Linear response for slow variables of deterministic or stochastic dynamics with time scale separation

Rafail V. Abramov *

Department of Mathematics, Statistics and Computer Science
University of Illinois at Chicago

Abstract

Many real-world numerical models are notorious for the time-scale separation of different subsets of variables and the inclusion of random processes. The existing algorithms of linear response to external forcing are vulnerable to the time-scale separation due to increased response errors at fast scales. Here we develop the linear response algorithm for slow variables in a multiscale deterministic or stochastic dynamical system, which has improved numerical stability and reduced computational expense.

1 Introduction

Recently, Majda and the author [1–4] developed and tested a novel computational algorithm for predicting the mean response of nonlinear functions of states of a chaotic dynamical system to small change in external forcing via the Fluctuation-Dissipation Theorem (FDT). This geometric algorithm (also called the short-time FDT (ST-FDT) algorithm in [2–4]) takes into account the fact that the dynamics of chaotic nonlinear forced-dissipative systems often reside on chaotic fractal attractors, where the classical quasi-Gaussian formula of the fluctuation-dissipation theorem often fails to produce satisfactory response prediction, especially in dynamical regimes with weak and moderate chaos and slower mixing. It has been discovered that the ST-FDT algorithm is an extremely precise response approximation for short response times, and can be blended with the classical quasi-Gaussian FDT algorithm (qG-FDT) for longer response

* Corresponding author's address: 851 S. Morgan st, Chicago, IL 60607, USA
Email address: `abramov@math.uic.edu` (Rafail V. Abramov).

times to alleviate negative effects of expanding Lyapunov directions. Additionally, in [1] the author developed a computationally inexpensive method for ST-FDT using the reduced-rank tangent map.

However, in multiscale dynamical systems with time scale separation the FDT methods can be vulnerable to the presence of the fast variables, especially when the response needs to be computed only for slow model variables (such as those in a climate system), due to increased response errors at fast scales. Moreover, it is often the case that there are only a few slow variables in the model and a large number of fast variables. Even if only the response of the slow variables is needed, the ST-FDT response operator has to be computed for all variables in the model, which can be computationally expensive (or even practically impossible for models with large sets of fast variables).

In the work, we develop a new response algorithm based on averaged dynamics of multiscale ODEs and SDEs [10, 12, 13]. The new method allows to compute the response operators directly at slow scales using existing FDT formulas, without involving fast scales at all, which improves numerical stability and reduces computational expense. In addition, the new approach allows to use the geometric response formula for slow variables of multiscale models with stochastically driven fast variables.

2 Averaged systems

Consider a dynamical system of the form

$$\begin{aligned} \dot{x} &= f(x,t) + h(y,t,t/\varepsilon), \\ \dot{y} &= \frac{1}{\varepsilon} g(y,t,t/\varepsilon) + q(x,y,t,t/\varepsilon), \end{aligned} \tag{1}$$

where $x = x(t) \in \mathbb{R}^{N_x}$, $y = y(t) \in \mathbb{R}^{N_y}$, f is a smooth nonlinear function, while the nonlinear functions g, h and q may optionally have stochastic terms. The constant parameter $0 < \varepsilon \ll 1$ sets the time scale separation between $x(t)$ and $y(t)$ into slow and fast variables, respectively. Observe that the explicit time dependence in (1) is treated as if there are two time scales – the slow time scale t and the fast time scale t/ε. This explicit time scale separation reflects the physical forcing originating from different sources; for instance, in the weather/climate dynamics this separate time-scale dependence could be drawn between the seasonal changes, as the axis of the Earth inclination changes its orientation relative to the Sun because of its motion around the Sun, and the daily changes due to the Earth's own rotation.

Next, we rescale the time in (1) as $t = \varepsilon\tilde{t}$. For the rescaled time $\tilde{t}$, the dynamical system in (1) becomes

$$\begin{aligned} \dot{x} &= \varepsilon f(x,t) + \varepsilon h(y,t,\tilde{t}), \\ \dot{y} &= g(y,t,\tilde{t}) + \varepsilon q(x,y,t,\tilde{t}). \end{aligned} \tag{2}$$

Above, as the parameter $\varepsilon \to 0$, the original unrescaled time t can be approximately treated as the constant parameter relative to $\tilde{t}$. Following [10, 12, 13], we write the *averaged* system of equations for (2) as

$$\begin{aligned} \dot{\bar{x}} &= \varepsilon f(\bar{x},t) + \varepsilon \bar{h}(t), \\ \dot{\bar{y}}_t &= g(\bar{y}_t,t,\tilde{t}), \end{aligned} \tag{3}$$

where the term $\bar{h}(t)$ is given by

$$\bar{h}(t) = \lim_{s\to\infty} \frac{1}{s} \int_0^s h(\bar{y}_t(\tilde{t}),t,\tilde{t})\mathrm{d}\tilde{t}. \tag{4}$$

For $\bar{h}(t)$ we further assume that it does not contain stochastic terms, as the stochastic terms in h from (1) have been eliminated due to time averaging. Rescaling the time back, for the slow variables we obtain

$$\dot{\bar{x}} = f(\bar{x},t) + \bar{h}(t). \tag{5}$$

For identical initial conditions and starting time t_0, and the time interval $t \sim 1$, the solution of (5) constitutes an approximation to the solution of the first equation in (1) [10, 12, 13].

