Math 215 - Introduction to Advanced Mathematics

Midterm Exam Take 2

Spring 2018

1. (5 points.) Let $S = \{a, b, c, d\}$ and $T = \{a, c, e\}$. What is $T - S$? What is $T \cap S$?

$$T - S = \{e\}$$

$$T \cap S = \{a, c\}$$

2. (5 points.) What is $\mathcal{P}(\emptyset)$?

$$\mathcal{P}(\emptyset) = \{\emptyset\}$$

3. (5 points.) Write the contrapositive for the statement "If $A \neq B$ and A is very loud, then it is not raining."

If it is raining, then either $A = B$ or A is not very loud.

4. (5 points.) Let $P(x, y)$ and $Q(x, y)$ be predicates. Give the negation of

$$\exists x \in S, \forall y \in S, (P(x, y) \vee \neg Q(x, y)).$$

$$\forall x \in S\ \exists y \in S\ (\neg P(x,y) \wedge Q(x,y))$$

5. (5 points.) List the elements of the set

$$\{(m, n) \in \mathbb{Z}^+ \times \mathbb{Z}^+ | \forall p \in \mathbb{Z}^+, m + n \leq p + 2\}.$$

$$\{(1,1), (1,2), (2,1)\}$$

6. (25 points.) Prove for any sets A, B, and C, that

$$(A \cap C) - B = (A - B) \cap C.$$

Let $x \in (A \cap C) - B$, then $x \in A \cap C$ and $x \notin B$. So $x \in A$. Therefore, $x \in A - B$. Also, $x \in C$. Hence, $x \in (A-B) \cap C$.

Conversely, let $x \in (A-B) \cap C$. Then $x \in A - B$ and $x \in C$. So $x \in A$ and $x \notin B$. Thus, $x \in A \cap C$. So $x \in (A \cap C) - B$ $\blacksquare$

7. (25 points.) Let $g : \mathbb{Z}^+ \to \mathbb{Z}^+$ be defined by $g(x) = x + 1$. Is g surjective, injective, both, or neither? Why or why not?

g is <u>not</u> surjective since $1 \in \mathbb{Z}^+$, but if $g(x) = 1$, then $x + 1 = 1$ so $x = 0$ and $0 \notin \mathbb{Z}^+$.

g is injective. Let $a, b \in \mathbb{Z}^+$ such that $g(a) = g(b)$. Then $a + 1 = b + 1$. Subtract 1 from both sides to see that $a = b$.

8. (25 points.) Let h_n be defined by $h_0 = 0$ and $h_n = 3h_{n-1} + 3^{n-1}$ for all integers $n \geq 1$. Prove **by induction** that $h_n = n3^{n-1}$ for all integers $n \geq 0$.

base case: $n=0$ then $n3^{n-1} = 0 \cdot 3^{-1} = 0 = h_0$ ✓

induction step: Assume that $h_k = k \cdot 3^{k-1}$ for some $k \geq 0$. Then

$$h_{k+1} = 3h_{(k+1)-1} + 3^{(k+1)-1}$$
$$= 3h_k + 3^k$$
$$= 3(k \cdot 3^{k-1}) + 3^k$$
$$= k \cdot 3^k + 3^k$$
$$= (k+1) \cdot 3^{(k+1)-1}$$