1. In general we need to find the biggest integer n such that $f(n) \leq t$ where t is in microseconds.

So for instance, when $f(n) = \lg n$, then

$$\lg n \leq t$$

$$\downarrow$$

$$n \leq 2^t \quad \text{since} \quad 2^{\lg n} = n$$

Now we can just plug in the time in microseconds we're interested in for t. So if it's a day, then

"microseconds"

$$t = 1\,\text{day} \cdot \frac{24 \text{ hours}}{1 \text{ day}} \cdot \frac{60 \text{ minutes}}{1 \text{ hour}} \cdot \frac{60 \text{ seconds}}{1 \text{ minute}} \cdot \frac{10^6 \mu s}{1\,s}$$

$$t = 86{,}400 \cdot 10^6 \ \mu s$$

$$t = 864 \times 10^8 \ \mu s$$

Therefore,

$$n \leq \underbrace{2^{(864 \times 10^8)}}$$

this is very big but it should be clear that it is an integer.

So $n = 2^{(864 \times 10^8)}$ is the largest n we can put into our problem if we want a solution in a day.

Here are the general bounds on n for different functions:

$f(n)$	max n
$\lg n$	2^t
$\sqrt{n}$	t^2
n	t
$n \lg n$	??
n^2	$\sqrt{t}$
n^3	$\sqrt[3]{t}$
2^n	$\lg t$
$n!$	??

a couple of these are not so easy to solve explicitly:

$$n \lg n \leq t \quad \text{and} \quad n! \leq t$$

At this point most of you are probably much better than me at writing good code, but here is the simplest approach to searching for n for a give time t:

$n \lg n \le t$

```
t := Input
n := 1
m := 0
while m ≤ t:
    m := n · lg n
    n++
Output n - 1
```

$n! \le t$

```
t := Input
n := 1
m := 1
while m ≤ t:
    m := m · n
    n++
Output n - 1
```

Of course, there's any number of ways to make these codes smarter. You can search w/ powers of 2 or 10 or whatever and then narrow your focus. Or you can note that in both cases $n < t$ since $t \log t > t$ and $t! > t$ so you could search backwards from there or something. Whatever works.

② $\boxed{A.1-1}$ *(I did not understand what he meant by a and b in the assignment so I only did each of these one way. Only later did I figure out what he meant. Sorry)

$$\sum_{k=1}^{n}(2k-1) = 2\underbrace{\sum_{k=1}^{n}k}_{\text{arithmetic}} - \sum_{k=1}^{n}1 = 2\left(\frac{1}{2}n(n+1)\right) - n = n^2+n-n = \underline{\underline{n^2}}$$

$\boxed{A.1-2}$

Note that $\displaystyle\sum_{k=1}^{n}\frac{1}{2k-1} = 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1}$, the "odd" terms of the harmonic series $\displaystyle\sum_{k=1}^{2n}\frac{1}{k}$.

So when added to the "even" terms we get

$$\sum_{k=1}^{n}\frac{1}{2k-1} + \underbrace{\sum_{k=1}^{n}\frac{1}{2k}}_{} = \sum_{k=1}^{2n}\frac{1}{k} = \ln(2n) + O(1)$$

$$\sum_{k=1}^{n}\frac{1}{2k} = \frac{1}{2}\sum_{k=1}^{n}\frac{1}{k}$$

$$= \frac{1}{2}\left(\ln(n) + O(1)\right)$$

So

$$\sum_{k=1}^{n}\frac{1}{2k-1} = \ln(2n) + O(1) - \frac{1}{2}\left(\ln(n) + O(1)\right)$$

(log property)

$$= \ln(2) + \ln(n) - \frac{1}{2}\ln(n) + O(1) - \frac{1}{2}O(1)$$

$$= \frac{1}{2}\ln(n) + \underbrace{\left[O(1) - \tfrac{1}{2}O(1) + \ln(2)\right]}_{}$$

(log property)

$$= \ln(\sqrt{n}) + O(1)$$

✓

$\boxed{A.1-3}$

we already know that $\sum_{k=0}^{\infty} k x^k = \frac{x}{(1-x)^2}$ when $0 < |x| < 1$

so $$\frac{d}{dx}\left(\sum_{k=0}^{\infty} k x^k\right) = \frac{d}{dx}\left(\frac{x}{(1-x)^2}\right)$$

$$\sum_{k=0}^{\infty} k^2 x^{k-1} = \frac{1}{(1-x)^2} + x \cdot -2 \cdot \frac{1}{(1-x)^3} \cdot -1$$

$$= \frac{(1-x) + 2x}{(1-x)^3} = \frac{1+x}{(1-x)^3}$$

$$\sum_{k=0}^{\infty} k^2 x^{k-1} = \frac{1+x}{(1-x)^3}$$

$$x\left(\sum_{k=0}^{\infty} k^2 x^{k-1}\right) = \frac{x(1+x)}{(1-x)^3}$$

$$\sum_{k=0}^{\infty} k^2 x^k = \frac{x(1+x)}{(1-x)^3}$$

$\boxed{\text{A.1-4}}$

$$\sum_{k=0}^{\infty} \frac{k-1}{2^k} = \sum_{k=0}^{\infty} k\left(\frac{1}{2}\right)^k - \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k$$

$$= \frac{(1/2)}{(1-1/2)^2} - \frac{1}{1-1/2} \Leftarrow$$

here use geometric series where $x = 1/2$

$$= \frac{1/2}{1/4} - 2 = 2 - 2 = 0$$

$\boxed{\text{A.1-5}}$

$$\sum_{k=1}^{\infty} (2k+1)x^{2k} = \frac{d}{dx}\left(\sum_{k=1}^{\infty} x^{2k+1} \right)$$

$$= \frac{d}{dx}\left(x \sum_{k=1}^{\infty} x^{2k} \right)$$

$$= \frac{d}{dx}\left(x \sum_{k=1}^{\infty} (x^2)^k \right)$$

(for $|x|<1$)

$$= \frac{d}{dx}\left(x \cdot \left(\frac{1}{1-x^2} - 1 \right) \right)$$

$$\cancel{= \frac{d}{dx}\Big(}$$

$$= \left(\frac{1}{1-x^2} - 1 \right) + x\left(-\frac{1}{(1-x^2)^2} \cdot (-2x) \right)$$

$$= \frac{2x^2}{(1-x^2)^2} + \frac{1}{1-x^2} - 1 = \frac{x^2(3-x^2)}{(1-x^2)^2}$$

$\boxed{A.1-6}$

$$\sum_{k=1}^{n} O(f_k(i)) = \sum_{k=1}^{n} g_k(i)$$

where $0 \le g_k(i) \le c_k f_k(i)$ for all $i \ge i_k$
for some positive constants c_k
and i_k (def of O on pg 47)

