by a sequence of ideals $(I_j)_{j\geq 1}$. If M is an R-module, a linear topology on M is given by a non-increasing sequence of submodules $(M_j)_{j\geq 1}$ such that for every j, we have $I_m M \subseteq M_j$ for $m \gg 0$. In this case M becomes a topological R-module, with a basis of open neighborhoods of $u \in M$ given by $\{u+M_j \mid j \geq 1\}$. If on R we have the $\mathfrak{a}$-adic topology, where $\mathfrak{a}$ is

[6] One can allow, more generally, the set of ideals to be indexed by an arbitrary ordered set. However, we will not need this level of generality.

an ideal in R, then the $\mathfrak{a}$-adic topology on M is given by $(\mathfrak{a}^j M)_{j\geq 1}$. In general, M is separated with respect to the topology defined by $(M_j)_{j\geq 1}$ if and only if $\cap_{j\geq 1} M_j = 0$. The *completion* of M is $\widehat{M} = \varprojlim_j M/M_j$. Note that $\widehat{R}$ is a ring and $\widehat{M}$ is naturally an $\widehat{R}$-module. Since we assume $I_j \neq R$ for $j \gg 0$, we have $\widehat{R} \neq 0$. On $\widehat{M}$ we consider the projective limit topology, where each M/M_j carries the discrete topology. In fact, we have canonical surjections $\widehat{M} \to M/M_j$ and if N_j denotes the kernel of this surjection, then $(N_j)_{j\geq 1}$ defines the projective limit topology on $\widehat{M}$.

Note that the canonical continuous morphism $\phi\colon R \to \widehat{R}$ is the completion of R, that is, $\widehat{R}$ is complete and separated and if $\psi\colon R \to S$ is another continuous morphism to a complete and separated topological ring S, then there is a unique continuous ring homomorphism $\widehat{\psi}\colon \widehat{R} \to S$ such that $\widehat{\psi} \circ \phi = \psi$. In particular, the morphism ϕ only depends on the topological ring R and not on the particular sequence of ideals $(I_j)_{j\geq 1}$. We have a similar characterization for the completion of an R-module M with a linear topology. In particular, we have the completion functor that takes M to $\widehat{M}$ from the category of R-modules with linear topology to itself.

Let $\mathfrak{m}$ be a maximal ideal in a ring R. Suppose that R carries a linear topology defined by the sequence of ideals $(I_j)_{j\geq 1}$, which is coarser than the $\mathfrak{m}$-adic topology. By assumption, there is j such that $I_j \neq R$ and for every such j there is N_j such that $\mathfrak{m}^{N_j} \subseteq I_j$. This implies that $I_j \subseteq \mathfrak{m}$. Let $\mathfrak{n}$ be the inverse image of $\mathfrak{m}/I_j$ via the canonical surjection $\widehat{R} \to R/I_j$ (note that this is independent of j). It is clear that $\mathfrak{n}$ is a maximal ideal of $\widehat{R}$, with residue field $R/\mathfrak{m}$. In fact, this is the unique maximal ideal of $\widehat{R}$: if $a \in \widehat{R} \smallsetminus \mathfrak{n}$, then the image of a in all R/I_j as above is invertible, hence a is invertible (note that R/I_j is a local ring). Since we assumed that the topology on R is coarser then the $\mathfrak{m}$-adic topology, it follows that the topology on $\widehat{R}$ is coarser than the $\mathfrak{n}$-adic topology (it is enough to note that whenever $\mathfrak{m}^{N_j} \subseteq I_j$, we have $\mathfrak{n}^{N_j} \subseteq \operatorname{Ker}(\widehat{R} \to R/I_j)$). Note also that the localization $R_{\mathfrak{m}}$ has a linear topology induced by the ideals $I_j R_{\mathfrak{m}}$. Since $R_{\mathfrak{m}}/I_j R_{\mathfrak{m}} \simeq R/I_j$, we see that $\widehat{R}$ is also canonically isomorphic to the completion of $R_{\mathfrak{m}}$ with respect to this topology.

We now turn to the class of rings that we will we concerned with. Let k be a fixed field. We denote by $\operatorname{Comp}(k)$ the category whose objects are local k-algebras $(R, \mathfrak{m})$ with residue field k, that carry a linear topology which is coarser than the $\mathfrak{m}$-adic topology and with respect to which R is separated and complete. The morphisms in $\operatorname{Comp}(k)$ are local continuous morphisms of k-algebras.

We also consider the full subcategory $\operatorname{Nil}(k)$ of $\operatorname{Comp}(k)$ consisting of *test rings*, that is, local k-algebras $(A, \mathfrak{m})$ with residue field k, such that $\mathfrak{m}^N = 0$ for some N, considered with the discrete topology.

Remark 10.9.1. Note that if R is an object in $\operatorname{Comp}(k)$, with the topology defined by the sequence of ideals $(I_j)_{j\geq 1}$, then $R \simeq \varprojlim_j R/I_j$ and each R/I_j is an object in $\operatorname{Nil}(k)$ whenever it is nonzero. It follows that if R' is any other object in $\operatorname{Comp}(k)$, then we have a canonical isomorphism

$$\mathrm{Hom}_{\mathrm{Comp}(k)}(R', R) \simeq \varprojlim_j \mathrm{Hom}_{\mathrm{Comp}(k)}(R', R/I_j).$$

Therefore it is a consequence of Yoneda's lemma that the natural contravariant functor

$$h\colon \mathrm{Comp}(k) \to \mathrm{Fun}(\mathrm{Nil}(k), \mathrm{Sets}), h(R) = \mathrm{Hom}_{\mathrm{Comp}(k)}(R, -)$$

is fully faithful.

Remark 10.9.2. If R is a topological k-algebra, with the topology defined by a sequence of ideals $(I_j)_{j\geq 1}$, and $(A, \mathfrak{m}_A)$ is a test ring, then a ring homomorphism $\phi\colon R \to A$ is continuous if and only if it factors through some R/I_j. Moreover, suppose that S is a k-algebra and $\mathfrak{m}$ is a maximal ideal in S, with residue field k. If S has the $\mathfrak{m}$-adic topology and $R = \widehat{S}$, then the morphisms $R \to A$ in $\mathrm{Comp}(k)$ are in natural bijection with the k-algebra homomorphisms $\psi\colon S \to A$ with $\psi(\mathfrak{m}) \subseteq \mathfrak{m}_A$.

Example 10.9.3. The following example is of particular importance for us. Let B be any ring. Consider the polynomial ring over B with variables indexed by a fixed set Λ, namely $S = B[x_i \mid i \in \Lambda]$. We consider the ideal $\mathfrak{m}_S = (x_i \mid i \in \Lambda)$ in S and give S the $\mathfrak{m}_S$-adic topology. It is clear that $S/\mathfrak{m}_S \simeq B$. We put $B[\![x_i \mid i \in \Lambda]\!] := \widehat{S}$. Note that every element $f \in \widehat{S}$ can be uniquely written as $f = \sum_\alpha c_\alpha x^\alpha$, where α runs over the maps $I \to \mathbb{Z}_{\geq 0}$ such that $\alpha(i) \neq 0$ for finitely many i, $c_\alpha \in B$, and we put $x^\alpha = \prod_i x_i^{\alpha(i)}$. The condition for f to be a well-defined element of $B[\![x_i \mid i \in \Lambda]\!]$ is that for every N, there are only finitely many α such that $c_\alpha \neq 0$ and $\sum_i \alpha(i) \leq N$. Of course, when Λ is finite set with n elements, we recover the usual formal power series ring over B in n variables.

In particular, if $B = k$ is a field, then $k[\![x_i \mid i \in \Lambda]\!]$ is an object in $\mathrm{Comp}(k)$. In this case, its maximal ideal consists of those $f = \sum_\alpha c_\alpha x^\alpha$ with $c_0 = 0$. Note that if $(A, \mathfrak{m}_A)$ is an object in $\mathrm{Comp}(k)$, then giving a morphism $\phi\colon k[\![x_i \mid i \in \Lambda]\!] \to A$ in $\mathrm{Comp}(k)$ is equivalent to giving elements $a_i \in \mathfrak{m}_A$ for every $i \in \Lambda$ (in which case $\phi(x_i) = a_i$).

We note that when Λ is an infinite set, the behavior of this power series ring is somewhat peculiar, even when $B = k$. For example, in this case the maximal ideal $\mathfrak{m}_{\widehat{S}}$ in $\widehat{S}$ is different from $\mathfrak{m}_S \cdot \widehat{S}$ (it is easy to check that if $\Lambda = \mathbb{Z}_{>0}$, then $f = \sum_{i\geq 1}(x_i)^i$ lies in $\mathfrak{m}_{\widehat{S}}$, but not in $\mathfrak{m}_S \cdot \widehat{S}$). Moreover, if Λ is infinite, then $\widehat{S}$ is not complete in the $\mathfrak{m}_{\widehat{S}}$-adic topology.

Remark 10.9.4. Suppose that $(R, \mathfrak{m})$ and $(S, \mathfrak{n})$ are objects in $\mathrm{Comp}(k)$, with the topologies defined by the sequences of ideals $(I_j)_{j\geq 1}$ and $(J_j)_{j\geq 1}$, respectively. On the tensor product $R \otimes_k S$ we have the topology induced by the sequence of ideals $\mathfrak{a}_\ell = I_\ell \otimes_k S + R \otimes_k J_\ell$. The completion of $R \otimes_k S$ with respect to this topology is denoted by $R \widehat{\otimes} S$. Note that this is an element of $\mathrm{Comp}(k)$. Indeed, we have in $R \otimes_k S$ the maximal ideal $\mathfrak{b} = \mathfrak{m} \otimes_k S + R \otimes_k \mathfrak{n}$, with residue field k, and the topology on $R \otimes_k S$ is coarser than the $\mathfrak{b}$-adic topology. It is easy to check that $R \widehat{\otimes} S$ is the coproduct of R and S in the category $\mathrm{Comp}(k)$.

Note that for every two sets Λ and Γ, we have a canonical isomorphism in $\mathrm{Comp}(k)$

$$k[\![x_i, y_j \mid i \in \Lambda, j \in \Gamma]\!] \simeq k[\![x_i \mid i \in \Lambda]\!] \widehat{\otimes} k[\![y_j \mid j \in \Gamma]\!].$$

This follows by considering morphisms to objects in $\mathrm{Comp}(k)$. It is also easy to see that for every set Γ, we have a morphism of k-algebras

$$\phi\colon T[\![y_j, j \in \Gamma]\!] \to k[\![x_i, y_j \mid i \in \Lambda, j \in \Gamma]\!],$$

where $T = k[\![x_i \mid i \in \Lambda]\!]$, given by

$$\phi\Big(\sum_{\beta}\Big(\sum_{\alpha} c_{\alpha,\beta}x^{\alpha}\Big)y^{\beta}\Big) = \sum_{\alpha,\beta} c_{\alpha,\beta}x^{\alpha}y^{\beta}.$$

This is not surjective if Γ is infinite and Λ is nonempty: for example, if $i_0 \in \Lambda$ and $\Gamma = \mathbb{Z}_{>0}$, then $\sum_{m\geq 1}(x_{i_0})^m y_m$ lies in $k[\![x_i, y_j \mid i \in \Lambda, j \in \Gamma]\!]$ but it is not in the image of ϕ. On the other hand, if Γ is finite, then ϕ is an isomorphism.

Example 10.9.5. We may also consider the following variant of the construction in Example 9.9.3. Suppose that $(R, \mathfrak{m}_R, k)$ is a local ring and on R we have a linear topology defined by the sequence of ideals $(I_j)_{j\geq 1}$, which is coarser than the $\mathfrak{m}$-adic topology. Given a set Λ, we consider $S = R[x_i \mid i \in \Lambda]$ and the ideal in S given by $\mathfrak{m}_S = \mathfrak{m}_R \cdot S + (x_i \mid i \in \Lambda)$. It is clear that $S/\mathfrak{m}_S \simeq k$. We give S the topology defined by the sequence of ideals $(J_j)_{j\geq 1}$, with

$$J_j = I_j \cdot S + (x_i \mid i \in \Lambda)^j.$$

By considering morphisms to test rings, it is easy to see that we have an isomorphism in $\mathrm{Comp}(k)$

$$\widehat{S} \simeq \widehat{R} \widehat{\otimes} k[\![x_i \mid i \in \Lambda]\!]. \tag{10.23}$$

We now turn to the Weierstrass division and preparation theorems. We will prove these in a slightly more general setting than is usually done, in order to be able to use them also in the setting of formal power series in infinitely many variables. The following easy lemma is the main ingredient in the proof.

Lemma 10.9.6. *Let R be a ring and $\mathfrak{a}$ an ideal in R. Suppose that R is complete and separated with respect to the linear topology given by a sequence of ideals $(I_j)_{j\geq 1}$, which is coarser than the $\mathfrak{a}$-adic topology. If M is an R-module which is separated with respect to the linear topology given by $(I_jM)_{j\geq 1}$ and if $u_1, \ldots, u_r \in M$ are such that $M/\mathfrak{a}M$ is generated over $R/\mathfrak{a}$ by $\overline{u_1}, \ldots, \overline{u_r}$, then M is generated over R by $u_1, \ldots, u_r$.*

Proof. For every $u \in M$, we can write by hypothesis

$$u = \sum_{i=1}^{r} a_{i,1}u_i + w_1, \tag{10.24}$$

with $a_{i,1} \in \mathfrak{a}$ for all i and $w_1 \in \mathfrak{a}M$. We now show by induction on $m \geq 1$ that we can write

$$u = \sum_{i=1}^{r} a_{i,m} u_i + w_m, \tag{10.25}$$

with $a_{i,m} - a_{i,m-1} \in \mathfrak{a}^m$ for all i and $w_m \in \mathfrak{a}^m M$ (where we put $a_{i,0} = 0$ for all i). The case $m = 1$ follows from (9.24), hence we only need to prove the induction step. Suppose that $a_{i,m}$ and w_m are as above. Since $w_m \in \mathfrak{a}^m M$, we can write $w_m = \sum_{j=1}^{d} b_{j,m} v_{j,m}$, with $b_{j,m} \in \mathfrak{a}^m$ and $v_{j,m} \in M$ for all j. Applying the hypothesis to each $v_{j,m}$, we write

$$v_{j,m} = \sum_{i=1}^{r} c_{i,j,m} u_i + v'_{j,m},$$

with $c_{i,j,m} \in \mathfrak{a}$ and $v'_{j,m} \in \mathfrak{a}M$. We thus have

$$u = \sum_{i=1}^{r} \left(a_{i,m} + \sum_{j=1}^{d} b_{j,m} c_{i,j,m} \right) + \sum_{j=1}^{d} b_{j,m} v'_{j,m}.$$

Since $\sum_{j=1}^{d} b_{j,m} c_{i,j,m} \in \mathfrak{a}^{m+1}$ and $\sum_{j=1}^{d} b_{j,m} v'_{j,m} \in \mathfrak{a}^{m+1} M$, this completes the proof of the induction step.

Note that for every i with $1 \leq i \leq r$, the sequence $(a_{i,m})_{m \geq 1}$ is Cauchy in the $\mathfrak{a}$-adic topology. This implies that it is also Cauchy in the topology given by $(I_j)_{j \geq 1}$, and therefore it converges in this topology to some $a_i \in R$. We claim that $u = \sum_{i=1}^{r} a_i u_i$. By the separatedness assumption on M, it is enough to show that $u - \sum_{i=1}^{r} a_i u_i \in I_j M$ for all j. By (9.25), for every m we have

$$u - \sum_{i=1}^{r} a_i u_i = \sum_{i=1}^{r} (a_{i,m} - a_i) u_i + w_m.$$

Since $w_m \in \mathfrak{a}^m M \subseteq I_j M$ for $m \gg 0$ and $a_{i,m} - a_i \in I_j$ for $m \gg 0$, we deduce that $u - \sum_{i=1}^{r} a_i u_i \in I_j M$. This completes the proof of the lemma. $\square$

Theorem 10.9.7. *Let $(R, \mathfrak{m}, k)$ be a local ring such that R is complete and separated with respect to the linear topology given by a sequence of ideals $(I_j)_{j \geq 1}$, which is coarser than the $\mathfrak{m}$-adic topology. If $f = \sum_{i \geq 0} a_i x^i \in R[\![x]\!]$ is such that for some nonnegative integer h, we have $a_h \notin \mathfrak{m}$ and $a_i \in \mathfrak{m}$ for $0 \leq i \leq h-1$, then $R[\![x]\!]/(f)$ is free over R, with basis $1, x, \ldots, x^{h-1}$.*

Proof. We give the argument in several steps.
Step 1. We show that if I is an ideal in R and $g = \sum_{i \geq 0} b_i x^i \in R[\![x]\!]$ is such that there is a polynomial $Q \in R[x]$ of degree $< h$ such that all coefficients of $fg - Q$ lie in I, then $b_i \in \cap_{\ell \geq 1} (I + \mathfrak{m}^\ell)$ for every $i \geq 0$. We prove this by induction on ℓ, the case $\ell = 0$ being trivial. Suppose that we know the assertion for ℓ. We show that $b_i \in I + \mathfrak{m}^{\ell+1}$ by induction on $i \geq 0$. Let us assume that $b_j \in I + \mathfrak{m}^{\ell+1}$ for $j < i$. By considering the coefficient of x^{h+i} in $fg - Q$, we see that $\sum_{j=0}^{h+i} a_j b_{h+i-j} \in I$. For $j < h$, we have $a_j \in \mathfrak{m}$ by hypothesis and $b_{h+i-j} \in I + \mathfrak{m}^\ell$ by the induction hypothesis on ℓ, hence $a_j b_{h+i-j} \in I + \mathfrak{m}^{\ell+1}$. On the other hand, for $j > h$ we have $b_{h+i-j} \in I + \mathfrak{m}^{\ell+1}$ by

the induction hypothesis on i. Therefore $a_h b_i \in I + \mathfrak{m}^{\ell+1}$ and since a_h is invertible, it follows that $b_i \in I + \mathfrak{m}^{\ell+1}$. This completes the proofs of both induction steps.
Step 2. The R-module $M = R[\![x]\!]/(f)$ is separated with respect to the topology defined by $(I_j M)_{j\geq 1}$. Indeed, suppose that $P \in R[\![x]\!]$ is such that $P \in (f) + I_j R[\![x]\!]$ for every $j \geq 1$. In this case we can write $P = fg_j + B_j$ for every $j \geq 1$, where $B_j \in I_j R[\![x]\!]$. In particular, for all j, all coefficients of $f(g_j - g_{j+1})$ lie in I_j. Note that $\mathfrak{m}^\ell \subseteq I_j$ for $\ell \gg 0$ by assumption, hence it follows from Step 1 that all coefficients of $g_j - g_{j+1}$ lie in I_j. Since R is complete with respect to the topology given by $(I_j)_{j\geq 1}$, it follows that there is $g \in R[\![x]\!]$ such that for every N and every j, if $\ell \gg 0$, then the coefficients of the monomials of degree $< N$ in $g - g_\ell$ are in I_j. In this case we have $P = fg$. Indeed, for every N and every j, we have $P - fg = f(g_\ell - g) + B_\ell$, hence for $\ell \gg 0$, all coefficients of the monomials of degree $< N$ in $P - fg$ are in I_j. Since R is separated in the topology given by $(I_j)_{j\geq 1}$, we conclude that $P = fg$. This completes the proof of the fact that M is separated.
Step 3. Note that $M/\mathfrak{m}M$ is generated over k by $1, x \ldots, x^{h-1}$. Indeed, $f \equiv \sum_{i\geq h} a_i x^i \pmod{\mathfrak{m}R[\![x]\!]}$. Since $\sum_{i\geq h} a_i x^i = x^h \cdot T(x)$, for some invertible $T \in R[\![x]\!]$, we have $M/\mathfrak{m}M \simeq R[\![x]\!]/((x^h) + \mathfrak{m}R[\![x]\!])$ and this is clearly generated by $1, \ldots, x^{h-1}$. By Step 2, we may thus apply Lemma 9.9.6 to conclude that M is generated over R by $1, x, \ldots, x^{h-1}$.
Step 4. We now show that these elements are linearly independent over R. Suppose there are $c_0, \ldots, c_{h-1} \in R$ such that $\sum_{i=0}^{h-1} c_i x^i = fg$ for some $g \in R[\![x]\!]$. It follows from Step 2 that all coefficients of g lie in $\cap_{\ell\geq 1} \mathfrak{m}^\ell$. However, this intersection is 0 since R is the separated in the topology given by $(I_j)_{j\geq 1}$, which is coarser than the $\mathfrak{m}$-adic one. Therefore $g = 0$, hence $c_i = 0$ for $0 \leq i \leq h-1$. This completes the proof of the theorem. □

Remark 10.9.8. For future reference, we note that by applying Step 1 of the above proof with $I = 0$, we deduce that if f is as in Theorem 9.9.7, then f is a non-zero divisor in $R[\![x]\!]$. Similarly, by taking $I = \mathfrak{m}^s$, with $s \geq 1$, we see that the image $\overline{f}$ of f in $(R/\mathfrak{m}^s)[\![t]\!]$ is a non-zero divisor.

Corollary 10.9.9 (Weierstrass division theorem). *Under the assumptions of Theorem 9.9.7, for every $p \in R[\![x]\!]$, there are unique $g, q \in R[\![x]\!]$, with q a polynomial of degree $< h$, such that $p = fg + q$.*

Proof. The existence of g and q, as well as the uniqueness of q, follow from Theorem 9.9.7. The uniqueness of g then follows from the fact that f is a non-zero divisor (see Remark 9.9.8). □

Corollary 10.9.10 (Weierstrass preparation theorem). *Under the assumptions of Theorem 9.9.7, we can uniquely write $f = uP$, with $u \in R[\![x]\!]$ invertible and $P = x^h + \sum_{i=0}^{h-1} c_i x^i$, with $c_i \in \mathfrak{m}$ for all i (such P is called a* Weierstrass polynomial*).*

Proof. We apply Corollary 9.9.9 with $p = x^h$ to write $x^h = fg + q$, where $q \in R[x]$ is a polynomial of degree $< h$. By mapping to $k[\![x]\!]$, we see that

$$x^h = x^h \overline{g}(x) \cdot \sum_{i=h}^{\infty} \overline{a_i} x^{i-h} + \overline{q}.$$

Since $\overline{q}$ has degree $< h$, we conclude that $\overline{q} = 0$ and $\overline{g}$ is invertible. This implies that g is invertible and we may take $u = g^{-1}$ and $P = x^h - q$.

Conversely, if $f = uP$ is as in the statement and $P = x^h + \sum_{i=0}^{h-1} c_i x^i$, then

$$x^h = u^{-1} f - \sum_{i=0}^{h-1} c_i x^i,$$

and the uniqueness of u and P follows from the uniqueness statement in Corollary 9.9.9. □

Example 10.9.11. If k is an infinite field and $f \in k[\![x_i \mid i \in \Lambda]\!]$ is nonzero, then there is $i_1 \in \Lambda$ and an automorphism

$$\phi\colon k[\![x_i \mid i \in \Lambda]\!] \to k[\![x_i \mid i \in \Lambda]\!] \simeq R[\![x_{i_1}]\!],$$

where $R = k[\![x_i \mid i \in \Lambda \smallsetminus \{i_1\}]\!]$, such that $\phi(f) = uP$, with u invertible and P a Weierstrass polynomial in x_{i_1}. In fact, we can find such ϕ given by a linear change of coordinates in finitely many variables. We write $f = \sum_{\ell \geq 0} f_\ell$, where each $f_\ell \in k[x_i \mid i \in \Gamma]$ is a homogeneous polynomial of degree ℓ. Let d be the smallest nonnegative integer such that $f_d \neq 0$ and suppose that $i_1, \ldots, i_r \in \Lambda$ are such that $f_d \in k[x_{i_1}, \ldots, x_{i_r}]$. After possibly applying a linear change of variables in $x_{i_1}, \ldots, x_{i_r}$, we may assume that $f_d(1, x_{i_2}, \ldots, x_{i_r}) \neq 0$. In this case, if $a_2, \ldots, a_r \in k$ are general, then $f_d(1, a_2, \ldots, a_r)$ is nonzero, hence the monomial $x_{i_1}^d$ appears with nonzero coefficient in $f_d(x_{i_1}, x_{i_2} + a_2 x_{i_1}, \ldots, x_{i_r} + a_r x_{i_1})$. Therefore we may assume that some power of x_{i_1} appears with nonzero coefficient in f, in which case Corollary 9.9.10 implies that we can write f as a product of an invertible element and a Weierstrass polynomial in x_{i_1}.

10.9.2 The formal arc theorem

In this section we prove a theorem concerning the completion of the arc space at a point that does not lie in the space of arcs of the non-smooth locus. This result provides a way to reduce the local ring of a point on X_∞, where X is a singular scheme, to the case of the local ring of a scheme of finite type and the local ring of an arc on a smooth variety. The twist comes from the fact that this is only true after passing to completions. We work over an arbitrary field k. If X is a scheme of finite type over k, we denote by X_{sm} the open subset consisting of the points where X is smooth over k and put $X_{\mathrm{sing}} = X \smallsetminus X_{\mathrm{sm}}$.

Theorem 10.9.12 (Formal arc theorem). *Let X be a scheme of finite type over k. If γ is a k-valued arc on X that does not lie in $J_\infty(X_{\mathrm{sing}})$, then there is a scheme of finite type Y over k and $y \in Y(k)$ such that we have an isomorphism in* $\mathrm{Comp}(k)$*:*

$$\widehat{\mathscr{O}_{J_\infty(X),\gamma}} \simeq \widehat{\mathscr{O}_{Y,y}} \,\widehat{\otimes}\, k[\![x_i \mid i \geq 1]\!].$$

Of course, the scheme Y in the theorem is not unique. If (Y,y) satisfies the condition in the theorem, then so does $(Y\times\mathbb{A}^1,(y,0))$. The theorem was first proved by Grinberg and Kazhdan in [GK00], over a field of characteristic 0. We present the proof following Drinfeld's note [Dri].

