1. $y'' + xy' + 2y = 0$, $x_0 = 0$.

Represent solution as series $y(x) = \sum_{k=0}^{\infty} a_k x^k$

$$\left.\begin{aligned} 2y &= \sum_{k=0}^{\infty} 2a_k x^k \\ xy' &= \sum_{k=0}^{\infty} a_k k x^k \\ y'' &= \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k \end{aligned}\right] \Rightarrow \sum_{k=0}^{\infty}\left[a_{k+2}(k+2)(k+1) + a_k(k+2)\right]x^k = 0$$

$$a_{k+2} = -\frac{a_k}{k+1}$$

a_0, a_1 arbitrary, $a_2 = -a_0$, $a_3 = -\frac{a_1}{2}$, $a_4 = \frac{a_0}{3}$

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2. $xy'' + (1-x)y' - y = 0$

$p_0 = \lim_{x \to 0} x\frac{Q(x)}{P(x)} = \lim_{x \to 0}(1-x) = 1.$

$q_0 = \lim_{x \to 0} x^2\frac{R(x)}{P(x)} = 0.$

Indicial equation: $x^2y'' + xy' = 0$

Assume $y = x^r$, then $[r(r-1)+r]x^r = 0$ $\quad r^2 = 0$, $r = 0$.

One of the two solutions: $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r} = \sum_{k=0}^{\infty} a_k x^k$

$(1-x)y' = \sum_{k=0}^{\infty} a_k k(x^{k-1} - x^k)$

$xy'' = \sum_{k=0}^{\infty} a_k(k-1)kx^{k-1}$. Thus, $\sum_{k=0}^{\infty} a_k k^2 x^{k-1} - \sum_{k=0}^{\infty} a_k(k+1)x^k = 0$

$$\sum_{k=0}^{\infty} a_{k+1}(k+1)^2x^k - \sum_{k=0}^{\infty} a_k(k+1)x^k = 0 \Rightarrow a_{k+1} = \frac{a_k}{k+1} \Rightarrow a_k = \frac{a_0}{k!}$$

Therefore, $y(x) = a_0 \sum_{k=0}^{\infty} \frac{x^k}{k!} = a_0 e^x$.

③ $\dot{x} = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} x$

Assume solution in the form $\vec{x}(t) = \vec{\xi} e^{\lambda t}$

For λ we have $\det\begin{pmatrix} 1-\lambda & -2 \\ 3 & -(4+\lambda) \end{pmatrix} = 0 \Rightarrow \lambda^2 + 3\lambda + 2 = 0$

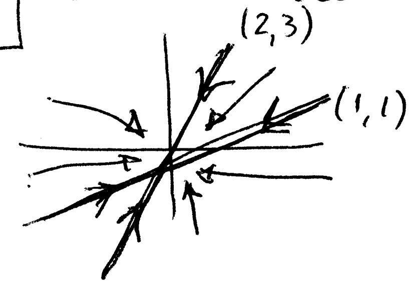
$\lambda_1 = -1, \ \lambda_2 = -2$

$\vec{\xi}_1 : \begin{pmatrix} 1+1 & -2 \\ 3 & -(4-1) \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ 3 & -3 \end{pmatrix} \Rightarrow \vec{\xi}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\vec{\xi}_2 : \begin{pmatrix} 1+2 & -2 \\ 3 & -(4-2) \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 3 & -2 \end{pmatrix} \Rightarrow \vec{\xi}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$\Rightarrow \vec{x}(t) = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{-2t}$

Stable node



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④ $\dot{x} = -x + 2xy$
$\dot{y} = y - (x^2 + y^2)$

Four fixed points: $(0,0), (0,1), (\pm\frac{1}{2}, \frac{1}{2})$

$J(0,0) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ – sadde node, linearly/nonlinearly unstable

$J(0,1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ – saddle node, linearly/nonlinearly unstable

$J(\frac{1}{2}, \frac{1}{2}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ – center, linearly stable, nonlinearly undetermined