At this point, consider the system in (1) perturbed at slow variables by a small deterministic forcing $w(x)\delta f(t)$:

$$\begin{aligned} \dot{x} &= f(x,t) + h(y,t,t/\varepsilon) + w(x)\delta f(t), \\ \dot{y} &= \frac{1}{\varepsilon} g(y,t,t/\varepsilon) + q(x,y,t,t/\varepsilon), \end{aligned} \tag{6}$$

where $w : \mathbb{R}^{N_x} \to [\mathbb{R}^{N_x} \times \mathbb{R}^K]$ is a matrix-valued function of x, while $f : T \to \mathbb{R}^K$ is a vector-valued function of time t for some integer K. Following the same steps as above, one obtains the averaged system for the slow variables of (6) as

$$\dot{\bar{x}} = f(\bar{x},t) + \bar{h}(t) + w(\bar{x})\delta f(t). \tag{7}$$

Just as for the unperturbed system of equations, for identical initial conditions and starting time t_0, and for the time intervals $t \sim 1$, the solution of (7) constitutes an approximation to the solution of the first equation in (6). Note that due to the initial arrangement of terms in (1), the averaged time-dependent forcing $\bar{h}(t)$ is identical for both (5) and (7), and does not depend on the slow variables

x, which are key properties for the linear response formulas developed further in this work.

For convenience, henceforth by t we will denote the elapsed interval of time starting at t_0. Observe that the averaged dynamics in both (5) and (7) do not contain stochastic terms, and, therefore, generate the flows $\bar{\phi}^{t_0,t}$ and $\bar{\phi}^{*t_0,t}$, respectively, such that, for $\bar{x}(t_0) = \bar{x}_0$, $\bar{x}(t_0 + t) = \bar{\phi}^{t_0,t}\bar{x}_0$ corresponds to the solution of the averaged equation in (5), and $\bar{x}^*(t_0 + t) = \bar{\phi}^{*t_0,t}x_0$ corresponds to the solution of the perturbed averaged equation in (7). We extensively use this observation in the next section.

3 Fluctuation-dissipation theorem for slow variables

Recently, Majda and Wang [9] developed a comprehensive linear response framework in the case of non-autonomous dynamics with time-periodic forcing (which also applies for general non-autonomous dynamics). Here we derive the approximate fluctuation-dissipation theorem for the slow variables of the multiscale dynamical system in (1) and its perturbed version in (6) for general non-autonomous dynamics, based on the averaged formulas (5) and (7), and then describe the practical FDT formulas for non-autonomous dynamics with time-periodic forcing and autonomous dynamics with invariant probability measure.

We start with a general assumption that the non-autonomous dynamical system in (1) posesses, for a suitable set of initial conditions, the family of time-dependent probability measures ρ_t, such that for any observable $A(x)$ its average value $\langle A\rangle(t)$ is given by

$$\langle A\rangle(t) = \rho_t(A) = \int A(x)\rho_t(\mathrm{d}x). \tag{8}$$

Similarly, for (6) we assume the existence of ρ_t^*.

At this point, assume that the small forcing in (6) is "turned on" at time t_0. Observe that $\delta f(t_0 + t) = 0$ when $t \le 0$, and, since (6) is the same as (1) for $t \le 0$, we apparently have $\rho^*_{t_0+t} = \rho_{t_0+t}$ for $t \le 0$. For $t > 0$, $\rho^*_{t_0+t}$ and ρ_{t_0+t} depart from each other. Thus, we define the *average response* of $A(x)$ to the small forcing in (6), starting at t_0, as

$$\delta\rho_{t_0+t}(A) = \rho^*_{t_0+t}(A) - \rho_{t_0+t}(A). \tag{9}$$

Observe that, for $t \sim 1$, $\rho_{t_0+t}(A) \approx \rho_{t_0}(A \circ \bar{\phi}^{t_0,t})$, and $\rho^*_{t_0+t}(A) \approx \rho_{t_0}(A \circ \bar{\phi}^{*t_0,t})$. With this, we now define the approximate average response of $A(x)$ to the small forcing in (6), starting at time t_0, as

$$\bar{\delta}\rho_{t_0+t}(A) = \rho_{t_0}(A \circ \bar{\phi}^{*t_0,t} - A \circ \bar{\phi}^{t_0,t}), \tag{10}$$

where $\bar{\phi}^{t_0,t}$ and $\bar{\phi}^{*t_0,t}$ are the flows generated by (5) and (7), respectively. Upon linearization, the response is given by

$$\bar{\delta}\rho_{t_0+t}(A) = \int \nabla A(\bar{\phi}^{t_0,t}x)\delta\bar{\phi}^{t_0,t}x\rho_{t_0}(\mathrm{d}x), \tag{11}$$

where $\delta\bar{\phi}^{t_0,t}x$ is given by

$$\delta\bar{\phi}^{t_0,t}x = \bar{\phi}^{*t_0,t}x - \bar{\phi}^{t_0,t}x. \tag{12}$$

