Extremal Problems on Directed Hypergraphs and the Erdős-Gyárfás Ramsey Problem Variant for Graphs

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Overview

Part 1: Extremal Problems on Directed Hypergraphs

- "Extremal Numbers for Directed Hypergraphs with Two Edges," The Electronic Journal of Combinatorics, 25(1), P1.56 (2018).
- "Extremal problems on generalized directed hypergraphs," arXiv:1607.04927 (2016).

Part 2: The Erdős-Gyárfás Ramsey Problem Variant for Graphs

- "A $(5,5)$-colouring of K_n with few colors" (with Emily Heath), to appear in Combinatorics, Probability & Computing (2018).
- "An explicit edge-coloring of K_n with six colors on every K_5," arXiv:1704.01156 (2017).

Part 1

Extremal Problems on Directed Hypergraphs

The Forbidden Subgraph Problem

Definition

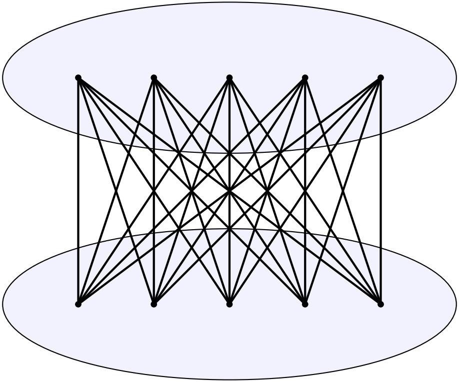
Given a forbidden subgraph F let $\text{ex}(n, F)$ denote the maximum number of edges that a graph on n vertices can have without containing F as a subgraph (not necessarily induced).

The Forbidden Subgraph Problem

For example, the number of edges in a triangle-free graph is at most $\frac{n^2}{4}$.

Theorem (Mantel, 1907)

$ex(n, K_3) = \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil$



The Forbidden Subgraph Problem

Mantel's Theorem extends to forbidden complete graphs of any size.

Theorem (P. Turán, 1941)

$ex(n, K_{r+1}) \approx \left(1 - \frac{1}{r}\right) \frac{n^2}{2}$ *for* $r \geq 2$.

The forbidden subgraph problem extends to hypergraphs.

Definition

Given a forbidden r-uniform hypergraph F let $\mathrm{ex}(n, F)$ denote the maximum number of hyperedges that an r-uniform hypergraph on n vertices can have without containing F as a subgraph (not necessarily induced).

The Forbidden Subgraph Problem

Definition

The Turán density of a forbidden r-uniform hypergraph F is the limit of the edge densities of the extremal F-free hypergraphs as the number of vertices increase,

$$\pi(F) = \lim_{n \to \infty} \frac{\mathrm{ex}(n, F)}{\binom{n}{r}}.$$

- For graphs it is well-known that the chromatic number of a forbidden graph determines its Turán density (Erdős-Stone Theorem, 1946).
- Determining the Turán density of particular forbidden r-uniform hypergraphs is difficult for $r \geq 3$.
- However, it is known that $\pi(F) = 0$ if and only if F is r-partite (Erdős, 1964).

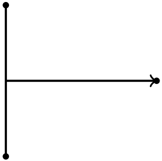
Extremal Digraph Problems

- In 1969, Brown and Harary established the extremal numbers for many "small" examples of forbidden digraphs and determined the extremal numbers for all tournaments and direct sums of tournaments.
- In 1973, Brown, Erdős, and Simonovits determined a general structure of extremal sequences for every forbidden family of digraphs analogous to the Turán graphs for simple graphs.
- In 2017, Kühn, Osthus, Townsend, and Zhao looked at forbidden oriented cycles.
- A nice survey: 'Extremal multigraph and digraph problems' by Brown and Simonovits (2002).

Extremal Directed Hypergraph Problems

- In 1984, Brown and Simonovits looked at r-uniform directed hypergraphs where each edge had a linear order on r vertices.
- In 1993, the graph theoretic properties of a more general definition of a nonuniform directed hypergraph were studied by Gallo, Longo, Pallottino, and Nguyen. They defined a directed hyperedge as some subset of vertices with a partition into head vertices and tail vertices.
- In 2009, Langlois, Mubayi, Sloan, and Turán studied extremal properties of certain small configurations in a directed hypergraph model. This model can be thought of as a $2 \to 1$ directed hypergraph where each edge has three vertices, two of which are "tails" and the third is a "head."

