A variation of the Ramsey problem: (p, q)-colorings.

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Definition

Definition (Erdős; 1981)

A (p, q)-*coloring* of a graph is coloring of the edges such that every copy of K_p contains at least q distinct colors. Let $f(n, p, q)$ denote the minimum number of colors needed to (p, q)-color of the complete graph on n vertices, K_n.

- $1 \leq q \leq \binom{p}{2}$.
- $f(n, p, 1) = 1$.
- $f\left(n, p, \binom{p}{2}\right) = \binom{n}{2}$ for $p \geq 4$.
- Determining $f(n, p, 2)$ is equivalent to the classic Ramsey problem.
- $f(n, 3, 3) \approx n$.

Background

Theorem (Erdős and Gyárfás; 1997)

The Local Lemma gives a general upper bound:

$$f(n,p,q) \le cn^{\frac{p-2}{\binom{p}{2}-q+1}}.$$

Theorem (Erdős and Gyárfás; 1997)

Fix p and let $q = \binom{p}{2} - p + 3$. Then $f(n,p,q) = \Theta(n)$ and $f(n,p,q-1) \le cn^{1-\frac{1}{p-1}}$.

Theorem (Erdős and Gyárfás; 1997)

Fix p and let $q = \binom{p}{2} - \lfloor p \rfloor + 2$. Then $f(n,p,q) = \Theta(n^2)$ and $f(n,p,q-1) \le cn^{2-\frac{4}{p}}$.

Background

Erdős and Gyárfás gave a simple induction argument which demonstrates that

$$n^{\frac{1}{p-2}} - 1 \leq f(n, p, p),$$

the smallest value of q for which they could find a polynomial lower bound. They also considered several cases for small p:

- $\frac{5(n-1)}{6} \leq f(n, 4, 5) \leq n$.
- $n^{1/2} - 1 \leq f(n, 4, 4) \leq cn^{2/3}$ - one of the "most interesting" cases.
- $f(n, 4, 3) \leq cn^{1/2}$ - the "most annoying" case since unsure if it is even polynomial at all.
- $cn \leq f(n, 5, 9) \leq cn^{3/2}$ - the other "most interesting" case to see whether this is linear or not.

$(5, 9)$-coloring

Theorem

$$\frac{11}{4}n - \frac{23}{4} \leq f(n, 5, 9) \leq 2n^{1+\frac{c}{\sqrt{\log n}}}$$

- Upper bound: Axenovich; 2000.
- Lower bound: Krop; 2008.

$(p, p-1)$-coloring

Theorem (Mubayi; 1998)

$$f(n,4,3) \leq e^{\sqrt{c \log n}(1+o(1))}.$$

Theorem (Conlon, Fox, Lee, and Sudakov; 2015)

$$f(n,p,p-1) \leq 2^{16p(\log n)^{1-1/(p-2)} \log \log n}.$$

This shows that $q = p$ is the threshold at which $f(n,p,q)$ becomes polynomial in n.

(4, 4)-coloring

Theorem (Mubayi; 2004)

$$f(n, 4, 4) \leq n^{1/2} e^{c\sqrt{\log n}}.$$

- This shows that $n^{1/2} \leq f(n, 4, 4) \leq n^{1/2+o(1)}$.
- Uses the product of two explicit colorings:
  - the construction showing that $f(n, 4, 3)$ is subpolynomial, and
  - an algebraic coloring which associates each vertex with a vector in $\mathbb{F}_q^2$ and uses a symmetric map $\mathbb{F}_q^2 \times \mathbb{F}_q^2 \to \mathbb{F}_q$ to color the edges.

(5, 5)-coloring

Theorem (C. and Heath; 2017)

$$f(n,5,5) \leq n^{1/3} 2^{c\sqrt{\log n}\log\log n}.$$

- This shows that $n^{1/3} \leq f(n,5,5) \leq n^{1/3+o(1)}$.
- Uses the product of two explicit colorings:
  - the construction by Conlon, Fox, Lee, and Sudakov (CFLS) using $n^{o(1)}$ colors, and
  - an algebraic coloring which associates each vertex with a vector in $\mathbb{F}_q^3$ and uses a symmetric map $\mathbb{F}_q^3 \times \mathbb{F}_q^3 \to \mathbb{F}_q$ to color the edges.