Following a standard derivation for $\delta\bar{\phi}^{t_0,t}x$ (see, for instance, [1,11]), we find that it obeys, after subtracting (5) from (7), expanding in Taylor series with respect to $\delta\bar{\phi}^{t_0,t}x$ and dropping higher order terms,

$$\begin{gathered}\frac{\partial}{\partial t}\delta\bar{\phi}^{t_0,t}x = J(\bar{\phi}^{t_0,t}x, t_0+t)\delta\bar{\phi}^{t_0,t}x + w(\bar{\phi}^{t_0,t}x, t_0+t)\delta f(t_0+t),\\ \delta\bar{\phi}^{t_0,0}x = 0,\end{gathered} \tag{13}$$

where $J = D_x f$ is the Jacobian of f from (1). In order to produce a computationally tractable formula for the solution of (13), we denote the tangent map of the averaged flow $\bar{\phi}^{t_0,t}x$ from (5) as

$$\bar{T}_x^{t_0,t} = \frac{\partial}{\partial x}\bar{\phi}^{t_0,t}x. \tag{14}$$

Substituting $\bar{\phi}^{t_0,t}x$ into (5) and differentiating it with respect to x, we obtain the evolution equation for the tangent map in (14)

$$\frac{\partial}{\partial t}\bar{T}_x^{t_0,t} = J(\bar{\phi}^{t_0,t}x, t_0+t)\bar{T}_x^{t_0,t}, \tag{15}$$

which can be solved numerically for $\bar{T}_x^{t_0,t}$ (see [1–4]), and, by the chain rule, we also have

$$\frac{\partial}{\partial t}\bar{T}_{\bar{\phi}^{t_0,\tau}x}^{t_0+\tau,t-\tau} = J(\bar{\phi}^{t_0,t}x, t_0+t)\bar{T}_{\bar{\phi}^{t_0,\tau}x}^{t_0+\tau,t-\tau}. \tag{16}$$

Then, the solution of (13) is given by

$$\delta\bar{\phi}^{t_0,t}x = \int_0^t \bar{T}_{\bar{\phi}^{t_0,\tau}x}^{t_0+\tau,t-\tau} w(\bar{\phi}^{t_0,\tau}x)\delta f(t_0+\tau)\mathrm{d}\tau, \tag{17}$$

which we check by direct substitution:

$$\begin{gathered}\frac{\partial}{\partial t}\delta\bar{\phi}^{t_0,t}x = \frac{\partial}{\partial t}\int_0^t \bar{T}_{\bar{\phi}^{t_0,\tau}x}^{t_0+\tau,t-\tau} w(\bar{\phi}^{t_0,\tau}x)\delta f(t_0+\tau)\mathrm{d}\tau =\\ = T_{\bar{\phi}^{t_0,t}x}^{t_0+t,0} w(\bar{\phi}^{t_0,t}x)\delta f(t_0+t) + \int_0^t \frac{\partial}{\partial t}\bar{T}_{\bar{\phi}^{t_0,\tau}x}^{t_0+\tau,t-\tau} w(\bar{\phi}^{t_0,\tau}x)\delta f(t_0+\tau)\mathrm{d}\tau =\\ = J(\bar{\phi}^{t_0,t}x, t_0+t)\int_0^t \bar{T}_{\bar{\phi}^{t_0,\tau}x}^{t_0+\tau,t-\tau} w(\bar{\phi}^{t_0,\tau}x)\delta f(t_0+\tau)\mathrm{d}\tau +\\ + w(\bar{\phi}^{t_0,t}x)\delta f(t_0+t) = J(\bar{\phi}^{t_0,t}x, t_0+t)\delta\bar{\phi}^{t_0,t}x + w(\bar{\phi}^{t_0,t}x)\delta f(t_0+t),\end{gathered} \tag{18}$$

where we take advantage of the fact that $T^{t_0+t,0}_{\bar{\phi}^{t_0,t}x}$ is the identity map. With (17), the linear response formula in (11) can be written as

$$\bar{\delta}\rho_{t_0+t}(A) = \int_0^t \bar{R}_{ST}(t_0,t,\tau)\delta f(t_0+\tau)\mathrm{d}\tau, \tag{19}$$

where the averaged short-time linear response operator (AST-FDT) $\bar{R}_{ST}(t_0,t,\tau)$ is given by

$$\bar{R}_{ST}(t_0,t,\tau) = \int \nabla A(\bar{\phi}^{t_0,t}x)\bar{T}^{t_0+\tau,t-\tau}_{\bar{\phi}^{t_0,\tau}x} w(\bar{\phi}^{t_0,\tau}x)\rho_{t_0}(\mathrm{d}x). \tag{20}$$

If the probability measure ρ_{t_0} is absolutely continuous with respect to the Lebesgue's measure $\mathrm{d}x$ for any t_0 (which is usually the case for the stochastic g, h and q in (1)), i.e. $\rho_{t_0}(\mathrm{d}x) = p_{t_0}(x)\mathrm{d}x$ with $p_{t_0}(x)$ being a smooth probability density function, then, using the approximation $\rho_{t_0}(A \circ \bar{\phi}^{t_0,t}) \approx \rho_{t_0+t}(A)$ for $t \sim 1$, one can integrate the formula in (20) by parts, obtaining the classical FDT formula

$$R_{class}(t_0,t,\tau) = -\int A(\bar{\phi}^{t_0+\tau,t-\tau}x)\mathrm{div}(w(x)p_{t_0+\tau}(x))\mathrm{d}x. \tag{21}$$

