NORM INFLATION FOR GENERALIZED NAVIER-STOKES EQUATIONS

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ABSTRACT. We consider the incompressible Navier-Stokes equation with a fractional power $\alpha \in [1,\infty)$ of the Laplacian in the three dimensional case. We prove the existence of a smooth solution with arbitrarily small in $\dot{B}^{-\alpha}_{\infty,p}$ $(2 < p \le \infty)$ initial data that becomes arbitrarily large in $\dot{B}^{-s}_{\infty,\infty}$ for all $s > 0$ in arbitrarily small time. This extends the result of Bourgain and Pavlović [1] for the classical Navier-Stokes equation which utilizes the fact that the energy transfer to low modes increases norms with negative smoothness indexes. It is remarkable that the space $\dot{B}^{-\alpha}_{\infty,\infty}$ is supercritical for $\alpha > 1$. Moreover, the norm inflation occurs even in the case $\alpha \ge 5/4$ where the global regularity is known.

KEY WORDS: fractional Navier-Stokes equation; norm inflation; Besov spaces; interactions of plane waves

CLASSIFICATION CODE: 76D03, 35Q35.

1. INTRODUCTION

In this paper we study the three dimensional incompressible Navier-Stokes equations with a fractional power of the Laplacian:

$$\begin{aligned} u_t + (u\cdot\nabla)u + \nabla p &= -\nu(-\Delta)^\alpha u,\\ \nabla\cdot u &= 0,\\ u(x,0) &= u_0, \end{aligned} \tag{1.1}$$

where $x \in \mathbb{R}^3$, $t \ge 0$, u is the fluid velocity, p is the pressure of the fluid and $\nu > 0$ is the kinematic viscosity coefficient. The initial data u_0 is divergence free. The power $\alpha = 1$ corresponds to the classical Navier-Stokes equations. A vast amount of literature has been devoted to these equations, for background we refer the readers to [5] and [10].

Solutions to the fractional Navier-Stokes equation (1.1) have the following scaling property. If $(u(x,t), p(x,t))$ solves system (1.1) with the initial data $u_0(x)$, then

$$u_\lambda(x,t) = \lambda^{2\alpha-1}u(\lambda x, \lambda^{2\alpha}t),\ p_\lambda(x,t) = \lambda^{2(2\alpha-1)}p(\lambda x, \lambda^{2\alpha}t)$$

solves the system (1.1) with the initial data

$$u_{0\lambda} = \lambda^{2\alpha-1}u_0(\lambda x).$$

A space that is invariant under the above scaling is called a critical space. The largest critical space in three dimension for the fractional NSE (1.1) is the Besov space $\dot{B}^{1-2\alpha}_{\infty,\infty}$ (see [2]).

The study of the Navier-Stokes equations in critical spaces has been a focus of the research activity since the initial work of Kato [6]. In 2001, Koch and Tataru [7] established the global well-posedness of the classical Navier-Stokes equations with small initial data in the space BMO^{-1}. Then the question whether this result can be extended to the largest

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The work of Alexey Cheskidov was partially supported by NSF Grant DMS–1108864.

critical space $\dot{B}^{-1}_{\infty,\infty}$ had become of great interest among researchers, but it still remains open.

The first indication that such an extension might not be possible came in the work by Bourgain and Pavlović [1] who showed the norm inflation for the classical Navier-Stokes equations in $\dot{B}^{-1}_{\infty,\infty}$. More precisely, they constructed arbitrarily small initial data in $\dot{B}^{-1}_{\infty,\infty}$, such that mild solutions with this data become arbitrarily large in $\dot{B}^{-1}_{\infty,\infty}$ after an arbitrarily short time. This result was later extended to generalized Besov spaces smaller than $B^{-1}_{\infty,p}$, $p > 2$ by Yoneda [11]. Moreover, in [4] Cheskidov and Shvydkoy proved the existence of discontinuous Leray-Hopf solutions of the Navier-Stokes equations in $\dot{B}^{-1}_{\infty,\infty}$ with arbitrarily small initial data.

Recently, Yu and Zhai [12] considered the fractional Navier-Stokes equations (1.1) with $\alpha \in (1/2, 1)$ and showed global well-posedness for small initial data in the largest critical space $\dot{B}^{1-2\alpha}_{\infty,\infty}$, conjecturing that the above mentioned ill-posedness results could not be extended to the hypodissipative case $\alpha < 1$.

