From Ramsey Theory to arithmetic progressions and hypergraphs

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Of three ordinary people, two must have the same sex

Ramsey Theory – total disorder is impossible

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Is it true for five people?

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$R(3, 3) = 6$

$R(4, 4) = 18$

$$43 \le R(5,5) \le 49$$

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How many possible situations with 49 people?

$$2^{\binom{49}{2}} = 2^{1176}$$

Ramsey's Theorem (finite case)

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No major improvements since the 1940's

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$$3 \qquad 7 \qquad 11 \qquad 15$$

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What if we want an AP of length 3?

9 numbers suffice but 8 do not!

1 2 3 4 5 6 7 8

What if we want an AP of length p?

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Van-der-Waerden's Theorem (1927)

$W(p)$ is finite for every p.

How big is $W(p)$?

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$$W(p) < ???$$

Big Functions

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EXP is obtained by applying DOUBLE x times starting at 1:

$$2^x = f_2(x) = 2 \cdot 2 \cdot 2 \cdots 2 \cdot 1 = f_1(f_1(f_1(\cdots f_1(f_1(1)))))$$

where we iterate x times.

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Shelah's Theorem

$$W(p) < f_4(5p)$$

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Lower Bound

$$W(p) > 2^p$$

Density Results

Erdős-Turán Conjecture

Fix $k \geq 2$ and $\epsilon > 0$. Then for n sufficiently large, every subset S of $\{1, 2, \ldots, n\}$ with $|S| > \epsilon n$ contains a k-term AP.

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How large is "sufficiently large" ??

Higher Dimensional Szemerédi Theorem

Multidimensional Szemerédi Theorem (Furstenberg-Katznelson)

For every $\epsilon > 0$, every positive integer r and every finite subset $X \subset \mathbb{Z}^r$ there is a positive integer n such that every subset S of the grid $\{1, 2, \ldots, n\}^r$ with $|S| > \epsilon n^r$ has a subset of the form $\vec{a} + dX$ for some positive integer d.

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The Furstenberg-Katznelson proof gave no actual bound on n.

Hypergraphs

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A k-simplex is the k-uniform hypergraph on $[k+1]$ which consists of all possible k-element sets (there are $\binom{k+1}{k} = k+1$ of them).

Graph Removal Lemmas

Question

Suppose we have a graph (= 2-uniform hypergraph) with few triangles (= 2-simplices). Can we delete few edges so that after removing the edges, there are no triangles?

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Rusza-Szemerédi (6,3) theorem

For every $a > 0$ there exists $c > 0$ with the following property. If G is any graph with n vertices and at most cn^3 triangles, then it is possible to remove at most an^2 edges from G to make it triangle-free.

Hypergraph Removal Lemma

Theorem (Frankl-Rödl, Rödl-Schacht, Gowers)

For every $a > 0$ there exists $c > 0$ with the following property. If H is any k-uniform hypergraph with n vertices and at most cn^{k+1} k-simplices, then it is possible to remove at most an^k edges from H to make it k-simplex-free.

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A corollary to the removal lemma above is that we get an effective bound for n in the Furstenberg-Katznelson theorem.