Extremal Problems on Directed Hypergraphs: Forbidden Subgraphs with Two Edges

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The Forbidden Subgraph Problem

Given a family of forbidden subgraphs, $\mathcal{F}$, let $ex(n, \mathcal{F})$ denote the maximum number of edges that a graph on n vertices can have without containing any member of $\mathcal{F}$ as a subgraph.

The Forbidden Subgraph Problem

For example, the number of edges in a triangle-free graph is at most $\frac{n^2}{4}$.

Theorem (Mantel, 1907)

$ex(n, K_3) = \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil$

And this generalizes to any size clique.

Theorem (P. Turán, 1941)

$ex(n, K_{r+1}) \approx \left(1 - \frac{1}{r}\right) \frac{n^2}{2}$ *for* $r \geq 2$.

Hypergraph, Digraph, etc Turán Problems

Apply this idea to other structures. For r-uniform hypergraphs you get a bunch of big open problems.

Extremal Digraph Problems

- 1969: Brown and Harary established the extremal numbers for many "small" examples of forbidden digraphs and determined the extremal numbers for all tournaments and direct sums of tournaments.
- A nice survey: 'Extremal multigraph and digraph problems' by Brown and Simonovits (2002).

Directed Hypergraphs

Definition

A $(k \to 1)$-uniform directed hypergraph is defined as $D = (V, E)$ where V is some finite vertex set and the edge set, E, is a family of pointed $(k+1)$-subsets of V. That is, each edge has $k+1$ elements, one of which is distinguished (the "head" vertex) from the others (the "tail" vertices).
For a given family, $\mathcal{F}$, of forbidden $(k \to 1)$-digraphs let the extremal number, $\mathrm{ex}_{k\to 1}(n, \mathcal{F})$, be the maximum number of edges that a $(k \to 1)$-digraph on n vertices could have without containing any member of $\mathcal{F}$ as a subgraph.

Extremal Problems Results

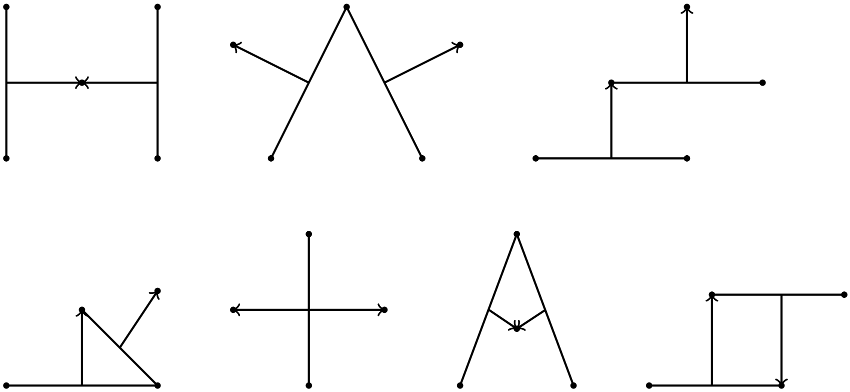
Combinatorial problems for this model of directed hypergraphs were considered by Langlois, Mubayi, Sloan, and Gy. Turán starting in 2009.

Theorem (Langlois, Mubayi, Sloan, and Gy. Turán, 2010)

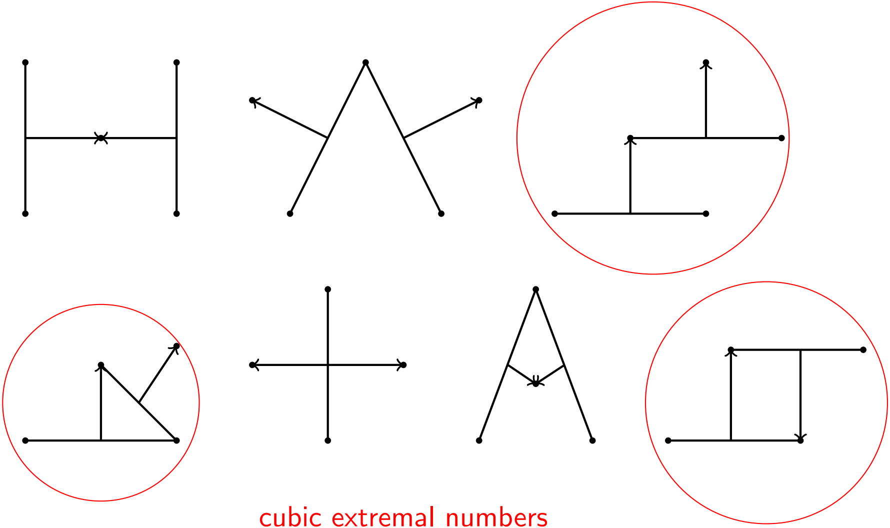
$$ex_{2\to 1}(n, F) = n\binom{\frac{n}{2}}{2}$$

where $V(F) = \{a, b, c, d\}$ *and* $E(F) = \{ab \to c, bc \to d\}$.