~~So~~ and

$$\sum_{k=1}^{n} g_k(i) \le \sum_{k=1}^{n} c_k f_k(i) \quad \forall\ i \ge i_0$$

where $i_0 = \max\{i_1, \ldots, i_k\}$

and

$$\sum_{k=1}^{n} c_k f_k(i) \le \sum_{k=1}^{n} c f_k(i) \quad \forall\ i \ge i_0$$

where $c = \max\{c_1, \ldots, c_k\}$

and

$$\sum_{k=1}^{n} c f_k(i) = c \sum_{k=1}^{n} f_k(i)$$

So

$$\sum_{k=1}^{n} O(f_k(i)) \le c \sum_{k=1}^{n} f_k(i) \quad \forall\ i \ge i_0$$

$$\Rightarrow \sum_{k=1}^{n} O(f_k(i)) = O\left(\sum_{k=1}^{n} f_k(i) \right)$$

$\square$

$\boxed{A.1-7}$

$$\prod_{k=1}^{n} 2\cdot 4^k = 2^{\lg\left(\prod_{k=1}^{n} 2\cdot 4^k\right)} = 2^{(n^2+2n)}$$

since

$$\lg\left(\prod_{k=1}^{n} 2\cdot 4^k\right) = \sum_{k=1}^{n} \lg(2\cdot 4^k) = \sum_{k=1}^{n}\left[\lg(2) + \lg(4^k)\right]$$

$$= n\lg(2) + \sum_{k=1}^{n} k\lg(4) = n + \sum_{k=1}^{n} 2k$$

$$= n + 2\left(\frac{1}{2}n(n+1)\right) = n^2 + 2n$$

$\boxed{A.1-8}$

$$\prod_{k=2}^{n}(1 - 1/k^2) = 2^{\lg\left(\prod_{k=2}^{n}(1-1/k^2)\right)} = 2^{\lg\left(\frac{n+1}{n}\right) - 1} = \frac{2^{\lg\left(\frac{n+1}{n}\right)}}{2}$$

$$\lg\left(\prod_{k=2}^{n}(1 - 1/k^2)\right) = \sum_{k=2}^{n} \lg(1 - 1/k^2) = \sum_{k=2}^{n} \lg\left(\frac{k^2-1}{k^2}\right)$$

$$= \sum_{k=2}^{n} \lg\left(\frac{(k+1)(k-1)}{k^2}\right) = \sum_{k=2}^{n}\left[\lg(k+1) + \lg(k-1) - \lg(k) - \lg(k)\right]$$

$$= \sum_{k=2}^{n}\left[\left(\lg(k+1) - \lg(k)\right) - \left(\lg(k) - \lg(k-1)\right)\right]$$

telescoping

$$= \lg(n+1) - \lg(n) - \lg(2) + \lg(1)$$

$$= \lg\left(\frac{n+1}{n}\right) - 1 + 0$$

$$= \frac{n+1}{2n}$$

③ $\boxed{\text{A.2-1}}$ The trick is to break this one up into parts.

Let $\ell = \lfloor \lg(n) \rfloor \Rightarrow \ell \le \lg(n) < \ell + 1$

$$\Downarrow$$

$$2^{\ell} \le n < 2^{\ell+1}$$

the parts we want are between each power of 2:

first part: 1 $\quad (2^0 \le x < 2^1)$

second part: $2, 3$ $\quad (2^1 \le x < 2^2)$

third part: $4, 5, 6, 7$ $\quad (2^2 \le x < 2^3)$

$\vdots$ and so on until

$(\ell+1)$th part: $2^{\ell}, 2^{\ell}+1, \dots, 2^{\ell+1}-1$

So each part has 2^i terms for $i = 0, 1, \dots, \ell$

since $2^{i+1} - 2^i = 2^i(2-1) = 2^i$.

(each term of)
a particular part of our sequence is bounded by its largest term:

$$\frac{1}{(2^i)^2} + \frac{1}{(2^i+1)^2} + \dots + \frac{1}{(2^{i+1}-1)^2} < 2^i \cdot \frac{1}{(2^i)^2}$$

$$= \frac{1}{2^i}$$

Therefore,

$$\sum_{k=1}^{n} 1/k^2 < \sum_{i=0}^{\ell} 1/2^i < \sum_{i=0}^{\infty} 1/2^i$$

← this is the geometric series for $x = 1/2$

$$= \frac{1}{1 - 1/2} = 2$$

So $\sum_{k=1}^{n} 1/k^2 < 2$ for any n.

$\boxed{A.2-2}$ ~~Again~~ let $\ell = \lceil \lg n \rceil$ (sorry about the scratch outs)

so $2^{\ell-1} \le n \le 2^{\ell}$

which means that $\left\lceil \frac{n}{2^k} \right\rceil \le \left\lceil \frac{2^{\ell}}{2^k} \right\rceil = 2^{\ell-k}$ (an integer since $k \le \lfloor \lg n \rfloor = \ell - 1$)

so $\sum_{k=0}^{\lfloor \lg n \rfloor} \left\lceil \frac{n}{2^k} \right\rceil \le \sum_{k=0}^{\ell-1} 2^{\ell-k} = \sum_{k=1}^{\ell} 2^k$

we can show that $\sum_{k=1}^{\ell} 2^k = O(2^{\ell})$

by showing that $\exists\ c > 0$ such that $\sum_{k=1}^{\ell} 2^k \le c \cdot 2^{\ell}$

for all $\ell > 0$.

<u>base case</u>: $\ell = 1 \Rightarrow 2 \le c \cdot 2$ so true for any $c \ge 1$.

<u>induction step</u>: assume $\sum_{k=1}^{\ell} 2^k \le c 2^{\ell}$

then $\sum_{k=1}^{\ell+1} 2^k \le c 2^{\ell} + 2^{\ell+1} = c \cdot 2^{\ell+1}\left(\frac{1}{2} + \frac{1}{c}\right)$

we want: $c \cdot 2^{\ell+1}\left(\frac{1}{2} + \frac{1}{c}\right) \le c \cdot 2^{\ell+1}$

so we need: $\left(\frac{1}{2} + \frac{1}{c}\right) \le 1$

which we get if $\frac{c+2}{2c} \le 1 \leftrightarrow c + 2 \le 2c \longleftrightarrow c \ge 2$.

Hence $\sum_{k=1}^{\ell} 2^k = O(2^{\ell})$ so $\sum_{k=0}^{\lfloor \lg n \rfloor} \lceil n/2^k \rceil = O(2^{\lceil \lg n \rceil})$.