Indeed, in the recent work [3] Cheskidov and Shvydkoy were able to prove the existence of discontinuous Leray-Hopf solutions in the largest critical space with arbitrarily small initial data for $\alpha \in [1, 5/4)$. However, the construction broke down for $\alpha < 1$.

In this paper we consider the case $\alpha \in [1, \infty)$ and demonstrate that the natural space for norm inflation is not critical, but $\dot{B}^{-\alpha}_{\infty,p}$. Note that it is critical only in the classical case $\alpha = 1$, and it is not scaling invariant otherwise. More precisely, we prove the existence of a smooth space-periodic solution with arbitrarily small in $\dot{B}^{-\alpha}_{\infty,p}$ $(2 < p \leq \infty)$ initial data that becomes arbitrarily large in $\dot{B}^{-s}_{\infty,\infty}$ for all $s > 0$ in arbitrarily small time. This recovers Bourgain and Pavlović's ill-posedness result in the case $\alpha = 1$, and shows the norm inflation in spaces $\dot{B}^{-s}_{\infty,p}$ for all $s \geq \alpha$, $p \in (2, \infty]$ in the case $\alpha \geq 1$. The case $\alpha > 1$ is particularly interesting since the norm inflation occurs not only in critical spaces, but even in supercritical spaces, which suggests that a small initial data result might be out of reach there. It is remarkable that the norm inflation holds even in the case $\alpha \geq 5/4$, where the global regularity is known. In that case the smooth solution that exhibits the norm inflation can be extended globally in time.

Our construction is similar to the one of Bourgain and Pavlović, but we have to deal with the lack of continuity of the bilinear operator corresponding to the fractional heat kernel on a modified Koch-Tataru adapted space. This result is also based on the fact that a backwards energy cascade, harmless as far as the regularity of a solution is concerned, results in the growth of Besov norms with negative smoothness indexes. In this construction we also make sure that the initial data is space-periodic and has a finite energy if viewed on a torus. Namely, we show that

Theorem 1.1. *Let $\alpha \geq 1$. For any $\delta > 0$ and $2 < p \leq \infty$ there exists a smooth space-periodic solution $u(t)$ of (1.1) with period 2π and the initial data*

$$\|u(0)\|_{\dot{B}^{-\alpha}_{\infty,p}} \lesssim \delta$$

that satisfies, for some $0 < T < \delta$ and all $s > 0$,

$$\|u(T)\|_{\dot{B}^{-s}_{\infty,\infty}} \gtrsim \frac{1}{\delta}.$$

We refer the reader to the beginning of section of Preliminaries for the definition of the symbol $\lesssim$.

Note that the homogeneous and non-homogeneous Besov norms are equivalent for periodic functions. Therefore, for the space-periodic solution in Theorem 1.1 we have

$$\|u(0)\|_{\dot{B}^{-s}_{\infty,p}} \lesssim \|u(0)\|_{B^{-s}_{\infty,p}} \lesssim \|u(0)\|_{\dot{B}^{-\alpha}_{\infty,p}} \qquad \text{for all} \qquad s \geq \alpha,$$

Also, since

$$\|u(T)\|_{\dot{B}^{-s}_{\infty,p}} \gtrsim \|u(T)\|_{\dot{B}^{-s}_{\infty,\infty}},$$

the norm inflation occurs in all the spaces $\dot{B}^{-s}_{\infty,p}$, $s \in [\alpha, \infty)$, $p \in (2, \infty]$. More precisely, we have the following.