At this point, for practical computation of the linear response operators we need to convert the above formulas from measure averages to time averages over a long-term trajectory of (1). In order to accomplish this in the presence of explicit time dependence in (1), here we assume that the family of probability measures ρ_t is $\mathcal{T}$-periodic, such that for any observable $A(x)$, where x is the set of slow variables in (1), and any t_0 and integer m

$$\rho_{t_0}(A) = \rho_{t_0+m\mathcal{T}}(A), \tag{22}$$

which can usually be achieved in the case of a periodic dependence of f and q on time (see Majda and Wang [9] for more details). Under this assumption of periodicity, the averaging with respect to ρ_{t_0} above can be replaced with the $\mathcal{T}$-averaging over the solution series of (1) as

$$\begin{aligned}\bar{R}_{ST}(t_0,t,\tau) = \lim_{M\to\infty}\frac{1}{M}\sum_{m=0}^{M} \nabla A(\bar{\phi}^{t_0+m\mathcal{T},t}x(t_0+m\mathcal{T}))\times \\ \times \bar{T}^{t_0+m\mathcal{T}+\tau,t-\tau}_{\bar{\phi}^{t_0+m\mathcal{T},\tau}x(t_0+m\mathcal{T})} w(\bar{\phi}^{t_0+m\mathcal{T},\tau}x(t_0+m\mathcal{T})),\end{aligned} \tag{23}$$

and

$$\begin{aligned}R_{class}(t_0,t,\tau) = -\lim_{M\to\infty}\frac{1}{M}\sum_{m=0}^{M} A(\bar{\phi}^{t_0+m\mathcal{T}+\tau,t-\tau}x(t_0+m\mathcal{T}+\tau)\times \\ \times \mathrm{div}(w(x(t_0+m\mathcal{T}+\tau))p_{t_0+\tau}(x(t_0+m\mathcal{T}+\tau))).\end{aligned} \tag{24}$$

For practical purposes of computation, $\bar{\phi}$ can be replaced with the time series $x(t)$ from (1) above and in (15). The direct substitution of the time series from

(1) yields

$$\bar{R}_{ST}(t_0,t,\tau) = \lim_{M\to\infty}\frac{1}{M}\sum_{m=0}^{M}\nabla A(x(t_0+t+m\mathcal{T}))\times \\ \times \bar{T}^{t_0+m\mathcal{T},t-\tau}_{x(t_0+\tau+m\mathcal{T})}w(x(t_0+\tau+m\mathcal{T})) \tag{25}$$

and

$$R_{class}(t_0,t,\tau) = -\lim_{M\to\infty}\frac{1}{M}\sum_{m=0}^{M}A(x(t_0+t+m\mathcal{T}))\times \\ \times Q_{class}(x(t_0+\tau+m\mathcal{T}),t_0+\tau), \tag{26}$$

where $Q_{class}(x,t)$ is given by

$$Q_{class}(x,t) = \operatorname{div}(w(x)) + w\nabla\log p_t(x). \tag{27}$$

Note that here we need to have an approximation of $p_{t_0+\tau}$ at all τ over which the computation proceeds in (19) (which in practice is a finite set due to time discretization), even though the response is computed from the initial state at t_0. For the special case when p_t is Gaussian, i.e.

$$p_t(x) = (2\pi)^{-N_x/2}\det(\sigma(t))^{-1}\exp\left(\frac{1}{2}(x-\rho_t(x))^T\sigma^{-2}(t)(x-\rho_t(x))\right), \tag{28}$$

where $\sigma^2(t)$ is the time-dependent covariance matrix of ρ_t, the quasi-Gaussian FDT formula is

$$R_{qG}(t_0,t,\tau) = -\lim_{M\to\infty}\frac{1}{M}\sum_{m=0}^{M}A(x(t_0+t+m\mathcal{T}))\times \\ \times Q_{qG}(x(t_0+\tau+m\mathcal{T}),t_0+\tau), \tag{29}$$

where $Q_{qG}(x,t)$ is given by

$$Q_{qG}(x,t) = \operatorname{div}(w(x)) + w(x)\sigma^{-2}(t)(x-\rho_t(x)). \tag{30}$$

For the authonomous case (i.e. without explicit time dependence in (1)), we have $\bar{\phi}^{t_0,t} = \bar{\phi}^t$ and, assuming that $\rho_{t_0} = \rho$ is the invariant probability measure for (1) [1–4], obtain

$$\bar{\delta}\rho_t(A) = \int_0^t \bar{R}_{ST}(t-\tau)\delta f(t_0+\tau), \\ \bar{R}_{ST}(t) = \lim_{s\to\infty}\frac{1}{s}\int_0^s \nabla A(x(t+\tau))\bar{T}^t_{x(\tau)}w(x(\tau))\mathrm{d}\tau. \tag{31}$$