$2 \to 1$ Directed Hypergraphs



Definition

A $(2 \to 1)$-uniform directed hypergraph is defined as $D = (V, E)$ where V is some finite vertex set and the edge set E is a family of pointed 3-subsets of V. That is, each edge has three elements, one of which is distinguished (the "head" vertex) from the others (the "tail" vertices). We say that a $(2 \to 1)$-graph is oriented if it has at most one edge on any three vertices.

$2 \to 1$ Directed Hypergraphs

Definition

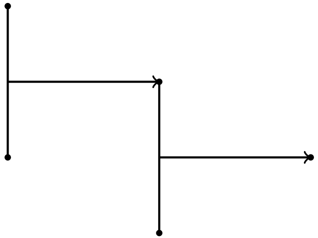
For a given forbidden $2 \to 1$ directed hypergraph F let $\text{ex}(n, F)$ denote the maximum number of edges that an F-free $(2 \to 1)$-graph on n vertices can have. Similarly, let $\text{ex}_o(n, F)$ denote the maximum number of edges that an oriented F-free $(2 \to 1)$-graph on n vertices can have.

Definition

The Turán density of a forbidden $(2 \to 1)$-graph F is the limit of the edge densities of the extremal F-free hypergraphs as the number of vertices increase,

$$\pi(F) = \lim_{n \to \infty} \frac{\text{ex}(n, F)}{3\binom{n}{3}}.$$

$\text{ex}(n, R_4)$



Theorem (Langlois, Mubayi, Sloan, and Turán, 2010)

For sufficiently large n,

$$ex(n, R_4) = \left\lfloor \frac{n}{3} \right\rfloor \binom{\left\lceil \frac{2n}{3} \right\rceil}{2}$$

where $V(R_4) = \{a, b, c, d, e\}$ *and* $E(R_4) = \{ab \to c, cd \to e\}$.

$\mathrm{ex}(n, R_4)$

Theorem (C. (alternate proof), 2018)

For all $n \geq 29$,

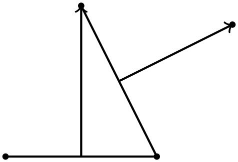
$$ex_o(n, R_4) = \left\lfloor \frac{n}{3} \right\rfloor \binom{\lceil \frac{2n}{3} \rceil}{2}$$

and for all $n \geq 56$,

$$ex(n, R_4) = \left\lfloor \frac{n}{3} \right\rfloor \binom{\lceil \frac{2n}{3} \rceil}{2}.$$

Moreover, in each case there is one unique extremal construction up to isomorphism when $n \equiv 0, 1$ *mod* 3 *and exactly two when* $n \equiv 2$ *mod* 3.

ex(n, R_3)



Theorem (Langlois, Mubayi, Sloan, and Turán, 2009)

Let R_3 denote the $(2 \to 1)$-graph on vertex set $\{a, b, c, d\}$ with edge set $\{ab \to c, bc \to d\}$, then $\pi(R_3) = \frac{1}{4}$.

ex(n, R_3)

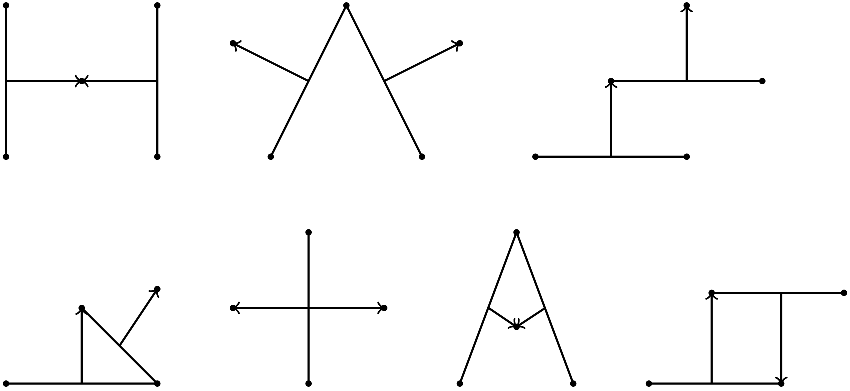
Theorem (C., 2018)

For all $n \geq 6$,

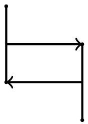
$$ex(n, R_3) = ex_o(n, R_3) = \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil \frac{n-2}{2}.$$

Moreover, there is one unique extremal R_3*-free construction up to isomorphism for each* n.

