MCS 421 - Combinatorics

Midterm Exam Take 2

Spring 2018

1. (20 points.) In how many ways can 15 people be seated at a round table if person B refuses to sit next to person A?

$$\underbrace{14!}_{\substack{\text{total \# of} \\ \text{seatings} \\ \text{(circle permutation} \\ \text{of 15 elements)}}} - \underbrace{2 \cdot 13!}_{\substack{\text{\# of seatings} \\ \text{in which A and B} \\ \text{are next to each other}}} = 12(13!)$$

(arrange everyone but B, then seat B either to the left or to the right of A)

2. (20 points.) Show that if $n+1$ integers are picked from a set $\{1, 2, \ldots, 2n\}$, then there must be two that differ by one.

Each picked integer comes from one of the following pairs : $\{1,2\}, \{3,4\}, \ldots, \{2n-1, 2n\}$. Since there are n such pairs and $n+1$ integers, then 2 integers must belong to the same pair by pigeonhole. The difference of these two integers is 1.

3. (20 points.) Let X be a set of n elements. How many antisymmetric relations can we define on X?

An antisymmetric relation $R \subseteq X \times X$ is one for which $(a,b), (b,a) \in R$
$$\Rightarrow a = b.$$

So for any pair of distinct elements $a, b \in X$, either $(a,b) \in R$, $(b,a) \in R$, or neither: 3 choices.

For any ordered pair of the form (a,a), we have 2 choices - either $(a,a) \in R$ or $(a,a) \notin R$.

So all together there are

$3^{\binom{n}{2}} \cdot 2^n$ possible antisymmetric relations on X.

4. (20 points.) State and prove the binomial theorem (any proof is fine as long as it is correct and I can follow the reasoning).

Binomial Thm: For any $n \in \mathbb{Z}^+$,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Proof: Can use induction on n or a combinatorial argument about the # of ways to pick k x's and $(n-k)$ y's from $\underset{1}{(x+y)}\underset{2}{(x+y)}\cdots\underset{n}{(x+y)}$ to get a term $x^k y^{n-k}$. (we did these proofs in class and they are in the book)

5. (20 points.) A bakery sells chocolate, cinnamon, and plain donuts and at a particular time has 6 chocolate, 6 cinnamon, and 3 plain available. If a box contains 12 donuts (and order doesn't matter), then how many different boxes can the bakery make?

Same as # of int. solns to

$$x_1 + x_2 + x_3 = 12$$

$$0 \le x_1 \le 6$$

$$0 \le x_2 \le 6$$

$$0 \le x_3 \le 3$$

For $i = 1, 2, 3$,

Let A_i be ~~#~~ set of int. solns to the same problem w/ $x_i \ge 7$ (if $i = 1, 2$) or $x_i \ge 4$ (if $i = 3$) and no upper bounds

Let U be set of int solns with no upper bounds.

Then by inclusion exclusion,

$$|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| = |U| - |A_1| - |A_2| - |A_3| + |A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3| - |A_1 \cap A_2 \cap A_3|$$

$$= \binom{12+3-1}{3-1} - \binom{5+3-1}{3-1} - \binom{5+3-1}{3-1} - \binom{8+3-1}{3-1} + 0 + \binom{1+3-1}{3-1} + \binom{1+3-1}{3-1} - 0$$

$$= \binom{14}{2} - 2\binom{7}{2} - \binom{10}{2} + 6 = 7 \cdot 13 - 7 \cdot 6 - 5 \cdot 9 + 6$$

$$= 49 - 45 + 6 = 10$$