The CLFS Coloring

Let $n = 2^{\beta^2}$ for some positive integer β. Associate each vertex of K_n with a unique binary string of length β^2:

$$V = \{0,1\}^{\beta^2}.$$

For any vertex $v \in V$, let $v^{(i)}$ denote the ith block of bits of length β in v so that

$$v = (v^{(1)}, \dots, v^{(\beta)})$$

where each $v^{(i)} \in \{0,1\}^{\beta}$.
Between two vertices $x, y \in V$, the CFLS coloring is defined by

$$\varphi_1(x,y) = \left(\left(i, \{x^{(i)}, y^{(i)}\} \right), i_1, \dots, i_\beta \right)$$

where i is the first index for which $x^{(i)} \neq y^{(i)}$, and for each $k = 1, \dots, \beta$, $i_k = 0$ if $x^{(k)} = y^{(k)}$ and otherwise is the first index at which a bit of $x^{(k)}$ differs from the corresponding bit in $y^{(k)}$.

CFLS Example

$$x = (0, 1, 1, 1, 0, 1, 0, 0, 1)$$
$$y = (0, 1, 1, 0, 0, 1, 0, 1, 1)$$
$$\varphi_1(x, y) = \left((2, \{(0, 0, 1), (1, 0, 1)\}\right), 0, 1, 2)$$

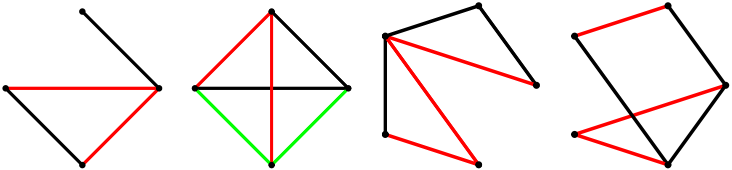
CFLS Facts

Two easy and important facts about the CFLS coloring:

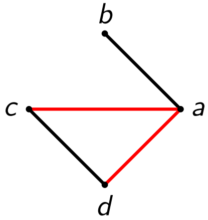
- Each color class is bipartite.
- Assume that all binary strings are ordered by the integers they represent so that the vertices are linearly ordered and also the individual blocks are ordered. Then, $a < b < c$ implies that $\varphi_1(a, b) \neq \varphi_1(b, c)$.

Avoided Configurations

Four configurations avoided by the CFLS coloring.



Avoided Configurations



Assume towards a contradiction that $\varphi_1(a, b) = \varphi_1(c, d) = \alpha$ and $\varphi_1(a, c) = \varphi_1(a, d) = \gamma$. Let $\alpha_0 = (i, \{x, y\})$. Without loss of generality, $a^{(i)} = c^{(i)} = x$ and $b^{(i)} = d^{(i)} = y$. Then $\gamma_i = 0$ since a and c agree at i, but $\gamma_i \neq 0$ as a and d differ at i, a contradiction.

Modified CFLS

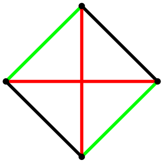


Figure: A striped K_4.

Let $x < y$. Define

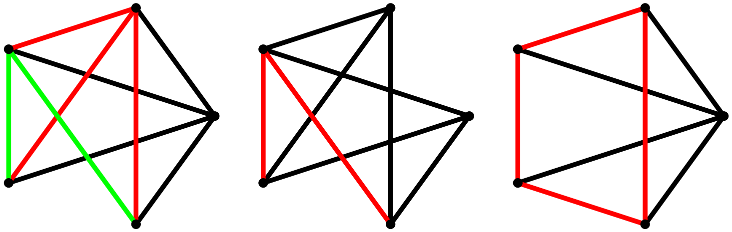
$$\varphi_2(x,y) = (\delta_1(x,y), \ldots, \delta_\beta(x,y))$$

where for each i,

$$\delta_i(x,y) = \begin{cases} -1 & x^{(i)} > y^{(i)} \\ +1 & x^{(i)} \leq y^{(i)} \end{cases}$$

Problem Configurations

Three configurations not avoided by the modified CFLS coloring.



MDP Coloring

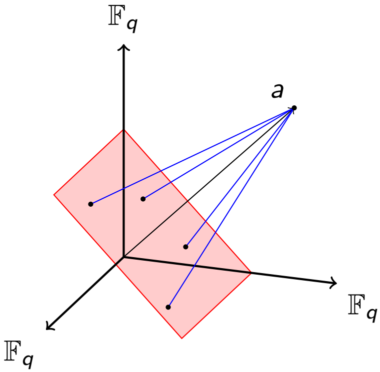
Let q be some odd prime power, and let $\mathbb{F}_q^*$ denote the nonzero elements of the finite field with q elements. The vertices of our graph will be the three-dimensional vectors over this set,

$$V = \left(\mathbb{F}_q^*\right)^3.$$

The explicit definition of the coloring is a bit technical, but it is essentially the inner product of the the two vectors with several modifications to take care of certain cases. I call it the Modified Dot Product (MDP) coloring.