Corollary 1.2. Let $\alpha \geq 1$. For any $s \geq \alpha$, $p \in (2, \infty]$, and $\delta > 0$ there exists a smooth space-periodic solution $u(t)$ of (1.1) with the initial data

$$\|u(0)\|_{\dot{B}^{-s}_{\infty,p}} \lesssim \delta,$$

that satisfies, for some $0 < T < \delta$,

$$\|u(T)\|_{\dot{B}^{-s}_{\infty,p}} \gtrsim \frac{1}{\delta}.$$

Moreover, due to the embedding of the Triebel-Lizorkin space $\dot{F}^{-s}_{\infty,p}$

$$\dot{B}^{-s}_{\infty,p} \subset \dot{F}^{-s}_{\infty,p} \subset \dot{F}^{-s}_{\infty,\infty} = \dot{B}^{-s}_{\infty,\infty},$$

Theorem 1.1 also gives norm inflation in Triebel-Lizorkin spaces $\dot{F}^{-s}_{\infty,p}$, $s \geq \alpha$, $p \in (2, \infty]$:

Corollary 1.3. Let $\alpha \geq 1$. For any $s \geq \alpha$, $p \in (2, \infty]$, and $\delta > 0$ there exists a smooth space-periodic solution $u(t)$ of (1.1) with the initial data

$$\|u(0)\|_{\dot{F}^{-s}_{\infty,p}} \lesssim \delta,$$

that satisfies, for some $0 < T < \delta$,

$$\|u(T)\|_{\dot{F}^{-s}_{\infty,p}} \gtrsim \frac{1}{\delta}.$$

We now recall some auxiliary concepts related to the plane waves, which are necessary in the sequel:

- The "diffusion" of a plane wave $v \cos(k \cdot x)$ in R^3 under the fractional Laplacian $-(-\Delta)^\alpha$ is given by
$$e^{-t(-\Delta)^\alpha} v \cos(k \cdot x) = e^{-|k|^{2\alpha} t} v \cos(k \cdot x)$$
Thus the magnitude of the diffusion of a plane wave dies down in time in the scale that is measured by $|k|^{2\alpha}$.
- It is easy to see that $u = e^{-|k|^{2\alpha} t} v \cos(k \cdot x)$ solves the system (1.1) when the wave vector k is orthogonal to the amplitude vector v.
- The nonlinear interaction of two such diffusions in the system (1.1) can be captured, and it produces only a slower diffusion if the two wave vectors are close.

We note that these observations are the basis of the original argument of Bourgain and Pavlović in [1]. We will use them to construct a combination of such "diffusions" with least nonlinear interactions yet producing enough slower "diffusions" to cause the norm inflation in short time.

The rest of the paper is organized as: in Section 2 we introduce some notations that shall be used throughout the paper and some auxiliary results; in Section 3 we describe how the diffusions of plane waves interact in the fractional NSE system; in Section 4 we devote to proving Theorem 1.1.

2. PRELIMINARIES

2.1. **Notation.** We denote by $A \lesssim B$ an estimate of the form $A \leq CB$ with some absolute constant C, and by $A \sim B$ an estimate of the form $C_1 B \leq A \leq C_2 B$ with some absolute constants C_1, C_2. For simplification of the notation, we denote $\|\cdot\|_p = \|\cdot\|_{L^p}$.

2.2. **Semigroup operator $e^{-t(-\Delta)^\alpha}$.** Consider the Cauchy problem of the n dimensional dissipative equation with a fractional power of the Laplacian,

$$\begin{aligned} u_t + (-\Delta)^\alpha u = 0, \\ u(x,0) = \phi(x), \end{aligned} \tag{2.2}$$

where $(x,t) \in \mathbb{R}^n \times [0,\infty)$. Denote by $\mathcal{F}$ and $\mathcal{F}^{-1}$ the Fourier transform and inverse Fourier transform respectively. Let $e^{-t(-\Delta)^\alpha}$ denote the semigroup generated by a fractional Laplacian:

$$e^{-t(-\Delta)^\alpha}\phi := \mathcal{F}^{-1}\left(e^{-t|\xi|^{2\alpha}}\mathcal{F}(\phi)(\xi)\right). \tag{2.3}$$

Then $u = e^{-t(-\Delta)^\alpha}\phi$ is a solution of (2.2).

Let $\mathbb{P}$ denote the projection on divergence-free vector fields, which acts on a function ϕ as

$$\mathbb{P}(\phi) = \phi + \nabla \cdot (-\triangle)^{-1}\mathrm{div}\phi.$$

We will use the following well-known estimate.

