Extremal Problems on Generalized Directed Hypergraphs

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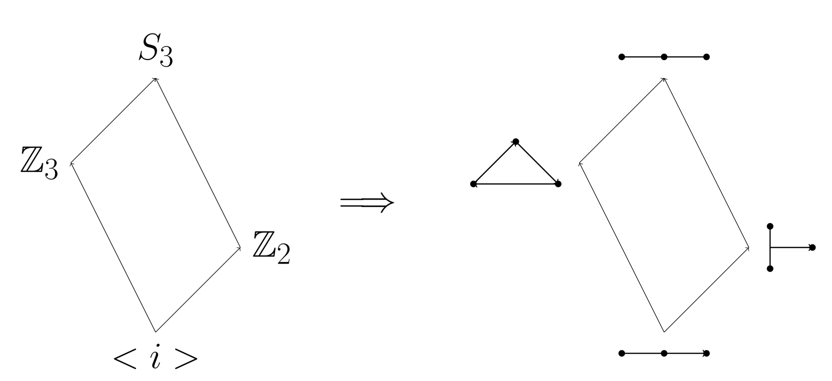
GDH definition

Definition 1 *Let $\mathcal{L} = \{E\}$, a language with one r-ary relation symbol E. Let T be an $\mathcal{L}$-theory that consists of a single sentence of the form*

$$\forall x_1 \cdots x_r E(x_1, \ldots, x_r) \implies \bigwedge_{i \neq j} x_i \neq x_j \wedge \bigwedge_{\pi \in J_T} E(x_{\pi(1)}, \ldots, x_{\pi(r)})$$

for some subgroup of the group of permutations on r elements, $J_T \subseteq S_r$. Call such a theory a generalized directed hypergraph theory *and any finite model of T is a* generalized directed hypergraph (GDH).

Lattice of GDH types when $r = 3$



Turán density and blowups

Definition 2 *Given a family of GDHs $\mathcal{F}$ and a positive integer n, let the nth* extremal number, *$ex_T(n, \mathcal{F})$, be defined as the maximum number of edges over all $\mathcal{F}$-free GDHs on n elements,*

$$ex_T(n, \mathcal{F}) := \max_{\mathcal{F}\text{-free } G_n} \{e_T(G_n)\}.$$

The Turán density *of $\mathcal{F}$ is defined as*

$$\pi_T(\mathcal{F}) := \lim_{n\to\infty} \frac{ex_T(n, \mathcal{F})}{\frac{r!}{m_T}\binom{n}{r}}.$$

Definition 3 *Let G be a GDH with $V_G = \{x_1, \ldots, x_n\}$, and let $t = (t_1, \ldots, t_n)$ be a tuple of positive integers. Define the t-blowup of G to be the $\mathcal{L}$-structure $G(t)$ where*

$$V_{G(t)} = \{x_{11}, \ldots, x_{1t_1}, \ldots, x_{n1}, \ldots, x_{nt_n}\}$$

and

$$(x_{i_1 j_1}, \ldots, x_{i_r j_r}) \in E_{G(t)} \iff (x_{i_1}, \ldots, x_{i_r}) \in E_G.$$

Supersaturation

Theorem 1 *Let F be a GDH on k elements. Let $\epsilon > 0$. For sufficiently large $n \geq n_0(F, \epsilon)$, any GDH G on n elements with density $d(G) \geq \pi_T(F) + \epsilon$ will contain at least $c\binom{n}{k}$ copies of F for some constant $c = c(F, \epsilon)$.*

Characterization of degenerate forbidden families

Theorem 2 *Let $\mathcal{F}$ be some family of GDHs, then $\pi_T(\mathcal{F}) = 0$ if and only if some member $F \in \mathcal{F}$ is a subGDH of the t-blowup of a single edge for some vector, $t = (t_1, \ldots, t_r)$, of positive integers. Otherwise, $\pi(\mathcal{F}) \geq \frac{m_T}{r^r}$.*

Jumps

Definition 4 *Let T be a GDH theory, then $\alpha \in [0, 1)$ is a jump for T if there exists a $c > 0$ such that for any $\epsilon > 0$ and any positive integer l, there exists a positive integer $n_0(\alpha, \epsilon, l)$ such that any GDH G on $n \geq n_0$ elements that has at least $(\alpha + \epsilon)\frac{r!}{m_T}\binom{n}{r}$ edges contains a subGDH on l elements with at least $(\alpha + c)\frac{r!}{m_T}\binom{l}{r}$ edges.*

Theorem 3 *The GDH theory T has a jump α if and only if there exists a finite family $\mathcal{F}$ of GDHs such that $\pi_T(\mathcal{F}) \leq \alpha$ and $b_T(F) > \alpha$ for each $F \in \mathcal{F}$.*

Jumps pass up the lattice

Theorem 4 *Let T and T' be two GDH theories such that $J_{T'} \subseteq J_T$. Then for any family $\mathcal{F}$ of T-graphs there exists a family $\mathcal{F}'$ of T'-graphs for which $\pi_{T'}(\mathcal{F}') = \pi_T(\mathcal{F})$. Moreover, if $\mathcal{F}$ is a finite family, then $\mathcal{F}'$ is also finite.*

The converse is false in general. For example, the permutation subgroup for the theory T' of $(2 \to 1)$-uniform directed hypergraphs is a subgroup of the permutation group for the theory T of undirected 3-graphs, S_3. The extremal number for the directed hypergraph is $R_4 = \{ab \to c, cd \to e\}$ is

$$ex_{T'}(n, R_4) = \left\lfloor \frac{n}{3} \right\rfloor \binom{\left\lceil \frac{2n}{3} \right\rceil}{2}$$

as shown in [2]. Therefore, the Turán density is $\pi_{T'}(R_4) = \frac{4}{27}$. However, it is well-known that no Turán densities exist for 3-graphs in the interval $\left(0, \frac{6}{27}\right)$.