7 Types of (Nontrivial) Intersection



The Escher Graph



Theorem (C., 2018)

For all n,

$$ex_o(n, E) = \binom{n}{3}$$

and there is exactly one extremal construction up to isomorphism.

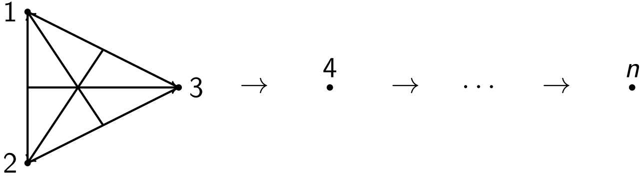
For all n,

$$ex(n, E) = \binom{n}{3} + 2$$

and there are exactly two extremal construction up to isomorphism for each $n \geq 4$.

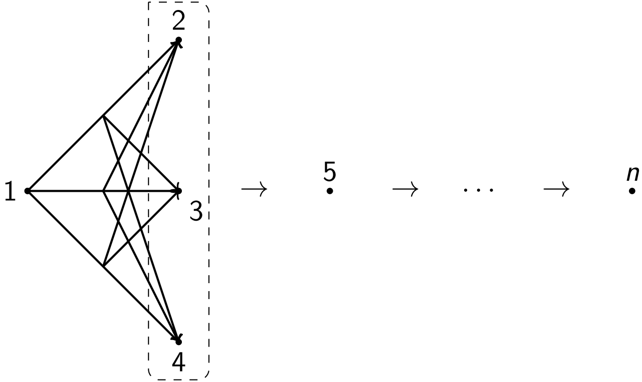
First Construction

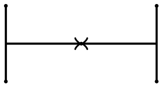
The first construction can be formed from the ordered construction in the oriented case by adding edges $\{1,3\} \to 2$ and $\{2,3\} \to 1$.



Second Construction

The second construction can be formed from the ordered construction in the oriented case by removing the edge $\{2,3\} \to 4$ and adding edges $\{1,3\} \to 2$, $\{1,4\} \to 2$ and $\{1,4\} \to 3$.





Theorem (C., 2018)

For each $n \geq 5$,

$$ex(n, I_0) = n(n-2)$$

and for each $n \geq 6$, *there are exactly* $(n-1)^n$ *different labeled* I_0*-free graphs that attain this maximum number of edges.*

The Degenerate Cases

Theorem (C., 2018)

For all $n \geq 9$,

$$ex_o(n, l_0) = \begin{cases} n(n-3) + \frac{n}{3} & n \equiv 0 \textit{ mod 3} \\ n(n-3) + \frac{n-4}{3} & n \equiv 1 \textit{ mod 3} \\ n(n-3) + \frac{n-5}{3} & n \equiv 2 \textit{ mod 3} \end{cases}$$

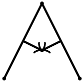
with exactly one extremal example up to isomorphism when $3|n$, exactly 18 non-isomorphic extremal constructions when

$$n \equiv 1 \textit{ mod 3},$$

and exactly 32 constructions when

$$n \equiv 2 \textit{ mod 3}.$$

The Degenerate Cases



Theorem (C., 2018)

For all $n \geq 4$*,*

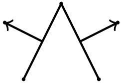
$$ex(n, I_1) = ex_o(n, I_1) = n \left\lfloor \frac{n-1}{2} \right\rfloor$$

and there are

$$\left(\frac{(n-1)!}{2^{\left\lfloor \frac{n-1}{2} \right\rfloor} \left\lfloor \frac{n-1}{2} \right\rfloor !} \right)^n$$

labeled graphs that attain this maximum in the standard case.