Lemma 2.1. *For any $\phi \in L^\infty$,*

$$\|\nabla e^{-t(-\Delta)^\alpha}\mathbb{P}\phi\|_\infty \lesssim t^{-\frac{1}{2\alpha}}\|\phi\|_\infty, \qquad t > 0.$$

2.3. **Norm of Besov spaces.** We recall the definitions of norms for the homogeneous and non-homogeneous Besov spaces $\dot{B}^{-s}_{\infty,\infty}$ and $B^{-s}_{\infty,\infty}$ (see [8]) for $s > 0$

$$\begin{aligned} \|f\|_{\dot{B}^{-s}_{\infty,\infty}} &= \sup_{t>0} t^{\frac{s}{2\alpha}}\|e^{-t(-\Delta)^\alpha}f\|_{L^\infty}, \\ \|f\|_{B^{-s}_{\infty,\infty}} &= \sup_{0<t<1} t^{\frac{s}{2\alpha}}\|e^{-t(-\Delta)^\alpha}f\|_{L^\infty}. \end{aligned} \tag{2.4}$$

Note that for periodic functions the homogeneous and non-homogeneous norms are equivalent (see [9]). Therefore, for periodic functions with some fixed period we have

$$\|f\|_{\dot{B}^{-s}_{\infty,\infty}} \lesssim \|f\|_{L^\infty}, \tag{2.5}$$

since $\|e^{-t(-\Delta)^\alpha}f\|_{L^\infty} \leq \|f\|_{L^\infty}$.

We also recall the norm in the Besov space $\dot{B}^{-s}_{\infty,p}$:

$$\|u\|_{\dot{B}^{-s}_{\infty,p}} = \|\{2^{-sq}\|\Delta_q u\|_\infty\}_{q\in\mathbb{Z}}\|_{l^p},$$

where $\Delta_q u$ is the Littlewood-Paley projection of u.

2.4. **Bilinear operator.** Define the bilinear operator

$$\mathcal{B}_\alpha(u,v)=\int_0^t e^{-(t-\tau)(-\Delta)^\alpha}\mathbb{P}\nabla\cdot(u\otimes v)\,d\tau. \tag{2.6}$$

As shown in [7] in the case of $\alpha=1$, the bilinear operator $\mathcal{B}_1$ continuously maps $X_T\times X_T$ into X_T, where X_T is the Koch-Tataru adapted space. In [1], the continuity of $\mathcal{B}_1$ on $X_T\times X_T$ plays an important role to estimate the higher order iterations (the part y in the paper) of the nonlinear term. Having to deal with the lack of continuity in the case $\alpha>1$, we use a relatively weaker estimate for the bilinear operator to control nonlinear interactions (cf. [11]):

Lemma 2.2. *For all $\alpha>0$ the bilinear operator satisfies*

$$\|\mathcal{B}_\alpha(u,v)\|_\infty\lesssim\int_0^t\frac{1}{(t-\tau)^{1/(2\alpha)}}\|u(\tau)\|_\infty\|v(\tau)\|_\infty d\tau. \tag{2.7}$$

Proof: By the definition (2.6) and Lemma 2.1 we have

$$\begin{aligned}\|\mathcal{B}_\alpha(u,v)\|_\infty&\lesssim\int_0^t\|e^{-(t-\tau)(-\Delta)^\alpha}\mathbb{P}\nabla\cdot(u\otimes v)(\tau)\|_\infty d\tau\\&\lesssim\int_0^t\frac{1}{(t-\tau)^{1/(2\alpha)}}\|u(\tau)\|_\infty\|v(\tau)\|_\infty d\tau.\end{aligned}$$

□

3. INTERACTIONS OF PLANE WAVES

3.1. **The first approximation of a mild solution.** Let u be a solution to (1.1). We write it in the form

$$u(t)=e^{-t(-\Delta)^\alpha}u_0-u_1(t)+y(t), \tag{3.8}$$

where

$$u_1(t)=\mathcal{B}_\alpha(e^{-t(-\Delta)^\alpha}u_0,e^{-t(-\Delta)^\alpha}u_0). \tag{3.9}$$