Proof of Theorem 9.9.12. The theorem is local, hence we may assume that X is affine. By Remark 9.9.1, it is enough to find (Y,y) as in the statement with the property that for every test ring $(A,\mathfrak{m}_A)$, we have a natural bijection

$$\mathrm{Hom}_{\mathrm{Comp}(k)}(\widehat{\mathscr{O}_{J_\infty(X),\gamma}},A)\simeq \mathrm{Hom}(\mathscr{O}_{Y,y},A)\times\mathfrak{m}_A^{\mathbb{Z}_{>0}}, \tag{10.26}$$

where the Hom set on the right-hand side is in the category of local k-algebras. On the other hand, since A is a local ring, it follows from Lemma 9.2.2 that the Hom set on the left-hand side is in natural bijection with the set of A-valued arcs on X that induce γ

Consider a closed embedding $X\hookrightarrow\mathbb{A}^N$. Let $x_\eta\in X$ be the image via γ of the generic point of $\mathrm{Spec}\,k[\![t]\!]$. By assumption, we have $x_\eta\in X_{\mathrm{sm}}$. If the dimension of X_{sm} at x_η is n and $r=N-n$, then there are $f_1,\ldots,f_r$ in the ideal of X such that if $W=V(f_1,\ldots,f_r)$, then $X=W$ at x_η and some r-minor of the Jacobian matrix of $f_1,\ldots,f_r$ does not vanish at x_η. We claim that the inclusion $J_\infty(X)\hookrightarrow J_\infty(W)$ induces an isomorphism $\widehat{\mathscr{O}_{J_\infty(X),\gamma}}\simeq\widehat{\mathscr{O}_{J_\infty(W),\gamma}}$. Let I_X and I_W be the ideals defining X and W, respectively, in $\mathbb{A}^N$. We need to show that for every test ring $(A,\mathfrak{m}_A)$ and every local k-algebra homomorphism $\delta^*\colon\mathscr{O}(\mathbb{A}^N)\to A[\![t]\!]$ which induces $\gamma^*\colon\mathscr{O}(\mathbb{A}^N)\to k[\![t]\!]$, if $\delta^*(I_W)=0$, then $\delta^*(I_X)=0$. Let us consider the ideal $\mathfrak{a}=\{g\in\mathscr{O}(\mathbb{A}^N)\mid g\cdot I_X\subseteq I_W\}$. Since $I_W=\mathscr{O}_X$ at x_η, it follows that $\mathfrak{a}=\mathscr{O}_{\mathbb{A}^N}$ at x_η, hence $\gamma^*(\mathfrak{a})\neq 0$. Therefore there is $h\in\mathfrak{a}$ such that $\delta^*(h)$ is a non-zero divisor (see Remark 9.9.8). On the other hand, we have $h\cdot\delta^*(I_X)\subseteq\delta^*(\mathfrak{a}\cdot I_X)\subseteq\delta^*(I_W)=0$. We conclude that $\delta^*(I_X)=0$, as claimed.

Therefore we may assume that $X=W$. In other words, we may assume that X is defined in $\mathrm{Spec}\,k[x_1,\ldots,x_n,y_1,\ldots,y_r]$ by $f_1(x,y),\ldots,f_r(x,y)$ and $\det(\frac{\partial f}{\partial y})$ does not vanish at x_η. In what follows, we denote the matrix $\left(\frac{\partial f}{\partial y}\right)$ by $B(x,y)$, its classical adjoint matrix by $\widehat{B}(x,y)$, and its determinant by $D(x,y)$. Therefore we have $B\cdot\widehat{B}=\widehat{B}\cdot B=D\cdot I_r$. We also denote by $f(x,y)$ the column vector $(f_1(x,y),\ldots,f(x,y))^T$.

The arc γ is given by some $(u_0,v_0)\in k[\![t]\!]^{\oplus n}\times k[\![t]\!]^{\oplus r}$. By hypothesis, $D(u_0,v_0)$ is a nonzero element of $k[\![t]\!]$. Let d be its order. Note that if $d=0$, then $\gamma\in J_\infty(X_{\mathrm{sm}})$, in which case the assertion in the theorem follows from the fact that after possibly replacing X with a suitable affine open subset, we may assume that $\Omega_{X/k}$ is trivial, hence $J_\infty(X)\simeq X\times\mathrm{Spec}\,k[x_i\mid i\in\mathbb{Z}_{>0}]$. In this case it is enough to use the isomorphism (9.23) in Remark 9.9.5. Therefore from now on we may and will assume $d>0$.

Suppose now that $(A,\mathfrak{m}_A)$ is a test ring and we want to describe the left-hand side of (9.26). This is in natural bijection with the set of A-valued arcs on X that induce γ, that is, with the set of those $(u,v)\in A[\![t]\!]^{\oplus n}\times A[\![t]\!]^{\oplus r}$ such that $f(u,v)=0$ and (u,v) is a lift of (u_0,v_0). Given such (u,v), note that $D(u,v)$ is a lift of $D(u_0,v_0)$, which has order d. We may thus apply the Weierstrass preparation theorem to write $D(u,v) \in A[\![t]\!]$ as αq, with α invertible and q a monic polynomial of degree d that is a lift of $t^d \in k[\![t]\!]$.

The key idea is to keep track also of q. In other words, we are interested in the set of those (u,v,q) such that $(u,v) \in A[\![t]\!]^{\oplus n} \times A[\![t]\!]^{\oplus r}$ is a lift of (u_0,v_0) that satisfies $f(u,v)=0$ and $q \in A[t]$ is a monic polynomial of degree d which is a lift of $t^d \in k[t]$ such that

$$D(u,v) \in qA[\![t]\!]. \tag{10.27}$$

Note that the conditions on q imply that it is a non-zero divisor in $A[\![t]\!]$ and moreover, its image in any $(A/\mathfrak{m}_A^i)[\![t]\!]$ is a non-zero divisor (see Remark 9.9.8). If (9.27) holds, then we can write $D(u,v) = \alpha q$ for a unique α, which is invertible since it is a lift of an invertible element in $k[\![t]\!]$. The uniqueness assertion in the Weierstrass preparation theorem therefore implies that such q is uniquely determined by (u,v).

The following lemma will allow us to isolate a finite set of equations. Let s be a fixed positive integer.

Lemma 10.9.13. *Suppose that $(A,\mathfrak{m}_A)$ is a test ring, $(u,v) \in A[\![t]\!]^{\oplus n} \times A[\![t]\!]^{\oplus r}$ is a lift of (u_0,v_0), and $q \in A[t]$ is a monic polynomial of degree d which is a lift of $t^d \in k[t]$ such that (9.27) holds, and furthermore, the following conditions are satisfied:*

$$f(u,v) \in (q^s A[\![t]\!])^{\oplus r} \quad \textit{and} \tag{10.28}$$

$$\widehat{B}(u,v) \cdot f(u,v) \in (q^{s+1}A[\![t]\!])^{\oplus r}. \tag{10.29}$$

In this case there is a unique $v' \in A[\![t]\!]^{\oplus r}$ that is a lift of v_0, with $v'-v \in (q^sA[\![t]\!])^{\oplus r}$, and such that $f(u,v')=0$.

Proof. By assumption, there is $e \geq 1$ such that $\mathfrak{m}_A^e = 0$. We prove the assertion by induction on e. If $e=1$, then $A=k$ and there is nothing to prove. Suppose now that $e \geq 2$. We may apply the induction hypothesis to $A/\mathfrak{m}_A^{e-1}$ to conclude that there is $w \in A[\![t]\!]^{\oplus r}$ which is a lift of v_0, such that $w - v \in (q^sA[\![t]\!])^{\oplus r}$ and $f(u,w) \in (\mathfrak{m}_A^{e-1})^{\oplus r}$.

We show that there is a unique $R \in A[\![t]\!]^{\oplus r}$ whose image in $(A/\mathfrak{m}_A^{e-1})[\![t]\!]^{\oplus r}$ is 0 such that if $v' = w + q^sR$, then $f(u,v')=0$. The Taylor expansion of f with respect to $y_1,\ldots,y_r$ gives

$$f(u,v') = f(u,w) + q^sB(u,w)\cdot R \tag{10.30}$$

(since the coefficients of all power series in R lie in $\mathfrak{m}_A^{e-1}$ and $2(e-1) \geq e$, the other terms in the Taylor expansion vanish). We now remark that it is enough to have $\widehat{B}(u,w)\cdot f(u,v')=0$. Indeed, if this is the case, then multiplying on the left by $B(u,w)$ gives $D(u,w)f(u,v')=0$. Since the image of $D(u,w)$ in $k[\![t]\!]$ is equal to $D(u_0,v_0)$, which is nonzero, it follows from Remark 9.9.8 that $D(u,w)$ is a non-zero divisor. Therefore in this case $f(u,v')=0$.

We deduce from (9.30) that

$$\widehat{B}(u,w)\cdot f(u,v') = \widehat{B}(u,w)\cdot f(u,w) + q^sD(u,w)\cdot R. \tag{10.31}$$

Since $w-v \in (q^sA[\![t]\!])^{\oplus r}$, it follows from (9.27) that q divides $D(u,w)$. Using again the fact that the image of $D(u,w)$ in $k[\![t]\!]$ is equal to $D(u_0,v_0)$, which has order d, we conclude that $D(u,w)=q\beta$, for some invertible $\beta \in A[\![t]\!]$. Since $w-v \in (q^sA[\![t]\!])^{\oplus r}$, it follows from (9.28) that $f(u,w) \in (q^sA[\![t]\!])^{\oplus r}$. Since $2s \geq s+1$, this together with (9.28) implies $\widehat{B}(u,w)\cdot f(u,w) \in (q^{s+1}A[\![t]\!])^{\oplus r}$. Let us write $\widehat{B}(u,w)\cdot f(u,w)=q^{s+1}S$ for some $S \in A[\![t]\!]^{\oplus r}$. Since $f_i(u,w) \in \mathfrak{m}_A^{e-1}$ for every i and since the class of q in $(A/\mathfrak{m}_A^{e-1})[\![t]\!]$ is a non-zero divisor, we conclude that the image of S in $(A/\mathfrak{m}_A^{e-1})[\![t]\!]$ is 0. We thus conclude that if $R=-\beta^{-1}S$, then the image of R in $(A/\mathfrak{m}_A^{e-1})[\![t]\!]$ is 0 and then $f(u,w+q^sR)=0$. Note also that there is a unique R that satisfies these two conditions. This is a consequence of (9.31) and of the fact that q is a non-zero divisor.

In order to prove the uniqueness of v', suppose that we also have $v'' \in A[\![t]\!]^{\oplus r}$ which is a lift of v_0 such that $v''-v \in (q^sA[\![t]\!])^{\oplus r}$ and $f(u,v'')=0$. Therefore we may write $v''-w=q^sR'$ for some $R' \in A[\![t]\!]^{\oplus r}$. By the induction hypothesis, we see that v' and v'' have the same image in $(A/\mathfrak{m}_A^{e-1})[\![t]\!]^{\oplus r}$, hence the image of q^sR' in $(A/\mathfrak{m}_A^{e-1})[\![t]\!]^{\oplus r}$ is 0. Using again the fact that the class of q in $(A/\mathfrak{m}_A^{e-1})[\![t]\!]$ is a non-zero divisor, we conclude that the image of R' in $(A/\mathfrak{m}_A^{e-1})[\![t]\!]^{\oplus r}$ is 0. In this case we have $R'=R$ by the uniqueness of R, hence $v'=v''$. $\square$

We now return to the proof of the theorem. It is clear that conditions (9.27), (9.28), and (9.29) only depend on the values of u and v mod q^{s+1}. Note that each $u \in A[\![t]\!]^{\oplus n}$ and $v \in A[\![t]\!]^{\oplus r}$ can be uniquely written as $u=u'q^{s+1}+u''$ and $v=v'q^{s+1}+v''$, with

$$u' \in A[\![t]\!]^{\oplus n},\ v' \in A[\![t]\!]^{\oplus r},\ u'' \in A[t]^{\oplus n},\ v'' \in A[t]^{\oplus r},$$

such that both u'' and v'' have all entries of degree $<(s+1)d$. Moreover, if we write similarly $u_0=u_0't^{(s+1)d}+u_0'$ and $v_0=v_0't^{(s+1)d}+v_0''$, with

$$u_0' \in k[\![t]\!]^{\oplus n},\ v_0' \in k[\![t]\!]^{\oplus r},\ u_0'' \in k[t]^{\oplus n},\ v_0'' \in k[t]^{\oplus r},$$

with u_0'' and v_0'' having all entries of degree $<(s+1)d$, then (u,v) is a lift of (u_0,v_0) if and only if (u',v') is a lift of (u_0',v_0') and (u'',v'') is a lift of (u_0'',v_0''). In particular, we see that the condition for (u',v') is that $u'-u_0'=\sum_{\imath\geq 0}\alpha_it^i$ and $v'-v_0'=\sum_{i\geq 0}\beta_it^i$, where $\alpha_i,\beta_i \in \mathfrak{m}_A$ for all $i \in \mathbb{Z}_{\geq 0}$.

Suppose that Y is the scheme of finite type over k such that for every k-algebra A, we have a natural bijection between $Y(A)$ and the set of triples (q,u,v), where $q \in A[t]$ is a monic degree d polynomial, $u \in A[t]^{\oplus n}$ and $v \in A[t]^{\oplus r}$ have all entries of degree $<(s+1)d$, such that conditions (9.27), (9.28), and (9.29) are satisfied. Note that since q is monic, each of these three divisibility conditions are algebraic conditions on the coefficients of q, u, and v. If $y \in Y(k)$ is the point corresponding to (t^d,u_0'',v_0''), we see that we have

$$\widehat{\mathscr{O}_{J_\infty(X),\gamma}} \simeq \widehat{\mathscr{O}_{Y,y}}\,\widehat{\otimes}\,k[\![x_i \mid i \geq 1]\!].$$

$\square$

Example 10.9.14. Suppose that $X \subseteq \mathbb{A}_k^n$ is defined by $\sum_{i=1}^n x_i^2 = 0$ (where we assume char$(k) \neq 2$). Let γ be the k-arc on X given by $(c_1 t, \ldots, c_n t)$, where $\sum_{i=1}^n c_i^2 = 0$ and $(c_1, \ldots, c_n) \neq (0, \ldots, 0)$. Suppose, for example, that $c_n \neq 0$. With the notation in the proof of the theorem, we have $d = 1$ and we take $s = 1$. In this case $q(t) = t - \alpha$ and it is more convenient to write each polynomial of degree < 2 as $u(t-\alpha) + v$. Therefore we may take Y to be the set of those $(\alpha, u_1, \ldots, u_n, v_1, \ldots, v_n) \in \mathbb{A}^{2n+1}$ such that $(t-\alpha)$ divides $u_n(t-\alpha) + v_n$ and $(t-\alpha)^2$ divides $\sum_{i=1}^n (u_n(t-\alpha) + v_n)^2$. Therefore

$$Y = \{(\alpha, u_1, \ldots, u_n, v_1, \ldots, v_n) \in \mathbb{A}^{2n+1} \mid v_n = 0, \sum_{i=1}^{n-1} u_i v_i = 0, \sum_{i=1}^{n-1} v_i^2 = 0\}.$$

Moreover, in this case $y = (0, c_1, \ldots, c_n, 0, \ldots, 0)$.

An easy computation then shows that in fact if $n = 2$, then we may take $Y = \mathrm{Spec}(k)$ and if $n = 3$, then we may take $Y = \mathrm{Spec}\, k[z]/(z^2)$.

10.9.3 The curve selection lemma

10.10 The Nash problem

This topic was started off by the influential paper [Nas95] of John Nash. While the paper was only published in 1995, it circulated in preprint form since the middle of the 1960s and it generated a lot of activity. After formulating the problem, we discuss some easy examples, including the case of toric varieties, then give an overview of the recent solution of the two-dimensional case, and end with a counterexample in dimension 3.

For simplicity, in this section we work over an algebraically closed field k, of characteristic 0. We will explicitly mention where the latter assumption is critical. Most of the time, however, it will only be used since we need to use resolutions of singularities. In particular, whenever we are in a setting where such resolutions are known to also exist in positive characteristic (for example, for surfaces or for toric varieties), most of what follows will carry through. In this section, by a resolution of singularities for a variety X we mean a projective, brational morphism $f : Y \to X$, with Y smooth.

10.10.1 The Nash map

Let X be a variety over k and Z a proper closed subset of X. In the usual setting for the Nash problem one often takes $Z = X_{\mathrm{sing}}$, but we prefer not to restrict to this case. We put

$$J_\infty^Z(X) = (\pi_\infty^X)^{-1}(Z) \subseteq J_\infty(X).$$

A *good component* of $J_\infty^Z(X)$ is an irreducible component which is not contained in $J_\infty(X_{\mathrm{sing}})$. Recall that by Proposition 9.2.15, $J_\infty^Z(X)$ has finitely many irreducible components.

Remark 10.10.1. Given X and Z as above, we also consider the set of k-valued points of $J_\infty^Z(X)$, that is,
$$X_\infty^Z := (\pi_\infty^X)^{-1}(Z) \subseteq X_\infty.$$
It follows from Proposition 9.2.15 that X_∞^Z is dense in $J_\infty^Z(X)$, hence we have a bijection between the irreducible components of $J_\infty^Z(X)$ and those of X_∞^Z, such that the good components of $J_\infty^Z(X)$ correspond to the good components of X_∞^Z, that is, to the irreducible components of this set that are not contained in $(X_{\mathrm{sing}})_\infty$. Therefore whenever describing the good components, we may restrict to the k-valued points.

Proposition 10.10.2. *Let X be a variety and Z a proper closed subset of X. If W is a good component of $J_\infty^Z(X)$, then the following hold:*

i) For every proper closed subset T of X, we have $W \not\subseteq J_\infty(T)$.
ii) If $f\colon Y \to X$ is a resolution of singularities, then there is a unique irreducible closed subset W_Y of $J_\infty(Y)$ such that $\overline{f_\infty(W_Y)} = W$. Moreover, there is a unique irreducible component Z' of $f^{-1}(Z)$ such that $W_Y = J_\infty^{Z'}(Y)$.

Proof. Given a resolution of singularities $f\colon Y \to X$, let B be a proper closed subset of X such that f is an isomorphism over $X \smallsetminus B$. Suppose that W is not contained in $J_\infty(B)$. Recall that by Proposition 9.2.8, f_∞ is surjective over $J_\infty(X) \smallsetminus J_\infty(B)$. Since $f_\infty^{-1}(J_\infty^Z(X)) = J_\infty^{f^{-1}(Z)}(Y)$, it follows that we can write
$$J_\infty^Z(X) = J_\infty^{Z\cap B}(B) \cup f_\infty(J_\infty^{f^{-1}(Z)}(Y)).$$
If $Z_1', \ldots, Z_r'$ are the irreducible components of $f^{-1}(Z)$, we obtain
$$J_\infty^Z(X) = J_\infty^{Z\cap B}(B) \cup \overline{f_\infty(J_\infty^{Z_1'}(Y))} \cup \ldots \cup \overline{f_\infty(J_\infty^{Z_r'}(Y))}.$$
Note that each $\overline{f_\infty(J_\infty^{Z_i'}(Y))}$ is irreducible. Since W is an irreducible component of $J_\infty^Z(X)$ that is not contained in $J_\infty(B)$, it follows that there is i such that $W = \overline{f_\infty(J_\infty^{Z_i'}(Y))}$. This implies, in particular, that for every proper closed subset B' of X, we have $W \not\subseteq J_\infty(B')$. Indeed, otherwise $J_\infty^{Z_i'}(Y) \subseteq J_\infty(f^{-1}(B'))$, contradicting the fact that a nonempty cylinder in the space of arcs of a smooth variety is not contained in the space of arcs of a proper closed subset (see Lemma 9.3.4). We also note that it is automatic that there is at most one irreducible closed subset W_Y of $J_\infty(Y)$ such that $\overline{f_\infty(W_Y)} = W$. Indeed, in this case f_∞ maps the generic point of W_Y to the generic point of W, which lies in the open subset $J_\infty(X) \smallsetminus J_\infty(B)$, over which f_∞ is injective.

Suppose now that we choose a resolution f as above that is an isomorphism over the smooth locus of X (hence we may take $B = X_{\mathrm{sing}}$). Since W is a good component, it follows that we may apply the above discussion. In particular, we obtain the assertion in i). This in turn implies that for every resolution f, we may apply the above argument and thus also deduce ii). $\square$

The following property shows that in the usual setting of the Nash problem, all components of $J_\infty^Z(X)$ are good. For this, the characteristic 0 assumption is crucial. This property, however, will not play an important role in what follows.

Proposition 10.10.3. *If X is a variety and $Z = X_{\mathrm{sing}}$, then all irreducible components of $J_\infty^Z(X)$ are good.*

Proof. Let $f\colon Y \to X$ be a resolution of singularities that is an isomorphism over $X \smallsetminus Z$. As in the proof of Proposition 9.10.2, we write

$$J_\infty^Z(X) = J_\infty(Z) \cup \overline{f_\infty(J_\infty^{f^{-1}(Z)}(Y))}$$

and $\overline{f_\infty(J_\infty^{f^{-1}(Z)}(Y))}$ is a union of irreducible closed subsets not contained in $J_\infty(Z)$. Therefore in order to prove the proposition, it is enough to show that $J_\infty(Z)$ is contained in the closure of $f_\infty(J_\infty^{f^{-1}(Z)}(Y))$. For this, we argue as in the proof of Theorem 9.2.10.

Let $Z_1,\ldots,Z_s$ be the irreducible components of Z, hence $J_\infty(Z) = \cup_{i=1}^s J_\infty(Z_i)$ by Lemma 9.2.7. For every i, let us choose W_i to be an irreducible component of $f^{-1}(Z_i)$ that dominates Z_i. By the generic smoothness theorem, we can find open subsets $U_i \subseteq Z_i$ and $V_i \subseteq W_i$ such that f induces a smooth surjective morphism $V_i \to U_i$. It follows from property 3) in Remark 9.2.3 that $J_\infty(U_i)$ is contained in the image of $J_\infty(V_i)$, hence in $f_\infty(J_\infty^{f^{-1}(Z)}(Y))$. Since each $J_\infty(Z_i)$ is irreducible by Theorem 9.2.10 and $J_\infty(U_i)$ is open in $J_\infty(Z_i)$, we conclude that each $J_\infty(Z_i)$ is contained in the closure of $f_\infty(J_\infty^{f^{-1}(Z)}(Y))$. Therefore the same holds for $J_\infty(Z)$. $\square$

Example 10.10.4. When $Z \neq X_{\mathrm{sing}}$, it is not necessarily true that all components of $J_\infty^Z(X)$ are good. Suppose for example that X is the hypersurface in $\mathbb{A}^3$ defined by $x^2 - y^2 z = 0$, where $\mathrm{char}(k) \neq 2$. We have seen in Example **??** that if Z consists of the origin, then $J_\infty^Z(Y)$ has two irreducible components, only one of which is good.

Example 10.10.5. The property in Proposition 9.10.3 can fail in positive characteristic. Suppose, for example, that X is the hypersurface in $\mathbb{A}^3$ given by $x^2 - y^2 z = 0$, with $\mathrm{char}(k) = 2$. Let $Z = X_{\mathrm{sing}}$. We have seen in Example 9.2.11 that $J_\infty(Z)$ contains an open subset of $J_\infty(X)$. Since $Z \simeq \mathbb{A}^1$, we deduce that $J_\infty(Z)$ is irreducible and therefore it is an irreducible component of $J_\infty^Z(X)$ which is not good.

It follows from Proposition 9.10.2 that given any resolution of singularities $f\colon Y \to X$, we can define a map $\mathcal{N}_{Y/X}^Z$ on the set of good components of $J_\infty^Z(X)$ and taking values in the set of irreducible components of $f^{-1}(Z)$ such that if $\mathcal{N}_{Y/X}^Z(W) = \widetilde{Z}$, then $W = \overline{f_\infty(J_\infty^{\widetilde{Z}}(Y))}$. It is clear from this formula that $\mathcal{N}_{Y/X}^Z$ is an injective map.

Remark 10.10.6. Suppose that X and Z are as above and $f\colon Y \to X$ and $g\colon Y' \to Y$ are such that both f and $f \circ g$ are resolutions of singularities. In this case, for every good component W of $J_\infty^Z(X)$, we have

$$\mathscr{N}_{Y/X}^Z(W) = g(\mathscr{N}_{Y'/X}^Z(W)).$$

Indeed, if $\widetilde{Z} = (\mathscr{N}_{Y/X}^Z(W))$ and $\widetilde{Z}' = (\mathscr{N}_{Y'/X}^Z(W))$, it follows from the uniqueness statement in Proposition 9.10.2 that

$$\overline{g_\infty(J_\infty^{\widetilde{Z}'}(Y'))} = J_\infty^{\widetilde{Z}}(Y).$$

Since $\pi_\infty^{Y'}(J_\infty^{\widetilde{Z}'}(Y')) = \widetilde{Z}'$ and $\pi_\infty^Y(J_\infty^{\widetilde{Z}}(Y)) = \widetilde{Z}$, we conclude that $\widetilde{Z}'$ dominates $\widetilde{Z}$, that is, $g(\widetilde{Z}') = \widetilde{Z}$.

Our next goal is to obtain a version of the map $\mathscr{N}_{Y/X}^Z$ that is independent of the resolution, mapping the good components of $J_\infty^Z(X)$ to certain divisors over X. Suppose that X is a variety and Z is a proper closed subset of X. An *essential divisor over X with respect to Z* is a divisor E over X such that for every resolution of singularities $f\colon Y \to X$, the center $c_Y(E)$ of E on Y is an irreducible component of $f^{-1}(Z)$. In particular, this implies that $c_X(E) \subseteq Z$. We simply say that E is an essential divisor over X if it is an essential divisor over X with respect to $X_{\rm sing}$.

Example 10.10.7. Suppose, for example, that E is a divisor over X such that $c_X(E) \subseteq Z$ and for every resolution of singularities $f\colon Y \to X$ the center $c_Y(E)$ is a divisor on Y. It is clear that in this case $c_Y(E)$ is an irreducible component of $f^{-1}(Z)$, hence E is an essential divisor over X with respect to Z.