7 Types of (Nontrivial) Intersection



7 Types of (Nontrivial) Intersection



cubic extremal numbers

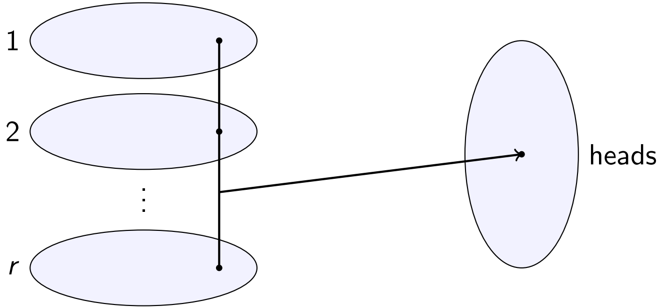
Theorem (C., 2015)

Determined the extremal numbers for every $(2 \to 1)$-graph with exactly two edges.

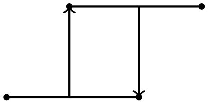
Turán density Result

Theorem (C., 2016)

For any forbidden $(r \to 1)$-graph, F, $\pi(F) = 0$ if and only if there is an $(r+1)$-coloring of the vertices of F such that every edge of F contains one vertex from each color class and all 'head' vertices of the edges belong to their own class.

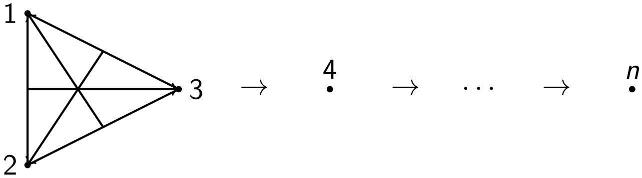


Two Lower Bound Constructions for The Escher Graph



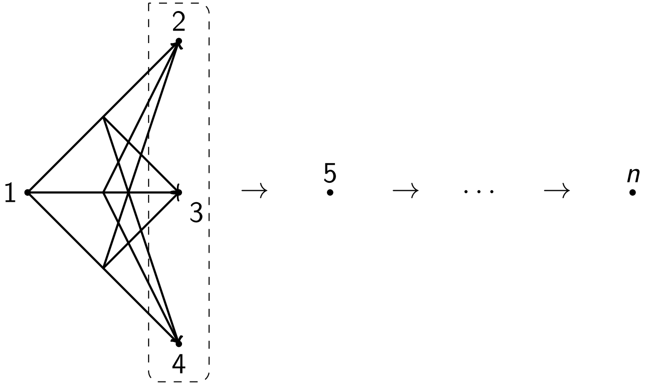
- The extremal number is $\binom{n}{3}$ for the 'oriented' version, and the unique lower bound construction is given where every triple of vertices have an edge oriented towards the largest vertex under some linear ordering of all of the vertices.
- However, there are exactly two distinct (up to isomorphism) constructions when we allow multiple edges on each triple, and the extremal number only increases by two: $\binom{n}{3} + 2$.

The first construction can be formed from the ordered construction in the oriented case by adding edges $\{1,3\} \to 2$ and $\{2,3\} \to 1$.



Second Construction

The second construction can be formed from the ordered construction in the oriented case by removing the edge $\{2,3\} \to 4$ and adding edges $\{1,3\} \to 2$, $\{1,4\} \to 2$ and $\{1,4\} \to 3$.



Upper Bound

Very deep mathematics that none of you have ever heard of

Questions

- What are the extremal numbers for tournaments?
- Conjecture: $ex_{2\to 1}(n, TT_4) = n\left(\frac{n-1}{2}\right)^2$.
- What are the exact extremal numbers for $(r \to 1)$-graphs with exactly two edges?
- Which results could carry over to other similar definitions of directed hypergraphs?