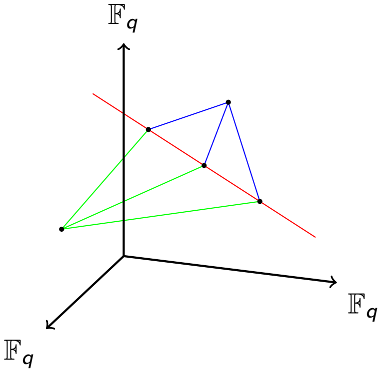
Monochromatic Neighborhoods

Given a vertex $a \in \left(\mathbb{F}_q^*\right)^3$ and a color $\alpha \in \mathbb{F}_q$, the monochromatic α-neighborhood of a is contained within an affine plane in $\mathbb{F}_q^3$.



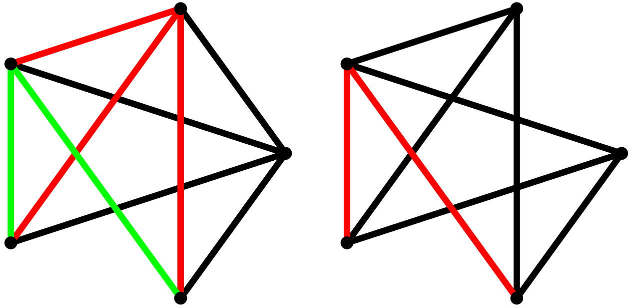
Intersection of Monochromatic Neighborhoods

"Most" of the time the intersection of two monochromatic neighborhoods defines a subset of an affine line.



First Two Problems

Therefore, if we knew that the construction induced a *proper* edge coloring on every affine line, then any three vertices in the intersection of two monochromatic neighborhoods must span three distinct edge colors. This would get us close to eliminating the first two problem configurations.



Affine Lines

The construction does *not* induce a proper edge coloring on an affine line, but it *almost* does.

$$s \cdot (s + \alpha t) = s \cdot (s + \beta t)$$
$$\alpha(s \cdot t) = \beta(s \cdot t)$$
$$s \cdot t = 0$$

The coloring is proper except at the vector that is orthogonal to the direction of the line. This only happens at one vector unless the direction of the line is isotropic.

Modification

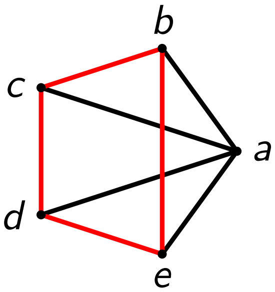
This leads us to the major modification of the coloring: when we get two vectors x and y such that

$$x \cdot (x - y) = 0 \text{ or } y \cdot (y - x) = 0,$$

then we replace the dot product with $x_1 + y_1$ if $x_1 \neq y_1$ and $x_2 + y_2$ if $x_1 = y_1$.

This makes the induced coloring on an affine line proper, and allows the coloring to retain the property that any vertex and any color define a subset of an affine plane.

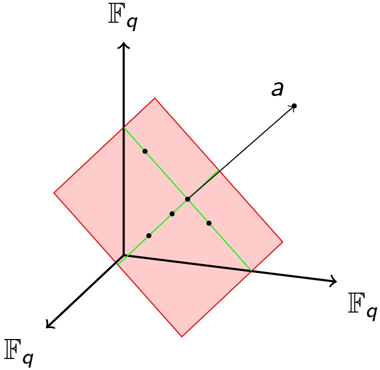
C_4 in a Monochromatic Neighborhood



This happens when the line defined by b and d is orthogonal to the line defined by c and e *and* these lines intersect at a scalar multiple of a.

Split the Coloring

This kind of thing happens a lot.



The solution is to partition the set of (non-isotropic) vectors in each linear plane into two sets so that no two in the same set are orthogonal. Then split each color into four colors based on these partitions.

Open Problems

- Is $f(n,p,p) \leq n^{1/(p-2)+o(1)}$ in general?
- Tighten other small cases:
    - $cn^{1/2} \leq f(n,5,6) \leq cn^{3/5}$
    - $cn^{2/3} \leq f(n,5,7) \leq cn^{3/4}$
- Generalizations and related:
    - Geometric version: minimize number of distinct distances between n points in $\mathbb{R}^d$
    - $r(G,H,q)$ - minimum number of colors such that every copy of H in G receives at least q colors
    - (p, q_1, q_2)-colorings
    - bipartite version
    - hypergraph version