Example 10.10.8. It was shown by Abhyankar (see [Abh56, Proposition 4]) that if $h\colon Y' \to Y$ is a proper, birational morphism of varieties, with Y smooth, and E is a prime divisor on Y' such that $\dim(h(E)) < \dim(E)$, then E is ruled, that is, it is birational to $Y_1 \times \mathbb{P}^1$ for some variety Y_1. This implies that if X is a variety, Z is a proper closed subset of X, and E is a divisor over X such that $c_X(E) \subseteq Z$ and E is not ruled (note that this assumption is independent on the model on which we view E), then for every resolution of singularities $f\colon Y \to X$, the center of E on Y is a divisor. Therefore E is an essential divisor over X with respect to Z.

Remark 10.10.9. If E is an essential divisor over X with respect to Z, then for every projective, birational morphism $g\colon X' \to X$, we have that E is an essential divisor over X', with respect to $g^{-1}(Z)$. Indeed, this simply follows from the fact that if $f\colon Y \to X'$ is a resolution of X', then $g \circ f$ is a resolution of X.

Remark 10.10.10. By putting conditions on the resolutions we consider, we can enlarge the class of essential divisors. For example, if in the definition of essential divisors we only consider resolutions $f\colon Y \to X$ such that $f^{-1}(Z)$ has pure codimension 1, then E is a *divisorially essential divisor over X with respect to Z*. Similarly, one can only consider, as in [IK03], resolutions of X that give an isomorphism over the smooth locus of X. Moreover, when Z is contained in the singular locus of X, one can only consider resolutions $f: Y \to X$ that give an isomorphism over the smooth locus of X and such that $f^{-1}(Z)$ has pure codimension 1. However, in what follows we will not make use of these variations.

Lemma 10.10.11. *Let E be a divisor over X and U an open subset of X such that $c_X(E) \cap U \neq \emptyset$. For every proper closed subset Z of X, E is an essential divisor over X with respect to Z if and only if E is an essential divisor over U, with respect to $Z \cap U$.*

Proof. Since $c_X(E) \cap U \neq \emptyset$, it follows that E can be considered as a divisor over U. If $f: Y \to X$ is a resolution of X, then the induced morphism $f^{-1}(U) \to U$ is a resolution of U. It is clear that $c_X(E)$ is an irreducible component of $f^{-1}(Z)$ if and only if $c_{f^{-1}(U)}(E) = c_Y(E) \cap f^{-1}(U)$ is an irreducible component of $f^{-1}(Z) \cap f^{-1}(U)$. In order to complete the proof, it is enough to show that given any resolution of singularities $g: V \to U$, there is a resolution of singularities $f: Y \to X$ and an isomorphism $V \simeq f^{-1}(U)$ over U. It follows from a theorem of Nagata and Deligne (see [Con07]) that we can factor the composition $V \to U \to X$ as $V \overset{j}{\hookrightarrow} \overline{V} \overset{h}{\to} X$, with h proper, $\overline{V}$ a variety, and j an open immersion. Since V is smooth, we can find a resolution of singularities $h': Y \to \overline{V}$ that is an isomorphism over V. It is clear that the composition $f = h \circ h'$ has the desired properties. □

Lemma 10.10.12. *Suppose that E is an essential divisor over X with respect to Z. For every resolution of singularities $f: Y \to X$, if $c_Y(E) = W$, then E is equal as a divisor over X with the unique irreducible component dominating W of the exceptional divisor on the blow-up of Y along W.*

Proof. It follows from Remark 9.10.9 that after replacing X by Y and Z by $f^{-1}(Z)$, we may assume that $Y = X$, in which case W is an irreducible component of Z. Furthermore, Lemma 9.10.11 implies that we may replace X by an open subset intersecting W, hence we may assume that W is smooth. Let $g: B \to Y$ be the blow-up of Y along W, and let F be the exceptional divisor. Since $c_B(E) \subseteq g^{-1}(W) = F$ is, by assumption, an irreducible component of $g^{-1}(Z)$, it follows that $c_B(E) = F$, hence $E = F$ as divisors over X. □

Corollary 10.10.13. *If $f: Y \to X$ is a resolution of X and Z is a proper closed subset of X, then any two distinct divisors over X that are essential with respect to Z have distinct centers on Y. In particular, there are at most finitely many essential divisors over X with respect to Z.*

Proof. It follows from Lemma 9.10.12 that if E is an essential divisor over X with respect to Z, then E is determined by its center on Y. This gives the first assertion in the corollary. The second assertion follows from the fact that the center of every essential divisor over X with respect to Z is an irreducible component of $f^{-1}(Z)$ and there are only finitely many such irreducible components. □

If $f\colon Y \to X$ is a resolution of X and Z is a proper closed subset of X, then the irreducible components of $f^{-1}(Z)$ that are centers on Y of essential divisors over X with respect to Z will be called *essential components of* $f^{-1}(Z)$. It follows from Lemma 9.10.12 that each essential component determines the corresponding divisor over X.

Proposition 10.10.14. *Let X be a variety and Z a proper closed subset of X. There is a unique injective map*

$$\mathcal{N}^Z\colon \{\textit{Good components of}\, J_\infty^Z(X)\} \to \{\textit{Essential divisors over}\, X \,\textit{with respect to}\, Z\}$$

such that for every resolution of singularities $f\colon Y \to X$, the center of $\mathcal{N}^Z(W)$ on Y is equal to $\mathcal{N}^Z_{Y/X}(W)$.

Proof. Let $h\colon \widetilde{X} \to X$ be a resolution of singularities such that $h^{-1}(Z)$ has all irreducible components of dimension 1. If W is a good component of $J_\infty^Z(X)$, we let $\mathcal{N}^Z(W)$ be the divisor over X corresponding to the prime divisor $\mathcal{N}^Z_{\widetilde{X}/X}(W)$ on $\widetilde{X}$. Since $\mathcal{N}^Z_{\widetilde{X}/X}$ is injective, it follows that $\mathcal{N}^Z$ is injective. If $f\colon Y \to X$ is any resolution of singularities, by considering a third resolution that dominates both Y and $\widetilde{X}$, we deduce using Remark 9.10.6 that the center of $\mathcal{N}^Z(W)$ on Y is equal to $\mathcal{N}^Z_{Y/X}(W)$. Therefore $\mathcal{N}^Z$ satisfies the property in the proposition. Moreover, by definition $\mathcal{N}^Z_{Y/X}(W)$ is an irreducible component of $f^{-1}(Z)$. We thus conclude that $\mathcal{N}^Z(W)$ is an essential divisor over X with respect to Z. $\square$

The map $\mathcal{N}^Z$ in Proposition 9.10.14 is the *Nash map* (of X, with respect to Z). The "classical" Nash map is obtained for $Z = X_{\mathrm{sing}}$.

10.10.2 The Nash problem. Examples

Let X be a variety and Z a proper closed subset of X. The *Nash problem* for X with respect to Z asks whether the Nash map $\mathcal{N}^Z$ is surjective, that is, whether every essential divisor over X with respect to Z is in the image of $\mathcal{N}^Z$. In the literature, one usually considers the special case of this question when $Z = X_{\mathrm{sing}}$.

Remark 10.10.15. The surjectivity of $\mathcal{N}^Z$ has the following interpretation. Suppose that $f\colon Y \to X$ is a resolution of singularities and $Z_1, \ldots, Z_r$ are the essential components of $f^{-1}(Z)$ (an important special case is when all irreducible components of $f^{-1}(Z)$ have codimension 1, hence each Z_i is a prime divisor). The Nash problem for $\mathcal{N}^Z$ has a positive answer (that is, $\mathcal{N}^Z$ is surjective) if and only if the closure of each $f_\infty(J_\infty^{Z_i}(Y))$ gives an irreducible component of $J_\infty^Z(X)$. Equivalently, this is the case if and only if

$$f_\infty(J_\infty^{Z_i}(Y)) \not\subseteq \overline{f_\infty(J_\infty^{Z_j}(Y))} \tag{10.32}$$

for every $i, j \leq r$, with $i \neq j$.

Remark 10.10.16. With the notation in the previous remark, suppose that $\mathscr{N}_Z$ is not surjective, and let $i \neq j$ be such that we have the inclusion (9.32). Let us assume, for simplicity, that X is affine. In this case, we see that for every nonzero $\phi \in \mathscr{O}(X)$, we have

$$\mathrm{ord}_{D_i}(\phi) \geq \mathrm{ord}_{D_j}(\phi),$$

where D_i and D_j are the divisors over X corresponding to Z_i and Z_j, respectively. Indeed, after possibly replacing f with another resolution, we may assume that both Z_i and Z_j are prime divisors. In this case the assertion follows from the fact that

$$\mathrm{ord}_{D_i}(\phi) = \min\{\mathrm{ord}_t\, \gamma^*(\phi \circ f) \mid \gamma \in J_\infty^{Z_i}(Y)\} = \min\{\mathrm{ord}_t\, \gamma^*(\phi) \mid \gamma \in f_\infty(J_\infty^{Z_i}(Y))\}$$

and the corresponding formula for D_j (see, for example, the proof of Theorem 9.6.2).

Remark 10.10.17. The Nash problem is of a local nature. More precisely, suppose that the variety X has a cover $X = \cup_{i=1}^s U_i$ by open subsets. If Z is a proper closed subset of X, then the map $\mathscr{N}^Z$ corresponding to X is surjective if and only if each map $\mathscr{N}^{Z \cap U_i}$ corresponding to U_i is surjective. This is an immediate consequence of the interpretation in Remark 9.10.15.

Example 10.10.18. Let us consider the easy case when X is a smooth variety. By taking in $f = 1_X$ in Remark 9.10.15, we see that for every Z, the Nash map $\mathscr{N}^Z$ is surjective. Moreover, in this case the essential divisors over X with respect to Z are in bijection with the irreducible components of Z: for each such component B, the corresponding divisor is the unique component dominating B of the exceptional divisor on the blow-up of X along B.

Proposition 10.10.19. *If X is a curve, then for every proper closed subset Z of X, the Nash map $\mathscr{N}^Z$ is surjective.*

Proof. By taking a suitable affine open cover of X, we see using Remark 9.10.17 that we may assume X is affine, Z consists of a single point $x_0 \in X$, and $X \smallsetminus \{x_0\}$ is smooth. Let $f\colon Y \to X$ be the normalization of X. Since this is the only resolution of X, it follows from definition that the essential divisors over X with respect to Z correspond to the points in the fiber $f^{-1}(x_0)$. Let $y_1, \ldots, y_r$ be these points. It follows from Remark 9.10.16 that if $\mathscr{N}_Z$ is not surjective, then we can find $i, j \leq r$, with $i \neq j$, such that for every nonzero $\phi \in \mathscr{O}(X)$ we have

$$\mathrm{ord}_{y_i}(\phi \circ f) \geq \mathrm{ord}_{y_j}(\phi \circ f).$$

Note that there is N such that for every nonzero $\phi \in \mathscr{O}(Y)$, if $\mathrm{div}_Y(\phi) \geq \sum_{i=1}^r N y_i$, then $\phi \in \mathscr{O}(X)$. In order to obtain a contradiction, it is enough to find a nonzero $\phi \in \mathscr{O}(Y)$ such that

$$\mathrm{div}_Y(\phi) \geq \sum_{i=1}^r N y_i \text{ and } \mathrm{ord}_{y_i}(\phi) < \mathrm{ord}_{y_j}(\phi). \tag{10.33}$$

Moreover, we may replace Y by any open subset containing $y_1, \ldots, y_r$. Let $\overline{Y}$ denote the smooth projective curve containing Y as an open subset. Let $D = \sum_{\ell=1}^r a_\ell y_\ell$ be a divisor on $\overline{D}$ such that $a_i < a_j$, $a_\ell \geq N$ for every ℓ, and $\sum_{\ell=1}^r a_\ell \geq 2g$, where g is the genus of $\overline{Y}$. In this case $\mathscr{O}_{\overline{Y}}(D)$ is globally generated, hence we can find an effective divisor E on $\overline{Y}$ such that $D \sim E$ and $y_\ell \notin \mathrm{Supp}(E)$ for every ℓ. After replacing Y by $Y \smallsetminus \mathrm{Supp}(E)$, we see that $D|_Y = \mathrm{div}_Y(\phi)$ for some nonzero $\phi \in \mathscr{O}(Y)$ and by the choice of D, (9.33) is satisfied. This gives a contradiction and thus completes the proof of the proposition. $\square$

We now prove, following [IK03], that the Nash map is surjective in the toric setting.

Theorem 10.10.20. *If X is a toric variety and Z is an invariant proper closed subset of X, then the Nash map $\mathscr{N}^Z$ is surjective.*

Proof. We may cover X by open subsets which are affine toric varieties, hence by Remark 9.10.17, it is enough to prove the theorem when $X = U_\sigma$, for some cone σ in $N_{\mathbb{R}}$. By Remark 9.10.1, in order to describe the good components of $J_\infty^Z(X)$, it is enough to consider the k-valued points of this set, that is, we may restrict to X_∞^Z. Recall that X_∞° denotes the arcs in X_∞ that do not lie in the space of arcs of any proper closed invariant subset of X. It follows from Proposition 9.10.2 that every good component of X_∞^Z has nonempty intersection with X_∞°. Therefore we have a bijection between the good components of X_∞^Z and the irreducible components of $X_\infty^Z \cap X_\infty^\circ$.

We make use of the description of X_∞° from Example 9.2.16. It is clear that $X_\infty^Z \cap X_\infty^\circ$ is preserved by the T_∞-action on X_∞. Let $\Lambda = \cup_\tau(\mathrm{Relint}(\tau) \cap N)$, where the union is over the faces τ of σ such that $V(\tau) \subseteq Z$. Note that if $v \in \sigma \cap N$, then $v \in \Lambda$ if and only if $T_\infty \cdot \gamma_v \subseteq X_\infty^Z \cap X_\infty^\circ$. Consider on $\sigma \cap N$ the order given by $v \geq w$ when $v - w \in \sigma$, and let S be the set of minimal elements in Λ with respect to this order relation. It follows from the discussion in Example 9.2.16 that the irreducible components of $X_\infty^Z \cap X_\infty^\circ$ are precisely the orbit closures $\overline{T_\infty \cdot \gamma_v}$, for $v \in S$. Note that each $v \in S$ is primitive by the minimality assumption and it is easy to see that $\mathscr{N}^Z$ maps the corresponding irreducible component of $J_\infty^Z(X)$ to the toric divisor D_v over X associated to v.

We turn to the essential divisors over X with respect to Z. Since there is a toric resolution of singularities $g\colon X' \to X$ such that $g^{-1}(Z)$ has all irreducible components of codimension 1, it follows that every essential divisor over X with respect to Z is toric. Let us choose such an essential divisor D_w corresponding to the primitive element $w \in \sigma \cap N$. Since $c_X(D_w) \subseteq Z$, it follows that $w \in \Lambda$. We assume, by way of contradiction, that $w \notin S$, that is, we can write $w = w_1 + w_2$, with $w_1 \in \Lambda$ and $w_2 \in \sigma \cap N$ nonzero. In order to get a contradiction, it is enough to construct a toric resolution of singularities $f\colon Y \to X$ corresponding to a fan Δ_Y refining σ, such that all irreducible components of $f^{-1}(Z)$ have codimension 1, but w does not lie on a ray in Δ_Y. For the facts about toric resolutions of singularities that we will use, we refer to [Ful93, Section 2.6].

Let us consider the 2-dimensional subcone σ_1 of σ generated by w_1 and w_2 and let Σ be its fan refinement giving the minimal resolution of the corresponding affine toric surface (see *loc.cit.*). It is known that the set of primitive generators for the rays in Σ gives the unique minimal system of generators for the semigroup $\sigma_1 \cap N$. Since $w = w_1 + w_2$ and w is primitive, it follows that w does not lie on any ray of Σ. Let v_1 and v_2 denote the primitive ray generators of the cone σ_2 in Σ that contains w. For every $w' \in N \cap (\sigma_1 \smallsetminus \mathbb{R}_{\geq 0} w_2)$, some multiple of w' can be written as $m_1 w_1 + m_2 w_2$, with $m_1, m_2 \in \mathbb{Z}_{\geq 0}$ and m_1 nonzero. This implies that if w' lies in a face τ of σ, then $w_1 \in \tau$, hence $V(\tau) \subseteq Z$ and we deduce that $w' \in \Lambda$. In particular, we conclude that at least one of v_1 and v_2, say v_1, lies in Λ.

We begin constructing a sequence of fans refining σ. Let Δ_1 be the star-division of Δ with respect to v_1 and Δ_2 the star-division of Δ_1 with respect to v_2. Note that σ_2 is a cone in Δ_2. We now construct a toric resolution of $X(\Delta_2)$, as follows. We first consider a succession of star-divisions resulting in a simplicial refinement Δ_3 of Δ_2. More precisely, at each step we pick a non-simplicial cone of smallest possible dimension and do a star-division with respect to a lattice point in the relative interior of this cone. After finitely many steps, we obtain the simplicial refinement Δ_3. We now do another succession of star-divisions, resulting in a regular fan Δ_4 refining Δ_3. At each step, we pick a singular cone of smallest possible dimension. If this cone τ has primitive ray generators $a_1, \ldots, a_s$, then there is $a = t_1 a_1 + \ldots + t_s a_s \in N$, with $0 \leq t_i < 1$ for all i; we apply the star-division with respect to a. After finitely many steps, the resulting fan Δ_4 is regular. Note that since the cone σ_2 is regular, it was not touched during this process. Therefore $\sigma_2 \in \Delta_4$. The final step is to apply a sequence of toric blow-ups in order to guarantee that the inverse image of Z has all irreducible components of codimension 1. Let $\tau_1, \ldots, \tau_d$ be the minimal cones in Δ_4 with the property that the corresponding irreducible invariant subvarieties of $X(\Delta_4)$ lie in the inverse image of Z. We first blow-up along $V(\tau_1)$, then blow-up along the proper transform of $V(\tau_2)$, and so on; after d steps, we obtain the fan Δ_5 refining Δ_4, which is still regular, and such that the inverse image of Z has codimension 1 irreducible components. Note that σ_2 is not a face of any of the τ_i: this is due to the fact that the divisor corresponding to $\mathbb{R}_{\geq 0} v_1$ lies in the inverse image of Z. Therefore σ_2 belongs to Δ_5, hence v does not lie on a ray of Δ_5. We thus achieved the desired contradiction. $\square$

10.10.3 *The Nash problem for surfaces*

10.10.4 *Counterexamples for the Nash problem*

The first counterexample to the higher-dimensional Nash problem was given in [IK03], in dimension 4. An example in dimension 3 was obtained in [dF13] and building on this, the paper [**?**] gave a series of such 3-dimensional examples. The moral is that such counterexamples are quite common. On the other hand, [**?**] proposes another formulation of the Nash's problem that might still hold in arbitrary dimensions.

In what follows we discuss the simplest counterexample to the Nash problem from [**?**], namely the hypersurface

$$X = V(x^2 + y^2 + z^2 + w^5) \subset \mathbb{A}^4. \tag{10.34}$$

Note that X has an isolated singular point at 0 and we take $Z = \{0\}$. Since X is a hypersurface, it is Gorenstein. Furthermore, it is normal, since it is Cohen-Macaulay and smooth in codimension 1. We begin with the following general result from [**?**], describing the irreducible components of $J_\infty^0(H)$ for certain hypersurfaces $H \subseteq \mathbb{A}^{n+2}$.

Lemma 10.10.21. *Let $f \in k[x_1, \ldots, x_n]$ be a polynomial with $\mathrm{ord}_0(f) = m \geq 2$. If*

$$H = V(uv + f(x_1, \ldots, x_2)) \subset \mathbb{A}^{n+2},$$

then H_∞^0 has m irreducible components $W_1, \ldots, W_{m-1}$ such that for a general $\gamma \in W_i$, we have $\mathrm{ord}_t(\gamma^*(u)) = i$ and $\mathrm{ord}_t(\gamma^*(v)) = m - i$.

Note that an obvious change of variable allows us to write the equation of X as $xy + z^2 + w^5 = 0$. Therefore the lemma implies that X_∞^0 is irreducible.

Proof of Lemma 9.10.21. We have

$$H_\infty^0 = \{(a, b, y_1, \ldots, y_n) \in (tk[[t]])^{n+2} \mid ab = f(y_1, \ldots, y_n)\}.$$

It is clear that for every $(a, b, y_1, \ldots, y_n) \in H_\infty^0$, we have $\mathrm{ord}_t(f(y_1, \ldots, y_n)) \geq m$, hence $\mathrm{ord}_t(a) + \mathrm{ord}_t(b) \geq m$. Moreover, the following open subset of H_∞^0

$$U := \{(a, b, y_1, \ldots, y_n) \in H_\infty^0 \mid \mathrm{ord}_t(f(y_1, \ldots, y_n)) = m\}$$

can be written as the union $U = U_1 \cup \ldots \cup U_{m-1}$, where

$$U_i = \{(a, b, y_1, \ldots, y_n) \in H_\infty^0 \mid \mathrm{ord}_t(a) = i, \mathrm{ord}_t(b) = m - i\}.$$

Since U_i consists of those $(a, b, y_1, \ldots, y_n) \in U$ with $\mathrm{ord}_t(a) \geq i$ and $\mathrm{ord}_t(b) \geq m - i$, it follows that U_i is closed in U. It is also clear that no U_i contains U_j for $i \neq j$. If we write $f = \sum_{i \geq m} f_i$, with each f_i homogeneous of degree i, then an element $\gamma = (a, b, y_1, \ldots, y_n) \in U_i$ is uniquely determined by $a = t^i a' \in t^i k[[t]]$ and the $y_i = t y_i' \in tk[[t]]$, for $1 \leq i \leq n$, with the condition $f_m(y_1', \ldots, y_n') \neq 0$. Therefore

$$U_i \simeq \{(a', y_1', \ldots, y_n') \in (k[[t]])^{n+1} \mid f_m(y_1', \ldots, y_n') \neq 0\}, \tag{10.35}$$

hence U_i is irreducible, since the right-hand side of (9.35) is an open subset of $(\mathbb{A}^{n+1})_\infty$. This implies that $U_1, \ldots, U_{m-1}$ are the irreducible components of U and we obtain the assertion in the lemma with $W_i = \overline{U_i}$ if we show that U is dense in H_∞^0.

Let us consider some $\gamma = (a, b, y_1, \ldots, y_n) \in (tk[[t]])^{n+2}$ with $ab = f(y_1, \ldots, y_n)$. We may and will choose i with $1 \leq i \leq m - 1$ such that $\mathrm{ord}_t(a) \geq i$ and $\mathrm{ord}_t(b) \geq$

$m-i$. We also choose some $c=(c_1,\ldots,c_n)\in k^n$ such that $f_m(c)\neq 0$. Consider the set

$$B=\{(\lambda,w)\in\mathbb{A}^1\times t^{m-i}k[\![t]\!]\mid f(y_1+\lambda c_1t,\ldots,y_n+\lambda c_nt)=(a+\lambda t^i)(b+w)\}.$$

It is easy to see that the projection onto the first component gives an isomorphism $B\simeq\mathbb{A}^1$, hence B is irreducible. On the other hand, we have the map

$$B\to H^0_\infty,\ (\lambda,w)\to(a+\lambda t^i,b+w,y_1+\lambda c_1t,\ldots,y_n+\lambda c_nt)$$

whose image intersects U_i and contains γ. Therefore $\gamma\in\overline{U}$. $\square$

Our goal is to show that there are two essential divisors over X. The key part of the argument will make use of log discrepancies (for the basic facts about relative canonical divisors that we will use, we refer to Section 3.1). We begin by describing a resolution of X. Let $\pi\colon X'\to X$ be the blow-up of X at 0. An easy computation in local charts shows that X' has a unique singular point p, which in a chart isomorphic to $\mathbb{A}^4$ is given by the equation $x^2+y^2+z^2+w^3=0$. As for X, we see that X' is normal and Gorenstein. If $\mathfrak{m}_0$ is the ideal of $0\in X$, then $\mathfrak{m}_0\cdot\mathscr{O}_{X'}=\mathscr{O}_{X'}(-E_1)$, where E_1 is a prime divisor on X', which in this chart is defined by (w). A computation based on the adjunction formula implies $K_{X'/X}=E_1$ (see Example 3.1.9).

Let $\pi'\colon X''\to X'$ be the blow-up at p. Again, a computation in local charts shows that X'' is smooth and the π'-exceptional divisor has a unique irreducible component E_2. Moreover, the proper transform of E_1 is smooth (we still denote it by E_1) and we have $(\pi')^*(E_1)=E_1+E_2$. Using the adjunction formula, we obtain $K_{X''/X'}=E_2$. The divisor E_2 has a unique singular point q, which does not lie on E_1. In fact, a computation in local charts shows that E_2 is the cone over a smooth plane conic, hence the blow-up $\pi''\colon\widetilde{X}\to X''$ of q gives a log resolution of X. Since X'' is smooth, we have $K_{\widetilde{X}/X''}=2E_3$, where E_3 is the exceptional divisor of π'', $(\pi'')^*(E_2)=E_2+2E_3$, and $(\pi'')^*(E_1)=E_1$. We thus conclude that

$$K_{\widetilde{X}/X}=E_1+2E_2+6E_3,$$

hence X has terminal singularities.

In particular, since X'' is smooth, it follows that $f=\pi\circ\pi'$ is a resolution of X, hence the essential divisors over X are among E_1 and E_2. We next show that E_1 is the divisor that lies in the image of the Nash map.

Lemma 10.10.22. *With the above notation, the Nash map of X (with respect to $\{0\}$) maps $J^0_\infty(X)$ to E_1.*

Proof. Since $J^0_\infty(X)$ is irreducible and we have the resolution f such that $f^{-1}(0)$ has only two irreducible components E_1 and E_2, we deduce that if the conclusion of the lemma fails, then

$$f_\infty(J^{E_1}_\infty(X''))\subseteq\overline{f_\infty(J^{E_2}_\infty(X''))}.$$

As pointed out in Remark 9.10.16, in this case we have $\mathrm{ord}_{E_1}(\phi)\geq\mathrm{ord}_{E_2}(\phi)$ for every nonzero $\phi\in\mathscr{O}(X)$. On the other hand, since $(\pi')^*(E_1)=E_1+E_2$, we see that

$\mathrm{ord}_{E_2}(\phi) \geq \mathrm{ord}_{E_1}(\phi)$ for every such ϕ. This implies that E_1 and E_2 define the same valuation, a contradiction. □

Therefore in order to show that X gives a counterexample to the Nash problem, it is enough to prove that E_2 is an essential divisor. Before achieving this, we need the following general lemma.