$\boxed{A.2-3}$ To show that $\sum_{k=1}^{n} \frac{1}{k} = \Omega(\lg n)$

we need to show that $\exists$ $c > 0$ and a n_0 such that

$$\sum_{k=1}^{n} \frac{1}{k} \geq c \lg n \quad \forall n \geq n_0.$$

Let $l = \lfloor \lg n \rfloor$

So we can break into sums:

$$1$$
$$1/2 + 1/3$$
$$1/4 + 1/5 + 1/6 + 1/7$$
$$\vdots$$
$$1/2^{l-1} + \cdots + 1/2^{l}-1$$
$$1/2^{l} + \cdots + 1/n$$

Since,

$$1/2^i + 1/2^{i}+1 + \cdots + 1/2^{i+1}-1 \geq 2^i \cdot \left(\frac{1}{2^{i+1}-1}\right) > \frac{2^i}{2^{i+1}} = \frac{1}{2}$$

then $\sum_{k=1}^{n} \frac{1}{k} > \sum_{i=0}^{l-1} \frac{1}{2} + \left(1/2^{l} + \cdots + 1/n\right) > \sum_{i=0}^{l-1} \frac{1}{2} = \frac{1}{2} l$

So $\sum_{k=1}^{n} \frac{1}{k} > \frac{1}{2} \lfloor \lg n \rfloor$

since $\lg n - \lfloor \lg n \rfloor < 1$ then $\lg n - 1 < \lfloor \lg n \rfloor$

So $\sum_{k=1}^{n} \frac{1}{k} > \frac{1}{2}(\lg n - 1)$

we need only find some constant c such that

$\frac{1}{2}(\lg n - 1) \geq c \lg n$ for "big enough" n

So $$c \le \frac{\frac{1}{2}\lg(n) - \frac{1}{2}}{\lg(n)} = \frac{1}{2} - \frac{1}{2\lg(n)}$$

since $\lg(n)$ is a strictly increasing function then this bound increases to $\frac{1}{2}$ as $n \to \infty$

So whatever, just take $n_0 = 4$ or something and we get that $c \le \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

Then for any $0 < c \le \frac{1}{4}$ and all $n \ge 4$ we have

$$\tfrac{1}{2}(\lg n - 1) \ge c \lg n$$

$$\Downarrow$$

$$\sum_{k=1}^{n} \tfrac{1}{k} > c \lg n$$

$$\Downarrow$$

$$\sum_{k=1}^{n} \tfrac{1}{k} = \Omega(\lg n) \qquad \square$$

$\boxed{\text{A.2-4}}$

$$\int_0^n x^3\,dx \le \sum_{k=1}^{n} k^3 \le \int_1^{n+1} x^3\,dx$$

$$\frac{1}{4}x^4 \Big|_0^n \le \sum_{k=1}^{n} k^3 \le \frac{1}{4}x^4 \Big|_1^{n+1}$$

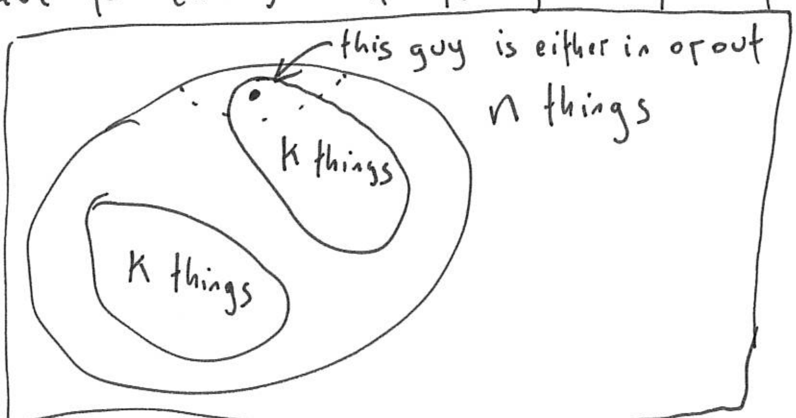
$$\frac{1}{4}n^4 \le \sum_{k=1}^{n} k^3 \le \frac{1}{4}(n+1)^4 - \frac{1}{4}$$

$\boxed{\text{A.2-5}}$ to avoid dividing by zero

4

a) Pascal's Identity says $\boxed{\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}}$

which I always remember by thinking about how many ways I can pick k things from a set of n things. If I grab one of the things, then if it is going to be in my k things then that leaves me with $\binom{n-1}{k-1}$ choices of how to grab the rest. If it isn't going to be in my k things, then I have to choose all k from the other n-1 things.



So now the problem:

induction on n

base case: $n = 0 \rightarrow 2^0 = 1$ and $\sum_{i=0}^{0} \binom{0}{i} = \binom{0}{0} = 1$ ✓

induction step: assume true for $n = k$

want to show that it's true for $n = k+1$

So we assume that

$$2^k = \binom{k}{0} + \binom{k}{1} + \cdots + \binom{k}{k}$$

multiply both sides by 2:

$$2^{k+1} = 2\left[\binom{k}{0} + \cdots + \binom{k}{k}\right]$$

$$2^{k+1} = \binom{k}{0} + \binom{k}{1} + \binom{k}{2} + \cdots + \binom{k}{k-1} + \binom{k}{k}$$
$$+ \binom{k}{0} + \binom{k}{1} + \cdots + \binom{k}{k-2} + \binom{k}{k-1} + \binom{k}{k}$$

apply Pascal's Id:

$$2^{k+1} = \binom{k}{0} + \binom{k+1}{1} + \binom{k+1}{2} + \cdots + \binom{k+1}{k-1} + \binom{k+1}{k} + \binom{k}{k}$$

Note that $\binom{k}{0} = \binom{k}{k} = \binom{k+1}{0} = \binom{k+1}{k+1} = 1$, so

$$2^{k+1} = \binom{k+1}{0} + \cdots + \binom{k+1}{k+1} = \sum_{i=0}^{k+1} \binom{k+1}{i} \quad \square$$

b) <u>base case</u>

$n = 0 \rightarrow \quad (x+y)^0 = 1$

and

$$\sum_{i=0}^{0} \binom{0}{i} x^i y^{0-i} = \binom{0}{0} x^0 y^0 = 1$$

✓

induction step

assume true for $n = k$:

$$(x+y)^k = \sum_{i=0}^{k} \binom{k}{i} x^i y^{k-i}$$

expand RHS and multiply both sides by $x+y$:

$$(x+y)^{k+1} = (x+y)\left[\binom{k}{0}y^k + \binom{k}{1}xy^{k-1} + \cdots + \binom{k}{k-1}x^{k-1}y + \binom{k}{k}x^k\right]$$

$$(x+y)^{k+1} = \left[\binom{k}{0}xy^k + \binom{k}{1}x^2y^{k-1} + \cdots + \binom{k}{k-1}x^k y + \binom{k}{k}x^{k+1}\right.$$
$$\left. + \binom{k}{0}y^{k+1} + \binom{k}{1}xy^k + \binom{k}{2}x^2y^{k-1} + \cdots + \binom{k}{k}x^k y\right]$$

$$(x+y)^{k+1} = \binom{k}{0}y^{k+1} + \left[\binom{k}{0} + \binom{k}{1}\right]xy^k + \left[\binom{k}{1} + \binom{k}{2}\right]x^2y^{k-1} + \cdots +$$
$$\left[\binom{k}{k-1} + \binom{k}{k}\right]x^k y + \binom{k}{k}x^{k+1}$$

apply pascal and note that $\binom{k}{0} = \binom{k+1}{0}$ and $\binom{k}{k} = \binom{k+1}{k+1}$:

$$(x+y)^{k+1} = \binom{k+1}{0}y^{k+1} + \binom{k+1}{1}xy^k + \cdots + \binom{k+1}{k+1}x^{k+1}$$

$$(x+y)^{k+1} = \sum_{i=0}^{k+1} \binom{k+1}{i} y^{k+1-i} x^i$$

$\square$

(5) $\boxed{3-2}$

a) I want to show that $\lg^k n = o(n^\varepsilon)$ for any $\varepsilon > 0$ and $k \geq 1$. This will imply that $\lg^k n = O(n^\varepsilon)$ and that $\lg^k n \neq \Omega(n^\varepsilon), \omega(n^\varepsilon), \Theta(n^\varepsilon)$.