The Degenerate Cases



Theorem (C., 2018)

For all $n \geq 6$,

$$ex_o(n, H_1) = \left\lfloor \frac{n}{2} \right\rfloor (n-2).$$

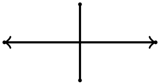
Theorem (C., 2018)

For all $n \geq 8$,

$$ex(n, H_1) = \binom{n+1}{2} - 3.$$

Moreover, there is one unique extremal construction up to isomorphism for each n.

The Degenerate Cases



Theorem (C., 2018)

For all $n \geq 5$,

$$ex(n, H_2) = ex_o(n, H_2) = \binom{n}{2}.$$

Moreover, there are $(n-2)^{\binom{n}{2}}$ *different labeled* H_2*-free graphs attaining this extremal number when in the standard version of the problem.*

GDH definition

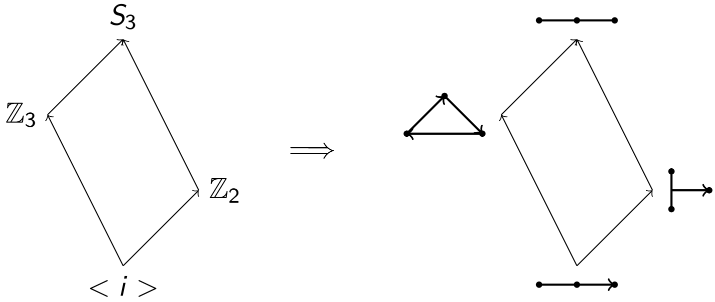
Definition

Let $\mathcal{L} = \{E\}$, a language with one r-ary relation symbol E. Let T be an $\mathcal{L}$-theory that consists of a single sentence of the form

$$\forall x_1 \cdots x_r E(x_1, \ldots, x_r) \implies \bigwedge_{i \neq j} x_i \neq x_j \wedge \bigwedge_{\pi \in J_T} E(x_{\pi(1)}, \ldots, x_{\pi(r)})$$

for some subgroup of the group of permutations on r elements, $J_T \subseteq S_r$. Call such a theory a generalized directed hypergraph theory and any finite model of T is a generalized directed hypergraph (GDH).

Lattice of GDH types when $r = 3$



Turán density

Definition

Given a family of GDHs $\mathcal{F}$ and a positive integer n, let the nth extremal number, $\text{ex}_T(n, \mathcal{F})$, be defined as the maximum number of edges over all $\mathcal{F}$-free GDHs on n elements,

$$\text{ex}_T(n, \mathcal{F}) := \max_{\mathcal{F}\text{-free } G_n} \{e_T(G_n)\}.$$

The Turán density of $\mathcal{F}$ is defined as

$$\pi_T(\mathcal{F}) := \lim_{n \to \infty} \frac{\text{ex}_T(n, \mathcal{F})}{\frac{r!}{m_T}\binom{n}{r}}.$$

Characterization of degenerate forbidden families

Definition

Let G be a GDH with $V_G = \{x_1, \ldots, x_n\}$, and let $t = (t_1, \ldots, t_n)$ be a tuple of positive integers. Define the t-blowup of G to be the $\mathcal{L}$-structure $G(t)$ where

$$V_{G(t)} = \{x_{11}, \ldots, x_{1t_1}, \ldots, x_{n1}, \ldots, x_{nt_n}\}$$

and

$$(x_{i_1 j_1}, \ldots, x_{i_r j_r}) \in E_{G(t)} \iff (x_{i_1}, \ldots, x_{i_r}) \in E_G.$$

Theorem (C., 2016)

Let $\mathcal{F}$ be some family of GDHs, then $\pi_T(\mathcal{F}) = 0$ if and only if some member $F \in \mathcal{F}$ is a subGDH of the t-blowup of a single edge for some vector, $t = (t_1, \ldots, t_r)$, of positive integers. Otherwise, $\pi(\mathcal{F}) \geq \frac{m_T}{r^r}$.

Open Questions

- What are the extremal numbers for tournaments? Conjecture:

$$\mathrm{ex}(n, TT_4) = n\left(\frac{n-1}{2}\right)^2.$$

- What are the exact extremal numbers for $(r \to 1)$-graphs with exactly two edges?
- Characterization of the difference between the standard and oriented extremal numbers.
- Extremal numbers of small cases for other directed hypergraph models .