Lemma 10.10.23. *If $f \in k[x_1, \ldots, x_n]$ is an irreducible polynomial and*

$$Y = V(uv + f(x_1, \ldots, x_n)) \subset \mathbb{A}^{n+2},$$

then $\mathscr{O}(Y)$ is factorial. In particular, Y is $\mathbb{Q}$-factorial.

Proof. Note that u is a non-zero divisor in $\mathscr{O}(Y)$ and let D be the effective Cartier divisor in Y defined by (u). By the assumption on f, this is a prime divisor in Y. Moreover, if Y_0 is the complement of D, then

$$Y_0 \simeq \operatorname{Spec} k[u, u^{-1}, v, x_1, \ldots, x_n]/(v + u^{-1} f(x)) \simeq (\mathbb{A}^1 \smallsetminus \{0\}) \times \mathbb{A}^n.$$

Therefore $\mathrm{Cl}(Y_0) = 0$ and the exact sequence

$$\mathbb{Z} \xrightarrow{\phi} \mathrm{Cl}(Y) \to \mathrm{Cl}(Y_0) \to 0, \ \phi(1) = [D]$$

implies that $\mathrm{Cl}(Y)$ is generated by the class of D. Since D is Cartier, it follows that every Weil divisor on Y is Cartier. □

We can now prove that X gives a counterexample to the Nash problem.

Proposition 10.10.24. *With the above notation, E_2 is an essential divisor over X.*

Proof. Let $g \colon Y \to X$ be a resolution of singularities. We need to show that $W := c_Y(E_2)$ is an irreducible component of $g^{-1}(0)$. This is clear if $\mathrm{codim}_Y(W) = 1$, hence we may assume that $\mathrm{codim}_Y(W) \geq 2$. By Lemma 9.10.23, X is $\mathbb{Q}$-factorial, hence all irreducible components of the exceptional locus $\mathrm{Exc}(g)$ have codimension 1 (see Remark 2.2.5). In particular, W is contained in at least one g-exceptional divisor.

If $\widetilde{g} \colon \widetilde{Y} \to Y$ is such that $\widetilde{Y}$ is a resolution of X that dominates X_2, then as we have seen, the coefficient of E_2 in $K_{\widetilde{Y}/X}$ is 2. On the other hand, we have

$$K_{\widetilde{Y}/X} = K_{\widetilde{Y}/Y} + \widetilde{g}^*(K_{Y/X})$$

and since X has terminal singularities, $K_{Y/X}$ is effective, and all g-exceptional divisors on Y have coefficient ≥ 1 in $K_{Y/X}$ (recall that $K_{Y/X}$ is an integral divisor since X is Gorenstein). On the other hand, it follows from Corollary 3.1.14 that the coefficient of E_2 in $K_{\widetilde{Y}/Y}$ is $\geq \mathrm{codim}_Y(W) - 1 \geq 1$. By putting these together, we conclude that W is a curve, there is a unique g-exceptional divisor F on Y that contains W, and the coefficient of F in $K_{Y/X}$ (hence also in $K_{\widetilde{Y}/X}$) is 1.

If $g(F)$ is a curve, then W is an irreducible component of $g^{-1}(0)$ and we are done. Therefore it is enough to consider the case when $g(F) = \{0\}$. In this case $F = E_1$ as divisors over X. Indeed, if $\operatorname{codim}_{X'}(c_{X'}(F)) \geq 2$, then the coefficient of F in

$$K_{\widetilde{Y}/X} = K_{\widetilde{Y}/X'} + (g')^*(K_{X'/X}) = K_{\widetilde{Y}/X'} + (g')^*(E_1)$$

is $\geq 1+1=2$. Here $g' \colon \widetilde{Y} \to X'$ is the induced morphism and we used the bound given by Corollary 3.1.14 and the fact that $c_{X'}(F) \subseteq E_1$. This gives a contradiction, hence $F = E_1$ as divisors over X.

Since $F \subseteq g^{-1}(0)$, we can write $\mathfrak{m}_0 \cdot \mathscr{O}_Y = \mathscr{O}_Y(-F) \cdot \mathfrak{a}$ for some ideal $\mathfrak{a}$ on Y. Recall that $\mathfrak{m}_0 \cdot \mathscr{O}_{X''} = \mathscr{O}_{X''}(-(\pi')^*(E_1)) = \mathscr{O}_{X''}(-E_1 - E_2)$. This implies that $\mathfrak{a} = \mathscr{O}_Y$ at the generic point of W. Let $h \colon \widetilde{Y} \to Y$ be the normalized blow-up of Y along $\mathfrak{a}$. By the universal property of the blowing-up, the rational map $h' \colon \widetilde{Y} \dashrightarrow X'$ is a morphism. The proper transform $\widetilde{W}$ of W on $\widetilde{Y}$ is mapped to $c_{X'}(E_2)$, which is a point. Since X' is $\mathbb{Q}$-factorial by Lemma 9.10.23, all irreducible components of $\operatorname{Exc}(h')$ have codimension 1 (see Remark 2.2.5). Therefore $\widetilde{W}$ is contained in an h'-exceptional divisor G. Since $h(G)$ is a g-exceptional divisor containing W, it follows that $h(G) = F$, hence $G = E_1$ as divisors over X. This contradicts the fact that G is h'-exceptional and completes the proof of the proposition. □

Remark 10.10.25. In fact, it is shown in [**?**] that for $m \geq 5$, the hypersurface $V(x^2 + y^2 + z^2 + w^m) \subset \mathbb{A}^4$ gives a counterexample to the Nash problem if and only if m is odd (when m is even, $z^2 + w^m$ is a reducible polynomial; in this case, it is shown in *loc. cit.* that there is a unique essential divisor).

Remark 10.10.26. Instead of only considering resolutions in the algebraic category, when working over $\mathbb{C}$ one can define essential divisors over X by also allowing resolutions in the analytic category. With this new definition, there is a better chance for the Nash problem to have a positive answer. In fact, there is an example such that the Nash problem has a positive answer in the analytic category, but a negative one in the algebraic category (see [dF13]). On the other hand, it is shown in [**?**] that the hypersurface X discussed above gives a counterexample for the Nash problem also in the analytic category.

Appendix A
Elements of convex geometry

In this appendix we review some of the basic properties of closed convex cones in finite-dimensional vector spaces. Let V be a finite-dimensional real vector space. We denote by V^* the dual vector space $\mathrm{Hom}_{\mathbb{R}}(V,\mathbb{R})$ and by $\langle -,- \rangle \colon V^* \times V \to \mathbb{R}$ the canonical pairing. Note that we have a canonical isomorphism $V \simeq (V^*)^*$. For a subset S of V^*, we put

$$S^{\perp} = \{v \in V \mid \langle u,v \rangle = 0 \text{ for all } u \in S\},$$

and dually, for a subset S of V, we obtain $S^{\perp} \subseteq V^*$.

A.1 Basic facts about convex sets and convex cones

Recall that a subset σ of V is a *cone* if $tv \in \sigma$ whenever $v \in \sigma$ and $t > 0$. A subset T of V is *convex* if for every $v_1, v_2 \in T$, and every real number $t \in [0,1]$, we have $tv_1 + (1-t)v_2 \in T$. We see that $\sigma \subseteq V$ is a *convex cone* if for every $v_1, v_2 \in \sigma$ and every $t_1, t_2 \in \mathbb{R}_{>0}$, we have $t_1v_1 + t_2v_2 \in \sigma$.

It follows from definition that an intersection of convex sets or of convex cones is again a convex set, respectively, a convex cone. Suppose now that $S \subseteq V$ is an arbitrary subset. The *convex hull* $\mathrm{conv}(S)$ of S is the intersection of all convex sets containing S, hence it is the smallest convex set which contains S. Similarly, the *convex cone generated by* S is the intersection of all convex cones containing S, hence it is the smallest convex cone which contains S. We denote it by $\mathrm{pos}(S)$. If $\sigma = \mathrm{pos}(S)$, we say that S is a set of generators of σ.

Lemma A.1.1. *For every subset S of V, we have*

$$\mathrm{pos}(S) = \left\{ \sum_{i=1}^{r} \lambda_i v_i \mid \lambda_i > 0, v_i \in S \right\}.$$

Proof. The right-hand side is a convex cone and contains S. Moreover, it is contained in every convex cone containing S, so it is equal to $\mathrm{pos}(S)$. □

We similarly have the following description of the convex hull of a set.

Lemma A.1.2. *For every subset S of V, we have*

$$\mathrm{conv}(S) = \left\{ \sum_{i=1}^{r} \lambda_i v_i \mid \lambda_i \geq 0, \sum_{i=1}^{r} \lambda_i = 1, v_i \in S \right\}.$$

If σ is a convex cone, then its closure $\overline{\sigma}$ is again a convex cone. It follows that if S is a non-empty subset of V, then the closed convex cone generated by S (that is, the smallest closed convex cone containing S) is equal to the closure of $\mathrm{pos}(S)$. We make the convention that all closed convex cones are non-empty, in which case they have to contain 0.

A *polytope* in V is the convex hull of finitely many vectors in V. A convex cone σ in V is *polyhedral* if it is the convex cone generated by a finite set.

A convex cone σ is *strongly convex* if whenever both v and $-v$ are in σ, we have $v = 0$ (equivalently, σ contains no nonzero linear subspaces). An arbitrary convex cone σ is (noncanonically) the product of a vector space and a strongly convex cone, as follows. Let

$$W = \sigma \cap (-\sigma) := \{v \in \sigma \mid -v \in \sigma\}$$

be the largest vector subspace of V which is contained in σ. If $p\colon V \to V/W$ is the canonical projection, then $p(\sigma)$ is a convex cone of V/W. Moreover, if we choose a splitting i of p, then we get an isomorphism $V \simeq W \times V/W$ that identifies σ with $W \times p(\sigma)$. Note that, by construction, $p(\sigma)$ is strongly convex. In addition, σ is closed or polyhedral if and only if $p(\sigma)$ has the same property.

Lemma A.1.3. *All polytopes and polyhedral convex cones are closed in V.*

Proof. For every polytope P there are $v_1, \ldots, v_N$ in V such that P is the image of the map

$$\left\{ \lambda = (\lambda_i) \in [0,1]^N \mid \sum_i \lambda_i = 1 \right\} \to V,$$

which takes λ to $\sum_i \lambda_i v_i$. Therefore P is compact, hence closed.

Suppose now that σ is a polyhedral convex cone. In order to show that σ is closed in V, we may assume that it is a strongly convex cone and that $\sigma \neq \{0\}$. Choose nonzero $v_1, \ldots, v_r$ in V such that σ is the convex cone generated by these vectors. Let P be the convex hull of $v_1, \ldots, v_r$. It follows from Lemmas A.1.2 and A.1.1 that $\sigma = \{\lambda v \mid v \in P, \lambda \geq 0\}$.

Suppose now that $\{\lambda_m v_m\}_m$ converges to w, where $\lambda_m \geq 0$ and $v_m \in P$. By the compactness of P, we may assume after passing to a subsequence that $\{v_m\}_m$ converges to some $v \in P$. Since σ is a strongly convex cone, 0 is not in P, hence $v \neq 0$. Therefore $\{\lambda_m\}_m$ is bounded and after passing again to a subsequence, we may assume that it converges to some $\lambda \geq 0$. Therefore $w = \lambda v$ is in σ, hence σ is closed. □

As the following examples show, in the non-polyhedral case some pathologies can occur.

Example A.1.4. It can happen that σ is a closed convex cone in V, $p\colon V \to W$ is a surjective linear map, and $p(\sigma)$ is not closed. For example, suppose that $V = \mathbb{R}^3$ with coordinates x_1, x_2, x_3, and K is the circle in the plane $x_3 = 1$ with center at $(0,1,1)$ and radius 1. If

$$\sigma = \{\lambda v \mid \lambda \geq 0, v \in K\},$$

then σ is a closed convex cone. On the other hand, if $p\colon \mathbb{R}^3 \to \mathbb{R}^2$ is the projection onto the first two coordinates, then

$$p(\sigma) = \{(u_1, u_2) \mid u_2 > 0\} \cup \{(0,0)\}$$

is not closed.

Example A.1.5. It can happen that σ and τ are closed convex cones in V, but

$$\sigma + \tau := \{v + w \mid v \in \sigma, w \in \tau\}$$

is not closed. Indeed, with the notation in Example A.1.4, let $\tau = \mathbb{R}_{\geq 0} \cdot (0,0,-1)$. In this case $\sigma + \tau$ is a convex cone containing $\ker(p)$, hence the fact that $p(\sigma + \tau) = p(\sigma)$ is not closed in $\mathbb{R}^2$ implies that $\sigma + \tau$ is not closed in $\mathbb{R}^3$.

If σ is a closed convex cone in V, then its *dimension*, denoted by $\dim(\sigma)$, is the dimension of the linear span of σ. It is clear that this can also be described as the maximum number of linearly independent elements of σ.

A.2 The dual of a closed convex cone

Let σ be a closed convex cone in V. The *dual cone* $\sigma^\vee$ is the subset of V^* given by

$$\sigma^\vee = \{u \in V^* \mid \langle u, v \rangle \geq 0\}.$$

It is clear that $\sigma^\vee$ is again a closed convex cone. The following is the fundamental result concerning duality of cones.

Proposition A.2.1. *If σ is a closed convex cone in V, then under the identification $V \simeq (V^*)^*$, we have $(\sigma^\vee)^\vee = \sigma$.*

Proof. The inclusion $\sigma \subseteq (\sigma^\vee)^\vee$ follows from definition, hence we only need to show that if $v \in V \smallsetminus \sigma$, then there is $u \in \sigma^\vee$ such that $\langle u, v \rangle < 0$. We fix a scalar product $(\cdot,\cdot)$ on V, which induces a metric d.

Since σ is closed, we can find $v' \in \sigma$ such that

$$d(v, v') = \min_{w \in \sigma} d(v, w). \tag{A.1}$$

Note that v' is different from v, as v is not in σ. It is enough to show that $(v'-v,y) \geq 0$ for every $y \in \sigma$, but $(v'-v,v) < 0$. For every $y \in \sigma$, we have $v'+ty \in \sigma$ for all $t > 0$, hence (A.1) gives

$$(v'-v,v'-v) \leq (v'-v+ty, v'-v+ty) = (v'-v,v'-v) + 2t(v'-v,y) + t^2(y,y).$$

We thus have $t^2(y,y) + 2t(v'-v,y) \geq 0$ for all $t > 0$. Dividing by t, and then letting t go to 0, we obtain $(v'-v,y) \geq 0$.

On the other hand, $\lambda v' \in \sigma$ for every $\lambda > 0$. Using one more time (A.1), we obtain

$$\begin{aligned}(v'-v,v'-v) &\leq ((\lambda-1)v' + (v'-v), (\lambda-1)v' + (v'-v)) \\ &= (v'-v,v'-v) + 2(\lambda-1)(v'-v,v') + (\lambda-1)^2(v',v'),\end{aligned}$$

hence $(\lambda-1)^2(v',v') + 2(\lambda-1)(v'-v,v') \geq 0$ for every $\lambda > 0$. We consider $\lambda < 1$, divide by $(\lambda-1)$, and then let λ go to 1, to deduce $(v'-v,v') \leq 0$. Since $v' \in \sigma$, it follows that $(v'-v,v') = 0$, and therefore

$$0 < (v'-v,v'-v) = -(v'-v,v),$$

as required. This completes the proof of the proposition. □

A.3 Faces of closed convex cones

Let σ be a closed convex cone in V. A *face* of σ is a subset of σ of the form

$$\sigma \cap u^{\perp} = \{v \in \sigma \mid \langle u,v \rangle = 0\}$$

for some $u \in \sigma^{\vee}$. Note, in particular, that σ is considered as a face of σ (for $u = 0$). A *proper face* of σ is a face of σ different from σ. It is clear that every face τ of σ is again a closed convex cone. In particular, its dimension is well-defined. Furthermore, a face τ of σ has the property that if $v_1, v_2 \in \sigma$, then $v_1 + v_2 \in \tau$ if and only if $v_1, v_2 \in \tau$.

It follows from definition that if τ is a face of σ, then τ is the intersection of σ and of the linear span of τ. Therefore each face of σ is determined by its linear span. In particular, if τ_1 and τ_2 are two faces of σ, with τ_1 strictly contained in τ_2, then $\dim(\tau_1) < \dim(\tau_2)$.

Suppose that we have $\tau_1 \subseteq \tau_2 \subseteq \sigma$, with τ_1 and τ_2 closed convex cones, such that τ_1 is a face of σ. It follows from definition that τ_1 is a face of τ_2. On the other hand, it is not true in general that if τ_2 is a face of σ, and τ_1 is a face of τ_2, then τ_1 is a face of σ (see Example A.4.5 below).

Remark A.3.1. If W is the linear span of a closed convex cone σ in V, then the faces of σ do not depend on whether we consider σ as a cone in V or W. This follows from definition after choosing a splitting for the inclusion $W \hookrightarrow V$, which induces a splitting of $V^* \to W^*$.

Lemma A.3.2. *If σ is a closed convex cone, then the intersection of a family of faces of σ is again a face of σ, and it is equal to the intersection of a finite subfamily.*

Proof. We first show that the intersection of a finite family of faces of σ is a face of σ. Let $\tau = \bigcap_{i=1}^{r} \tau_i$, where each τ_i is a face of σ. If we write $\tau_i = \sigma \cap u_i^{\perp}$, with $u_i \in \sigma^{\vee}$ for each i, then $u = \sum_{i=1}^{r} u_i \in \sigma^{\vee}$ and $\sigma \cap u^{\perp} = \tau$. Therefore τ is a face of σ.

Consider now an arbitrary family $(\tau_i)_{i \in I}$ of faces of σ, and let $J \subseteq I$ be a finite subset such that $\cap_{i \in J} \tau_i$ has minimal dimension. This minimality assumption implies that for every $j \in I$ we have $\dim(\cap_{i \in J} \tau_i) = \dim(\cap_{i \in J \cup \{j\}} \tau_i)$, and since both these intersections are faces of σ, it follows that $\cap_{i \in J} \tau_i = \cap_{i \in J \cup \{j\}} \tau_i$. Therefore $\cap_{i \in J} \tau_i = \cap_{i \in I} \tau_i$, which completes the proof of the lemma. $\square$

It follows from the above lemma that if σ is a closed convex cone and S is a subset of σ, then there is a unique smallest face of σ containing S, the *face generated by S*.

Let σ be a closed convex cone in V. The *relative interior* $\mathrm{Relint}(\sigma)$ of σ is the topological interior of σ as a subset of its linear span. It is clear that $\mathrm{Relint}(\sigma)$ is a convex cone.

Lemma A.3.3. *If σ is a closed convex cone, then the relative interior of σ is non-empty.*

Proof. Let W be the linear span of σ and let $d = \dim(W)$. We can find linearly independent $v_1, \ldots, v_d$ in σ. It is clear that the set

$$\{t_1 v_1 + \ldots + t_d v_d \mid t_1, \ldots, t_d > 0\}$$

is open in W and contained in σ, hence it is contained in $\mathrm{Relint}(\sigma)$. $\square$

Proposition A.3.4. *If σ is a closed convex cone, then*

$$\mathrm{Relint}(\sigma) = \sigma \smallsetminus \bigcup_{\tau \subsetneq \sigma} \tau,$$

where the union is over all proper faces of σ.

Proof. Suppose first that $v \in \mathrm{Relint}(\sigma)$ and that τ is a face of σ containing v. If W is the linear span of σ, then by assumption there is a ball in W centered in v that is contained in σ. This implies that the whole ball is contained in τ (recall that if $v_1, v_2 \in \sigma$ are such that $v_1 + v_2 \in \tau$, then $v_1, v_2 \in \tau$). Therefore W is contained in the linear span of τ, hence $\tau = \sigma$. This proves the inclusion "$\subseteq$" in the proposition.

In order to prove the reverse inclusion, let us assume that $v \in \sigma \smallsetminus \mathrm{Relint}(\sigma)$. After replacing V by the linear span of σ, we may assume that this linear span is V. Since v is not in the interior of σ, there are $v_n \in V$ with $\lim_{n \to \infty} v_n = v$ such that $v_n \notin \sigma$. It follows from Proposition A.2.1 that we can find $u_n \in \sigma^{\vee}$ such that $\langle u_n, v_n \rangle < 0$. Furthermore, after possibly passing to a subsequence, we may assume that $\lim_{n \to \infty} u_n = u$ for some nonzero $u \in V^*$ (for example, after rescaling the u_n we may assume that they lie on a sphere centered at the origin, with respect to a suitable norm on V^*). Since $\sigma^\vee$ is closed, we have $u \in \sigma^\vee$, while by passing to limit we get $\langle u, v\rangle \leq 0$. Therefore $v \in \sigma \cap u^\perp$, which is a proper face of σ, since the linear span of σ is V. This completes the proof of the proposition. $\square$

Corollary A.3.5. *If σ is a closed convex cone, $v \in \sigma$, and $w \in \mathrm{Relint}(\sigma)$, then $v + w \in \mathrm{Relint}(\sigma)$.*

Proof. We use the description of the relative interior of σ given in Corollary A.3.4. Suppose that $v + w \notin \mathrm{Relint}(\sigma)$, so that there is a proper face τ of σ, with $v + w \in \tau$. In this case both v and w lie in σ. In particular, $w \notin \mathrm{Relint}(\sigma)$, a contradiction. $\square$

Corollary A.3.6. *If σ is a closed convex cone, then σ is the closure of* $\mathrm{Relint}(\sigma)$.

Proof. Recall first that $\mathrm{Relint}(\sigma)$ is non-empty by Lemma A.3.3. Suppose now that we have $v \in \sigma$ and let us choose some $v' \in \mathrm{Relint}(\sigma)$. It follows from Corollary A.3.5 that $v + \frac{1}{m}v' \in \mathrm{Relint}(\sigma)$ for every positive integer m, hence v lies in the closure of $\mathrm{Relint}(\sigma)$. $\square$

Remark A.3.7. Suppose that σ and σ' are closed convex cones, with $\sigma' \subseteq \sigma$. If $v \in \mathrm{Relint}(\sigma')$, then a face τ of σ contains v if and only if it contains σ'. Indeed, $\tau \cap \sigma'$ is a face of σ', and by Proposition A.3.4 this contains v if and only if it is equal to σ'.

Proposition A.3.8. *If σ is a closed convex cone, then there is an order-reversing bijection between the faces of σ and those of $\sigma^\vee$, that takes a face τ of σ to the face $\sigma^\vee \cap \tau^\perp$ of $\sigma^\vee$. The inverse map takes a face τ' of $\sigma^\vee$ to the face $\sigma \cap (\tau')^\perp$ of σ.*

Proof. If S is any subset of σ, then

$$\sigma^\vee \cap S^\perp = \bigcap_{v \in S} (\sigma^\vee \cap v^\perp)$$

is an intersection of faces of $\sigma^\vee$, hence it is a face of $\sigma^\vee$ by Lemma A.3.2. Since the two maps are given by the same formula, in order to show that they are mutual inverses it is enough to show that for every face τ of σ, we have

$$\tau = \sigma \cap (\sigma^\vee \cap \tau^\perp)^\perp. \tag{A.2}$$

The inclusion "$\subseteq$" is trivial. For the reverse inclusion, let us write $\tau = \sigma \cap u^\perp$, for some $u \in \sigma^\vee$. In particular, $u \in \sigma^\vee \cap \tau^\perp$, hence every element in the right-hand side of (A.2) lies in $\sigma \cap u^\perp = \tau$. We thus have the equality in (A.2). The fact that the two inverse maps reverse inclusions is clear. $\square$

Remark A.3.9. If τ is a face of the closed convex cone σ and $v \in \mathrm{Relint}(\tau)$, then $\sigma^\vee \cap \tau^\perp = \sigma^\vee \cap v^\perp$. Indeed, the inclusion "$\subseteq$" is trivial. For the reverse inclusion, note that if $u \in \sigma^\vee \cap v^\perp$, then v is contained in the face $\tau \cap u^\perp$ of τ.

Remark A.3.10. Via the bijection in Proposition A.3.8, the largest face of σ (namely σ itself) corresponds to the smallest face of $\sigma^\vee$, namely $\sigma^\perp$. Note that $\sigma^\perp = \sigma^\vee \cap (-\sigma^\vee)$ is the largest linear subspace contained in $\sigma^\vee$. This shows that $\sigma^\vee$ is strongly convex if and only if $\{0\}$ is a face of $\sigma^\vee$. Furthermore, this is the case if and only if $\sigma^\perp = \{0\}$, that is, the linear span of σ is V. Of course, the same applies with the roles of σ and $\sigma^\vee$ reversed.

A.4 Extremal subcones

Definition A.4.1. If σ is a closed, convex cone, then an *extremal subcone* of σ is a closed convex cone $\tau \subseteq \sigma$ with the property that whenever $v_1, v_2 \in \sigma$, if $v_1 + v_2 \in \tau$, then $v_1, v_2 \in \tau$. An *extremal ray* is an extremal subcone of the form $\mathbb{R}_{\geq 0} \cdot v$, for some nonzero $v \in V$.

Remark A.4.2. It is clear that if τ is an extremal subcone of σ, then τ contains the largest linear subspace of σ, namely $\sigma \cap (-\sigma)$. In particular, in order for σ to have extremal rays, σ has to be strongly convex. It will follow from Proposition A.4.6 that this condition is also sufficient.

Lemma A.4.3. *If σ is a closed convex cone, τ is an extremal subcone of σ, and τ' is an extremal subcone of τ, then τ' is an extremal subcone of σ.*

Proof. Suppose that $v_1, v_2 \in \sigma$ are such that $v_1 + v_2 \in \tau'$. Since τ is an extremal subcone of σ and $v_1 + v_2 \in \tau$, it follows that $v_1, v_2 \in \tau$. Using now the fact that $v_1 + v_2 \in \tau'$, which is an extremal subcone of τ, it follows that $v_1, v_2 \in \tau'$. □

Remark A.4.4. It follows from definition that every face of a closed convex cone σ is an extremal subcone. The converse does not hold in general (see Example A.4.5 below).