$J(-\frac{1}{2}, \frac{1}{2}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ – center, linearly stable, nonlinearly undetermined

⑤ $x^2y'' - xy' + y = 0, \quad y(1) = 0, \; y(2) = 1$

Assume $y = x^r$, then

$$[r(r-1) - r + 1]x^r = 0, \quad r^2 - 2r + 1 = 0, \quad r = 1$$

$$y = (C_1 + C_2 \ln x)x. \quad y(1) = 0 \Rightarrow C_1 = 0.$$

$$y(2) = 1 \Rightarrow C_2 = 1/\ln 4$$

$$y(x) = \frac{1}{\ln 4} x \ln x$$

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⑥ $f(x) = 1 - |x|$ – even function. $-1 < x < 1$

Fourier series: $f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos k\pi x$

$$a_0 = 2\int_0^1 (1-x)dx = 1$$

$$a_n = 2\int_0^1 (1-x)\cos n\pi x \, dx = \frac{2}{n\pi}\sin n\pi x \Big|_0^1 - 2\frac{\partial}{\partial(n\pi)}\int_0^1 \sin n\pi x \, dx =$$

$$= 2\frac{\partial}{\partial(n\pi)}\left[\frac{1}{n\pi}\cos n\pi x\right]_0^1 = 2\frac{\partial}{\partial(n\pi)}\left[\frac{\cos n\pi - 1}{n\pi}\right] =$$

$$= 2\left[\frac{-n\pi \sin n\pi - \cos n\pi + 1}{n^2\pi^2}\right] = \frac{2}{n^2\pi^2}\left[(-1)^{n+1} + 1\right] = \Big/ n = 2k+1 \Big/ =$$

$$= \frac{4}{(2k+1)^2\pi^2} = a_{2k+1}$$

Then, $f(x) = \frac{1}{2} + \sum_{k=0}^{\infty} \frac{4}{(2k+1)^2\pi^2}\cos(2k+1)\pi x.$

⑦ $u_t = u_{xx} + \sin\pi x \cdot e^{-t}, \quad u(t,0) = \overbrace{\cancel{0}}^{\sin 2\pi x,}, \ u(t,1)=0, \ u(t,0)=0.$

Eigenfunctions: $u_t = u_{xx}, \quad X'' = -\omega^2 X$

$$X_k = C_1 \sin\omega_k x + C_2 \cos\omega_k x$$

Apply boundary conditions, get $X_k = \sin k\pi x$, $\omega = k\pi$

$u(t,x) = \sum_{k=1}^{\infty} C_k(t)\sin k\pi x$, substitute in the equation

$$\sum_{k=1}^{\infty} C_k'(t)\sin k\pi x = -\sum_{k=1}^{\infty} C_k(t)\,\omega_k^2 \sin k\pi x + \sin\pi x \cdot e^{-t}$$

$k=1$: $C_1'(t) = -C_1(t)\pi^2 + e^{-t}$, $C_1(0)=0$, $C_1(t) = \frac{1}{\pi^2-1}\left[e^{-t} - e^{-\pi^2 t}\right]$

Check: $C_1(t) = \int_0^t e^{-\pi^2(t-\tau)} e^{-\tau}\,d\tau = \frac{e^{-\pi^2 t}}{\pi^2-1} e^{(\pi^2-1)\tau}\Big|_0^t = \frac{1}{\pi^2-1}\left[e^{-t} - e^{-\pi^2 t}\right]$

$k=2$: $C_2'(t) = -C_2(t)\,4\pi^2$, $C_2(0) = 1 \Rightarrow C_2(t) = e^{-4\pi^2 t}$

$k>2$: $C_k'(t) = -C_k(t)\,k^2\pi^2$, $C_k(0) = 0 \Rightarrow C_k(t) = 0$.

Solution: $u(t,x) = \frac{1}{\pi^2-1}\left[e^{-t} - e^{-\pi^2 t}\right]\sin\pi x + e^{-4\pi^2 t}\sin 2\pi x$