So need to show that for any $c > 0$, $\exists\ n_0$ st

$$\lg^k n \leq c n^\varepsilon \quad \forall\ n \geq n_0.$$

Let $c > 0$, then $\sqrt[k]{c} > 0$ and $\varepsilon > 0 \rightarrow \varepsilon/k > 0$

so $\lg^k n \leq c n^\varepsilon\ \forall n \geq n_0 \longleftrightarrow \lg n \leq \sqrt[k]{c} \cdot n^{\varepsilon/k}\ \forall n \geq n_0$.

Therefore, we need only solve the case for $k=1$:

$$\lg n \leq c n^\varepsilon \quad \forall\ n \geq n_0$$

Well,

$$\lim_{n \to \infty} \frac{\lg n}{c n^\varepsilon} = \lim_{n \to \infty} \frac{\frac{1}{\ln(2) \cdot n}}{c \varepsilon n^{\varepsilon - 1}} = \lim_{n \to \infty} \frac{1}{\ln(2) c \varepsilon n^\varepsilon} = 0$$

L'Hopital since $\lg n \to \infty$ and $c n^\varepsilon \to \infty$ as $n \to \infty$

Since the ratio goes to zero ~~as~~ as $n \to \infty$ then at some n_0 it must be the case that $\lg n \leq c n^\varepsilon$, $\forall\ n \geq n_0$ since otherwise the ratio $\frac{\lg n}{c n^\varepsilon}$ would have an infinite # of n's st $\frac{\lg n}{c n^\varepsilon} \geq 1$, contradicting the limit

(note: to use L'Hopital I'm actually considering real-valued functions $\lg x$ and $c x^\varepsilon$ for $x \in \mathbb{R}$. But $\mathbb{Z} \subseteq \mathbb{R}$ so it's ok)

b) Same deal: $n^k = o(c^n)$ (for $c > 1$ and $k \geq 1$)

$\Rightarrow n^k = O(c^n)$ and $n^k \neq \Omega(c^n), \omega(c^n), \Theta(c^n)$

Let $d > 0$, then we want a n_0 such that $n^k \leq d \cdot (c^n) \quad \forall n \geq n_0$

$$\Longleftrightarrow$$

$$\lg(n^k) \leq \lg(d \cdot (c^n)) \quad \forall n \geq n_0$$

$$k \lg(n) \leq \lg(d) + \lg(c) \cdot n \quad \forall n \geq n_0$$

Let $c' = \lg(c)/k$ and $d' = \lg(d)/k$

then we need only that $\lg(n) \leq c'n + d' \quad \forall n \geq n_0$

By part a we already know that $\exists\, n_0$ st $\lg(n) \leq c'n \quad \forall n \geq n_0$

so if $d' \geq 0$ then we're done.

Otherwise $d' < 0$ but is still just a constant so we can use the same trick as before:

$$\lim_{n\to\infty} \frac{\lg(n)}{c'n + d'} = \lim_{n\to\infty} \frac{1/n}{c'} = \lim_{n\to\infty} \frac{1}{c'n} = 0.$$

So $n^k = o(c^n)$ $\square$

c) There is no asymptotic relationship between $\sqrt{n}$ and $n^{\sin n}$ since $n^{\sin n}$ "bounces" back and forth between n and $1/n$ as $\sin n$ bounces between 1 and -1. So whenever $\sin n > 1/2$ then $\sqrt{n} < n^{\sin n}$ and when $\sin n < 1/2$ then $\sqrt{n} > n^{\sin n}$.

$\sin x > 1/2$ for $2\pi k + \frac{\pi}{6} < x < 2\pi k + \pi - \frac{\pi}{6}$.

$$\left(2\pi k + \pi - \frac{\pi}{6}\right) - \left(2\pi k + \frac{\pi}{6}\right) = \pi - \frac{2\pi}{6} = \pi \cdot \frac{2}{3} > 2$$

and $2\pi - \frac{2}{3}\pi = \frac{4}{3}\pi > 4$

so both ranges allow room for integers. Hence, there are infinitely many $n \geq 0$ for which $\sin n > 1/2$ and infinitely many for which $n \leq 1/2$.

Since $n^{1/2} = o(n^{\alpha}), \omega(n^{\beta})$ for $\beta < 1/2 < \alpha$

then for any $c > 0$ and any n_0, $\exists\, n_1, n_2$ such that $n^{1/2} > c(n_1)^{\sin n_1}$ and $n^{1/2} < c(n_2)^{\sin n_2}$.

So $n^{1/2} \neq O(n^{\sin n}), \Omega(n^{\sin n}), o(n^{\sin n}), \omega(n^{\sin n}), \Theta(n^{\sin n})$

d) Suppose $2^n \geq c \cdot 2^{n/2}$ for $c > 0$

then $2^{n/2} \geq c$ which is true for all n st

$$n/2 \geq \lg c$$
$$n \geq 2 \lg c$$

So for each $c > 0$, let $n_0 = \lceil 2\lg c \rceil$, then $\forall\, n \geq n_0$, $2^n \geq c \cdot 2^{n/2}$. So $2^n = \omega(2^{n/2}) \rightarrow 2^n = \Omega(2^{n/2})$ and $2^n \neq O(2^{n/2}), o(2^{n/2}), \Theta(2^{n/2})$.

e) These are actually the same function since

$$\lg(n^{\lg c}) = (\lg c)\cdot(\lg n) = \lg(c^{\lg n})$$

$$\Rightarrow n^{\lg c} = c^{\lg n} \quad (\lg x \text{ is one-to-one}).$$

Therefore, $n^{\lg c} = O(c^{\lg n}), \Omega(c^{\lg n}), \Theta(c^{\lg n})$

and $n^{\lg c} \neq o(c^{\lg n}), \omega(c^{\lg n})$.

f) $\lg(n!) = \Theta(\lg(n^n))$ since $\lg(n!) = O(\lg(n^n))$

and $\lg(n!) = \Omega(\lg(n^n))$.

$O(\lg(n^n))$

easy since

$$\lg(n!) = \lg(1) + \lg(2) + \cdots + \lg(n)$$

$$\lg(n^n) = \underbrace{\lg(n) + \lg(n) + \cdots + \lg(n)}_{n \text{ times}}$$

and $\lg(i) \le \lg(n)$ for $i = 1, 2, \ldots, n$

So $\lg(n!) \le \lg(n^n)$.