Example A.4.5. Let $V = \mathbb{R}^3$ with coordinates x_1, x_2, x_3, and K_1 the convex set in the plane $x_3 = 1$ which is the union of

$$\mathrm{conv}\{(0,0,1), (2,0,1), (0,2,1), (2,2,1)\}$$

and of the right semicircle of radius 1 with center at $(2,1,1)$. Let K_2 be the line segment with vertices $(0,0,1)$ and $(2,0,1)$, and $K_3 = \{(2,0,1)\}$. If

$$\sigma_i = \{\lambda v \mid \lambda \geq 0, v \in K_i\},$$

for $i = 1,2,3$, then it is clear that σ_1 is a closed convex cone, σ_2 is a face of σ_1, and σ_3 is a face of σ_2, but not of σ_1. In particular, we see that σ_3 is an extremal subcone of σ_1, but not a face.

Proposition A.4.6. *If σ is a closed, strongly convex cone, then σ is generated as a convex cone (not just as a closed convex cone) by its extremal rays.*

Proof. We prove the assertion by induction on $\dim(\sigma)$, the case $\dim(\sigma) \leq 1$ being trivial. We assume that $\dim(\sigma) \geq 2$ and let C denote the convex cone generated by the extremal rays of σ. Suppose first that there is a proper face τ of σ that is not contained in C. By the inductive assumption, τ is the convex cone generated by its extremal rays, hence there is an extremal ray R of τ that is not contained in C. However, R is also an extremal ray of σ by Lemma A.4.3, hence it should be contained in C, a contradiction. Therefore all proper faces of σ are contained in C and by Proposition A.3.4, it is enough to show that also the relative interior of σ is contained in C.

Suppose that this is not the case and let $v_1 \in \mathrm{Relint}(\sigma) \smallsetminus C$. We also choose $v_2 \in \sigma$ linearly independent from v_1 such that, if $C \neq \{0\}$, then $v_2 \in C$. Since v_1 lies in the relative interior of σ, it follows that $v_1 - tv_2 \in \sigma$ for $0 \leq t \ll 1$. On the other hand, $v_1 - tv_2 \not\in \sigma$ for $t \gg 0$; indeed, otherwise $\frac{1}{t}v_1 - v_2 \in \sigma$ for all $t \gg 0$, and by letting t go to infinity, we obtain $-v_2 \in \sigma$, a contradiction with the fact that $v_2 \in \sigma$ is nonzero and σ is strongly convex. Therefore

$$t_0 := \sup\{t \geq 0 \mid v_1 - tv_2 \in \sigma\} \in \mathbb{R}_{>0},$$

and since σ is a closed convex cone, we see that for $t \geq 0$, we have $v_1 - tv_2 \in \sigma$ if and only if $t \leq t_0$. Therefore $v_1 - t_0v_2$ lies in $\sigma \smallsetminus \mathrm{Relint}(\sigma)$. It follows from Corollary A.3.4 that there is a proper face σ' of σ such that $v_1 - t_0v_2 \in \sigma'$. However, we have seen that $\sigma' \subseteq C$. If $C \neq \{0\}$, then $v_2 \in C$ and we conclude that $v_1 \in C$, a contradiction. On the other hand, if $C = \{0\}$, we conclude that v_2 and v_1 are linearly dependent, giving again a contradiction. Therefore $\mathrm{Relint}(\sigma) \subseteq C$ and we conclude that $C = \sigma$. $\square$

A.5 Polyhedral cones

In this section we discuss the special features of polyhedral cones. If $V = M_{\mathbb{R}}$, where M is a finitely generated, free abelian group, then a convex cone is *rational polyhedral* if it is generated by finitely many element in $M_{\mathbb{Q}}$ (or equivalently, in M).

Suppose that σ is a polyhedral cone, and let $v_1, \ldots, v_r$ be such that we have $\sigma = \mathrm{pos}(\{v_1, \ldots, v_r\})$. If τ is a face of σ and $a_1, \ldots, a_r \in \mathbb{R}_{\geq 0}$, then $a_1v_1 + \ldots + a_rv_r \in \tau$ if and only if $v_i \in \tau$ for all i with $a_i \neq 0$. Therefore τ is the convex cone generated by those $v_i \in \tau$. In particular, we see that σ has only finitely many faces and each of them is a polyhedral cone. If σ is rational polyhedral, then all faces have the same property.

Proposition A.5.1. *If σ is a polyhedral cone, then the extremal subcones of σ are precisely the faces of σ. In particular, a face of a face of σ is a face of σ.*

Proof. We only need to show that if τ is an extremal subcone of σ, then τ is a face of σ. Consider the convex cone

$$\gamma = \sigma - \tau := \{u_1 - u_2 \mid u_1 \in \sigma, u_2 \in \tau\}.$$

This is clearly polyhedral, hence closed, and $\gamma^\vee = \sigma^\vee \cap \tau^\perp$. Let $u \in \mathrm{Relint}(\gamma^\vee)$, so that $u \in \sigma^\vee$ and $\tau \subseteq \sigma \cap u^\perp$. Furthermore, we have

$$\gamma \cap u^\perp = \gamma \cap (-\gamma) = (\sigma - \tau) \cap (\tau - \sigma).$$

It follows that if $v \in \sigma \cap u^\perp \subseteq \gamma \cap u^\perp$, then we can write $v = v_1 - v_2$, with $v_1 \in \tau$ and $v_2 \in \sigma$. Since τ is an extremal subcone and $v + v_2 \in \tau$, we conclude that $v \in \tau$. We have shown that $\tau = \sigma \cap u^\perp$, with $u \in \sigma^\vee$, hence τ is a face of σ. $\square$

A *facet* of σ is a maximal proper face of σ. If σ is a strongly convex polyhedral cone, a *ray* of σ is a 1-dimensional face of σ.

Proposition A.5.2. *If σ is a polyhedral cone, then for every facet τ of σ, we have* $\dim(\tau) = \dim(\sigma) - 1$. *More generally, if $\tau_1 \subsetneq \tau_2$ are faces of σ such that there is no other face in between, then* $\dim(\tau_1) = \dim(\tau_2) - 1$.

Proof. We may assume that the linear span of σ is the vector space V. Let $u \in \sigma^\vee$ be such that $\tau = \sigma \cap u^\perp$. Suppose that $\dim(\tau) \leq \dim(\sigma) - 2$. In this case there is w linearly independent from u such that $\tau \subseteq w^\perp$. Let $v_1, \ldots, v_r$ generate σ. After possibly replacing w by $-w$, we may assume that $\langle w, v_i \rangle < 0$ for some i. If

$$t_0 := \max\{t \in \mathbb{R}_{\geq 0} \mid \langle u + tw, v_j \rangle \geq 0 \text{ for all } j\},$$

then $u + t_0 w$ is nonzero, lies in $\sigma^\vee$, and $\sigma \cap (u + t_0 w)^\perp$ is a proper face of σ strictly containing τ. This contradiction implies that $\dim(\tau) = \dim(\sigma) - 1$. The last assertion in the proposition is a consequence of the first one and of the fact that τ_1 is a facet of τ_2 (this is a consequence of the hypothesis, since every face of τ_2 is also a face of σ by Proposition A.5.1). $\square$

Suppose now that σ is a polyhedral cone whose linear span is V. It follows from Proposition A.5.2 that each facet of σ can be written as $\sigma \cap u_\tau^\perp$, where u_τ is unique up to multiplication by an element of $\mathbb{R}_{>0}$. Note that if $V = M_\mathbb{R}$ and σ is rational polyhedral, then we may choose $u_\tau \in M$.

Lemma A.5.3. *With the above notation, we have*

$$\sigma = \{v \in V \mid \langle u_\tau, v \rangle \geq 0 \textit{ for all facets } \tau \textit{ of } \sigma\}.$$

Proof. The inclusion "$\subseteq$" is clear. On the other hand, if $v \notin \sigma$ and we consider $w \in \mathrm{Relint}(\sigma)$, then it follows from Proposition A.3.4 that

$$t_0 := \max\{t \in [0,1] \mid tv + (1-t)w \in \sigma\}$$

has the property that $v' := t_0 v + (1 - t_0) w \in u_\tau^\perp$ for some facet τ. Since $\langle u_\tau, w \rangle > 0$, we conclude that $\langle u_\tau, v \rangle < 0$. This proves the equality in the lemma. $\square$

The following proposition says that a cone is (rational) polyhedral if and only if it is the intersection of finitely many (rational) half-spaces.

Proposition A.5.4 (Farkas). *If σ is a (rational) polyhedral cone, then $\sigma^\vee$ has the same property.*

Proof. We may assume that the linear span of σ is the ambient vector space V. In this case, Lemma A.5.3 implies that σ is the dual of the (rational) polyhedral cone γ generated by the u_τ. Since $\sigma^\vee = \gamma$ by Proposition A.2.1, this completes the proof. $\square$

If we interpret a polyhedral cone as the intersection of finitely many half-spaces, we obtain the following two corollaries.

Corollary A.5.5. *The intersection of finitely many (rational) polyhedral cones is (rational) polyhedral.*

Corollary A.5.6. *Let $\phi\colon A \to B$ be a group homomorphism, where A and B are finitely generated, free abelian groups, and let $\phi_{\mathbb{R}}\colon A_{\mathbb{R}} \to B_{\mathbb{R}}$ be the corresponding linear map. If σ is a rational polyhedral cone in $B_{\mathbb{R}}$, then $\phi^{-1}(\sigma)$ is a rational polyhedral cone in $A_{\mathbb{R}}$.*

Proposition A.5.7 (Carathéodory). *If σ is the convex cone generated by a set T, then σ is the union of the convex cones generated by subsets of T that are linearly independent.*

Proof. It is enough to show that if $v = \lambda_1 v_1 + \ldots + \lambda_r v_r$, with $\lambda_i > 0$ for all i, and if $v_1, \ldots, v_r$ are not linearly independent, then v can be written as a linear combination with nonnegative coefficients of $r-1$ of the v_i. For this, consider a relation $a_1 v_1 + \ldots + a_r v_r = 0$, where some of the a_i are nonzero. After possibly multiplying the relation by (-1), we may assume that there is j such that $a_j > 0$.

Let i be such that $a_i > 0$ and $\lambda_i/a_i = \min\{\lambda_j/a_j \mid a_j > 0\}$. Note that we have $\lambda_j - \lambda_i \frac{a_j}{a_i} \geq 0$ for all j. Therefore we can write

$$v = \sum_{j \neq i} \left(\lambda_j - \lambda_i \frac{a_j}{a_i} \right) v_j$$

and all coefficients are nonnegative. $\square$

Corollary A.5.8. *If $V = W_{\mathbb{R}}$, where W is a finite-dimensional vector space over $\mathbb{Q}$, and if σ is the convex cone generated in V by the vectors $w_1, \ldots, w_d \in W$, then for every $u \in \sigma \cap W$, there are $\lambda_1, \ldots, \lambda_d \in \mathbb{Q}_{\geq 0}$ such that $u = \sum_{i=1}^d \lambda_i w_i$.*

Proof. It follows from Proposition A.5.7 that after possibly ignoring some of the w_i, we may assume that $w_1, \ldots, w_d$ are linearly independent. By assumption, we can write $u = \sum_{i=1}^d \lambda_i w_i$, with $\lambda_i \in \mathbb{R}_{\geq 0}$. Since the w_i can be completed to a basis of W and by assumption $u \in W$, we conclude that $\lambda_i \in \mathbb{Q}$ for all i. $\square$

A.6 Monoids and cones

Recall that a *monoid* is a set S endowed with a binary operation $+$ (we only use the additive notation), which is commutative, associative, and has a unit element 0. If S is a monoid, a subset $T \subseteq S$ is a *submonoid* if $0 \in T$ and $u+v \in T$ whenever $u, v \in T$. A monoid S is *finitely generated* if there are $u_1, \ldots, u_m \in S$ such that every $u \in S$ can be written as $u = a_1u_1 + \ldots + a_mu_m$, for some $a_1, \ldots, a_m \in \mathbb{Z}_{\geq 0}$ (in this case one says that $u_1, \ldots, u_m$ *generate* S).

In this section we only consider subsemigroups of finitely generated, free abelian groups. If M is such a group and S is a submonoid of M, one says that S is *saturated* (in M) if for every $u \in M$ such that $mu \in S$ for a positive integer m, we have $u \in S$. Given an arbitrary submonoid S of M, there is a smallest saturated submonoid that contains S, the *saturation* of S, namely

$$S^{\mathrm{sat}} := \{u \in M \mid mu \in S \text{ for some } m \geq 1\}.$$

From now on, we fix a finitely generated, free abelian group M and let $V = M_{\mathbb{R}}$.

Lemma A.6.1 (Gordan). *If σ is a rational polyhedral cone in V, then $\sigma \cap M$ is a finitely generated, saturated submonoid of M.*

Proof. The fact that $S = \sigma \cap M$ is saturated is clear. In order to see that S is finitely generated, consider generators $v_1, \ldots, v_r \in M$ of σ. The set

$$K := \left\{ \sum_{i=1}^{r} \lambda_i v_i \mid \lambda_i \in [0,1] \text{ for all } i \right\}$$

is compact and we have $v_i \in K$ for all i. Since M is discrete in V, its intersection with K is finite. Let $w_1, \ldots, w_s$ be the elements of $K \cap M$. If $v \in S$ and if we write $v = \sum_{i=1}^{r} \alpha_i v_i$, with $\alpha_i \geq 0$, then there is j such that $v = \sum_{i=1}^{r} \lfloor \alpha_i \rfloor v_i + w_j$. Therefore S is generated as a monoid by $w_1, \ldots, w_s$. □

For a submonoid S of M, we denote by $\mathbb{R}_{\geq 0}S$ the convex cone generated by S.

Proposition A.6.2. *If S is a finitely generated submonoid of M, then we have $\mathbb{R}_{\geq 0}S \cap M = S^{\mathrm{sat}}$, and this is a finitely generated submonoid of M.*

Proof. The inclusion "$\supseteq$" is clear. For the reverse inclusion, we use the fact that if S is generated by $v_1, \ldots, v_d$ and $v \in \mathbb{R}_{\geq 0}S \cap M$, then by Corollary A.5.8, we can find $\lambda_1, \ldots, \lambda_d \in \mathbb{Q}_{\geq 0}$ such that $v = \sum_{i=1}^{d} \lambda_i v_i$. If m is a positive integer such that $m\lambda_i \in \mathbb{Z}$ for all i, then $mv \in S$, hence $v \in S^{\mathrm{sat}}$. This proves the first assertion in the proposition and the second one follows from Lemma A.6.1 □

Remark A.6.3. If $S \subseteq M$ is a finitely generated submonoid, then there is a positive integer d such that $du \in S$ for every $u \in S^{\mathrm{sat}}$. Indeed, it follows from Proposition A.6.2 that there are finitely many elements $u_1, \ldots, u_r \in S^{\mathrm{sat}}$ that generate this monoid, and by definition, there is a positive integer d such that $du_i \in S$ for all i. Therefore $du \in S$ for every $u \in S^{\mathrm{sat}}$.

The following is now an immediate consequence of Lemma A.6.1, Proposition A.6.2, and the definitions.

Proposition A.6.4. *The map $S \to \mathbb{R}_{\geq 0}S$ gives a bijection between finitely generated, saturated subsemigroups of M and rational polyhedral cones in $V = M_{\mathbb{R}}$, whose inverse is given by $\sigma \to \sigma \cap M$.*

Remark A.6.5. If σ is a strongly convex, rational polyhedral cone, $S' = \sigma \cap M$, and $S \subseteq S'$ is the monoid generated by the primitive elements on the rays of σ, then using Proposition A.6.4 we see that $S' = S^{\mathrm{sat}}$. It follows from Remark A.6.3 that there is a positive integer d such that $du \in S$ for every $u \in S'$.

Remark A.6.6. If S is a finitely generated submonoid of M and $C = \mathbb{R}_{\geq 0}S$, then

$$C \cap M_{\mathbb{Q}} = \left\{ \frac{1}{m} \cdot u \mid u \in S, m \geq 1 \right\}.$$

This is an immediate consequence of the fact that $C \cap M = S^{\mathrm{sat}}$.

Corollary A.6.7. *If M is a finitely generated, free abelian group and S and T are saturated, finitely generated subsemigroups of M, then $S \cap T$ is a saturated, finitely generated submonoid of M.*

Proof. If σ and τ are the cones generated by S and T, respectively, then these are rational polyhedral cones. The intersection $\sigma \cap \tau$ is rational polyhedral by Corollary A.5.5 and therefore $\sigma \cap \tau \cap M = S \cap T$ is a saturated, finitely generated submonoid of M by Lemma A.6.1. □

Corollary A.6.8. *Let $\phi \colon A \to B$ be a morphism of finitely generated, free abelian groups. If T is a saturated, finitely generated submonoid of B, then $S := \phi^{-1}(T)$ is a finitely generated, saturated submonoid of A.*

Proof. We consider the induced linear map $\phi_{\mathbb{R}} \colon A_{\mathbb{R}} \to B_{\mathbb{R}}$. If τ is the convex cone generated by T, then τ is a rational polyhedral cone, hence $\phi_{\mathbb{R}}^{-1}(\tau)$ is a rational polyhedral cone by Corollary A.5.6. We thus conclude that

$$S = \phi^{-1}(T) = \phi^{-1}(\tau \cap B) = \phi_{\mathbb{R}}^{-1}(\tau) \cap A$$

is finitely generated and saturated by Lemma A.6.1. □

A.7 Fans and fan refinements

Let V be a finite-dimensional real vector space. A *fan* in V is a finite collections of polyhedral convex cones in V such that the following conditions hold:

i) If $\sigma \in \Delta$ and τ is a face of σ, then $\tau \in \Delta$.

ii) If $\sigma_1, \sigma_2 \in \Delta$, then $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2.

Note that unlike in toric geometry, we do not require that the cones in Δ are strongly convex. If $V = M_{\mathbb{R}}$, for a free, finitely generated abelian group and the cones in Δ are rational, we say that Δ is a *rational fan*. The *support* $|\Delta|$ of a fan Δ is the union of the cones in Δ. We note that if $|\Delta|$ is convex, then it is a polyhedral convex cone, being generated by the union of the generators of the cones in Δ.

Example A.7.1. If $\mathscr{C}$ is a finite collection of polyhedral convex cones in V such that for every $\sigma_1, \sigma_2 \in \mathscr{C}$, the intersection $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ, then it is straightforward to check that the set $\Delta(\mathscr{C})$ of all faces of the cones in $\mathscr{C}$ is a fan.

Lemma A.7.2. *If $C, C_1, \ldots, C_r$ are closed convex cones in V such that $C = C_1 \cup \ldots \cup C_r$, then C can be also written as the union of those C_i with* $\dim(C_i) = \dim(C)$.

Proof. Let $n = \dim(C)$ and let us fix linearly independent elements $v_1, \ldots, v_n \in C$. Suppose that $C_1, \ldots, C_s$ are the C_i of dimension n and that we have $v \in C \smallsetminus (C_1 \cup \ldots \cup C_s)$. Since the C_i are closed, it follows that there is $\varepsilon > 0$ such that $v + \sum_{i=1}^n a_i v_i \not\in (C_1 \cup \ldots \cup C_s)$ if $|a_j| < \varepsilon$ for $1 \leq j \leq n$. We choose a set of vectors $w_m = v + \sum_{j=1}^n a_{j,m} v_j$ for $1 \leq m \leq (r-s)(n-1)+1$ such that $|a_{j,m}| < \varepsilon$ for all j and m and such that every n of these vectors are linearly independent. Since

$$w_m \in C \smallsetminus (C_1 \cup \ldots \cup C_s) \subseteq (C_{s+1} \cup \ldots \cup C_r),$$

we conclude that there are at least n of the w_m that lie in the same cone C_j, with $j > s$, contradicting the fact that $\dim(C_j) < n$. $\square$

Corollary A.7.3. *If Δ is a fan such that $|\Delta|$ is convex, then all maximal cones in Δ have dimension equal to the dimension of the linear span of $\Delta|$.*

Proof. It is clear that $|\Delta|$ is the union of the maximal cones in Δ. Moreover, in this union we cannot leave out any maximal cone: otherwise, by property ii) in the definition of a fan, some maximal cone in Δ would be equal to the union of its proper faces, a contradiction. Therefore the assertion in the corollary follows from Lemma A.7.2. $\square$

We say that a fan Σ in V *refines* another fan Δ (or that Δ is coarser than Σ) if $|\Delta| = |\Sigma|$ and every cone $\sigma \in \Sigma$ is contained in some cone in Δ.

Lemma A.7.4. *If Δ and Σ are fans in V such that Σ is a refinement of Δ, then every cone in Δ is a union of cones in Σ.*

Proof. We need to show that for every $\sigma \in \Delta$, we have $\sigma = \bigcup_{\tau \in \Sigma, \tau \subseteq \sigma} \tau$. Note that if this holds for σ, then the corresponding formula holds for every face of σ. Therefore we may assume that σ is a maximal cone in Δ. Furthermore, it is enough to show that for every $v \in \mathrm{Relint}(\sigma)$, there is $\tau \in \Sigma$ such that $v \in \tau$ and $\tau \subseteq \sigma$. Since $|\Delta| = |\Sigma|$, there is $\tau \in \Sigma$ such that $v \in \tau$. By assumption, there is $\sigma' \in \Delta$ such that $\tau \subseteq \sigma'$. In this case, v lies in $\sigma \cap \sigma'$, which is a face of σ. Since $v \in \mathrm{Relint}(\sigma)$, it follows that $\sigma \cap \sigma' = \sigma$. On the other hand, σ is a maximal cone in Δ and therefore $\sigma = \sigma' \supseteq \tau$. $\square$

Corollary A.7.5. *If Σ is a fan refining Δ, then $\#\Sigma \geq \#\Delta$.*

Proof. We have a map $f\colon \Sigma \to \Delta$, such that $f(\tau)$ is the smallest cone in Δ that contains τ. Given $\sigma \in \Delta$, if $v \in \mathrm{Relint}(\sigma)$ and $\tau \in \Sigma$ is such that $v \in \tau$, then $\sigma = f(\tau)$. We thus see that f is onto, which implies the inequality in the corollary. □

Lemma A.7.6. *If Δ_1 and Δ_2 are fans with the same support, then there is a unique coarsest fan Σ that refines both Δ_1 and Δ_2.*

Proof. Let $\mathscr{C}$ be the collection of all intersections $\sigma_1 \cap \sigma_2$, where $\sigma_1 \in \Delta_1$ and $\sigma_2 \in \Delta_2$. It is easy to check that if $\sigma_1, \sigma_1' \in \Delta_1$ and $\sigma_2, \sigma_2' \in \Delta_2$, then $(\sigma_1 \cap \sigma_1') \cap (\sigma_2 \cap \sigma_2')$ is a face of both $\sigma_1 \cap \sigma_2$ and $\sigma_1' \cap \sigma_2'$. It thus follows from Example A.7.1 that the set $\Delta(\mathscr{C})$ consisting of all faces of the cones in $\mathscr{C}$ is a fan with the same support as Δ_1 and Δ_2. It is straightforward to check that $\Delta(\mathscr{C})$ refines both Δ_1 and Δ_2 and that it is the coarsest fan with these properties. □

Remark A.7.7. In general, given a family $(\Delta_i)_{i\in I}$ of fans with the same support, there is no fan refining all Δ_i. However, if there is one such fan, then there is a unique coarsest one. Indeed, suppose that Σ refines all Δ_i. For every finite subset $J \subseteq I$, consider the unique coarsest fan Δ_J that refines all Δ_i, with $i \in J$. Since Σ refines all Δ_J, it follows from Corollary A.7.5 that $\#\Delta_J \leq \#\Sigma$ for every J. If J_0 is such that $\#\Delta_{J_0}$ is maximal, it is clear that $\Delta_J = \Delta_{J_0}$ for every $J \supseteq J_0$. It is then clear that Δ_{J_0} is the coarsest fan refining all Δ_i.

Corollary A.7.8. *Given a fan Δ in V such that $|\Delta|$ is convex and $u_1, \ldots, u_d \in V^*$, there is a fan Σ refining Δ such that for every cone $\sigma \in \Sigma$ and every i, with $1 \leq i \leq d$, we have either $u_i \in \sigma^\vee$ or $-u_i \in \sigma^\vee$.*

Proof. It is enough to prove the corollary when $d = 1$ since we can then iterate the construction for $u_1, u_2, \ldots, u_d$. We may assume, of course, that u_1 is nonzero. Consider the following two polyhedral convex cones

$$C_1 = |\Delta| \cap \{v \mid \langle u_1, v\rangle \geq 0\} \quad \text{and} \quad C_2 = |\Delta| \cap \{v \mid \langle u_1, v\rangle \leq 0\}.$$

Since it is clear that $C_1 \cap C_2$ is a face of both C_1 and C_2, it follows from Example A.7.1 that the set Δ' consisting of all faces of C_1 and C_2 is a fan with support $C_1 \cup C_2 = |\Delta|$. We may thus apply Lemma A.7.6 to conclude that there is a common refinement Σ of Δ and Δ'. It is clear that this has the desired property. □

Corollary A.7.9. *Given a fan Δ in V such that $|\Delta|$ is convex and given polyhedral, convex cones $C_1, \ldots, C_r \subseteq |\Delta|$, there is a fan Σ refining Δ such that each C_i is a union of cones in Σ.*

Proof. For every i, we choose $u_{i,1}, \ldots, u_{i,m_i}$ that generate C_i as a convex cone. We apply Corollary A.7.8 to construct a fan Σ refining Δ such that for every $\sigma \in \Sigma$ and every i, j, we have either $u_{i,j} \in \sigma^\vee$ or $-u_{i,j} \in \sigma^\vee$. We claim that this satisfies the condition in the corollary. Indeed, suppose that $v \in C_i$ and let $\sigma \in \Sigma$ be such that $v \in \mathrm{Relint}(\sigma)$. It is enough to show that in this case $\sigma \subseteq C_i$. If this is not the case, then there is j such that $u_{i,j} \notin \sigma^\vee$. By assumption, we have $-u_{i,j} \in \sigma^\vee$. Using the fact that v lies in C_i, we deduce $\langle u_{i,j}, v\rangle = 0$. Therefore v lies on a proper face of σ, contradicting the fact that it lies in the relative interior. □

Remark A.7.10. If $V = M_{\mathbb{R}}$, for a free, finitely generated abelian group and the fans Δ_1 and Δ_2 in Lemma A.7.6 are rational, it follows from the proof that the fan Σ is rational, too. As a consequence, if the fan Δ in Corollary A.7.8 is rational and $u_1,\dots,u_d \in M^*_{\mathbb{Q}}$, then the fan Σ can be taken to be rational. This is turn implies that if in Corollary A.7.9 both the fan Δ and the cones $C_1,\dots,C_r$ are rational, then also the fan Σ can be taken rational.

