Introduction
In 1827, botanist Robert Brownian, through a microscope, observed something curious about the motion of tiny particles suspended in water: the particles, having been ejected from their pollen grain host, moved in a strangely jagged way. Such work attracted the attention of many, and inspired interest in developing a mathematical construct of their behavior.

In this project, we studied the properties of Brownian motion on Riemannian manifolds. In particular, we simulated several realizations of the Wiener process (the name for the mathematical construct of the Brownian motion) on the surface of a sphere, in order to examine Birkhoff’s Ergodic Theorem.

Brownian Motion
A Brownian motion $X_t$ on $t \geq 0$ is a continuous-time stochastic process with the following characteristics:
1. Independent increments. $X_{t+s}-X_t$ is independent of $\sigma\{X_u|u \leq t\}$ for $s \geq 0$.
2. Gaussian increments. $X_{t+s}-X_t \sim N(0, s)$.
3. Continuous paths. With probability 1, $X_t$ is continuous with respect to $t$.

Birkhoff’s Ergodic Theorem
In the context of this project, Birkhoff’s Ergodic Theorem tells us the following: For any real-valued measurable function $f: M \rightarrow \mathbb{R}$, $\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t f(X_s)ds = \int_M f(x)\mu(dx)$ for any compact manifold $M$ and $\mu$ is the normalized volume measure.

Approach
We simulated in both R and Python several Brownian motions on the surface of a sphere in two ways. The following outlines the approach used for the simulations:
1. On the plane tangent to the point $(0, r, 0)$ (referred to as “North pole”) on a sphere centered at the origin, generate a single step of a Brownian motion starting at the point at which the plane and the sphere are in contact.
2. “Smooth” the step onto the sphere in such a way that the length of the step is preserved.
3. Rotate the sphere so that the end of the step is positioned at the North pole.
4. Repeat 1-3 until the desired number of steps is achieved.

Results
Two remarks:
- The images to the bottom-left (Brownian Motion on $S^2$) show that after a sufficient amount of time, the Brownian motion will have visited on every part of the sphere uniformly.
- The graph below also verifies that, after a sufficient amount of time, the time-average of the Brownian motion’s trajectory over $f$ converges to the space average.

Convergence: Testing for Uniform Coverage

Brownian Motion as a Diffusion Process
Initial Simulation: 2 spheres connected at ends of a cylinder
- Radius of Sphere: 1
- Speed Parameter (step size): 0.00083
- Number of steps combined across objects: 30000

Results from the simulation after 10000, 30000, 50000, and 120000 steps

Future Research
Future study branching from this project might have such foci as studying the convergence of a Brownian motion in a diffusion scenario or the convergence of a Brownian motion on Gabriel’s Horn—an object whose volume is finite but whose surface area is infinite (will the motion become “trapped” near the mouthpiece?).

In fact, effort has already been made toward the former of such foci. Considering an object composed of two spheres connected by a bridge whose radius tapers from the sphere-bridge connection to the center of the bridge, we ask: in what way does the rate of convergence of the motion change when 1) the length of the bridge or 2) the opening of the sphere-bridge connection is modified?

It would also be convenient to convert the code used to run the simulations for this project into a tool for studying and/or continuing to develop methods relating to Brownian motion on manifolds.

References