The Erdős-Gyárfás Ramsey Problem Variant for Graphs

Classical Ramsey Theory

- Color the edges of a complete graph on n vertices red and blue in any way.
- Given two integers $s, t \geq 2$, what is the minimum number of vertices N for which any such coloring of the edges of K_N must yield a red K_s or a blue K_t.
- We say that $N = R(s, t)$, the Ramsey number for s, t.

Classical Ramsey Theory

- This question generalizes to more than 2 colors. Let $R(s_1, \ldots, s_k)$ denote the minimum number of vertices N for which a coloring of the edges of K_N with k colors results in either an s_1-clique in the first color, or an s_2-clique in the second color, etc.
- Diagonal case: Let $R_k(s)$ denote the minimum number of vertices at which any edge coloring with k colors of the complete graph is forced to contain a monochromatic s-clique.

Definition

Definition (Erdős and Shelah; 1975)

A (p, q)-coloring of a graph is coloring of the edges such that every copy of K_p contains at least q distinct colors. Let $f(n, p, q)$ denote the minimum number of colors needed to (p, q)-color of the complete graph on n vertices, K_n.

- $1 \leq q \leq \binom{p}{2}$.
- $f(n, p, 1) = 1$.
- $f\left(n, p, \binom{p}{2}\right) = \binom{n}{2}$ for $p \geq 4$.
- $f(n, 3, 3) \approx n$.

Background

Theorem (Erdős and Gyárfás; 1997)

The Local Lemma gives a general upper bound:

$$f(n,p,q) \le cn^{\frac{p-2}{\binom{p}{2}-q+1}}.$$

Theorem (Erdős and Gyárfás; 1997)

Fix p and let $q = \binom{p}{2} - p + 3$. Then $f(n,p,q) = \Theta(n)$ and $f(n,p,q-1) \le cn^{1-\frac{1}{p-1}}$.

Theorem (Erdős and Gyárfás; 1997)

Fix p and let $q = \binom{p}{2} - \lfloor p \rfloor + 2$. Then $f(n,p,q) = \Theta(n^2)$ and $f(n,p,q-1) \le cn^{2-\frac{4}{p}}$.

Background

Erdős and Gyárfás gave a simple induction argument which demonstrates that

$$n^{\frac{1}{p-2}} - 1 \leq f(n,p,p),$$

the smallest value of q for which they could find a polynomial lower bound. They also considered several cases for small p:

- $\frac{5(n-1)}{6} \leq f(n,4,5) \leq n$.
- $n^{1/2} - 1 \leq f(n,4,4) \leq cn^{2/3}$ - one of the "most interesting" cases.
- $f(n,4,3) \leq cn^{1/2}$ - the "most annoying" case since unsure if it is even polynomial at all.
- $cn \leq f(n,5,9) \leq cn^{3/2}$ - the other "most interesting" case to see whether this is linear or not.

(5, 9)-coloring

Theorem

$$\frac{11}{4}n - \frac{23}{4} \leq f(n, 5, 9) \leq 2n^{1+\frac{c}{\sqrt{\log n}}}$$

- Upper bound: Axenovich; 2000.
- Lower bound: Krop; 2008.

$(p, p-1)$-coloring

Theorem (Mubayi; 1998)

$$f(n,4,3) \leq e^{\sqrt{c \log n}(1+o(1))}.$$

Theorem (Conlon, Fox, Lee, and Sudakov; 2015)

$$f(n,p,p-1) \leq 2^{16p(\log n)^{1-1/(p-2)} \log \log n}.$$

This shows that $q = p$ is the threshold at which $f(n,p,q)$ becomes polynomial in n.

$(4,4)$-coloring

Theorem (Mubayi; 2004)

$$f(n,4,4) \leq n^{1/2} e^{c\sqrt{\log n}}.$$

- This shows that $n^{1/2} \leq f(n,4,4) \leq n^{1/2+o(1)}$.
- Uses the product of two explicit colorings:
  - the construction showing that $f(n,4,3)$ is subpolynomial, and
  - an algebraic coloring which associates each vertex with a vector in $\mathbb{F}_q^2$ and uses a symmetric map $\mathbb{F}_q^2 \times \mathbb{F}_q^2 \to \mathbb{F}_q$ to color the edges.