A.8 Convex functions

Let V be a finite-dimensional real vector space. If T is a convex subset of V, a function $\phi\colon T \to \mathbb{R}$ is *convex* if

$$\phi(tu_1+(1-t)u_2) \leq t\phi(u_1)+(1-t)\phi(u_2) \text{ for all } u_1,u_2 \in T \text{ and } t \in [0.1]. \quad \text{(A.3)}$$

If $V = W_{\mathbb{R}}$ for a $\mathbb{Q}$-vector space W and ϕ is only defined on the rational points of T, then ϕ is *convex* if (A.3) holds under the extra assumption that $u_1,u_2 \in T\cap W$ and $t \in \mathbb{Q}$.

Proposition A.8.1. *If T is an open convex subset of V, then every convex function $\phi\colon T \to \mathbb{R}$ is continuous.*

Proof. Let us show that ϕ is continuous at a point $x \in T$. We choose a basis of V that gives an isomorphism $V \simeq \mathbb{R}^n$. We consider a box

$$P = x + \{u = (u_1,\dots,u_n) \in \mathbb{R}^n \mid |u_i| \leq \eta \text{ for } 1 \leq i \leq n\}$$

for some $\eta > 0$ such that $P \subseteq T$. We denote by ∂P the boundary of this box, that is,

$$\partial P = x + \{u = (u_1,\dots,u_n) \in \mathbb{R}^n \mid \max_{1\leq i\leq n} |u_i| = \eta\}.$$

We first note that there is M such that $\phi(u) \leq M$ for all $u \in P$. Indeed, since ϕ is a convex function, the values of ϕ on any line segment are bounded above by the maximum of the values at the end points of the segment. This implies that $\sup_{u\in P}\phi(u) \leq \sup_{u\in\partial P}\phi(u)$. Repeating this, we see that we may take M to be the maximum value of ϕ at the vertices of P.

Suppose now that $z \neq x$ is a point in $P \setminus \partial P$, and let $y \in \partial P$ be such that $z = \lambda y + (1-\lambda)x$ for some $\lambda \in (0,1)$. Since ϕ is convex, we have

$$\phi(z) \leq \lambda\phi(y) + (1-\lambda)\phi(x)$$

and therefore

$$\phi(z) - \phi(x) \leq \lambda(\phi(y) - \phi(x)) \leq \lambda(M - \phi(x)). \tag{A.4}$$

Note that the second point where the line through x and z intersects ∂P is $2x - y$. Since we can write $x = \frac{1}{1+\lambda} z + \frac{\lambda}{\lambda+1}(2x - y)$, using one more time the convexity of ϕ, we obtain

$$(1+\lambda)\phi(x) \leq \phi(z) + \lambda\phi(2x - y).$$

Therefore

$$\phi(z) \geq \phi(x) + \lambda(\phi(x) - \phi(2x - y)) \geq \phi(x) + \lambda(\phi(x) - M). \tag{A.5}$$

By combining (A.4) and (A.5), we obtain

$$|\phi(z) - \phi(x)| \leq \lambda(M - \phi(x)). \tag{A.6}$$

If we have a sequence $(z_m)_{m\geq 1}$ with $\lim_{m\to\infty} z_m = x$, then the corresponding λ_m satisfy $\lim_{m\to\infty} \lambda_m = 0$, hence (A.6) implies $\lim_{m\to\infty} \phi(z_m) = \phi(x)$. This completes the proof of the proposition. □

Remark A.8.2. Suppose now that $V = W_{\mathbb{Q}}$, for a $\mathbb{Q}$-vector space W, and ϕ is a function defined on the rational points of an open subset T of V. Applying verbatim the argument in the proof of Proposition A.8.1 (by taking $\eta \in \mathbb{Q}$ and only dealing with the rational points in P), we see that also in this setting the convexity of ϕ implies the fact that it is continuous.

Remark A.8.3. If the set T in Proposition A.8.1 is not open, the conclusion can fail. For example, if ϕ is a convex function on a closed interval $[a,b]$ in $\mathbb{R}$, we can replace $\phi(a)$ by any larger value, without affecting the convexity of the function.

A.9 Convex piecewise linear functions

Let V be a finite-dimensional real vector space, C a closed convex cone in V, and $\phi\colon C \to \mathbb{R}$ a function.

Definition A.9.1. We say that ϕ is *piecewise linear* if there is a fan Δ with $|\Delta| = C$ and for every cone $\sigma \in \Delta$ there is a linear function $\ell_\sigma\colon V \to \mathbb{R}$ such that $\phi(v) = \ell_\sigma(v)$ for all $v \in \sigma$. It is clear that for this to hold, C has to be polyhedral.

Remark A.9.2. Note that if $\phi\colon C \to \mathbb{R}$ is piecewise linear, then it satisfies $\phi(tv) = t\phi(v)$ for every $v \in C$ and every $t \in \mathbb{R}_{\geq 0}$. We also note that if this condition is satisfied, then ϕ is convex if and only if $\phi(u+v) \leq \phi(u) + \phi(v)$ for every $u, v \in C$.

Proposition A.9.3. *If C is a polyhedral convex cone in V and $\phi\colon \sigma \to \mathbb{R}$ is a function, then the following are equivalent:*

i) The function ϕ is convex and there are closed convex cones $C_1, \ldots, C_r$ and linear functions $\ell_i\colon V \to \mathbb{R}$ such that $C = C_1 \cup \ldots \cup C_r$ and $\phi(v) = \ell_i(v)$ for every $v \in C_i$.

ii) The are linear functions $\alpha_1,\ldots,\alpha_r\colon V\to\mathbb{R}$ such that for every $v\in C$, we have

$$\phi(v)=\max\{\alpha_i(v)\mid 1\le i\le r\}. \tag{A.7}$$

iii) The function ϕ is convex and piecewise linear.

Proof. After replacing V by the linear span of C, we may assume that this linear span is equal to V. Suppose first that ϕ satisfies i). It follows from Lemma A.7.2 that we may assume that $\dim(C_j)=\dim(V)$ for all j. In particular, the linear maps $\ell_1,\ldots,\ell_r$ are uniquely determined. We will show that

$$\phi(v)=\max\{\ell_i(v)\mid 1\le i\le r\} \tag{A.8}$$

for every $i\in v$. Since $C=\cup_{i=1}^r C_i$ and $\phi=\ell_i$ on C_i, the inequality "$\le$" in (A.8) is clear. Let us show now that $\phi(v)\ge\ell_j(v)$ for every $v\in C$ and every j. Let i be such that $v\in C_i$. Since $\dim(C_j)=n$, the interior of C_j is nonempty (see Remark A.3.3) and we choose a point w in the interior of C_j and $t\in(0,1)$ such that $v'=tv+(1-t)w\in C_j$. Using the convexity of ϕ, we obtain

$$t\ell_j(v)+(1-t)\ell_j(w)=\ell_j(v')=\phi(v')\le t\phi(v)+(1-t)\phi(w)=t\phi(v)+(1-t)\ell_j(w).$$

Since $t>0$, we conclude that $\ell_j(v)\le\phi(v)$. This completes the proof of "i)$\Rightarrow$ii)".

Suppose now that we have linear functions $\alpha_1,\ldots,\alpha_r$ as in ii). It is clear in this case that $\phi(tv)=t\phi(v)$ for every $t\ge 0$ and every $v\in C$. Moreover, given $v,w\in C$, if i is such that $\phi(v+w)=\alpha_i(v+w)=\alpha_i(v)+\alpha_i(w)$, then $\phi(v+w)\le\phi(v)+\phi(w)$. Therefore ϕ is convex. In order to show that it is also piecewise linear, for every i with $1\le i\le r$, we put

$$\sigma_i=C\cap\{v\mid \alpha_i(v)\ge\alpha_j(v)\ \text{for}\ 1\le i\le r\}.$$

Since C is polyhedral, it follows that each σ_i is polyhedral. Moreover, $\sigma_i\cap\sigma_j$ is clearly a face of both σ_i and σ_j, hence the cones σ_i and their faces form a fan Δ with support C (of course, it might happen that $\sigma_i=\sigma_j$ for some $i\neq j$). It is then clear that ϕ is equal to a linear function on each of the cones σ_i, hence it is piecewise linear. Since the implication iii)$\Rightarrow$i) is trivial, this completes the proof of the proposition. □

Corollary A.9.4. *If C is a polyhedral convex cone in V and $\phi\colon C\to\mathbb{R}$ is a piecewise linear, convex function, then there is a coarsest fan that satisfies the condition in Definition A.9.1; more precisely, every other fan that satisfies this condition is a refinement of Δ.*

Proof. After replacing V by the linear span of C, we may assume that this linear span is equal to V. We use the notation in the proof of Proposition A.9.3. Note first that by the proposition, there are linear functions $\alpha_1,\ldots,\alpha_r$ on V such that (A.7) holds. We claim that the fan Δ constructed using these functions is minimal with the property that it satisfies Definition A.9.1 (only minimality is left to prove). Suppose that Σ is another fan that satisfies Definition A.9.1. Let τ be a maximal cone in Σ

and let $\ell\colon V \to \mathbb{R}$ be a linear map such that $\phi(v) = \ell(v)$ for $v \in \tau$. Note that τ has dimension equal to $\dim(V)$ by Corollary A.7.3, hence ℓ is uniquely determined. It follows from (A.7) that there is i such that $\ell = \alpha_i$. Indeed, otherwise we can find for every i an element $v_i \in \tau$ such that $\ell(v_i) > \alpha_i(v_i)$ (note that by (A.7), we also have $\ell(v_i) \geq \alpha_j(v_i)$ for all j). We thus have

$$\ell(v_1 + \ldots + v_r) > \alpha_j(v_1 + \ldots + v_r)$$

for $1 \leq j \leq r$, contradicting (A.7). Therefore we can find i such that $\ell = \alpha_i$, which gives $\tau \subseteq \sigma_i$. We conclude that Σ is a refinement of Δ. $\square$

Remark A.9.5. Suppose that C is a polyhedral convex cone in V and $(\phi_i)_{i\in I}$ is a family of piecewise linear, convex functions on C. If there is a fan Δ with $|\Delta| = C$ such that each ϕ_i is linear on the cones of Δ, then there is a unique coarsest fan with this property. Indeed, this is the coarsest fan refining all Δ_i (see Remark A.7.7), where Δ_i is the coarsest fan such that ϕ_i is linear on all cones of Δ_i.

Suppose now that M is a finitely generated, free abelian group, $V = M_{\mathbb{R}}$, and $C \subseteq V_{\mathbb{R}}$ is a rational polyhedral cone. Given $\phi\colon C \cap M_{\mathbb{Q}} \to \mathbb{R}$, we say as in Definition A.9.1 that ϕ is *piecewise linear* if there is a rational fan Δ with $|\Delta| = C$ and for every cone $\sigma \in \Delta$ there is a linear function $\ell_\sigma\colon M_{\mathbb{Q}} \to \mathbb{Q}$ such that $\phi(v) = \ell_\sigma(v)$ for all $v \in \sigma \cap M_{\mathbb{Q}}$. It is clear that Proposition A.9.3 and Corollary A.9.4 have variants in this setting.

The next proposition gives examples of piecewise linear, convex functions. We make use of this result in studying the consequences of finite generation for section rings associated to several line bundles.

Proposition A.9.6. *Let M be a finitely generated, free abelian group and C the convex cone generated by $v_1, \ldots, v_d \in M_{\mathbb{R}}$. For every $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{Q}^d_{\geq 0}$, if a function $\phi_\alpha\colon C \cap M_{\mathbb{Q}} \to \mathbb{R}$ satisfies*

$$\phi_\alpha(v) = \min\left\{ \sum_{j=1}^{d} \lambda_j \alpha_j \mid \lambda_1, \ldots, \lambda_d \in \mathbb{Q}_{\geq 0}, \sum_{j=1}^{d} \lambda_j v_j = v \right\} \quad \text{for all} \quad v \in C \cap M_{\mathbb{Q}}, \tag{A.9}$$

then ϕ_α is convex and piecewise linear. Furthermore, there is a rational fan Δ with support C such that each ϕ_α as above is linear on the cones in Δ.

Proof. Let us first check that ϕ_α is convex. It is clear that $\phi_\alpha(tv) = t \cdot \phi_\alpha(v)$ for every $t \in \mathbb{Q}_{\geq 0}$ and $v \in C$. Moreover, if $v = \sum_j \lambda_j v_j$ and $v' = \sum_j \lambda'_j v_j$ are such that $\phi_\alpha(v) = \sum_j \lambda_j \alpha_j$ and $\phi_\alpha(v') = \sum_j \lambda'_j \alpha_j$, then $v + v' = \sum_j (\lambda_j + \lambda'_j) v_j$ and by definition

$$\phi_\alpha(v + v') \leq \sum_j (\lambda_j + \lambda'_j)\alpha_j = \phi_\alpha(v) + \phi_\alpha(v').$$

We next show that if $v \in C$ and $v = \sum_j \lambda_j v_j$ is such that $\phi_\alpha(v) = \sum_j \lambda_j \alpha_j$, then we may assume that the v_j for which $\lambda_j \neq 0$ are linearly independent. The argument for this follows closely the proof of Proposition A.5.7. We may assume that the number of nonzero λ_j is minimal among those $(\lambda_1,\dots,\lambda_d)\in\mathbb{Q}^d_{\geq 0}$ such that $v=\sum_j \lambda_j v_j$ and $\phi_\alpha(v)=\sum_j\lambda_j\alpha_j$. Let $J=\{j\mid \lambda_j\neq 0\}$. Suppose that there is a relation $\sum_{j\in J}a_jv_j=0$ such that not all a_j are 0. After possibly multiplying this relation by (-1), we may assume that $\sum_{j\in J}a_j\alpha_j\geq 0$ and $a_j>0$, for some $j\in J$ (we use the fact that $\alpha_j\geq 0$ for all j). Let $i\in I$ be such that $\lambda_i/a_i=\min\{\lambda_j/a_j\mid j\in J, a_j>0\}$. In this case we can write

$$v=\sum_{j\in J\smallsetminus\{i\}}\left(\lambda_j-\lambda_i\frac{a_j}{a_i}\right)v_j$$

and

$$\sum_{j\in J}\left(\lambda_j-\lambda_i\frac{a_j}{a_i}\right)\alpha_j=\phi_\alpha(v)-\frac{\lambda_i}{a_i}\sum_{j\in J}a_j\alpha_j\leq\phi_\alpha(v),$$

a contradiction with the minimality in the choice of J.

Let $\Lambda_1,\dots,\Lambda_r$ be the subsets of $\{v_1,\dots,v_d\}$ that consist of linearly independent vectors and let C_j be the convex cone generated by Λ_j. We apply Corollary A.7.8 to construct a rational fan Δ with $|\Delta|=C$ such that each C_j is a union of cones in Δ. We claim that every ϕ_α is linear on the cones in Δ. Indeed, for every $\sigma\in\Delta$, let $J(\sigma)$ be the set consisting of those j such that $\sigma\subseteq C_j$. For every j, we have linear functions $u_{j,i}\in M^*_{\mathbb{Q}}$, for $i\in\Lambda_j$, such that for every $v\in C_j$, we have $v=\sum_{i\in\Lambda_j}\langle u_{j,i},v\rangle v_i$. Let $L_j\in M^*_{\mathbb{Q}}$ be given by $L_j=\sum_{i\in\Lambda_j}\alpha_i u_{j,i}$. Note that if v lies on a proper face $C_{j'}$ of C_j (that is, if some $\langle u_{j,i},v\rangle$ are zero) and if we run the same process with respect to $C_{j'}$, then $L_j(v)=L_{j'}(v)$. We thus conclude that for every $v\in\sigma$, we have

$$\phi_\alpha(v)=\min_{j\in J(\sigma)}L_j(v).$$

Since each L_j is a linear function, we deduce from Proposition A.9.3 that $-\phi_\alpha$ is convex on σ. On the other hand, we have seen that ϕ_α is a convex function. Therefore ϕ_α is linear on σ and this completes the proof of the proposition. □

Remark A.9.7. Proposition A.9.6 has a variant for a finite-dimensional real vector space V. More precisely, if C is the cone generated by $v_1,\dots,v_d\in V$ and if for $\alpha\in\mathbb{R}^d_{\geq 0}$ we have a function ϕ_α that satisfies

$$\phi_\alpha(v)=\min\left\{\sum_{j=1}^d\lambda_j\alpha_j\mid\lambda_1,\dots,\lambda_d\in\mathbb{R}_{\geq 0},\sum_{j=1}^d\lambda_jv_j=v\right\}\quad\text{for all}\quad v\in C,$$

then there is a fan Δ with support C such that each ϕ_α as above is linear on the cones in Δ.

Appendix B
Birational maps and resolution of singularities

In the first section we collect a few elementary facts that are used elsewhere in the book. We then discuss birational maps and exceptional loci and in the last section we review the terminology concerning various types of resolutions of singularities and give the existence statements. All schemes are assumed to be separated and of finite type over a ground field k.

B.1 A few basic facts

We begin with the following easy lemma.

Lemma B.1.1. *Let $f\colon Y \to X$ be a surjective morphism between complete varieties, with* $\dim(Y) > \dim(X)$. *If H is an ample, effective Cartier divisor on Y, then $f(H) = X$.*

Proof. Indeed, if $f(H) \neq X$ and $x \in X \smallsetminus f(H)$, then $(H \cdot C) = 0$ for every curve C contained in $f^{-1}(x)$, contradicting the fact that H is ample. $\square$

Corollary B.1.2. *If $f\colon Y \to X$ is a surjective morphism between complete schemes, then for every irreducible, closed subset Z of X, there is an irreducible, closed subset W of Y such that $f(W) = Z$ and* $\dim(W) = \dim(Z)$. *Moreover, given a dense open subset U of Y, which dominates X, then we may assume that $W \cap U \neq \emptyset$.*

Proof. We may replace f by $f^{-1}(Z) \to Z$ and thus assume that $Z = X$. We argue by induction of $\dim(Y)$. After replacing Y by an irreducible component that dominates X, we may also assume that Y is irreducible. Note that $\dim(Y) \geq \dim(X)$ and if we have equality, then there is nothing to prove.

Suppose now that $\dim(Y) > \dim(X)$. By Chow's lemma, we have a surjective morphism $g\colon Y' \to Y$, with Y' irreducible and projective and $\dim(Y') = \dim(Y)$. If we can find a closed, irreducible closed subset W' in Y' with $f(g(W')) = X$ and $\dim(W') = \dim(X)$ (and, in the presence of U, such that $W' \cap g^{-1}(U) \neq \emptyset$), then $W =$

$g(W')$ satisfies the required conditions (note that $\dim(f(g(W')) \leq \dim(g(W')) \leq \dim(W')$, hence both these are equalities). Therefore we may assume that Y is projective. Let H be an ample effective Cartier divisor on Y (which we may, in the presence of U, assume that intersects U). It follows from Lemma B.1.1 that H surjects onto X. The assertion now follows by induction. □

Remark B.1.3. In Lemma B.1.1, the same result holds if we only assume that f is a proper, surjective morphism of varieties and H is f-ample. This implies that in Corollary B.1.2 it is enough to assume that f is proper and surjective.

Proposition B.1.4. *If X is a connected, complete scheme, then for every two (closed) points $x \neq y$ in X, there is a connected, 1-dimensional closed subscheme Z of X containing x_1 and x_2 in its support.*

Proof. We argue by induction on $n = \dim(X)$. Since X is connected, we can find irreducible components $X_1, \ldots, X_r$ of X such that $x \in X_1$, $y \in X_r$, and $X_i \cap X_{i+1} \neq \emptyset$ for $1 \leq i \leq r-1$. If we choose $z_i \in X_i \cap X_{i+1}$, then it is enough to prove the assertion in the proposition for each of the pairs $(x,z_1), (z_1,z_2), \ldots, (z_{r-1},y)$ that consist of distinct points. Therefore we may assume that X is irreducible. Of course, after replacing X by X_{red}, we may assume that X is also reduced.

Note that if $f\colon X' \to X$ is a surjective, proper, generically finite morphism and if $x' \in f^{-1}(x)$ and $y' \in f^{-1}(y)$, then for every subscheme Z' of X' that satisfies the conclusion of the proposition for x' and y', its image $f(Z)$ satisfies the conclusion for x and y. Therefore, after applying Chow's lemma and then taking the normalization, we may assume that X is normal and projective. If $n = 1$, then we may take $Z = X$. On the other hand, if $n \geq 2$, then we consider a very ample, effective Cartier divisor H on X such that $x, y \in H$. Since H is connected by [Har77, Cor. III.7.9] and $\dim(X) = n-1$, we can apply the inductive hypothesis to complete the proof. □

Corollary B.1.5. *If X is a connected, complete scheme and Y is a proper, nonempty subset of X, then there is a curve C in X that is not contained in Y, but meets Y.*

Proof. Let $x_1 \in Y$ and $x_2 \in X \smallsetminus Y$. If Z is a closed subscheme of X as in Proposition B.1.4, then some irreducible component C of Z satisfies the conditions in the corollary. □

Remark B.1.6. If the ground field is algebraically closed, then one can do better then in Proposition B.1.4: if X is any irreducible scheme, any two distinct points $x, y \in X$ lie on a curve C on X, that is, the scheme in the proposition can be taken to be irreducible. In order to prove this, we argue by induction. By Chow's lemma, we may assume that X is a quasi-projective variety and by taking the closure in a suitable projective space, we may assume that X is projective. If $\dim(X) \geq 2$, then we consider the blow-up $\pi\colon X' \to X$ along $\{x,y\}$ and an effective, very ample Cartier divisor H on X'. By taking H general in the corresponding linear system, we may assume that H is irreducible by a version of Bertini's theorem, see [Jou83, Théorème 6.3] (it is here that we use the assumption that k is algebraically closed, since we need X' to be geometrically irreducible). Since H is ample, it intersects both $\pi^{-1}(x)$ and $\pi^{-1}(y)$. Therefore there are points $x', y' \in H$ lying over x and y, respectively. By induction, there is a curve C' on H containing x' and y' and $C = \pi(C')$ is a curve that contains both x and y.

B.2 Birational maps and exceptional loci

Suppose that $f\colon Y \to X$ is a proper birational morphism between varieties over a field k. Let U be the largest open subset of X on which the inverse rational map f^{-1} is defined. Equivalently, U is the largest open subset of X such that f is an isomorphism over U. The closed subset $Y \smallsetminus f^{-1}(U)$ is the *exceptional locus* of f, that we denote by $\mathrm{Exc}(f)$. We say that a Weil divisor on Y is *exceptional* if its support is contained in $\mathrm{Exc}(f)$. We denote by $\mathrm{ExcDiv}(f)$ the sum of the exceptional divisors of f.

Note that if X is normal, then $\mathrm{codim}(X \smallsetminus U, X) \geq 2$. In particular, a prime divisor E on Y is exceptional if and only if $\dim(f(E)) < \dim(E)$. We also note that in this case, every prime divisor D on X intersects U, hence its strict transform $\widetilde{D}$ is well-defined as a prime divisor on Y. If $\Delta = \sum_i a_i D_i$ is an $\mathbb{R}$-divisor on X, we put $\widetilde{\Delta} = \sum_i a_i \widetilde{D}_i$.

Suppose now that X is normal and Δ is an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. It follows from definition that the difference $f^*(\Delta) - \widetilde{\Delta}$ is an effective exceptional $\mathbb{R}$-divisor.

Lemma B.2.1. *If $g\colon Z \to Y$ and $f\colon Y \to X$ are proper, birational morphisms, then*

$$\mathrm{Exc}(f \circ g) = g^{-1}(\mathrm{Exc}(f)) \cup \mathrm{Exc}(g).$$

Proof. Suppose first that $x \notin g^{-1}(\mathrm{Exc}(f)) \cup \mathrm{Exc}(g)$. In this case there are open neighborhoods U of $f(g(x))$ and V of $g(x)$ such that f^{-1} is defined on U and g^{-1} is defined on V. In this case $(f \circ g)^{-1}$ is defined on $U \cap (f^{-1})^{-1}(V)$. Therefore $x \notin \mathrm{Exc}(f \circ g)$, proving the inclusion "$\subseteq$" in the lemma.

On the other hand, if $x \notin \mathrm{Exc}(f \circ g)$, then $(f \circ g)^{-1}$ is defined in some neighborhood W of $f(g(x))$. In this case $f^{-1} = g \circ (f \circ g)^{-1}$ is defined on W and $g^{-1} = (f \circ g)^{-1} \circ f$ is defined on the open neighborhood $f^{-1}(W)$ of $g(x)$. Therefore $x \notin g^{-1}(\mathrm{Exc}(f)) \cup \mathrm{Exc}(g)$, completing the proof of the lemma. □

Lemma B.2.2. *If $f\colon Y \to X$ is a proper birational morphism between two varieties and $y \in Y$ lies on an irreducible component of $f^{-1}(f(y))$ of positive dimension, then $y \in \mathrm{Exc}(f)$. The converse holds if X is normal.*

Proof. The first assertion is clear. For the converse, note that if Y is normal, then f is a fiber space. In particular, f has connected fibers by Zariski's Main Theorem. Suppose that $y \in Y$ is such that $\{y\}$ is a zero-dimensional component of $f^{-1}(f(y))$. In this case there is an open neighborhood V of y such that every $y' \in V$ has the same property (see [Har77, Exer. II.3.22]). The connectedness of each $f^{-1}(f(y'))$

then implies that for every $y' \in V$, we have $f^{-1}(f(y')) = \{y'\}$. We deduce that if $W = f(V)$, then $V = f^{-1}(W)$. Since f is closed, this implies that $W = X \smallsetminus f(Y \smallsetminus V)$ is open. The morphism $V \to W$ is a bijective fiber space, hence an isomorphism, and we see that $y \not\in \mathrm{Exc}(f)$. $\square$

Lemma B.2.3. *If $f\colon Y \to X$ is a proper birational morphism between smooth varieties, then* $\mathrm{Exc}(f)$ *is an effective divisor. In fact, if $K_{Y/X}$ is the effective divisor defined by the nonzero morphism of line bundles $\pi^*(\omega_X) \to \omega_Y$, then* $\mathrm{Supp}(K_{Y/X}) = \mathrm{Exc}(f)$.