$\Omega(n^n)$

We can use Sterling's Approximation for $n!$

So

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1+\Theta(1/n)\right)$$

$$\lg(n!) = \lg\left(\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1+\Theta(1/n)\right)\right)$$

$$= \frac{1}{2}\lg(2\pi n) + n\lg(n/e) + \lg(1+\Theta(1/n))$$

$$\geq \frac{1}{2}\lg(2\pi n) + n\lg(n/e) + \lg(1+d/n)$$

(for some constant $d > 0$)

Since $2\pi n \geq 1$ for $n \geq 1$

and $1 + d/n \geq 1$ for $n \geq 1$

then $\lg(2\pi n), \lg(1+d/n) \geq 0$ so

$$\lg(n!) \geq n\lg(n/e).$$

we need a constant $c > 0$ such that $n\lg(n/e) \geq c\lg(n^n)$ for all $n \geq n_0$ (for some n_0).

So $$\frac{n\lg(n/e)}{n\lg(n)} \geq c \quad \leftarrow \cancel{\lg(e)}$$

$$\Rightarrow \frac{n\lg(n) - n\lg(e)}{n\lg(n)} \geq c$$

$$1 - \frac{\lg(e)}{\lg(n)} \geq c$$

Note that $1 < \lg(e) < 2$, $\lg(16) = 4$, and $1 - \frac{\lg(e)}{\lg(n)}$ is monotone increasing (but bounded by 1) as $n \to \infty$.

Hence, $1 - \frac{\lg(e)}{\lg(16)} \geq 1/2 \quad \forall \; n \geq 16$

So $\lg(n!) \geq \frac{1}{2}\lg(n^n) \quad \forall \; n \geq 16$

Thus, $\lg(n!) = \Omega(\lg(n^n))$.

To Summarize:

A	B	O	o	Ω	ω	Θ
$\lg^k n$	n^ε	Yes	Yes	No	No	No
n^k	c^n	Yes	Yes	No	No	No
$\sqrt{n}$	$n^{\sin n}$	No	No	No	No	No
2^n	$2^{n/2}$	No	No	Yes	Yes	No
$n^{\lg c}$	$c^{\lg n}$	Yes	No	Yes	No	Yes
$\lg(n!)$	$\lg(n^n)$	Yes	No	Yes	No	Yes

(6) want to show that $\exists$ $c > 0$ and n_0 st

$$T(n) \leq cn \quad \forall\ n \geq n_0$$

since $T(0) = 0$, then this is true for any c.

Suppose we have found such a c that works for all $k < n$

then since $\lfloor n/3 \rfloor, \lfloor n/5 \rfloor, \lfloor n/7 \rfloor < n$ it follows that

$$T(\lfloor n/3 \rfloor) \leq c\left\lfloor \tfrac{n}{3} \right\rfloor, \quad T(\lfloor n/5 \rfloor) \leq c\left\lfloor \tfrac{n}{5} \right\rfloor, \quad T(\lfloor n/7 \rfloor) \leq c\left\lfloor \tfrac{n}{7} \right\rfloor$$

so

$$T(n) \leq c\left\lfloor \tfrac{n}{3} \right\rfloor + c\left\lfloor \tfrac{n}{5} \right\rfloor + c\left\lfloor \tfrac{n}{7} \right\rfloor + n \leq c\left(\tfrac{n}{3} + \tfrac{n}{5} + \tfrac{n}{7}\right) + n$$

(since $\left\lfloor \tfrac{n}{3} \right\rfloor \leq \tfrac{n}{3}$ etc)

We want $$c\left(\frac{n}{3} + \frac{n}{5} + \frac{n}{7}\right) + n \leq cn$$

so $$cn\left(\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{c}\right) \leq cn$$

* assuming I didn't make a mistake

$$\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{c} \leq 1 \rightsquigarrow \frac{105}{34} \leq c$$

So if we pick any $c \geq \frac{105}{34}$, then $T(n) \leq cn$ $\forall\ n$.

Hence, $T(n) = O(n)$ $\square$

(7)

$n=0$ is a counterexample for the statement for both $\mathbb{R}$ and $\mathbb{Z}$. So I assume this means <u>positive</u> numbers in both cases, $\mathbb{R}^+$ and $\mathbb{Z}^+$:

a) <u>counterexample:</u> $n = 1.9$ $\quad n+1 = 2.9$

then let $x = \lg(1.9)$ and $y = \lg(2.9)$

So $2^x = 1.9$ and $2^y = 2.9$

Therefore, $0 < x < 1$ and $1 < y < 2$

so $\lfloor x \rfloor = 0$ and $\lceil y \rceil = 2$

$\Rightarrow \lceil \lg(2.9) \rceil \neq \lfloor \lg(1.9) \rfloor + 1$.

b) true: Let $2^i \leq n < 2^{i+1}$ for some $i = 0, 1, 2, \ldots$

then $2^i < n+1 \leq 2^{i+1}$

so $\lfloor \lg(n) \rfloor = i$ and $\lceil \lg(n+1) \rceil = i+1$

Hence, $\lceil \lg(n+1) \rceil = \lfloor \lg(n) \rfloor + 1$ $\square$

8

a) $$\sum_{i=1}^{n} i\cdot 3^i = 3\sum_{i=1}^{n} i\cdot 3^{i-1} = 3\, d/dx\left(\sum_{i=1}^{n} x^i\right) \quad \text{for } x=3$$

$$= 3\, d/dx\left(\frac{x^{n+1}-1}{x-1} - 1\right) \qquad \text{since } 1+x+x^2+\cdots+x^n = \frac{x^{n+1}-1}{x-1}$$

$$= 3\left(\frac{(n+1)x^n(x-1) - (x^{n+1}-1)}{(x-1)^2}\right)$$

$$= 3\left(\frac{(n+1)\cdot 3^n(3-1) - (3^{n+1}-1)}{(3-1)^2}\right)$$

$$= \underline{\underline{3/4\left(2(n+1)3^n - 3^{n+1} + 1\right)}}$$

b) $$\sum_{i=1}^{n} i\cdot 3^i = \frac{1}{2}\sum_{i=1}^{n} i\cdot\left(3^{i+1}-3^i\right) \qquad \text{since } 3^{i+1}-3^i = 3^i(3-1) = 2\cdot 3^i$$

$$= \frac{1}{2}\left[\sum_{i=1}^{n} i\cdot 3^{i+1} - \sum_{i=1}^{n} i\cdot 3^i\right]$$

$$= 1/2\left[\sum_{i=1}^{n} i\cdot 3^{i+1} - \sum_{i=0}^{n}(i+1)\cdot 3^{i+1}\right] = 1/2\left[\sum_{i=1}^{n-1}(i-i-1)\cdot 3^{i+1} + n3^{n+1} - 3\right]$$

$$= 1/2\left[n3^{n+1} - 3 - \sum_{i=1}^{n-1} 3^{i+1}\right] = 1/2\left[n3^{n+1} - \sum_{i=1}^{n} 3^i\right]$$

$$= 1/2\left[n3^{n+1} - \left(\frac{3^{n+1}-1}{3-1} - 1\right)\right] = \frac{1}{2}\left[n3^{n+1} - \frac{1}{2}3^{n+1} + \frac{3}{2}\right]$$

$$= \underline{\frac{3}{4}\left(2n\cdot 3^n - 3^n + 1\right)}$$

equals above in part a since $2(n+1)3^n - 3^{n+1} = 3^n(2n+2-3)$
$= 3^n(2n-1) = 2n3^n - 3^n$.