Similarly, for the classical and quasi-Gaussian FDT, and $p_{t_0}(x) = p(x)$ we obtain

$$R_{class}(t) = -\lim_{s\to\infty} \frac{1}{s} \int_0^s A(x(t+\tau))\times \\ \times [\operatorname{div}(w(x(\tau)) + w(x(\tau))\nabla \log p(x(\tau))] \,\mathrm{d}\tau, \tag{32}$$

and

$$R_{qG}(t) = -\lim_{s\to\infty} \frac{1}{s} \int_0^s A(x(t+\tau))\times \\ \times \left[\operatorname{div}(w(x(\tau)) + w(x(\tau))\sigma^{-2}(x(\tau) - \rho(x)\right] \mathrm{d}\tau. \tag{33}$$

Advantages of the averaged FDT algorithm

- By design, the algorithm is a direct and straightforward application of the existing (and, possibly, future) FDT methods separately onto the slow variables, which guarantees easy practical implementation;
- There is a significant computational advantage in the case of $N_x \ll N_y$, since all the FDT formulas (including the tangent map) are computed for slow variables x only (even though the time series $x(t)$ has to be computed from (1));
- The tangent map $\bar{T}_x^t$ is not too sensitive to the presence of large Lyapunov exponents at fast variables y, and the numerical stability of its computation is largely restricted by the characteristic Lyapunov time of the slow variables x;
- Remarkably, the AST-FDT formula in (25) can be used with stochastic dynamics in (1), which means that the blended response algorithm from [2, 4], computed for averaged FDT approximations above, can also be used with stochastic dynamics;
- Only the AST-FDT formula in (25) constitutes an averaged linear response approximation for $t \sim 1$: observe that the classical formula in (26) is formally a valid linear response approximation for times beyond 1 since it does not contain the averaged tangent map;
- Therefore, if the blended response algorithm [2,4] for AST-FDT has the blending cut-off time ~ 1, the averaged blended FDT response approximation is formally valid for times beyond 1.

4 Numerical setup and results

In this section we present some preliminary tests of the new linear response algorithm for a model with time scale separation.

The full Lorenz 96 (L96) model [5–7] is given by

$$\begin{aligned} \dot{X}_k &= X_{k-1}(X_{k+1} - X_{k-2}) - dX_k + F - \lambda \sum_{j=1}^{J} Y_{k,j}, \\ \dot{Y}_{k,j} &= \frac{1}{\varepsilon}[Y_{k,j+1}(Y_{k,j-1} - Y_{k,j+2}) - dY_{k,j} + F] + \lambda X_k, \\ &\qquad \varepsilon > 0, \lambda > 0, \end{aligned} \tag{34}$$

where $1 \leq k \leq K$, $1 \leq j \leq J$. Originally in [5–7] there is no F term in the equation for Y-variables in (34), however, in its absense the behavior of Y-variables is strongly dissipative [1], and here we add F in the right-hand side of the second equation in (34) to induce strongly chaotic behavior of Y-variables with large positive Lyapunov exponents. The following notations are adopted above:

- X is a set of slow variables of size K. The following periodic boundary conditions hold for $\vec{X}$: $X_{k+K} = X_k$.
- Y is a set of fast variables of size $K \times J$. The following boundary conditions hold for $\vec{Y}$: $X_{k+K,j} = X_{k,j}$ and $X_{k,j+J} = X_{k+1,j}$.
- F is the external forcing parameter;
- d is the dissipation parameter, set to 1 for $F \neq 0$, and 0 for $F = 0$;
- ε is the time scale separation parameter;
- λ is the coupling parameter.

In the case of zero F and d, the full L96 model in (34) preserves the quadratic energy of the form

$$E = \frac{1}{2} \sum_{k=1}^{K} \left(X_k^2 + \sum_{j=1}^{J} Y_{k,j}^2 \right), \tag{35}$$

and possesses the Liouville (incompressibility) property such that the equilibirum statistical state for (34) approaches the classical Gibbs equilibrium state with zero mean state and uniform energy spectrum as the number of variables tends to infinity. As a result, the classical FDT formula with Gaussian $p(x)$ (which we call the quasi-Gaussian FDT, or qG-FDT) is a good response approximation for the L96 model without forcing and dissipation.

Here we study the response of the mean state $\rho(X)$ of the slow variables, such that $A(X) = X$, of the L96 model to a small constant external perturbation $\delta f \in \mathbb{R}^K$ (such that $w = I$). Under the above assumptions, the linear response for the slow variables of the full L96 model without forcing and dissipation is given by

$$
\begin{aligned}
\delta\rho_t(X) &= \mathcal{R}(t)\delta f, \\
\mathcal{R}_{ST}(t) &= \lim_{s\to\infty}\frac{1}{s}\int_0^s \mathrm{d}\tau \int_0^t T^r_{X(\tau)}\mathrm{d}r, \\
\bar{\mathcal{R}}_{ST}(t) &= \lim_{s\to\infty}\frac{1}{s}\int_0^s \mathrm{d}\tau \int_0^t \bar{T}^r_{X(\tau)}\mathrm{d}r, \\
\mathcal{R}_{QG}(t) &= -\lim_{s\to\infty}\frac{1}{s}\int_0^s \mathrm{d}\tau \int_0^t X(r+\tau)\sigma^{-2}\left(X(\tau)-\rho(X)\right)\mathrm{d}r,
\end{aligned}
\tag{36}
$$

where σ^2 is the statistical covariance matrix. Below we study the errors in response produced by the response operators $\mathcal{R}_{ST}$, $\bar{\mathcal{R}}_{ST}$ and $\mathcal{R}_{QG}$. The errors are determined by comparison with the full ideal response operator $\mathcal{R}_I$, which is obtained by perturbing the model and measuring the response directly [2–4, 8]. The following parameters are used in the computation:

- $K = 8$, $J = 8$ (72 variables in total);
- $\varepsilon = 0.1$ (weak time scale separation), $\varepsilon = 0.01$ (strong time scale separation);
- $\lambda = 0.1$ (weak coupling), $\lambda = 0.5$ (intermediate coupling), $\lambda = 1$ (strong coupling);
- No forcing and dissipation, the time series are generated on a constant energy sphere of radius 1;
- Forced-dissipative case, $F = 6$, $d = 1$.

Here we call $\lambda = 0.5$ an intermediate coupling, however, for the L96 model its is in fact strong. The reason is the following: observe that, for the case without forcing and dissipation, the absolute value of every X_k and $Y_{k,j}$ cannot be greater than 1. Therefore, the quadratic term (which consists of slow variables) in the first equation in (34) provides rather weak contribution, while the linear term $F - dX_k$ is zero. On the other hand, the contribution from the fast variables in the same equation is the sum of $J = 8$ fast variables, which can be much greater than the contribution from the slow variables even when $\lambda = 0.5$. Similar argument can be made for the forced-dissipative set up: in this case both X_k and $Y_{k,j}$ variables have a nonzero mean state. The mean states of X_k cancel out in the nonlinear part for the slow variables, while the mean states of $Y_{k,j}$ add up in the sum of the fast variables in the first equation in (34), thus on average providing a strong contribution from the fast variables.

During the course of computations, the observed speed-up of AST-FDT over standard ST-FDT was about 200 times. In Figure 1 we show the relative errors between the ideal response operator and the ST-FDT, AST-FDT, and qG-FDT response operators for the slow variables of the L96 model without forcing and dissipation for two values of the time scaling parameter ε, 0.01 and 0.1, and two values of the coupling parameter λ, 0.1 and 0.5. Observe that the error and blow-up time of the ST-FDT operator strongly depends of the value of ε: for $\varepsilon = 0.1$

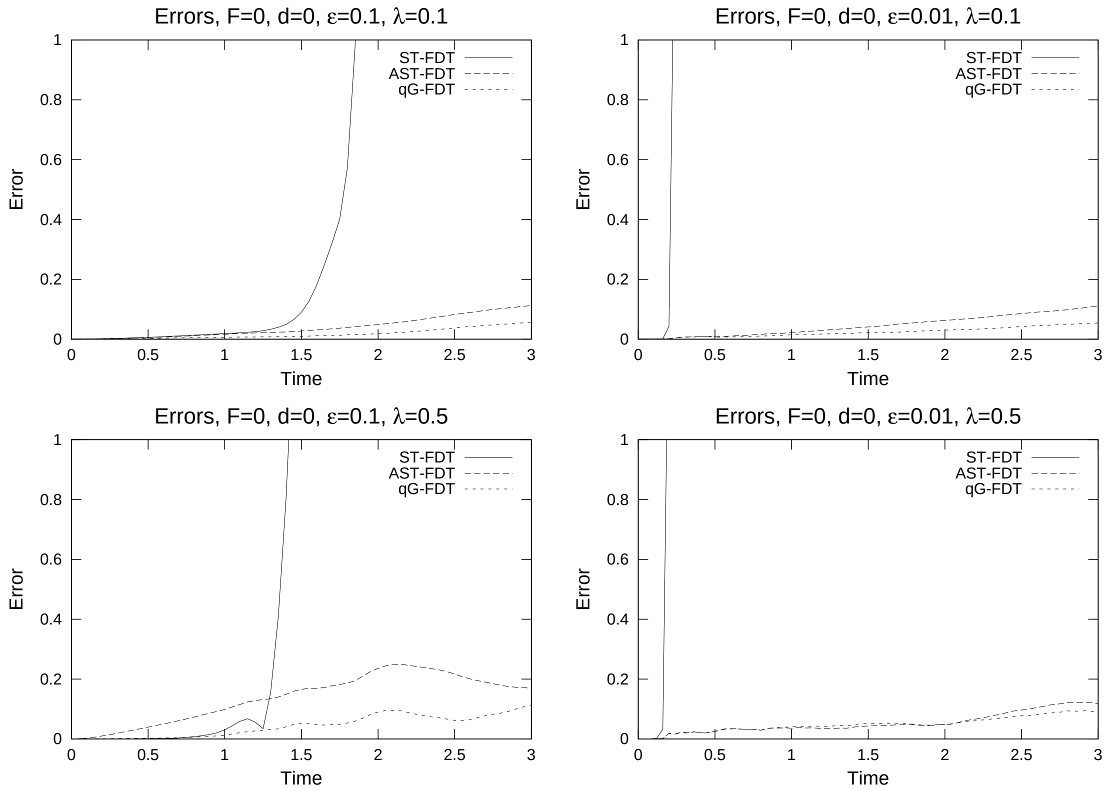

Fig. 1. The relative L_2 errors between the ideal and various FDT response operators for the L96 model without forcing and dissipation.