A simple calculation shows that

$$y(t)=-\int_0^t e^{-(t-\tau)(-\Delta)^\alpha}[G_0(\tau)+G_1(\tau)+G_2(\tau)]d\tau, \tag{3.10}$$

where

$$\begin{aligned}&G_0=\mathbb{P}[(e^{-t(-\Delta)^\alpha}u_0\cdot\nabla)u_1+(u_1\cdot\nabla)e^{-t(-\Delta)^\alpha}u_0+(u_1\cdot\nabla)u_1]\\&G_1=\mathbb{P}[(e^{-t(-\Delta)^\alpha}u_0\cdot\nabla)y+(u_1\cdot\nabla)y+(y\cdot\nabla)e^{-t(-\Delta)^\alpha}u_0+(y\cdot\nabla)u_1]\\&G_2=\mathbb{P}[(y\cdot\nabla)y].\end{aligned} \tag{3.11}$$

Note that G_0 does not depend on y, G_1 is linear, and G_2 is quadratic in y.

In this section we show how the diffusions of plane waves interact in the fractional NSE system. These interactions are the basis for the constructions of initial data to produce the norm inflation.

3.2. **Diffusion of a plane wave.** As a first step, we consider the initial data being one single plane wave. Suppose $k \in \mathbb{R}^3$, $v \in \mathbb{S}^2$ and $k \cdot v = 0$. Let

$$u_0 = v\cos(k \cdot x).$$

Then $\nabla \cdot u_0 = 0$ and

$$e^{-t(-\Delta)^\alpha} v\cos(k\cdot x) = e^{-|k|^{2\alpha}t} v \cos(k \cdot x). \tag{3.12}$$

In fact the "diffusion" $e^{-t(-\Delta)^\alpha} v\cos(k\cdot x)$ of a plane wave solves (1.1) with vanishing pressure. And it is important to notice that for $s \geq 0$

$$\|v\cos(k\cdot x)\|_{\dot{B}^{-s}_{\infty,\infty}} \sim |k|^{-s}.$$

3.3. **Interaction of plane waves.** Now we consider the interaction of two different single plane waves. Suppose $k_i \in \mathbb{R}^3$, $v_i \in \mathbb{S}^2$ and $k_i \cdot v_i = 0$, for $i = 1, 2$. Let

$$u_1 = \cos(k_1 \cdot x)v_1, \;\; u_2 = \cos(k_2 \cdot x) v_2.$$

To simplify our calculations we assume that $k_2 \cdot v_1 = \frac{1}{2}$. It then follows from a straightforward calculation that

$$\begin{aligned}
&e^{-t(-\Delta)^\alpha} u_1 \cdot \nabla (e^{-t(-\Delta)^\alpha} u_2)\\
&= -e^{-(|k_1|^{2\alpha}+|k_2|^{2\alpha})t} v_2 \cos(k_1\cdot x)\sin(k_2\cdot x)(k_2\cdot v_1)\\
&= -\frac{1}{4} e^{-(|k_1|^{2\alpha}+|k_2|^{2\alpha})t} v_1 (\sin((k_2-k_1)\cdot x) + \sin((k_1+k_2)\cdot x)).
\end{aligned}$$

Hence

$$\begin{aligned}
&\mathcal{B}_\alpha(e^{-t(-\Delta)^\alpha}u_1, e^{-t(-\Delta)^\alpha}u_2)\\
=&\frac{1}{4} v_1 \sin((k_2-k_1)\cdot x) \int_0^t e^{-(|k_1|^{2\alpha}+|k_2|^{2\alpha})\tau} e^{-|k_2-k_1|^{2\alpha}(t-\tau)} d\tau\\
&+ \frac{1}{4} v_1 \sin((k_1+k_2)\cdot x) \int_0^t e^{-(|k_1|^{2\alpha}+|k_2|^{2\alpha})\tau} e^{-|k_1+k_2|^{2\alpha}(t-\tau)} d\tau.
\end{aligned}$$

Therefore, the interaction of the two plane waves is small in $\dot{B}^{-s}_{\infty,\infty}$ if neither the sum nor the difference of their wave vectors is small in magnitude. In the contrary, the interaction is sizable in $\dot{B}^{-s}_{\infty,\infty}$ if either the sum or the difference of their wave vectors is small in magnitude.