9

want to show that $\sum_{i=1}^{n} i \cdot 3^i = \frac{3}{4}\left([2n-1] \cdot 3^n + 1\right)$
by induction:

base: $n=1 \rightarrow 1 \cdot 3^1 = 3$
and
$3/4\left([2 \cdot 1 - 1] \cdot 3^1 + 1\right) = 3$ ✓

induction step: assume $\sum_{i=1}^{n} i \cdot 3^i = 3/4\left([2n-1] \cdot 3^n + 1\right)$ for some $n \geq 1$

then $\sum_{i=1}^{n+1} i \cdot 3^i = 3/4\left([2n-1] \cdot 3^n + 1\right) + (n+1) \cdot 3^{n+1}$

$= 3/4\left([2n-1] \cdot 3^n + 1 + 4(n+1) \cdot 3^n\right)$

$= 3/4\left([2n-1+4n+4] \cdot 3^n + 1\right) = 3/4\left([6n+3] \cdot 3^n + 1\right)$

$= 3/4\left([2n+1] \cdot 3^{n+1} + 1\right) = 3/4\left([2(n+1)-1] \cdot 3^n + 1\right)$ $\square$

(10)

$$\sum_{i=1}^{n} i^2 \cdot 2^i = \sum_{i=1}^{n} i^2 \cdot \left(2^{i+1} - 2^i\right)$$

since $2^{i+1} - 2^i = 2^i(2-1) = 2^i$

$$= \sum_{i=1}^{n} i^2 \cdot 2^{i+1} - \sum_{i=1}^{n} i^2 \cdot 2^i = \sum_{i=1}^{n} i^2 \cdot 2^{i+1} - \sum_{i=0}^{n-1} (i+1)^2 \cdot 2^{i+1}$$

$$= \sum_{i=1}^{n-1} \left(i^2 - (i+1)^2\right) \cdot 2^{i+1} + n^2 \cdot 2^{n+1} - 2$$

$$= -\sum_{i=1}^{n-1} (2i+1) \cdot 2^{i+1} + n^2 \cdot 2^{n+1} - 2$$

$$= n^2 \cdot 2^{n+1} - 2 - \sum_{i=1}^{n-1} 2^{i+1} - 2\sum_{i=1}^{n-1} i \cdot 2^{i+1}$$

$$= n^2 \cdot 2^{n+1} - \sum_{i=1}^{n} 2^i - 2\left(\sum_{i=1}^{n-1} i \cdot 2^{i+1}\right)$$

$$\sum_{i=1}^{n} 2^i = \left(\frac{2^{n+1}-1}{2-1} - 1\right) = 2^{n+1} - 2$$

$\boxed{\text{SBP again}}$

$$\sum_{i=1}^{n-1} i \cdot 2^{i+1} = \sum_{i=1}^{n-1} i \cdot \left(2^{i+2} - 2^{i+1}\right) = \sum_{i=1}^{n-1} i\, 2^{i+2} - \sum_{i=1}^{n-1} i\, 2^{i+1}$$

$$= \sum_{i=1}^{n-1} i\, 2^{i+2} - \sum_{i=0}^{n-2} (i+1) \cdot 2^{i+2} = \sum_{i=1}^{n-2} (i-i-1)\, 2^{i+2} + (n-1) \cdot 2^{n+1} - 4$$

$$= -\sum_{i=1}^{n-2} 2^{i+2} + (n-1) \cdot 2^{n+1} - 4 = (n-1) \cdot 2^{n+1} - \sum_{i=2}^{n} 2^i$$

$$= (n-1) \cdot 2^{n+1} - \left(\frac{2^{n+1}-1}{2-1} - 1 - 2\right) = (n-1) \cdot 2^{n+1} - 2^{n+1} + 4$$

$$= 2^{n+1}(n-2) + 4$$

So all together: $n^2 \cdot 2^{n+1} - 2^{n+1} + 2 - 2\left(2^{n+1}(n-2) + 4\right) = \underline{\underline{2^{n+1}(n^2 - 2n + 3) - 6}}$

(11)

a) need $c > 0$ and n_0 st $0 \le 100n \le cn^2 \ \forall n \ge n_0$

Since $100n \le n^2 \quad \forall n \ge 100$

then let $n_0 = 100$ and $c = 1$. $\Rightarrow$ $0 \le 100n \le n^2 \ \forall n \ge 100$

So $100n = O(n^2)$.

b) same thing so note that $3x^4 - 2x^3 \le 3x^4 \quad \forall x > 0$

and $3x^4 - 2x^3 = x^3(3x-2) \ge 0$

for all $x \ge 2/3$

So $0 \le 3x^4 - 2x^3 \le 3x^4$

$\forall x \ge 1$

Next, $x^2 - 100 \le x^2 \le x^4 \quad \forall x \ge 1$

and $x^2 - 100 = (x-10)(x+10) \ge 0$

for all $x \ge 10$

So for all $x \ge 10$

$0 \le 3x^4 - 2x^3 + x^2 - 100 \le 3x^4 + x^4 = 4x^4$

So $3x^4 - 2x^3 + x^2 - 100 = O(x^4)$.

(12)

Since $\ln x$ is monotone increasing then

$$\int_{a-1}^{b} \ln x \, dx \le \sum_{i=a}^{b} \ln i \le \int_{a}^{b+1} \ln x \, dx$$

Here we want $\sum_{i=1}^{n} \ln n$ but $\int_0^n \ln x \, dx$ gives us trouble since $\ln(0)$ is undefined. However, note that $\ln 1 = 0$ so

$\sum_{i=1}^{n} \ln i = \sum_{i=2}^{n} \ln i$. Thus,

$$\int_{1}^{n} \ln x \, dx \le \sum_{i=1}^{n} \ln i \le \int_{2}^{n+1} \ln x \, dx$$

$$x \ln x - x \Big|_1^n \le \sum_{i=1}^{n} \ln i \le x \ln x - x \Big|_2^{n+1}$$

$$n \ln n - n - \ln 1 + 1 \le \sum_{i=1}^{n} \ln i \le (n+1)\ln(n+1) - (n+1) - 2\ln 2 + 2$$

$$n \ln n - n + 1 \le \sum_{i=1}^{n} \ln i \le (n+1)\ln(n+1) - (n+1) - 2\ln 2 + 2$$

This implies that $\sum_{i=1}^{n} \ln i = \Theta(n \ln n)$.

