Answer Key

MCS 421 - Combinatorics

Midterm Exam

Spring 2018

1. (20 points.) A bakery sells six different kinds of pastry. If the bakery has at least a dozen of each kind, how many different options are there to fill a box with a dozen pastries?

Same as: How many 12-combinations of $\{12 \cdot a_1, 12 \cdot a_2, \ldots, 12 \cdot a_6\}$.

We know ("stars and bars" or similar) that this is $\binom{12+6-1}{6-1} = \binom{17}{5} = \binom{17}{12}$

2. (20 points.) There are 100 people at a party. Each person has a positive even number of friends at the party.

(a) Prove that there are three people at the party with the same number of friends.

(b) Prove that there are three people at the party with the same number of friends if each person has an even (not necessarily positive) number of friends at the party. You should assume that friendship is a symmetric relation.

a) Each person has $2, 4, \ldots, 98$ friends.
So 49 possibilities for each person.
By PH principle, there are $\lceil \frac{100}{49} \rceil = 3$
people with the same # of friends.

b) Now each person has either $0, 2, 4, \ldots,$ or 98
friends - 50 possibilities. The only way
for no 3 people to have the same # is if
each # is associated to exactly 2 people.
Therefore, 2 people have zero friends. But this
implies that no one will have 98 friends, a
contradiction.

3. (20 points.) Evaluate the following expression for every integer $n > 1$:

$$\binom{n}{1} - 2\binom{n}{2} + 3\binom{n}{3} + \cdots + (-1)^{n-1} n \binom{n}{n}.$$

The binomial theorem gives

$$(x+1)^n = \binom{n}{0} + \binom{n}{1}x + \cdots + \binom{n}{n}x^n.$$

Differentiating both sides gives

$$n(x+1)^{n-1} = \binom{n}{1} + 2\binom{n}{2}x + 3\binom{n}{3}x^2 + \cdots + n\binom{n}{n}x^{n-1}.$$

Let $x=-1$ to get

$$\underline{\underline{0}} = \binom{n}{1} - 2\binom{n}{2} + 3\binom{n}{3} - \cdots + (-1)^{n-1} n\binom{n}{n}.$$

answer.

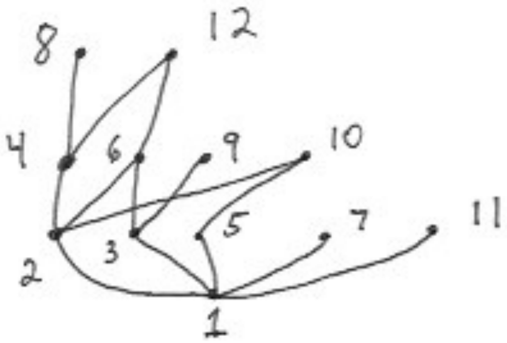
4. (20 points.) Consider the poset $([12], |)$ of the first twelve positive integers ordered by "divides."

   (a) Draw the Hasse diagram for the poset.

   (b) Determine a chain of largest size and a partition of [12] into the smallest number of antichains.

   (c) Determine an antichain of largest size and a partition of [12] into the smallest number of chains.

(a)



(b) largest chain size is 4

partition into antichains

$A_1 = \{1\}$

$A_2 = \{2, 3, 5, 7, 11\}$

$A_3 = \{4, 6, 9, 10\}$

$A_4 = \{8, 12\}$

(c) largest antichain size 6 (4, 6, 9, 5, 7, 11)

6 chains partition:

$C_1 = \{1, 3, 6, 12\}$

$C_2 = \{2, 4, 8\}$

$C_3 = \{5, 10\}$

$C_4 = \{9\}$

$C_5 = \{7\}$

$C_6 = \{11\}$

5. (20 points.) Let n be a positive integer. How many ways can we pick sets $A_1, A_2, \ldots, A_k$ such that

$$\emptyset \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq A_k \subseteq [n]?$$

Prove your answer is correct.

For each $x \in [n]$ we have $k+1$ choices: ① $x \in A_1$ and everything after

② $x \notin A_1$, $x \in A_2$ and everything after,

$\vdots$

ⓘ $x \notin A_1, A_2, \ldots, A_{i-1}$, $x \in A_i, A_{i+1}, \ldots, A_k$

$\vdots$

ⓚ $x \notin A_1, \ldots, A_{k-1}$, $x \in A_k$

or ($k+1$) $x \notin A_1, \ldots, A_k$.

Since there are n elements this gives $(k+1)^n$ ways we can pick the sets.