4. PROOF OF THEOREM 1.1

In this section we follow the idea from [1] to construct initial data that produce norm inflation for solutions to the fractional Navier-Stokes equation. From the discussions in Subsection 3.3 it is clear that the interaction of two plane waves is not enough to produce the norm inflation, which actually requires a large number of waves. We also make sure that the initial data is space-periodic and smooth, which ensures the local existence of a smooth periodic solution to the fractional NSE. As we control its L^∞ norm, the solution will remain smooth until the time of the norm inflation.

4.1. **Construction of initial data for the fractional NSE system.** For a fixed small number $\delta > 0$, the initial data will be chosen as follows:

$$u_0 = r^{-\beta} \sum_{i=1}^{r} |k_i|^\alpha (v \cos(k_i \cdot x) + v' \cos(k_i' \cdot x)), \tag{4.13}$$

where $\beta > 0$. We expect for each i the interaction of the two plane waves $v \cos(k_i \cdot x)$ and $v' \cos(k_i' \cdot x)$ to be sizable in $\dot{B}^{-s}_{\infty,\infty}$, while the interactions of plane waves corresponding to different indexes i to be small. Hence, we choose

- Wave vectors: Let $\zeta = (1, 0, 0)$ and $\eta = (0, 0, 1)$. The wave vectors $k_i \in \mathbb{Z}^3$ are parallel to ζ. Let K be a large integer dependent on r. The magnitude of k_i is defined by

  $$|k_i| = 2^{i-1}K, \quad i = 1, 2, 3, ..., r. \tag{4.14}$$

  The wave vectors $k_i' \in \mathbb{Z}^3$ are defined by

  $$k_i' = k_i + \eta. \tag{4.15}$$

- Amplitude vectors: Let

  $$v = (0, 0, 1), \quad v' = (0, 1, 0). \tag{4.16}$$

  Hence

  $$k_i \cdot v = k_i' \cdot v' = 0,$$

  which ensures that the initial data is divergence free.

We first point out the following simple facts to further motivate the choices of the parameters.

Lemma 4.1. *Let $\gamma > 0$, $\alpha \geq 1$. With the choices (4.14)-(4.16), the following holds:*

$$k_i \cdot v' = 0, \quad k_i' \cdot v = 1, \quad \forall \quad i = 1, 2, \ldots, r, \tag{4.17}$$

$$\sum_{j<i} |k_j|^\alpha \sim |k_{i-1}|^\alpha \quad \textit{and} \quad \sum_{j<i} |k_j'|^\alpha \sim |k_{i-1}'|^\alpha, \tag{4.18}$$

$$\sum_{i=1}^{r} |k_i|^\gamma e^{-|k_i|^{2\alpha} t} \lesssim t^{-\frac{\gamma}{2\alpha}} \quad \textit{and} \quad \sum_{i=1}^{r} |k_i'|^\gamma e^{-|k_i'|^{2\alpha} t} \lesssim t^{-\frac{\gamma}{2\alpha}}. \tag{4.19}$$

Proof: The first conclusion (4.17) is obvious due to (4.14)-(4.16). By the definition (4.14), it is clear that $|k_{i-1}|^\alpha < \frac{1}{2}|k_i|^\alpha$, which immediately implies (4.18). Thanks to (4.14), we have that $|k_i|^\alpha \sim |k_i|^\alpha - |k_{i-1}|^\alpha$. Thus,

$$\sum_{i=1}^{r} |k_i|^\gamma e^{-|k_i|^{2\alpha} t} \sim \sum_{i=1}^{r} |k_i|^{\gamma-\alpha} (|k_i|^\alpha - |k_{i-1}|^\alpha) e^{-|k_i|^{2\alpha} t},$$

while the latter one can be considered as a finite Riemman summation of the function $x^{\gamma/\alpha - 1} e^{-x^2 t}$. Therefore, for $\gamma > 0$ and $\alpha > 0$,

$$\sum_{i=1}^{r} |k_i|^\gamma e^{-|k_i|^{2\alpha} t} \lesssim \int_0^\infty x^{\gamma/\alpha-1} e^{-x^2 t} dx = t^{-\frac{\gamma}{2\alpha}} \int_0^\infty y^{\gamma/\alpha-1} e^{-y^2} dy \lesssim t^{-\frac{\gamma}{2\alpha}}.$$

$\square$

Next we estimate the norms of the initial data.

