Answer Key

Math 215 - Introduction to Advanced Mathematics

Midterm Exam

Spring 2018

1. (5 points.) Let $A = \{1, 2\}$ and $B = \{2, 3, 4\}$. What is $A \cup B$? What is $B \cap A$?

$$A \cup B = \{1, 2, 3, 4\}$$
$$B \cap A = \{2\}$$

2. (5 points.) Let $A = \{3, 9, 27\}$. What is the power set $\mathcal{P}(A)$?

$$\mathcal{P}(A) = \{\emptyset, \{3\}, \{9\}, \{27\}, \{3, 9\}, \{3, 27\}, \{9, 27\}, A\}$$

3. (5 points.) Write the converse for the statement "If $a|b$ or $a|c$, then $a|bc$."

If $a | bc$, then $a|b$ or $a|c$.

4. (5 points.) Let $P(x, y)$ and $Q(x, y)$ be predicates. Give the negation of

$$\forall x \in S, \exists y \in S, (P(x, y) \vee Q(x, y)).$$

$$\exists x \in S, \forall y \in S, (\neg P(x,y) \wedge \neg Q(x,y))$$

5. (5 points.) List the elements of the set

$$\{m \in \mathbb{Z}^+ | \forall n \in \mathbb{Z}^+, m - 1 \leq n\}.$$

$$\{1, 2\}$$

6. (25 points.) Let A, B, and C be sets, and let $b \in B$. Prove by **contradiction** that if $A \cap B \subseteq C$, then $b \notin A - C$.

Let $A \cap B \subseteq C$ and let $b \in B$. Assume, towards a contradiction, that $b \in A - C$. Then $b \in A$ and $b \notin C$. Since $b \in A$ and $b \in B$, then $b \in A \cap B$. Since $A \cap B \subseteq C$, then $b \in C$. So $b \in C$ and $b \notin C$, a contradiction. Hence, $b \notin A - C$ ∎

7. (25 points.) Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = 2x - 3$. Prove that g is a bijection.

<u>g is injective</u>: Let $x_1, x_2 \in \mathbb{R}$ such that $g(x_1) = g(x_2)$. Then $2x_1 - 3 = 2x_2 - 3$. So $2x_1 = 2x_2$. Therefore, $x_1 = x_2$. Hence, g is injective.

<u>g is surjective</u>: Let $y \in \mathbb{R}$. Then $\frac{y+3}{2} \in \mathbb{R}$, and

$$g\left(\frac{y+3}{2}\right) = 2\left(\frac{y+3}{2}\right) - 3 = y.$$

Thus, g is surjective.

Since g is injective and surjective, then it follows that g is bijective $\blacksquare$

8. (25 points.) Prove **by induction** on n that for all $n \in \mathbb{Z}^+$,

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

(base case) Let $n = 1$. Then $\sum_{i=1}^{1} i = 1$

and $\frac{1(1+1)}{2} = 1$. No problem.

(induction step) Assume for some $k \geq 1$ that

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2}.$$

Then

$$\sum_{i=1}^{k+1} i = \left(\sum_{i=1}^{k} i\right) + (k+1)$$

$$= \frac{k(k+1)}{2} + k+1$$

$$= \frac{k(k+1) + 2k+2}{2}$$

$$= \frac{k^2+3k+2}{2} = \frac{(k+1)(k+2)}{2} \quad \boxtimes$$