Proof. If $n = \dim(X) = \dim(Y)$, then we have a morpshim of rank n vector bundles $f^*(\Omega_X) \to \Omega_Y$, which is an isomorphism over an open subset of Y. By taking the top exterior powers, we obtain an injective map of line bundles $f^*(\omega_X) \to \omega_Y$, which corresponds to a nonzero section of $\omega_Y \otimes f^*(\omega_X)^{-1}$. The zero-locus of this section is $K_{Y/X}$. Therefore

$$\mathrm{Supp}(K_{Y/X}) = \{y \in Y \mid f \text{ is not étale at } y\}.$$

It follows from the definition of the exceptional locus that $\mathrm{Supp}(K_{Y/X}) \subseteq \mathrm{Exc}(\pi)$. The reverse inclusion follows from Lemma B.2.2. $\square$

Example B.2.4. If Z is a smooth, closed subvariety of a smooth variety X, of codimension r, and $f\colon Y \to X$ is the blow-up of X along Z, with exceptional divisor E, then $K_{Y/X} = (r-1)E$. Indeed, this can be checked in local charts. Suppose that we have coordinates $x_1, \ldots, x_n$ on an affine open subset U of X, such that $Z \cap U$ is defined by $(x_1, \ldots, x_r)$. A typical chart on $\pi^{-1}(U)$ has local coordinates $x_i, y_1, \ldots, \widehat{y_i}, \ldots, y_r, x_{r+1} \ldots, x_n$, for some i with $1 \le i \le r$, and where $x_j = x_i y_j$ for all j with $1 \le j \le r$, $j \neq i$. Note that in this chart E is defined by (x_i). Since $dx_j = x_i dy_j + y_j dx_i$ for $1 \le j \le r$, with $j \neq i$, one can easily check that

$$dx_1 \wedge \ldots \wedge dx_n = \pm x_i^{r-1} \cdot dx_i \wedge dy_1 \wedge \ldots \wedge \widehat{dy_i} \wedge \ldots \wedge dy_r \wedge dx_{r+1} \wedge \ldots \wedge dx_n$$

and we see that in this chart we have $K_{Y/X} = (r-1)E$.

The following lemma is an easy, but often useful fact.

Lemma B.2.5. *If $f\colon Y \to X$ is a proper birational morphism of normal varieties, D is a Cartier divisor on X, and E is an effective exceptional divisor on Y, then we have an equality $\mathscr{O}_X(D) = f_*\mathscr{O}_Y(f^*(D)+E)$ of subsheaves of the function field of X. In particular, we have $\mathscr{O}_X = f_*(\mathscr{O}_Y(E))$.*

Proof. We need to show that if U is an open subset of X and ϕ is a nonzero rational function on X, then $\mathrm{div}_X(\phi) + D$ is effective on U if and only if

$$\mathrm{div}_Y(\phi) + f^*(D) + E = f^*(\mathrm{div}_X(\phi) + D) + E$$

is effective on $f^{-1}(U)$. The "only if" part is clear since E is effective. On the other hand, if F is a prime divisor on X intersecting U whose coefficient a_F in $\mathrm{div}_X(\phi) + D$

is negative, then also the coefficient of $\widetilde{F}$ in $\mathrm{div}_Y(\phi) + f^*(D) + E$ is negative, being equal to a_F. This completes the proof of the lemma. □

Corollary B.2.6. *If $f\colon Y \to X$ is a proper, birational morphism of smooth varieties, then we have a canonical isomorphism $f_*(\omega_Y) \simeq \omega_X$.*

Proof. We have seen in Lemma B.2.3 that there is an effective, exceptional divisor $K_{Y/X}$ on Y such that $\omega_Y \simeq f^*(\omega_X) \otimes \mathscr{O}_Y(K_{Y/X})$. It follows from Lemma B.2.5 and the projection formula that

$$f_*(\omega_Y) \simeq \omega_X \otimes f_*(\mathscr{O}_Y(K_{Y/X})) \simeq \omega_X.$$

□

B.3 Resolutions of singularities

In this section we assume that all varieties are defined over a field k of characteristic zero. For a variety X over k we denote by X_{sm} the smooth locus of X.

Definition B.3.1. Given a variety X, a *resolution of singularities* of X is a projective birational morphism $f\colon Y \to X$, with Y a smooth variety.

The following is a fundamental result of Hironaka [Hir64].

Theorem B.3.2. *Every variety over k has a resolution of singularities $f\colon Y \to X$ which is an isomorphism over X_{sm}.*

In several instances one defines invariants of an algebraic variety in terms of a resolution of singularities. In each such case, one needs to check independence of the chosen resolution. The following proposition allows to compare two such resolutions.

Proposition B.3.3. *If $f_1\colon Y_1 \to X$ and $f_2\colon Y_2 \to X$ are two resolutions of singularities of X, then there is a third resolution dominating both of them, that is, there is a smooth variety Y and projective, birational morphisms $g_1\colon Y \to Y_1$ and $g_2\colon Y \to Y_2$ such that $f \circ g_1 = f \circ g_2$.*

Proof. Let $W = Y_1 \times_X Y_2$ and $p_1\colon W \to Y_1$ and $p_2\colon W \to Y_2$ the canonical projections. Since f_1 and f_2 are birational, it follows that there is an open subset U of X such that the induced map $h\colon W \to X$ is an isomorphism over U. With the reduced scheme structure, $W_0 := \overline{h^{-1}(U)}$ is a variety such that $p_1|_{W_0}$ and $p_2|_{W_0}$ are projective birational morphisms. If $g\colon Y \to W_0$ is a resolution of singularities, then $g_1 = p_1 \circ g$ and $g_2 = p_2 \circ g$ satisfy the requirements in the proposition. □

We will also need the following version of resolution of singularities for a divisor, which is also due to Hironaka [Hir64].

Theorem B.3.4. *If X is a smooth variety and Δ is an effective divisor on X, then there is a projective morphism $f\colon Y \to X$, with Y smooth, which is an isomorphism over $X \smallsetminus \mathrm{Supp}(\Delta)$, and such that $f^*(\Delta)$ has simple normal crossings.*

We also consider two extensions of the above notion. In the first one we treat nonzero ideals on arbitrary varieties.

Definition B.3.5. If $\mathfrak{a}$ is a nonzero ideal on the variety X, then a *log resolution* of $(X,\mathfrak{a})$ (or simply of $\mathfrak{a}$) is a projective birational morphism $f\colon Y \to X$ such that

i) Y is smooth,
ii) $\mathfrak{a}\cdot\mathscr{O}_Y = \mathscr{O}_Y(-D)$ for an effective divisor D, and
iii) the divisor $D+\mathrm{Exc}(f)$ has simple normal crossings.

Corollary B.3.6. *Given a nonzero ideal $\mathfrak{a}$ on the variety X, there is a log resolution $f\colon Y \to X$ of $(X,\mathfrak{a})$ which is an isomorphism over $X_{\mathrm{sm}} \smallsetminus Z(\mathfrak{a})$ and such that $\mathrm{Exc}(f)$ is an effective divisor.*

Proof. We first take $f_1\colon X_1 \to X$ be the blow-up of X along $\mathfrak{a}$, hence $\mathfrak{a}\cdot\mathscr{O}_{X_1}$ is the ideal of an effective Cartier divisor. Note that f_1 is an isomorphism over the complement of $Z(\mathfrak{a})$. We then apply Theorem B.3.2 to get a resolution of singularities $f_2\colon X_2 \to X_1$ of X_1 which is an isomorphism over the smooth locus of X_1. We do not know much about the exceptional locus W of $f_1\circ f_2$, so we repeat the previous process in order to get the exceptional locus be a divisor: we let $f_3\colon X_3 \to X_2$ be the blow-up of X_2 along W, and $f_4\colon X_4 \to X_3$ a resolution of singularities of X_3 that is an isomorphism over the smooth locus of X_3. In particular, it follows from Lemma B.2.1 that the exceptional locus of the composition $g\colon X_4 \to X$ is the support of the divisor $E=(f_3\circ f_4)^{-1}(W)$. Let Δ be the effective Cartier divisor on X_4 such that $\mathfrak{a}\cdot\mathscr{O}_{X_4} = \mathscr{O}_{X_4}(-\Delta)$. We apply Theorem B.3.4 to find a projective morphism $f_5\colon Y \to X_4$, with Y smooth, which is an isomorphism over $X \smallsetminus \mathrm{Supp}(\Delta+E)$, and such that $f_5^*(\Delta+E)$ is a divisor with simple normal crossings. Let $f\colon Y \to X$ be the composition of the above maps. Note first that f is an isomorphism over $X_{\mathrm{sm}} \smallsetminus Z(\mathfrak{a})$. It follows from construction and Lemma B.2.3 that $\mathrm{Exc}(f_5)$ is a divisor with support contained in $\mathrm{Supp}(f_5^*(\Delta+E))$. Furthermore, we deduce from Lemma B.2.1 that

$$\mathrm{Exc}(f) = \mathrm{Exc}(f_5)\cup f_5^{-1}(\mathrm{Exc}(g)),$$

hence this is a divisor with support contained in $\mathrm{Supp}(f_5^*(\Delta+E))$. It is now clear that f satisfies the conditions for being a log resolution of $(X,\mathfrak{a})$. □

Remark B.3.7. If $\mathfrak{a}_1,\ldots,\mathfrak{a}_r$ are nonzero ideals on the variety X, then we may consider a log resolution $f\colon Y \to X$ for $(X,\mathfrak{a}_1\cdot\ldots\cdot\mathfrak{a}_r)$. It is easy to see that if a product of nonzero coherent ideal sheaves on an integral scheme is locally principal, then each of the ideals is locally principal. It follows that for every i we can write $\mathfrak{a}_i\cdot\mathscr{O}_Y = \mathscr{O}_Y(-D_i)$ for an effective divisor D_i on Y, and that $\mathrm{ExcDiv}(f)+D_1+\ldots+D_r$ has simple normal crossings.

Remark B.3.8. If Δ is an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on the variety X, then we may consider log resolutions for the pair (X,Δ), as follows. If we write $\Delta = \sum_i a_i F_i$, where the F_i are effective Cartier divisors and $a_i \in \mathbb{R}_{\geq 0}$, then a log resolution of (X,Δ) is provided by a log resolution $f\colon Y \to X$ of the product $\prod_i \mathscr{O}_X(-F_i)$. Note that this has the property that $f^*(\Delta) + \mathrm{ExcDiv}(f)$ is a simple normal crossings $\mathbb{R}$-divisor.

We will also consider a version of log resolutions in the presence of a Weil divisor.

Definition B.3.9. Suppose that X is a normal variety, $\Delta = \sum_i a_i F_i$ is an $\mathbb{R}$-divisor on X, and $\mathfrak{a}$ is a nonzero ideal on X. A *log resolution* of $(X,\Delta,\mathfrak{a})$ is a projective birational morphism $f\colon Y \to X$ such that

i) Y is smooth,
ii) we have $\mathfrak{a} \cdot \mathscr{O}_Y = \mathscr{O}_Y(-D)$ for an effective divisor D, and
iii) the divisor $D + \mathrm{ExcDiv}(f) + \sum_i \widetilde{F_i}$ has simple normal crossings.

Corollary B.3.10. *Given a nonzero ideal $\mathfrak{a}$ on the normal variety X, and an $\mathbb{R}$-divisor $\Delta = \sum_i a_i F_i$ on X, there is a log resolution $f\colon Y \to X$ of $(X,\Delta,\mathfrak{a})$ which is an isomorphism over $X_{\mathrm{sm}} \smallsetminus (Z(\mathfrak{a}) \cup \mathrm{Supp}(\Delta))$ and such that $\mathrm{Exc}(f)$ is an effective divisor.*

Proof. We first apply Theorem B.3.2 to get a resolution of singularities $g\colon X_1 \to X$ which is an isomorphism over X_{sm}. Let $\mathfrak{a}'$ denote the ideal of the reduced effective divisor $\sum_i \widetilde{F_i}$. If $\mathfrak{a}''$ is the ideal defining $\mathrm{Exc}(g)$ (with reduced structure), then we let f be the composition of g with a log resolution $h\colon Y \to X_1$ of $(X_1, \mathfrak{a}\cdot\mathfrak{a}'\cdot\mathfrak{a}'')$, which is an isomorphism over $X_1 \smallsetminus Z(\mathfrak{a}\cdot\mathfrak{a}'\cdot\mathfrak{a}'')$. We take $f = g \circ h$. By Lemma B.2.1, we have $\mathrm{Exc}(f) = \mathrm{Exc}(h) \cup h^{-1}(Z(\mathfrak{a}''))$ and it is clear that f is a log resolution of $(X,\Delta,\mathfrak{a})$, and that it is an isomorphism over the complement of $X_{\mathrm{sm}} \smallsetminus (Z(\mathfrak{a}) \cup \mathrm{Supp}(\Delta))$. ☐

Remark B.3.11. If instead of one ideal $\mathfrak{a}$ in Corollary B.3.10 we have several nonzero ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$, then we can proceed as in Remark B.3.7 by taking a log resolution for $(X,\Delta,\mathfrak{a}_1 \cdot \ldots \cdot \mathfrak{a}_r)$.

Remark B.3.12. Arguing as in Remark B.3.3, we see that any two log resolutions (for example, in the setting in Corollary B.3.10) can be dominated by a third one.

The known results on resolution of singularities offer more information on the resolutions, that are sometimes useful. We only mentions two such stronger versions, that we will need.

Remark B.3.13. In the context of Theorem B.3.2, one can construct the resolution $f\colon Y \to X$ as a composition of blow-ups of subschemes (in fact, smooth subvarieties) lying over $X \smallsetminus X_{\mathrm{sm}}$:

$$Y = Y_m \xrightarrow{f_m} Y_{m-1} \xrightarrow{f_{m-1}} \ldots \to Y_1 \xrightarrow{f_1} X.$$

Note that if f_i is the blow-up along the subscheme $Z_{i-1} \hookrightarrow Y_{i-1}$ and if $E_i = f_i^{-1}(Z_i)$, then E_i is an effective Cartier divisor on Y_i such that $\mathscr{O}_{Y_i}(-E_i)$ is f_i-ample. Using Proposition 1.6.15, we deduce that there is an effective Cartier divisor E on Y with $\operatorname{Supp}(E) \subseteq f^{-1}(X \smallsetminus X_{\mathrm{sm}})$ such that $\mathscr{O}_Y(-E)$ is f-ample.

Remark B.3.14. In the context of Theorem B.3.4, if U is an open subset of X such that $\Delta|_U$ has simple normal crossings, then one can construct the morphism $f\colon Y \to X$ such that it is an isomorphism over U. Moreover, f can be taken to be a composition of blow-ups with smooth centers, all centers lying above $X \smallsetminus U$. The fact that f can be taken to be an isomorphism over U is useful, for example, when compactifying pairs (X, Δ), where X is a smooth quasiprojective variety and Δ is a simple normal crossing divisor on X. More precisely, we can find an open immersion $X \hookrightarrow X'$, where X' is a smooth projective variety, and a divisor Δ' on X' such that

i) $X' \smallsetminus X$ is a divisor E.
ii) $\Delta'|_X = \Delta$ and Δ' has no common components with E.
iii) $\Delta' + E$ has simple normal crossings.

Indeed, we can first embed X as an open subset of a projective variety W. After possibly replacing W by its blow-up along $W \smallsetminus X$, we may assume that $W \smallsetminus X$ is the support of an effective Cartier divisor F. By Theorem B.3.2, we may construct a resolution of singularities $f\colon Y \to W$ that is an isomorphism over X. If $\Delta = \sum_i a_i \Delta_i$ and $\Delta_Y = \sum_i a_i \overline{\Delta_i}$ is the corresponding divisor on Y, then we consider a projective and birational morphism $g\colon X' \to Y$ such that X' is smooth, $g^*(\Delta_Y + F)$ has simple normal crossings, and g is an isomorphism over $f^{-1}(X)$. It is then clear that on X' we can choose Δ' that satisfies i), ii), and iii) above.

Appendix C
Finitely generated graded rings

In this appendix we collect some standard facts concerning finite generation for graded rings. In what follows we consider rings generated by semigroups. We do not aim for the most general statements and sometimes make restrictive hypotheses if these simplify the proofs and they are satisfied in the cases of interest for us. We refer to Section A.6 for the definitions related to semigroups. We denote by $\mathbb{N}$ the monoid $(\mathbb{Z}_{\geq 0}, +)$.

If S is a monoid, an S-graded ring is a ring[1] R with a direct sum decomposition $R = \bigoplus_{u \in S} R_u$, where each R_u is an abelian subgroup of R, such that $1 \in R_0$ and $R_u \cdot R_v \subseteq R_{u+v}$ for all $u, v \in S$. It is clear that in this case R_0 is a ring, each R_u is an R_0-module, and R is an R_0-algebra. If $f \in R_u$ is nonzero, then we put $\deg(f) = u$. Elements of R_u, for $u \in S$, are called *homogeneous*. If k is a fixed field, an S-graded k-algebra is a k-algebra that has a decomposition as above, such that each R_u is a k-vector subspace. In particular, R_0 is a k-algebra. A *graded subring* of $R = \bigoplus_{u \in S} R_u$ is a subring R' of R such that $R' = \bigoplus_{u \in S} (R' \cap R_u)$.

Remark C.0.1. If R is an S-graded ring as above and S is a submonoid of a monoid T, then we may consider R in a natural way as a T-graded ring.

Remark C.0.2. Suppose that S is a finitely generated submonoid of a finitely generated, free abelian group M. If S has no nonzero invertible elements, then the convex cone σ generated by S in $M_{\mathbb{R}}$ is strongly convex. Therefore there is a group homomorphism $\ell \colon M \to \mathbb{Z}$ such that $\ell(u) > 0$ for every nonzero $u \in S$. Given an S-graded ring $R = \bigoplus_{u \in S} R_u$, we can use ℓ to put on R a structure of $\mathbb{N}$-graded ring, by writing $R = \bigoplus_{m \in \mathbb{N}} R_m$, where $R_m = \bigoplus_{u \in S, \ell(u) = m} R_u$. This can be sometimes used to deduce properties of S-graded rings from the $\mathbb{N}$-graded case.

We now list some basic results about the finite generation of graded rings.

Lemma C.0.3. *If S is a monoid and $R = \bigoplus_{u \in S} R_u$ is an S-graded domain that is finitely generated as an R_0-algebra, then the submonoid $T = \{u \in S \mid R_u \neq 0\}$ of S is finitely generated.*

[1] All rings will be assumed commutative, with unit $1 \neq 0$.

Proof. Note first that indeed T is a submonoid of S, since R is a domain. Let $f_1,\ldots,f_n \in R$ be a system of generators of R as an R_0-algebra. We may clearly assume that all f_i are homogeneous and nonzero. If $u_i = \deg(f_i) \in S$, then it is straightforward to see that $u_1,\ldots,u_m$ generate T. $\square$

If R is an S-graded ring and T is a submonoid of S, then the *restriction of R to T* defined by $R|_T := \bigoplus_{u\in T} R_u$ is a T-graded ring.

Lemma C.0.4. *Let R be an S-graded ring, where S is a monoid. If S is the union of the submonoids $S_1,\ldots,S_r$ and each $R|_{S_i}$ is a finitely generated R_0-algebra, then R is a finitely generated R_0-algebra.*

Proof. The assertion is clear, since $R = R|_{S_1} + \ldots + R|_{S_r}$ as R_0-modules. $\square$

Lemma C.0.5. *If S is a submonoid of a finitely generated, free abelian group and R is an S-graded ring, then R is a domain if and only if for every two homogeneous nonzero elements $f,g \in R$, we have $fg \neq 0$.*

Proof. Since S is a submonoid of a finitely generated, free abelian group A, we can put on S a total order that is compatible with addition. For example, choose an isomorphism $A \simeq \mathbb{Z}^n$, and consider on $\mathbb{Z}^n$ the lexicographic order. Suppose that $f,g \in R$ are nonzero elements such that $fg = 0$. Writing $f = \sum_{u\in S} f_u$ and $g = \sum_{u\in S} g_u$, with $f_u, g_u \in R_u$, let

$$v = \max\{u \in S \mid f_u \neq 0\} \quad \text{and} \quad w = \max\{u \in S \mid g_u \neq 0\}.$$

Since $f_v g_w$ is the component of fg of degree $v+w$, it follows that $f_v g_w = 0$. This gives the assertion in the lemma. $\square$

Proposition C.0.6. *Let S be a monoid and $T \subseteq S$ a submonoid, such that for every $u \in S$, there is $m \in \mathbb{Z}_{>0}$ such that $mu \in T$. We consider an S-graded ring R such that R_0 is Noetherian. If R is a finitely generated R_0-algebra, then $R|_T$ has the same property. Furthermore, the converse holds if R is a domain and S is finitely generated.*

Proof. Suppose that R is a finitely generated R_0-algebra, with generators $f_1,\ldots,f_N$. We may and will assume that each f_i is nonzero and homogeneous. If m_i is a positive integer such that $m_i \cdot \deg(f_i) \in T$, then $g_i = f_i^{m_i} \in R|_T$. The R_0-algebra $R' = R_0[g_1,\ldots,g_N]$ is finitely generated over R_0, hence it is Noetherian. We have the ring extensions $R' \hookrightarrow R|_T \hookrightarrow R$ and since R is finite over R', it follows that also $R|_T$ is finite over R'. Since R' is a finitely generated R_0-algebra, we deduce that $R|_T$ has the same property.

Conversely, suppose that $R|_T$ is finitely generated over R_0, hence it is Noetherian, and that R is a domain Given $u \in S$, let $M_u := \bigoplus_{w\in T} R_{u+w}$. It is clear that M_u is an $R|_T$-submodule of R and we claim that it is finitely generated. This is trivial if $M_u = 0$. Otherwise, there is $h \in M_u$ nonzero. Let $q \in \mathbb{Z}_{>0}$ be such that $qu \in T$. Since R is a domain, multiplication by h^{q-1} induces an injective $R|_T$-linear map $M_u \hookrightarrow R|_T$.

Since $R|_T$ is Noetherian, we conclude that M_u is a finitely generated $R|_T$-module, as claimed.

Consider now generators $u_1,\ldots,u_r$ of S, and let m be a positive integer such that $mu_i \in T$ for all i. It follows that for every $u \in S$, there are $a_1,\ldots,a_r \in \{0,\ldots,m-1\}$ and $w \in T$ such that $u = a_1u_1 + \ldots + a_ru_r + w$. Therefore $R = \sum_u M_u$, where u varies over the finite set

$$\{a_1u_1 + \ldots + a_ru_r \mid 0 \leq a_i \leq m-1 \text{ for all } i\}.$$

This implies that R is a finitely generated $R|_T$-module. Since $R|_T$ is a finitely generated R_0-algebra, we conclude that R has the same property. $\square$

Remark C.0.7. If in Proposition C.0.6 we drop the assumption that R is a domain, it can happen that $R|_T$ is a finitely generated R_0-algebra, but R does not have this property. Suppose, for example, that R is the following $\mathbb{N}$-graded ring: R_0 is a field, $R_{2m} = 0$ for $m \geq 1$, and $R_{2m-1} = R_0\varepsilon_m$, for $m \geq 1$, with $\varepsilon_i \cdot \varepsilon_j = 0$ for all $i, j \geq 1$. It is clear that $R|_{2\mathbb{N}} = R_0$, but R is not finitely generated as an R_0-algebra.

We also have the following variant of the first assertion in Proposition C.0.6.

Proposition C.0.8. *Let S be a submonoid of a finitely generated, free abelian group M. If $R = \bigoplus_{u\in S} R_u$ is an S-graded ring that is a finitely generated R_0-algebra, with R_0 Noetherian, then for every finitely generated submonoid $T \subseteq S$, the R_0-algebra $R|_T$ is finitely generated.*

Proof. We may replace S by M and, by Proposition C.0.6, T by T^{sat}, hence we may assume that T is saturated in M. Let $y_1,\ldots,y_n$ be generators of R as an R_0-algebra. We may and will assume that each y_i is nonzero, homogeneous, of degree $u_i \in S$. Therefore we have a surjective morphism of R_0-algebras $f\colon A = R_0[x_1,\ldots,x_n] \to R$, with $f(x_i) = y_i$. We also consider the morphism of free abelian groups $\phi\colon \mathbb{Z}^n \to M$ given by $\phi(e_i) = u_i$, where $e_1,\ldots,e_n$ is the standard basis of $\mathbb{Z}^n$. It is clear that if we consider A with the natural $\mathbb{N}^n$-graded ring structure, then $f(A_u) \subseteq R_{\phi(u)}$ for every $u \in \mathbb{N}^n$. If $L = \phi^{-1}(T) \cap \mathbb{N}^n$, then L is a finitely generated monoid by Corollaries A.6.7 and A.6.8, hence $R_0[x_1,\ldots,x_n]|_L$ is a finitely generated R_0-algebra. Therefore its image via f, which is equal to $R|_T$, is a finitely generated R_0-algebra. $\square$

The following proposition is very useful when dealing with finitely generated $\mathbb{N}$-graded rings.

Proposition C.0.9. *If $R = \bigoplus_{m\in\mathbb{N}} R_u$ is an $\mathbb{N}$-graded ring which is a finitely generated R_0-algebra, then there is a positive integer d such that $R' := \bigoplus_{m\in\mathbb{N}} R_{dm}$ is generated as an R_0-algebra in degree 1.*

Proof. Let $y_1,\ldots,y_n$ be generators of R as an R_0-algebra. We may assume that each y_i is nonzero and homogeneous of degree $a_i \geq 1$. We divide $I = \{1,\ldots,n\}$ into subsets $I_1,\ldots,I_r$, such that all a_i in a set I_j are equal to some α_j and the α_j are mutually distinct. We argue by induction on r. If $r = 1$, then we are done by taking

d to be the common value of the positive a_i (if I is empty, then $R = R_0$, and the assertion in the proposition is trivial).