Lemma 4.2. *Let u_0 be given in (4.13) and $\alpha > 0$. Then*

$$\|u_0\|_{\dot{B}^{-\alpha}_{\infty,p}} \lesssim r^{1/p-\beta}, \qquad 1 \le p \le \infty. \tag{4.20}$$

Proof: Due to (3.12), we have that,

$$e^{-t(-\Delta)^\alpha} u_0 = r^{-\beta} \sum_{i=1}^{r} |k_i|^\alpha (v \cos(k_i \cdot x) e^{-|k_i|^{2\alpha} t} + v' \cos(k_i' \cdot x) e^{-|k_i'|^{2\alpha} t}). \tag{4.21}$$

Hence by Lemma 4.1,

$$\|u_0\|_{\dot{B}^{-\alpha}_{\infty,\infty}} \sim r^{-\beta} \sup_{0<t<1} t^{\frac{1}{2}} \sum_{i=1}^{r} |k_i|^\alpha \left(e^{-|k_i|^{2\alpha} t} + e^{-|k_i'|^{2\alpha} t} \right) \lesssim r^{-\beta}.$$

A direct computation also gives

$$\|u_0\|_{\dot{B}^{-\alpha}_{\infty,p}} \lesssim r^{-\beta} \left(\sum_{i=1}^{r} 1^p \right)^{1/p} = r^{1/p-\beta}, \qquad p \ge 1.$$

$\square$

Lemma 4.3. *Let u_0 be given in (4.13). Then*

$$\|e^{-t(-\Delta)^\alpha} u_0\|_\infty \lesssim r^{-\beta} t^{-1/2}.$$

Proof: By (4.21) and Lemma 4.1, we infer that

$$\|e^{-t(-\Delta)^\alpha} u_0\|_\infty \lesssim r^{-\beta} \sum_{i=1}^{r} |k_i|^\alpha \left(e^{-|k_i|^{2\alpha} t} + e^{-|k_i'|^{2\alpha} t} \right) \lesssim r^{-\beta} t^{-1/2}.$$

$\square$

4.2. **Analysis of u_1.**

As demonstrated in Subsection 3.1 we consider the decomposition

$$u = e^{-t(-\Delta)^\alpha} u_0 - u_1 + y.$$

Recall the definition (3.9)

$$u_1 = \mathcal{B}_\alpha(e^{-t(-\Delta)^\alpha} u_0, e^{-t(-\Delta)^\alpha} u_0).$$

By (4.16), (4.17), (4.21) and a straightforward calculation, it follows that

$$\begin{aligned}
&(e^{-t(-\Delta)^\alpha} u_0 \cdot \nabla) e^{-t(-\Delta)^\alpha} u_0 \\
&= - r^{-2\beta} \sum_{i=1}^{r} \sum_{j=1}^{r} |k_i|^\alpha |k_j|^\alpha e^{-(|k_i|^{2\alpha} + |k_j'|^{2\alpha}) t} v' \cos(k_i \cdot x) \sin(k_j' \cdot x) \\
&= - \frac{r^{-2\beta}}{2} \sum_{i=1}^{r} |k_i|^{2\alpha} e^{-(|k_i'|^{2\alpha} + |k_i|^{2\alpha}) t} \sin(\eta \cdot x) v' \\
&\quad - \frac{r^{-2\beta}}{2} \sum_{i \neq j}^{r} |k_i|^\alpha |k_j|^\alpha e^{-(|k_i|^{2\alpha} + |k_j'|^{2\alpha}) t} \sin((k_j' - k_i) \cdot x) v' \\
&\quad - \frac{r^{-2\beta}}{2} \sum_{i=1}^{r} \sum_{j=1}^{r} |k_i|^\alpha |k_j|^\alpha e^{-(|k_i|^{2\alpha} + |k_j'|^{2\alpha}) t} \sin((k_j' + k_i) \cdot x) v' \\
&\equiv E_0 + E_1 + E_2,
\end{aligned} \tag{4.22}$$

where we used the formula $\cos x \sin y = [\sin(x+y) - \sin(x-y)]/2$.