the blow-up time is roughly 1.5 time units, while for $\varepsilon = 0.01$ it is about 0.2 time units. This happens due to the fact that the Lyapunov characteristic time for the full set of variables $X \times Y$ is roughly ten times shorter for $\varepsilon = 0.01$ than that for $\varepsilon = 0.1$ (the fast variables are roughly ten times "faster"). The qG-FDT response operator is "exact" for the L96 model without forcing and dissipation as the equilibrium state of the model approaches the Gaussian distribution, and produces the smallest error among the computed FDT response operators. In the case of the weak coupling $\lambda = 0.1$ the AST-FDT operator produces comparable errors to the qG-FDT operator for both the weak ($\varepsilon = 0.1$) and strong ($\varepsilon = 0.01$) time scale separation. Remarkably, in the case of the intermediate coupling $\lambda = 0.5$ and weak time scale separation $\varepsilon = 0.1$ the AST-FDT operator produces significantly larger errors than the qG-FDT operator, but as the time scale separation becomes strong ($\varepsilon = 0.01$), the AST-FDT becomes roughly as precise as the qG-FDT operator.

In Figure 2 we show the relative errors between the ideal response operator and the ST-FDT, AST-FDT, and qG-FDT response operators for the slow variables of the L96 model with $F = 6$ for two values of the time scaling parameter ε, 0.01 and 0.1, and three values of the coupling parameter λ, 0.1, 0.5 and 1. Again, observe that the error and blow-up time of the ST-FDT operator strongly depends of the value of ε: for $\varepsilon = 0.1$ the blow-up time is roughly 1.5 time units, while

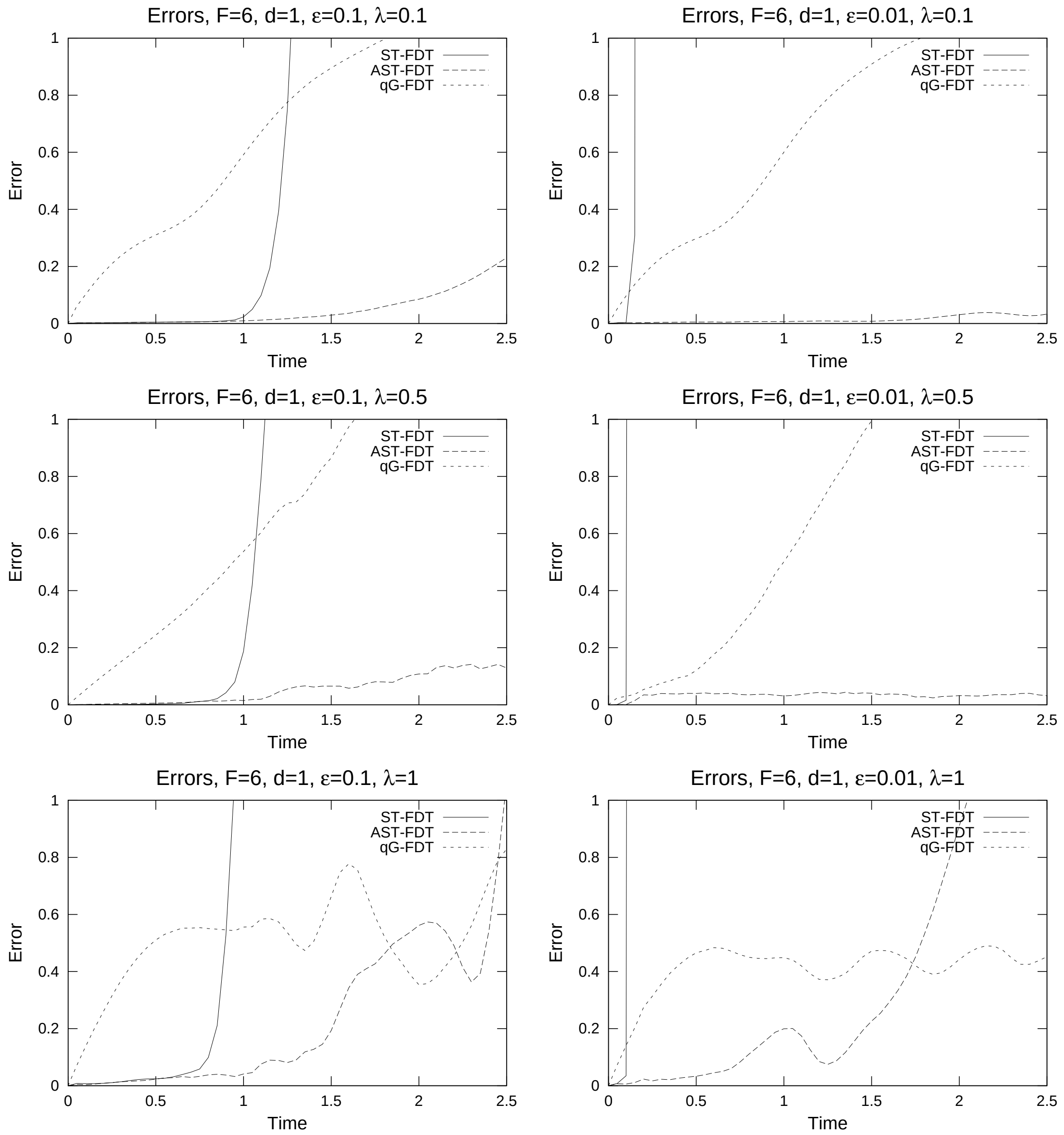

Fig. 2. The relative L_2 errors between the ideal and various FDT response operators for the L96 model with $F = 6$ and $d = 1$.