We now prove the induction step. It is enough to show that after possibly replacing R by $R|_{\ell\mathbb{N}}$, for some positive integer ℓ, the value of r goes down. In fact, we will show that $\ell = \mathrm{lcm}(\alpha_j \mid 1 \leq j \leq r)$ has this property.

By assumption, we have a surjective ring homomorphism

$$f\colon A = R_0[x_1, \ldots, x_n] \to R,\ f(x_i) = y_i \quad \text{for all} \quad i.$$

If we consider A to be $\mathbb{N}$-graded, such that $\deg(x_i) = a_i$ for all i, it is clear that $f(A_m) \subseteq R_m$ for all $m \in \mathbb{N}$. Since f is surjective, it follows that it is enough to prove that $A|_{\ell\mathbb{N}}$ is generated as an R_0-algebra by elements of degrees $\ell, 2\ell, \ldots, (r-1)\ell$. Therefore, it is enough to show that if

$$L_m = \{(u_1, \ldots, u_n) \in \mathbb{N}^n \mid a_1u_1 + \ldots + a_nu_n = m\},$$

then every element in $L_{m\ell}$, with $m \geq r$, can be written as the sum of two elements, lying in $L_{(m-1)\ell}$ and L_ℓ, respectively.

Suppose that $u = (u_1, \ldots, u_n) \in L_{m\ell}$, hence

$$\sum_{j=1}^{r} \alpha_j \cdot \sum_{i \in I_j} u_i = m\ell.$$

If there is j such that $\sum_{i \in I_j} u_i \geq \frac{\ell}{\alpha_j}$, then we can write $u = v + w$, with $v \in L_{(m-1)\ell}$ and $w \in L_\ell$ (simply take v with $v_i = u_i$ for $i \in I \smallsetminus I_j$ and $v_i \leq u_i$ for $i \in I_j$ such that $\sum_{i \in I_j} v_i = -\frac{\ell}{\alpha_j} + \sum_{i \in I_j} u_i$). On the other hand, since we assume $m \geq r$, there is always such j: otherwise

$$m\ell = \sum_{j=1}^{r} \alpha_j \cdot \sum_{i \in I_j} u_i < \sum_{j=1}^{r} \alpha_j \cdot \frac{\ell}{\alpha_j} = r\ell.$$

This completes the proof of the proposition. □

Proposition C.0.10. *Let S be a finitely generated submonoid of a finitely generated, free abelian group M, such that S contains no nonzero invertible elements. If $R = \bigoplus_{u \in S}$ is an S-graded domain which is a finitely generated R_0-algebra, with R_0 Noetherian, then for every nonzero $v_1, \ldots, v_m \in S$, the $\mathbb{N}^m$-graded ring*

$$T := \bigoplus_{(a_1,\ldots,a_m) \in \mathbb{N}^m} R_{a_1v_1 + \ldots + a_mv_m}$$

is a finitely generated R_0-algebra.

Proof. Note that the multiplication in R induces a multiplication on T which makes it an $\mathbb{N}^m$-graded ring such that $T_0 = R_0$. After possibly replacing S by S^{sat}, we may assume that S is saturated. It follows from Proposition C.0.8 that it is enough to prove that the $S \times \mathbb{N}^m$-graded ring

$$\widetilde{T} := \bigoplus_{(a,u)\in S\times\mathbb{N}^m} R_{u+a_1v_1+\ldots+u_mv_m}$$

is a finitely generated R_0-algebra. On the other hand, $\widetilde{T}$ is a obtained from R by iterating m times the construction for $m=1$. It follows that arguing by induction on m, it is enough to show that $\widetilde{T}$ is a finitely generated R_0-algebra when $m=1$ (in which case we write $v=v_1$). Note that since R is a domain, we deduce that $\widetilde{T}$ is a domain using Lemma C.0.5.

We first prove the assertion about $\widetilde{T}$ when $S=\mathbb{N}$. By Proposition C.0.9, in this case we can find a positive integer d such that $R|_{d\mathbb{N}}$ is generated by elements $x_1,\ldots,x_r$ of degree d. This implies that $\widetilde{T}|_{d\mathbb{N}\times d\mathbb{N}}$ is generated as an R_0-algebra by $x_1,\ldots,x_r\in\widetilde{T}_{(0,1)}$ and by all monomials of degree v in $x_1,\ldots,x_r$, considered as elements of $\widetilde{T}_{(1,0)}$. Since we know that $\widetilde{T}$ is a domain, this implies that $\widetilde{T}$ is finitely generated by Proposition C.0.6.

In the general setting, we use Remark C.0.2 to reduce to the $\mathbb{N}$-graded case. Let $\ell\colon M\to\mathbb{Z}$ be a group homomorphism such that $\ell(u)>0$ for all nonzero $u\in S$. We may consider R to be $\mathbb{N}$-graded by writing is as $R=\bigoplus_{i\in\mathbb{N}}R_{(i)}$, where

$$R_{(i)} = \bigoplus_{u\in S,\ell(u)=i} R_u.$$

By applying what we have already proved to this $\mathbb{N}$-graded ring and to $\ell(v)$, we conclude that the $\mathbb{N}^2$-graded ring

$$T' = \bigoplus_{(i,a)\in\mathbb{N}^2} R_{(i+a\ell(v))}, \text{ where } R_{(i+a\ell(v)} = \bigoplus_{u\in S,\ell(u)=i+a\ell(v)} R_u,$$

is a finitely generated R_0-algebra. On the other hand, we may consider T' as an S'-graded ring, where

$$S' = \{(i,a,u)\in\mathbb{N}\times\mathbb{N}\times S \mid \ell(u)=i+a\ell(v)\}.$$

It is easy to see that S' is finitely generated: note that S' is the intersection of the saturated submonoid $\mathbb{N}^2\times S$ of $\mathbb{Z}^2\times M$ with a linear subspace and the assertion follows from Proposition A.6.1. Moreover, we may consider $\mathbb{N}\times S$ as a submonoid of S' by the injective map that takes (a,u) to $(\ell(u),a,u+av)$. Since $\widetilde{T}$ is the restriction of T to this submonoid, we may apply Proposition C.0.8 to conclude that $\widetilde{T}$ is finitely generated. $\square$

Appendix D
Integral closure of ideals

In order to discuss the consequences of the finite generation of the section ring of several line bundles, we need some preparations regarding the integral closure of ideals. In introducing this concept we follow the geometric approach from [Laz04b, Chapter 9.6.A]. Let X be a normal variety[1]. Given a nonzero coherent ideal $\mathfrak{a}$ on X, consider a proper birational morphism $f\colon Y \to X$, with Y normal and such that $\mathfrak{a}\colon \mathscr{O}_Y = \mathscr{O}_Y(-F)$ for an effective Cartier divisor F on Y (for example, we could take Y to be the normalization of the blow-up of X along $\mathfrak{a}$). With this notation, the *integral closure* $\overline{\mathfrak{a}}$ of $\mathfrak{a}$ is given by $f_*(\mathscr{O}_Y(-F))$. Note that since X is normal, we have $f_*(\mathscr{O}_Y(-F)) \subseteq \phi_*(\mathscr{O}_Y) = \mathscr{O}_X$, hence $\overline{\mathfrak{a}}$ is a coherent ideal sheaf on X, which clearly contains $\mathfrak{a}$.

Lemma D.0.1. *The definition of $\overline{\mathfrak{a}}$ is independent of the choice of the morphism f.*

Proof. Since any two such morphisms are dominated by a third one, it is enough to consider another proper birational morphism $g\colon Z \to Y$, with Z normal, and show that $f_*(\mathscr{O}_Y(F)) = (f \circ g)_*(-g^*(F))$. Note that since Y is normal we have $g_*(\mathscr{O}_Z) \simeq \mathscr{O}_Y$ and the projection formula gives

$$g_*(\mathscr{O}_Y(-g^*(F)) \simeq \mathscr{O}_Y(-F) \otimes g_*(\mathscr{O}_Z) \simeq \mathscr{P}_Y(-F).$$

$\square$

Corollary D.0.2. *If X is a normal, affine variety and $\mathfrak{a}$ is a nonzero ideal on X, then*

$$\Gamma(X, \overline{\mathfrak{a}}) = \{\phi \in \mathscr{O}_X(U) \mid \mathrm{ord}_E(\phi) \geq \mathrm{ord}_E(\mathfrak{a}) \textit{for all divisors} E \textit{ over} X\}.$$

Proof. With the notation in the definition, it is clear that $\Gamma(X, \overline{\mathfrak{a}})$ is equal to the set of those $\phi \in \mathscr{O}_X(U)$ such that $\mathrm{ord}_E(\phi) \geq \mathrm{ord}_E(\mathfrak{a})$ for all prime divisors E on Y. SInce the definition is independent of the choice of f, we obtain the description in the corollary. $\square$

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[1] In fact, in this subsection we do not need to work over a ground field; everything that follows holds without any change if X is a normal, integral, Noetherian scheme.

For $\mathfrak{a}=0$, we put $\overline{\mathfrak{a}}=0$. One says that an ideal $\mathfrak{a}$ is integrally closed if $\overline{\mathfrak{a}}=\mathfrak{a}$. We collect in the next proposition some basic properties of integral closure.

Proposition D.0.3. *Let X be a normal variety and $\mathfrak{a},\mathfrak{b}$ coherent ideals on X.*

i) We have $\mathrm{ord}_E(\mathfrak{a})=\mathrm{ord}_E(\overline{\mathfrak{a}})$ for every divisor E over X.
ii) The ideal $\overline{fa}$ is integrally closed.
iii) We have $\overline{\mathfrak{a}}\subseteq\overline{\mathfrak{b}}$ if and only if $\mathrm{ord}_E(\mathfrak{a})\geq\mathrm{ord}_E(\mathfrak{b})$ for every divisor E over X. In particular, we have $\overline{\mathfrak{a}}=\overline{\mathfrak{b}}$ if and only if $\mathrm{ord}_E(\mathfrak{a})=\mathrm{ord}_E(\mathfrak{b})$ for every divisor E over X.
iv) If $\mathfrak{a}\subseteq\mathfrak{b}$, then $\overline{\mathfrak{a}}\subseteq\overline{\mathfrak{b}}$.

Proof. All assertions are local, hence we may assume that X is affine. Let E be a divisor over X. Since $\mathfrak{a}\subseteq\overline{\mathfrak{a}}$, we have $\mathrm{ord}_E(\mathfrak{a})\geq\mathrm{ord}_E(\overline{\mathfrak{a}})$. Since the reverse inclusion follows from Corollary D.0.2, this proves i). The assertion in ii) now follows from i) and the description of integral closure in Corollary D.0.2.

If $\overline{\mathfrak{a}}\subseteq\overline{\mathfrak{b}}$, then i) implies

$$\mathrm{ord}_E(\mathfrak{a})=\mathrm{ord}_E(\overline{\mathfrak{a}})\geq\mathrm{ord}_E(\overline{\mathfrak{b}})=\mathrm{ord}_E(\mathfrak{b})$$

for every divisor E over E. Conversely, if $\mathrm{ord}_E(\mathfrak{a})\geq\mathrm{ord}_E(\mathfrak{b})$ for every divisor E over X, then Corollary D.0.2 implies $\overline{\mathfrak{a}}\subseteq\overline{\mathfrak{b}}$. We thus have iii). The remaining assertions are immediate consequences. □

Corollary D.0.4. *Let X be a normal variety. If $\mathfrak{a}$ and $\mathfrak{b}$ are coherent ideals on X such that one of the following conditions holds:*

i) $\mathfrak{a}^m\subseteq\overline{\mathfrak{b}}^m$ for some $m\geq 1$.
ii)

$$\mathfrak{a}^m\subseteq\mathfrak{c}\cdot\mathfrak{b}^m$$

for some nonzero ideal $\mathfrak{c}$ and all $m\gg 0$,

, then $\overline{\mathfrak{a}}\subseteq\overline{\mathfrak{b}}$.

Proof. Let E be a divisor over X. In case i), we have by Proposition D.0.3

$$\mathrm{ord}_E(\mathfrak{a})=\frac{1}{m}\mathrm{ord}_E(\mathfrak{a}^m)\geq\frac{1}{m}\mathrm{ord}_E(\overline{b}^m)=\mathrm{ord}_E(\overline{\mathfrak{b}})=\mathrm{ord}_E(\mathfrak{b}).$$

In case ii), for every $m\geq 1$, we have

$$\mathrm{ord}_E(\mathfrak{c})+m\cdot\mathrm{ord}_E(\mathfrak{b})=\mathrm{ord}_E(\mathfrak{c}\cdot\mathfrak{b}^m)\leq\mathrm{ord}_E(\mathfrak{a}^m)=m\cdot\mathrm{ord}_E(\mathfrak{a}). \qquad \text{(D.1)}$$

Since $\mathrm{ord}_E(\mathfrak{c})$ is finite, dividing (D.1)i) by m and letting m go to infinity gives $\mathrm{ord}_E(\mathfrak{b})\leq\mathrm{ord}_E(\mathfrak{a})$. We thus conclude that in both cases we have $\mathrm{ord}_E(\mathfrak{b})\leq\mathrm{ord}_E(\mathfrak{a})$ for all divisors E over X and Proposition D.0.3iii) implies $\overline{\mathfrak{a}}\subseteq\overline{\mathfrak{b}}$. □

Corollary D.0.5. *Let X be a normal variety. For every coherent ideals $\mathfrak{a}$ and $\mathfrak{b}$ on X, we have*

$$\overline{\mathfrak{a}\cdot\mathfrak{b}}=\overline{\overline{\mathfrak{a}}\cdot\overline{\mathfrak{b}}}.$$

Proof. If E is a divisor over X, then using Proposition D.0.3i) we obtain

$$\mathrm{ord}_E(\mathfrak{a}\cdot\mathfrak{b}) = \mathrm{ord}_E(\mathfrak{a}) + \mathrm{ord}_E(\mathfrak{b}) = \mathrm{ord}_E(\overline{\mathfrak{a}}) + \mathrm{ord}_E(\overline{\mathfrak{b}}) = \mathrm{ord}_E(\overline{\mathfrak{a}}\cdot \mathrm{ord}_E(\mathfrak{b})).$$

Since $\mathfrak{a}\cdot\mathfrak{b}$ and $\overline{\mathfrak{a}}\cdot\overline{\mathfrak{b}}$ have the same order of vanishing along every E, the two ideals have the same integral closure by Proposition D.0.3iii). □

The following proposition gives another description for the integral closure of an ideal which explains its name. This is usually taken as the definition in the algebraic approach to this concept.

Proposition D.0.6. *Let $X = \mathrm{Spec}(R)$ be a normal, affine variety and $\mathfrak{a}$ an ideal in R. Given $f \in R$, we have $f \in \overline{\mathfrak{a}}$ if and only if f satisfies an equation of the form*

$$f^n + \alpha_1 f^{n-1} + \ldots + \alpha_n = 0$$

where $\alpha_i \in \mathfrak{a}^i$ for $1 \le i \le n$.

We first prove the following lemma.

Lemma D.0.7. *If X is a normal variety and $\mathfrak{a}$ is a coherent ideal on X, then for every $m \gg 0$ we have $\overline{\mathfrak{a}}^{m+1} = \mathfrak{a}\cdot\overline{\mathfrak{a}^m} = \overline{\mathfrak{a}}\cdot\overline{\mathfrak{a}^m}$.*

Proof. The equalities hold trivially if $\mathfrak{a} = 0$, hence from now on we assume that $\mathfrak{a} \neq 0$. Let $f\colon Y \to X$ be the normalization of the blow-up of X along $\mathfrak{a}$, with $\mathfrak{a}\cdot\mathscr{O}_Y = \mathscr{O}_Y(-F)$. Note that $\mathscr{O}_Y(-F)$ is f-ample. Note that we clearly have $\mathscr{O}_Y(-F)\cdot\mathscr{O}_Y(-mF) \subseteq \mathscr{O}_Y(-(m+1)F)$, hence $\overline{\mathfrak{a}}\cdot\overline{\mathfrak{a}^m} \subseteq \overline{\mathfrak{a}^{m+1}}$ for every m. Since we know that $\mathfrak{a} \subseteq \overline{\mathfrak{a}}$, in order to complete the proof of the lemma, it is enough to show that $\overline{\mathfrak{a}}^{m+1} \subseteq \mathfrak{a}\cdot\overline{\mathfrak{a}^m}$ for $m \gg 0$. By considering a finite affine open cover of X, we see that we may assume that X is affine. Let $f_1,\ldots,f_r$ be generators of $\mathfrak{a}$, hence we have a surjective morphism $\mathscr{O}_X^{\oplus r} \to \mathfrak{a}$. This induces a surjective morphism $\mathscr{O}_Y^{\oplus r} \to \mathscr{O}_Y(-F)$ on Y. Since $\mathscr{O}_Y(-F)$ is f-ample, we see that after tensoring this morphism with $\mathscr{O}_Y(-mF)$, for $m \gg 0$, and applying f_*, the induced morphism is again a surjection. This means that $f_*\mathscr{O}_Y(-(m+1)F) \subseteq \mathfrak{a}\cdot f_*\mathscr{O}_Y(-mF)$, hence $\overline{\mathfrak{a}}^{m+1} \subseteq \mathfrak{a}\cdot\overline{\mathfrak{a}^m}$ for $m \gg 0$. □

Proof of Proposition D.0.6. We may assume that $\mathfrak{a}$ is nonzero, since otherwise the assertion is clear. Suppose first that $f \in \overline{\mathfrak{a}}$. It follows from Lemma D.0.7 that there is $m > 0$ such that $f\cdot\overline{\mathfrak{a}^m} \subseteq \mathfrak{a}\cdot\overline{\mathfrak{a}^m}$. We now use the "determinant trick": if $u_1,\ldots,u_n$ are generators of $\overline{\mathfrak{a}^m}$ and we write

$$f\cdot u_i = \sum_{j=1}^n b_{i,j}u_j \text{ for } 1 \le i \le n, \text{ with } b_{i,j} \in \mathfrak{a},$$

then $\det(fI_n - B) \in \mathrm{Ann}(\overline{\mathfrak{a}^m}) = 0$, where $B = (b_{i,j})_{1\le i,j\le n}$. By expanding the determinant, we see that

$$f^n + \alpha_1 f^{n-1} + \ldots + \alpha_n = 0 \tag{D.2}$$

for suitable $\alpha_i \in \mathfrak{a}^i$.

Conversely, suppose that f satisfies (D.2). If E is a divisor over X, since ord_E is a valuation, we deduce that $\mathrm{ord}_E(f^n) \geq \mathrm{ord}_E(\alpha_i f^{n-i})$ for some i, with $1 \leq i \leq n$. Therefore

$$i \cdot \mathrm{ord}_E(f) \geq \mathrm{ord}_E(\alpha_i) \geq \mathrm{ord}_E(\mathfrak{a}^i) = i \cdot \mathrm{ord}_E(\mathfrak{a}).$$

Since $\mathrm{ord}_E(f) \geq \mathrm{ord}_E(\mathfrak{a})$ for all divisors E over X, we conclude that $f \in \overline{\mathfrak{a}}$ by Proposition D.0.3iii). $\square$

Corollary D.0.8. *If X is a normal variety and $\mathfrak{a}$ is a coherent ideal on X, then the normalization of* $\mathrm{Spec}\left(\bigoplus_{m\geq 0}\mathfrak{a}^m\right)$ *is* $\mathrm{Spec}\left(\bigoplus_{m\geq 0}\overline{\mathfrak{a}^m}\right)$.

Proof. The assertion is clear if $\mathfrak{a} = 0$, hence we may and will assume that $\mathfrak{a}$ is nonzero. By considering a finite affine open cover of X, we see that it is enough to prove the corollary when $X = \mathrm{Spec}(R)$ is affine. Let $f \colon Y \to X$ be the normalization of the blow-up of X along $\mathfrak{a}$, with $\mathfrak{a}\cdot\mathscr{O}_Z = \mathscr{O}_Z(-F)$. In general, if Z is a projective normal scheme over an affine scheme and $\mathscr{L}$ is an ample line bundle on Z, then it is well-known that the ring $\oplus_{m\geq 0}\Gamma(Z,\mathscr{L}^m)$ is normal. Applying this with $Z = Y$ and $\mathscr{L} = \mathscr{O}_Y(-F)$, we obtain that the ring $\bigoplus_{m\geq 0}\overline{\mathfrak{a}^m}$ is integrally closed.

On the other hand, the ring extension

$$R_1 = \bigoplus_{m\geq 0}\mathfrak{a}^m \hookrightarrow R_2 = \bigoplus_{m\geq 0}\overline{\mathfrak{a}^m}$$

is integral. Indeed, if $f \in \overline{\mathfrak{a}^m}$, then Proposition D.0.6 implies that f satisfies an equation

$$f^n + \sum_{i=1}^{n}\alpha_i f^{n-i} = 0,$$

where $\alpha_i \in \mathfrak{a}^{I'm}$ for $1 \leq i \leq n$. This implies that f, considered as a homogeneous element of degree m of R_2 is integral over R_1. We thus conclude that R_2 is the normalization of R_1. $\square$

Corollary D.0.9. *If $\mathfrak{a}$ is an ideal on X, then the normalizations of the blow-ups of X along $\mathfrak{a}$ and $\overline{\mathfrak{a}}$ are canonically isomorphic.*

Proof. The assertion is an immediate consequence of Corollary D.0.8 and of the fact that $\overline{\overline{\mathfrak{a}}^n} = \overline{\mathfrak{a}^n}$ for every n (this equality can be easily deduced from Corollary D.0.2). $\square$

Appendix E
Constructible sets

In this section we review some basic facts about constructible sets. Recall that if X is a Noetherian scheme, a subset $A \subseteq X$ is constructible if it can be written as a finite union of locally closed subsets of X. It is easy to check that in fact, the union can be taken to be disjoint. Furthermore, any constructible subset W of X contains an open dense subset of $\overline{W}$ (see [Har77, Exercise II.3.18]). It is clear from definition that the constructible subsets of X form an algebra of subsets, that is, any finite union or intersection of constructible subsets, as well as the difference of two constructible subsets, are again constructible.

Suppose that X is as above and A is a subset of X. If Z is a subset of A, closed with respect to the induced topology, then $\overline{Z} \cap A = Z$. Therefore any chain of closed subsets $Z_1 \subsetneq Z_2 \subsetneq \ldots \subsetneq Z_r$ of A induces a corresponding chain of closed subsets $\overline{Z_1} \subsetneq \overline{Z_2} \subsetneq \ldots \subsetneq \overline{Z_r}$. Since X is Noetherian, it follows that A is Noetherian, and $\dim(A) \leq \dim(X)$.

If X is a Noetherian scheme and W is a constructible subset of X, then W is a Noetherian topological space, hence it has a decomposition into irreducible components $W = W_1 \cup \ldots \cup W_n$. Since each $\overline{W_i}$ is closed and irreducible in X and $\overline{W_i} \not\subseteq \overline{W_j}$ for $i \neq j$ (otherwise we would get $W_i = \overline{W_i} \cap W \subseteq \overline{W_j} \cap W = W_j$), it follows that $\overline{W} = \overline{W_1} \cup \ldots \cup \overline{W_n}$ is the irreducible decomposition of $\overline{W}$.

It is clear from definition that if $f\colon X \to Y$ is a morphism of Noetherian schemes, then $f^{-1}(B) \subseteq X$ is constructible if $B \subseteq Y$ is constructible. The importance of the concept of constructible sets comes from the following theorem of Chevalley: if f is, in addition, of finite type, then $f(A) \subseteq Y$ is contractible for every constructible subset A of X.

Suppose now that X is a scheme of finite type over a field k. In particular, every locally closed subset of X has finite dimension. We first note that if A is a constructible subset of X, then $\dim(A) = \dim(\overline{A})$. Indeed, the inequality "$\leq$" holds since A is a subspace of $\overline{A}$, while the reverse inequality follows from the fact that A contains a dense open subset U of $\overline{A}$, hence $\dim(A) \geq \dim(U) = \dim(\overline{A})$. This implies that if we have $A = A_1 \cup \ldots \cup A_m$, with each A_i constructible in X, then $\dim(A) = \max_i \dim(A_i)$, since this holds after taking closures.

Let X be a scheme of finite type over a field k and A a constructible subset of X, with $\dim(A) = d$. Suppose that we have a disjoint decomposition $A = A_1 \sqcup \ldots \sqcup A_r$, with each A_i locally closed (or, more generally, constructible). This induces a decomposition $\overline{A} = \overline{A_1} \cup \ldots \cup \overline{A_r}$. If Z is an irreducible component of some A_i and $Z' \neq Z$ is an irreducible component of some A_j, then $\overline{Z} \cap A \neq \overline{Z'} \cap A$. This implies that if $\dim(Z) = d$, then $\overline{Z} \cap A$ is an irreducible component of A of dimension d. Furthermore, it is clear that every irreducible component of A of dimension d is of this form, for a unique i and a unique irreducible component Z of A_i.

Suppose now that $f\colon X \to Y$ is a morphism of schemes of finite type over k and $A \subseteq X$ and $B \subseteq Y$ are constructible subsets such that f induces a piecewise trivial vibration $g\colon A \to B$, with fiber F. By definition, there is a disjoint decomposition $B = B_1 \sqcup \ldots \sqcup B_r$ such that each B_i is locally closed in Y, each $A_i := g^{-1}(B_i)$ is locally closed in X, and we have an isomorphism $A_i \simeq B_i \times F$ for every i (where on A_i and B_i we consider the reduced scheme structures). Since $\dim(A_i) = \dim(B_i) + \dim(F)$ for every i, we conclude that $\dim(A) = \dim(B) + \dim(F)$. Furthermore, if k is algebraically closed and F is irreducible, we see that A and B have the same number of irreducible components of maximal dimension.

Proposition E.0.1. *If X is a scheme of finite type over an uncountable field k, then for every descending sequence $A_1 \supseteq A_2 \supseteq \ldots$ of nonempty constructible subsets of X, we have $\cap_{m \geq 1} A_m \neq \emptyset$.*

Proof. $\square$

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