$\Omega(n \ln n)$

NTS: $\exists\, c, n_0$ st $c n \ln(n) \le n \ln(n) - n + 1 \quad \forall n \ge n_0$

$$\text{so} \quad c \le \frac{n \ln(n) - n + 1}{n \ln(n)} = 1 - \frac{1}{\ln(n)} + \frac{1}{n \ln(n)}$$

$$= 1 - \left(\frac{1}{\ln(n)} - \frac{1}{n \ln(n)} \right)$$

$\frac{1}{\ln(n)} - \frac{1}{n \ln(n)} > 0$ but approaches 0

so as $n \to \infty$ the bound on c increases toward one

Let $c = 1/e$ then for $n \ge 3$ the statement works since $e < 3$

so $\underbrace{1 - \left(\frac{1}{\ln(e)} - \frac{1}{e \ln(e)} \right)}_{= 1/e} < 1 - \left(\frac{1}{\ln(n)} - \frac{1}{n \ln(n)} \right) \quad \forall n \ge 3$

$\underline{O(n\ln n)}$

NTS: $\exists\ c, n_0$ st $\quad c n \ln n \geq (n+1)\ln(n+1) - (n+1) - 2\ln 2 + 2$

$\forall n \geq n_0$

So we need:

$$c \geq \frac{n\ln(n+1) + \ln(n+1) - n + (1 - 2\ln 2)}{n \ln n}$$

$$\Uparrow$$

$$c \geq \frac{n\ln(n+1)}{n\ln(n)} + \frac{\ln(n+1)}{n\ln(n)} \quad \text{since } -n+1-2\ln 2 < 0 \ \ \forall n > 0$$

$$c \geq \frac{\ln(n+1)}{\ln(n)} + \frac{\ln(n+1)}{n\ln(n)}$$

Note that $\lim_{n\to\infty} \frac{\ln(n+1)}{n\ln(n)} = \lim_{n\to\infty} \frac{1/n+1}{1+\ln n} = \lim_{n\to\infty} \frac{1}{(n+1)(1+\ln n)} = 0$

and $\lim_{n\to\infty} \frac{\ln(n+1)}{\ln(n)} = \lim_{n\to\infty} \frac{1/n+1}{1/n} = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1} = 1$

So $\exists$ an n_0 for which $\quad 2 > \frac{\ln(n+1)}{\ln(n)} + \frac{\ln(n+1)}{n\ln(n)} \quad \forall n \geq n_0$

So let $c = 2$ and we're done.

$$\sum_{i=1}^{n} \ln i = \Theta(n\ln n)$$

13

$$T(n) = 2 \cdot T(\lfloor n/2 \rfloor) + n \quad , \quad T(1) = 1$$

a) <u>repeated substitution</u>

assume ~~even~~ n is a power of 2 — $n = 2^i$ for some $i = 0, 1, 2, \ldots$

then
$$\begin{aligned} T(2^i) &= 2T(2^{i-1}) + 2^i \\ &= 2(2T(2^{i-2}) + 2^{i-1}) + 2^i \\ &= 2^2 T(2^{i-2}) + 2^i + 2^i \\ &= 2^2(2 \cdot T(2^{i-3}) + 2^{i-2}) + 2^i + 2^i \\ &= 2^3 \cdot T(2^{i-3}) + 2^i + 2^i + 2^i \end{aligned}$$

notice the pattern $\Rightarrow$
$$\begin{aligned} &= \ldots \\ &= 2^i T(1) + i \cdot 2^i = (i+1) \cdot 2^i \end{aligned}$$

So we can show this formally by induction now that we've found the pattern.

for all $i \in \mathbb{Z}^{\geq 0}$ $\quad T(2^i) = (i+1) \cdot 2^i$

base: $i = 0 \rightarrow T(1) = 1$

and

$(0+1) \cdot 2^0 = 1$ ✓

induction step: assume true for i

then
$$\begin{aligned} T(2^{i+1}) &= 2T(2^i) + 2^{i+1} \\ &= 2((i+1)2^i) + 2^{i+1} \\ &= (i+1)2^{i+1} + 2^{i+1} = (i+2)2^{i+1} \end{aligned}$$

$\square$

b) <u>telescoping series method</u>

$$T(2^i) = 2T(2^{i-1}) + 2^i$$

we want
the T( ) of the RHS to
be in the same "kind" of term
as the T( ) on the LHS
so that we can telescope through

divide both sides by 2^i to get

$$\frac{T(2^i)}{2^i} = \frac{T(2^{i-1})}{2^{i-1}} + 1$$

$$\frac{T(2^{i-2})}{2^{i-2}} + 1$$

$$\frac{T(2^{i-3})}{2^{i-3}} + 1$$

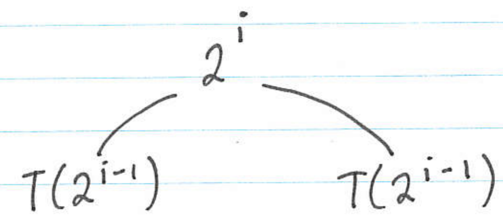
$$\vdots$$

$$\frac{T(1)}{1} + 1$$

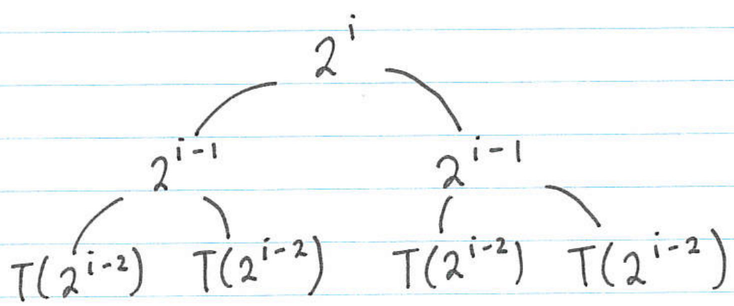
So $\dfrac{T(2^i)}{2^i} = 1 + i \longrightarrow T(2^i) = (i+1)2^i$ as before.

