A variation on the Ramsey problem: (p, q)-colorings

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Background: Let K_n denote the complete graph on n vertices. Define a (p, q)-coloring of a graph to be any edge-coloring such that every p-clique contains edges of at least q different colors. Let $f(n, p, q)$ denote the minimal number of colors needed to (p, q)-color K_n.

In 1981, Paul Erdős originally proposed the problem of determining the values of $f(n, p, q)$ [1]. Of course, when $q = 2$ the problem is equivalent to the classical Ramsey problem for graphs and is therefore very difficult. However, it is far less difficult to see that $f(n, 3, 3)$ is equal to n for odd n and $n - 1$ for even n since a $(3, 3)$-coloring is equivalent to a proper edge-coloring.

In 1997, Erdős and András Gyárfás used the Local Lemma to give a general upper bound [2]:

$$f(n, p, q) = O\left(n^{\frac{p-2}{1-q+\binom{p}{2}}}\right).$$

They also determined that the smallest q for which $f(n, p, q)$ is linear in n (for a fixed p) is

$$q = \binom{p}{2} - p + 3,$$

and the smallest q for which $f(n, p, q)$ is quadratic in n is

$$q = \binom{p}{2} - \left\lfloor \frac{p}{2} \right\rfloor + 2$$

for all $p \geq 4$.

Tighter bounds for most cases with small p and q were left open although they did find that

$$\frac{5(n-1)}{6} \leq f(n, 4, 5) \leq n.$$

Erdős and Gyárfás also made special mention of three particular cases. They called the problem of determining $f(n, 4, 3)$ "annoying" and claimed that $f(n, 4, 4)$ and $f(n, 5, 9)$ are the "most interesting" of the small cases.

In 1998, Dhruv Mubayi constructed an $(4, 3)$-coloring of K_n that uses $O\left(e^{\sqrt{\log n}}\right)$ colors [3]. This improves the above probabilistic upper bound though it is still unknown how close to the actual order of magnitude it is. He called this construction the "Symmetric Subset Ranking coloring (SSR)." The actual definition is somewhat involved, but the gist is that each vertex is associated with a t-subset of $[m]$ for some fixed integer m, the subsets of each t-set are then given some linear ordering, and the color of an edge between two vertices is based on properties of the symmetric difference of the corresponding t-sets as well as the ranks of their intersection.

In 2004, Mubayi gave an explicit edge-coloring that shows that $f(n, 4, 4) = n^{\frac{1}{2}+o(1)}$ [4]. An easy argument shows that $f(n, 4, 4) = \Omega\left(n^{\frac{1}{2}}\right)$. Therefore, Mubayi conjectured that the real value is $f(n, 4, 4) = \Theta\left(n^{\frac{1}{2}}\right)$.

This construction is even more involved. Loosely speaking, it combines the SSR coloring with something he calls a "Divided Algebraic coloring (DAC)." The DAC construction is a slight modification of an "Algebraic coloring" in which each vertex is associated with a vector from $\mathbb{F}_q^2$ where

$\mathbb{F}_q$ is a finite field for some odd prime power q. The color between two vertices is then based on a symmetric map from the associated vectors to the underlying field.

While these constructions take some room to describe, the math behind them does not go beyond basic abstract algebra.

Questions: There is simple induction argument on p that shows that

$$f(n,p,p) = \Omega\left(n^{\frac{1}{p-2}}\right).$$

Since there are known (p,p)-colorings for $p = 3,4$ that use *almost* $n^{\frac{1}{p-2}}$ colors, it seems natural to ask whether we might be able to find similar constructions for $p \geq 5$.

Question 1. *Can we construct a $(5,5)$-coloring of K_n that uses $O\left(n^{\frac{1}{3}}\right)$ colors or at least fewer than $n^{\frac{1}{2}}$ colors? Alternatively, can we show a better lower bound to improve on $f(n,5,5) = \Omega\left(n^{\frac{1}{3}}\right)$?*

Question 2. *More generally, can we beat the current upper and lower bounds,*

$$\Omega\left(n^{\frac{1}{p-2}}\right) \leq f(n,p,p) \leq O\left(n^{\frac{2}{p-1}}\right)$$

for any p?

Question 3. *Can we determine any $f(n,p,q)$ for any other small values of q and p?*

References

[1] P Erdős. Solved and unsolved problems in combinatorics and combinatorial number theory. *European Journal of Combinatorics*, 2:1–11, 1981.

[2] Paul Erdős and András Gyárfás. A variant of the classical ramsey problem. *Combinatorica*, 17(4):459–467, 1997.

[3] Dhruv Mubayi. Note–edge-coloring cliques with three colors on all 4-cliques. *Combinatorica*, 18(2):293–296, 1998.

[4] Dhruv Mubayi. An explicit construction for a ramsey problem. *Combinatorica*, 24(2):313–324, 2004.