for $\varepsilon = 0.01$ it is about 0.2 time units, which happens due to the fact that the Lyapunov characteristic time for the full set of variables $X \times Y$ is roughly ten times shorter for $\varepsilon = 0.01$ than that for $\varepsilon = 0.1$ (the fast variables are roughly ten times "faster"). The qG-FDT response operator does not produce a good response approximation for all considered values of ε and λ, which is due to the fact that the equilibrium statistical state of the model is no longer Gaussian. On the other hand, for weak and intermediate coupling $\lambda = 0.1, 0.5$ the AST-FDT operator yields small errors for weak time scale separation ($\varepsilon = 0.1$) and further improves for strong time scale separation ($\varepsilon = 0.01$). For the strong coupling $\lambda = 1$, the AST-FDT operator again provides the smallest errors among all compared operators until the time $T = 1.8$, however later exhibits relatively early blow-up (as compared to $\lambda = 0.1, 0.5$ where no blow-up is observed).

5 Conclusions

In the work we developed a new response algorithm based on the approximate averaged dynamics of multiscale ODEs and SDEs. The new method allows to compute the response operators directly at slow variables using existing FDT formulas, improving numerical stability and reducing computational expense, as well as allowing to use the geometric AST-FDT algorithm for the response of slow variables in multiscale models with stochastically driven fast variables. The new method is tested on the multiscale Lorenz 96 model with explicit time scale separation of variables through a small parameter ε. The model is run in two regimes: one is without forcing and dissipation, where the quasi-Gaussian FDT provides a valid approximation; another regime is with forcing and dissipation, where the AST-FDT approximation is necessary to calculate a good approximation to the linear response. In both cases, the new AST-FDT algorithm is observed to be superior to the standard ST-FDT in both numerical stability and computational expense.

Acknowledgment. The author thanks Andrew Majda for the reference to his joint work with Xiaoming Wang on the fluctuation dissipation theorem for non-autonomous dynamics with time-periodic forcing. Also, the author thanks Eric Vanden-Eijnden for important remarks and Ibrahim Fatkullin for interesting and useful discussions. This work is supported by the NSF CAREER grant DMS-0845760 and the ONR grant N000140610286.

References

[1] R. Abramov. Short-time linear response with reduced-rank tangent map. *Chin. Ann. Math.*, 2009. accepted and in press.

[2] R. Abramov and A. Majda. Blended response algorithms for linear fluctuation-dissipation for complex nonlinear dynamical systems. *Nonlinearity*, 20:2793–2821, 2007.

[3] R. Abramov and A. Majda. New approximations and tests of linear fluctuation-response for chaotic nonlinear forced-dissipative dynamical systems. *J. Nonlin. Sci.*, 18(3):303–341, 2008.

[4] R. Abramov and A. Majda. New algorithms for low frequency climate response. *J. Atmos. Sci.*, 66:286–309, 2009.

[5] D. Crommelin and E. Vanden-Eijnden. Subgrid scale parameterization with conditional Markov chains. *J. Atmos. Sci.*, 65:2661–2675, 2008.

[6] I. Fatkullin and E. Vanden-Eijnden. A computational strategy for multiscale systems with applications to Lorenz 96 model. *J. Comp. Phys.*, 200:605–638, 2004.

[7] E. Lorenz. Predictability: A problem partly solved. In *Proceedings of the Seminar on Predictability*, Shinfield Park, Reading, England, 1996. ECMWF.

[8] A. Majda, R. Abramov, and M. Grote. *Information Theory and Stochastics for Multiscale Nonlinear Systems*, volume 25 of *CRM Monograph Series of Centre de Recherches Mathématiques, Université de Montréal*. American Mathematical Society, 2005. ISBN 0-8218-3843-1.

[9] A. Majda and X. Wang. Linear response theory for statistical ensembles in complex systems with time-periodic forcing. *Comm. Math. Sci.*, 2008. accepted and in press.

[10] G. Papanicolaou. Introduction to the asymptotic analysis of stochastic equations. In R. DiPrima, editor, *Modern modeling of continuum phenomena*, volume 16 of *Lectures in Applied Mathematics*. American Mathematical Society, 1977.

[11] D. Ruelle. General linear response formula in statistical mechanics, and the fluctuation-dissipation theorem far from equilibrium. *Phys. Lett. A*, 245:220–224, 1998.

[12] E. Vanden-Eijnden. Numerical techniques for multiscale dynamical systems with stochastic effects. *Comm. Math. Sci.*, 1:385–391, 2003.

[13] V. Volosov. Averaging in systems of ordinary differential equations. *Russian Math. Surveys*, 17:1–126, 1962.