Corollary 1 *Let T and T' be two GDH theories such that $J_{T'} \subseteq J_T$. If α is a jump for T', then it is also a jump for T.*

Jumps do not pass down the lattice

Definition 5 *Let $\alpha \in [0, 1)$. Call α a* demonstrated nonjump *for a GDH theory T if there exists an infinite sequence of GDHs, $\{G_n\}$, such that $b_T(G_n) > \alpha$ for each G_n in the sequence and for any positive integer l there exists a positive integer n_0 such that whenever $n \geq n_0$ then any subGDH $H \subseteq G_n$ on l or fewer vertices has blowup density $b_T(H) \leq \alpha$.*

Theorem 5 *Let T and T' be GDH theories such that $J_{T'} \subseteq J_T$. Let α be a demonstrated nonjump for T. Then $\frac{km_{T'}}{m_T}\alpha$ is a demonstrated nonjump for T' for $k = 1, \ldots, \frac{m_T}{m_{T'}}$.*

Constructions of sequences of undirected r-graphs which show that $\frac{5r!}{2r^r}$ is a demonstrated nonjump for each $r \geq 3$ were given in [1]. This gives the following corollary.

Corollary 2 *Let T be an r-ary GDH theory for $r \geq 3$. Then $\frac{5m_T k}{2r^r}$ is a nonjump for T for $k = 1, \ldots, \frac{r!}{m_T}$.*

This in turn shows that the set of jumps for a theory T' is a proper subset of the set of jumps for T for any T such that $J_{T'} \subseteq J_T$ and $m_T \geq 3m_{T'}$.

Corollary 3 *Let T and T' be r-ary GDH theories such that $J_{T'} \subseteq J_T$ and $m_T \geq 3m_{T'}$. Then there exists an α that is a nonjump for T' and a jump for T.*

Question

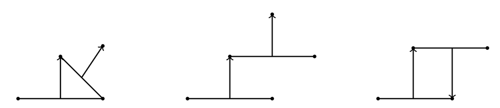
Let T' and T be r-ary GDH theories for $r \geq 3$ such that $J_{T'} \subseteq J_T$ and $m_T = 2m_{T'}$. Does there exists some $\alpha \in [0, 1)$ for which α is a jump for T but not for T'?

Extremal numbers for $(2 \to 1)$-graphs with exactly two edges

A $(2 \to 1)$-graph H is degenerate if its vertices can be partitioned into three sets, $V(H) = T_1 \cup T_2 \cup K$ such that every edge of $E(H)$ is of the form $t_1 t_2 \to k$ for some $t_1 \in T_1$, $t_2 \in T_2$, and $k \in K$.

There are nine different $(2 \to 1)$-graphs with exactly two edges. The extremal numbers for two of these - the one with two nonintersecting edges and the one with two completely intersecting edges - are trivial to find. Of the other seven, four are degenerate and three are nondegenerate.

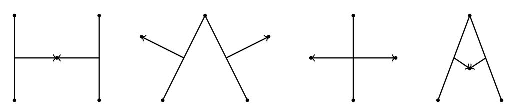
Extremal numbers for the nondegenerate cases



R_3 R_4 E

- For all $n \geq 6$,
$$ex(n, R_3) = \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil \frac{n-2}{2}.$$
Moreover, there is one unique extremal R_3-free construction up to isomorphism for each n.
- For all $n \geq 70$,
$$ex(n, R_4) = \left\lfloor \frac{n}{3} \right\rfloor \binom{\left\lceil \frac{2n}{3} \right\rceil}{2}.$$
Moreover, in each case there is one unique extremal construction up to isomorphism when $n \equiv 0, 1 \bmod 3$ and exactly two when $n \equiv 2 \bmod 3$.
- For all n,
$$ex(n, E) = \binom{n}{3} + 2$$
and there are exactly two extremal construction up to isomorphism for each $n \geq 4$.

Extremal numbers for the degenerate cases



I_0 H_1 H_2 I_2

- For each $n \geq 5$, $\text{ex}(n, I_0) = n(n-2)$.
- For all $n \geq 8$,
$$\text{ex}(n, H_1) = \binom{n+1}{2} - 3.$$
- For all $n \geq 5$,
$$\text{ex}(n, H_2) = \binom{n}{2}.$$
- For all $n \geq 4$,
$$\text{ex}(n, I_1) = n\left\lfloor \frac{n-1}{2} \right\rfloor.$$

References

[1] P. Frankl, Y. Peng, V. Rödl, and J. Talbot. A note on the jumping constant conjecture of Erdős. *Journal of Combinatorial Theory, Series B*, 97(2):204–216, 2007.

[2] M. Langlois. *Knowledge representation and related problems.* PhD thesis, University of Illinois at Chicago, 2010.