c) <u>recursion-tree method</u>

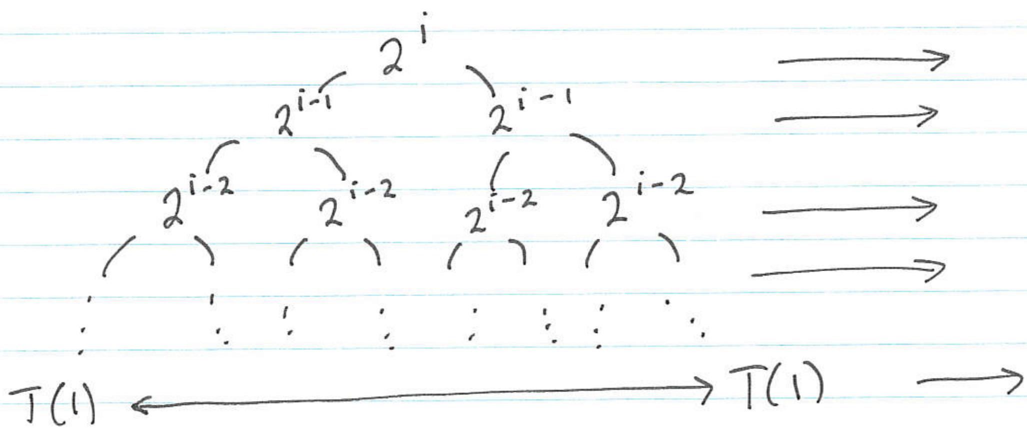
$T(2^i)$



$\Downarrow$



$\Downarrow$



2^i

$2 \cdot 2^{i-1}$

$4 \cdot 2^{i-2}$

$8 \cdot 2^{i-3}$

$\vdots$

$2^i \cdot T(1)$

Sum $= 2^i \cdot (i+1)$

as before

or in terms of $n \rightarrow T(n) = n \cdot (\lg n + 1)$

when n is a power of 2

d) <u>substitution method</u>

when $n = 2^i$ we know that $T(n) = n \lg n + n$

since $n \lg n \geq n$ then $\Rightarrow$ $T(n) \leq 2n\lg(n)$ for $n = 2^i$
so my guess is that $T(n) = O(n \lg n)$ in general.

NTS: $\exists$ $c > 0$ and a n_0 st

$$T(n) \leq c n \lg n \quad \forall \; n \geq n_0$$

suppose it is true for $\lfloor n/2 \rfloor$, then

$$T(\lfloor n/2 \rfloor) \leq c \lfloor n/2 \rfloor \lg(\lfloor n/2 \rfloor)$$

so

$$T(n) = 2T(\lfloor n/2 \rfloor) + n \leq 2c \lfloor n/2 \rfloor \lg(\lfloor n/2 \rfloor) + n$$

since $2 \cdot \lfloor n/2 \rfloor = n$ or $n-1$, then $2 \cdot \lfloor n/2 \rfloor \leq n$ so

$$T(n) \leq cn \lg(\lfloor n/2 \rfloor) + n$$

So we'd like to show that $cn\lg(\lfloor n/2 \rfloor) + n \leq cn\lg(n)$.

This would be true if $\lg(\lfloor n/2 \rfloor) + \frac{1}{c} \leq \lg(n)$.

Since $\lg(n) - \lg(\lfloor n/2 \rfloor) = \lg\left(\frac{n}{\lfloor n/2 \rfloor}\right) = \begin{cases} \lg(2) & \text{if } n \text{ is even} \\ \lg\left(2 + \frac{2}{n-1}\right) & \text{if } n \text{ is odd} \end{cases}$

then $\lg(n) - \lg(\lfloor n/2 \rfloor) \geq \lg(2) = 1$

So we need only pick c so that $1/c \leq 1 \rightarrow 1 \leq c$.

So $T(\lfloor n/2 \rfloor) \leq c \lfloor \frac{n}{2} \rfloor \lg(\lfloor n/2 \rfloor) \longrightarrow T(n) \leq cn\lg(n)$
for any $c \geq 1$.

We need only check the base case:

$T(1) = 1$ actually doesn't work for any c
since $c \cdot 1 \cdot \lg 1 = 0 < 1 = T(1)$.

but that's ok since we only need it true for all n past a certain n_0. So clearly $n_0 > 1$ for us.

$T(2) = 4$ and $c 2 \lg 2 = 2c$ so $c \geq 2$
and $T(3) = 2T(\lfloor 3/2 \rfloor) + 3$
$= 5$ so $c 3 \lg(3) \geq 5$

$$c \geq \frac{5}{3\lg(3)}$$

since $6\lg(3) > 6\lg(2) = 6 > 5$
then $c \geq 2$ works again

And this is all we need to do since for any n $\exists$ some i st

$$2^i \leq n < 2^{i+1} \rightarrow 2^{i-1} \leq \lfloor n/2 \rfloor < 2^i$$

So our induction base case need only be some set $\{2^i, 2^i+1, \ldots, 2^{i+1}-1\}$ which $\{2, 3\}$ is.

So we're done!

$$T(n) \leq 2 \cdot n \lg n \quad \forall\ n \geq 2$$

$$\Rightarrow T(n) = O(n \lg n) \quad \square$$

e) <u>Master method</u>

to use this method

We need $T(n) = aT(\lfloor n/b \rfloor) + f(n)$ for $a \geq 1$ and $b > 1$

Here, $a = b = 2$.

Then we must determine if $f(n) = O(n^{\log_b a - \varepsilon})$ for any $\varepsilon > 0$

or

$f(n) = \Theta(n^{\log_b a})$

or

$f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some $\varepsilon > 0$

Well, $n^{\log_b a} = n^{\log_2 2} = n$ so $f(n) = \Theta(n)$

since $f(n) = n$.

Then $T(n) = \Theta(n \lg n)$ by the Master Theorem.

(14)

$$f_n = f_{n-1} + 6 f_{n-2} \qquad f_0 = 0, \quad f_1 = 1$$

a)

$$f_n = \alpha^n \rightarrow \alpha^n = \alpha^{n-1} + 6\alpha^{n-2}$$
$$\alpha^2 = \alpha + 6$$
$$\alpha^2 - \alpha - 6 = 0$$
$$(\alpha - 3)(\alpha + 2) = 0$$
$$\alpha = -2, 3$$

So $f_n = a(-2)^n + b(3)^n$ works

since $f_0 = 0$ and $f_1 = 1$, then $\begin{cases} 0 = a + b \\ 1 = -2a + 3b \end{cases}$

$$\downarrow$$
$$5b = 1 \rightarrow b = 1/5$$
$$\rightarrow a = -1/5$$

So $f_n = -\frac{1}{5}(-2)^n + \frac{1}{5}(3)^n$

b) $f_n = O(3^n)$

assume $f_{n-1} \le c3^{n-1}$ and $f_{n-2} \le c3^{n-2}$

then $f_n = f_{n-1} + 6f_{n-2} \le c3^{n-1} + 6 \cdot c3^{n-2}$
$= c \cdot 3^{n-1}(1+2) = c \cdot 3^n$

and $f_0 = 0 \le c \cdot 3^0 = c$ for any $c > 0$
$f_1 = 1 \le c \cdot 3^1$ for any $c \ge 1/3$

So let $c = 1$ and $n_0 = 0$ and we're done.

(15) n is an integer and x is a real number

a) $n < x \Rightarrow n < \lceil x \rceil$

Let $n < x$. Since $\lceil x \rceil$ is defined as the integer m such that $m-1 < x \leq m$, then $n < x \leq m$. So $n < m = \lceil x \rceil$.

b) $n < \lceil x \rceil \Rightarrow n < x$

Let $n < \lceil x \rceil$, then $n < m$ so $n \leq m-1 < x$.

So $n < x$. $\square$