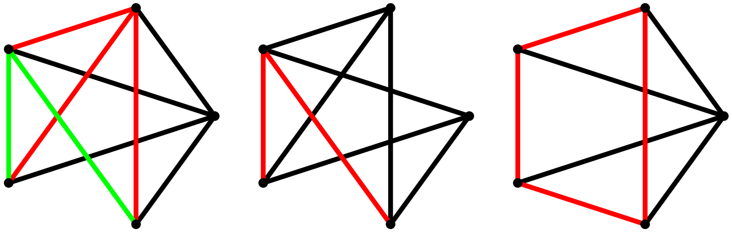
(5, 5)-coloring

Theorem (C. and Heath; 2017)

$$f(n,5,5) \leq n^{1/3} 2^{c\sqrt{\log n}\log\log n}.$$

- This shows that $n^{1/3} \leq f(n,5,5) \leq n^{1/3+o(1)}$.
- Uses the product of two explicit colorings:
  - the construction by Conlon, Fox, Lee, and Sudakov (CFLS) using $n^{o(1)}$ colors, and
  - an algebraic coloring which associates each vertex with a vector in $\mathbb{F}_q^3$ and uses a symmetric map $\mathbb{F}_q^3 \times \mathbb{F}_q^3 \to \mathbb{F}_q$ to color the edges.

Three configurations not avoided by the modified CFLS coloring.



Modified Inner Product (MIP) Coloring

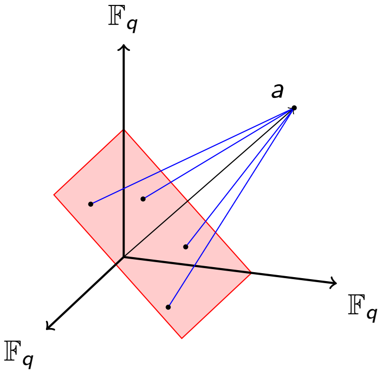
Let q be some odd prime power, and let $\mathbb{F}_q^*$ denote the nonzero elements of the finite field with q elements. The vertices of our graph will be the three-dimensional vectors over this set,

$$V = \left(\mathbb{F}_q^*\right)^3.$$

The explicit definition of the coloring is a bit technical, but it is essentially the inner product of the the two vectors with several modifications to take care of certain cases.

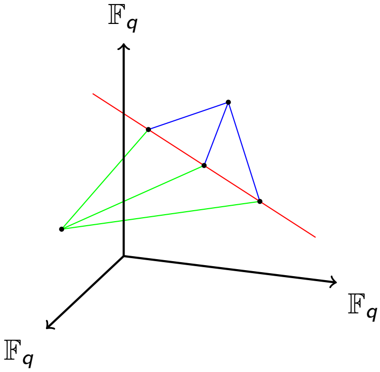
Monochromatic Neighborhoods

Given a vertex $a \in \left(\mathbb{F}_q^*\right)^3$ and a color $\alpha \in \mathbb{F}_q$, the monochromatic α-neighborhood of a is contained within an affine plane in $\mathbb{F}_q^3$.



Intersection of Monochromatic Neighborhoods

"Most" of the time the intersection of two monochromatic neighborhoods defines a subset of an affine line.



$(5,6)$-coloring

Theorem (C., 2017)

As $n \to \infty$,

$$\left(\frac{5}{6}n - \frac{95}{144}\right)^{1/2} \le f(n,5,6) \le n^{1/2}2^{O\left(\sqrt{\log n}\log\log n\right)}.$$

- This shows that $n^{1/2} \le f(n,5,6) \le n^{1/2+o(1)}$.
- Combines an adjusted version of the algebraic part of Mubayi's $(4,4)$-coloring with the modified version of the CFLS coloring.

Open Problems

- Is $f(n, p, p) \leq n^{1/(p-2)+o(1)}$ in general?
- Hypergraph version
- Tighten other small cases: $cn^{2/3} \leq f(n, 5, 7) \leq cn^{3/4}$