Recall that $\eta \cdot v' = 0$, $(k_j' + k_i) \cdot v' = 0$ and $(k_j' - k_i) \cdot v' = 0$ for all i, j due to (4.17). Hence E_0, E_1 and E_2 are divergence free vectors. Thus we can write

$$\begin{aligned} u_1 = &\int_0^t e^{-(t-\tau)(-\Delta)^\alpha} E_0(\tau)d\tau + \int_0^t e^{-(t-\tau)(-\Delta)^\alpha} E_1(\tau)d\tau \\ &+ \int_0^t e^{-(t-\tau)(-\Delta)^\alpha} E_2(\tau)d\tau \equiv u_{10} + u_{11} + u_{12}. \end{aligned} \tag{4.23}$$

We have the following estimates.

Lemma 4.4. *Let* u_{10} *be defined in (4.23) and* $s > 0$*. Then*

$$\begin{aligned} &\|u_{10}(\cdot,t)\|_{\dot{B}^{-s}_{\infty,\infty}} \gtrsim r^{1-2\beta}, \qquad \text{for all} \qquad K^{-2\alpha} \le t \le 1, \\ &\|u_{10}(\cdot,t)\|_\infty \lesssim r^{1-2\beta}, \qquad \text{for all} \qquad t > 0. \end{aligned}$$

Proof: From (4.22) and (4.23) it follows by a straightforward calculation

$$\begin{aligned} u_{10} &= -\frac{r^{-2\beta}}{2} \int_0^t \sum_{i=1}^r |k_i|^{2\alpha} e^{-(|k_i'|^{2\alpha}+|k_i|^{2\alpha})\tau} e^{-|\eta|^{2\alpha}(t-\tau)} \sin(\eta \cdot x) v' d\tau \\ &= -\frac{r^{-2\beta}}{2} \sin(\eta \cdot x) v' \sum_{i=1}^r |k_i|^{2\alpha} e^{-t} \frac{1 - e^{-(|k_i'|^{2\alpha}+|k_i|^{2\alpha}-1)t}}{|k_i'|^{2\alpha} + |k_i|^{2\alpha} - 1} \\ &\sim -\frac{r^{-2\beta}}{2} \sin(\eta \cdot x) v' \sum_{i=1}^r e^{-t}(1 - e^{-|k_i|^{2\alpha} t}). \end{aligned}$$

Hence for $K^{-2\alpha} \le t \le 1$ and $s > 0$,

$$\|u_{10}(\cdot,t)\|_{\dot{B}^{-s}_{\infty,\infty}} \gtrsim r^{-2\beta} \cdot r \sup_{0<\tau<1} \tau^{\frac{s}{2\alpha}} e^{-|\eta|^{2\alpha}\tau} \gtrsim r^{1-2\beta}.$$

On the other hand,

$$\|u_{10}(\cdot,t)\|_\infty \lesssim \frac{r^{-2\beta}}{2} \cdot r \lesssim r^{1-2\beta},$$

for all $t > 0$.

□

Lemma 4.5. *Let* u_{11} *and* u_{12} *be defined in (4.23). Then*

$$\|u_{11}(\cdot,t)\|_\infty \lesssim r^{-2\beta}, \quad \|u_{12}(\cdot,t)\|_\infty \lesssim r^{-2\beta},$$

for all $t > 0$*.*

Proof: Thanks to (4.22) and (4.23), it follows that

$$\begin{aligned} u_{11} = &\frac{r^{-2\beta}}{2} \int_0^t \sum_{i \neq j}^r |k_i|^\alpha |k_j|^\alpha e^{-(|k_i|^{2\alpha}+|k_j'|^{2\alpha})\tau} e^{-|k_j'-k_i|^{2\alpha}(t-\tau)} \sin((k_j' - k_i) \cdot x) v' d\tau \\ \sim &\frac{r^{-2\beta}}{2} \sum_{i=1}^r \sum_{j<i} |k_i|^\alpha |k_j|^\alpha e^{-|k_i-k_j'|^{2\alpha}t} \frac{1 - e^{-(|k_i|^{2\alpha}+|k_j'|^{2\alpha}-|k_i-k_j'|^{2\alpha})t}}{|k_i|^{2\alpha} + |k_j'|^{2\alpha} - |k_i - k_j'|^{2\alpha}} \\ &\cdot \sin((k_j' - k_i) \cdot x) v' \\ \sim &\frac{r^{-2\beta}}{2} \sum_{i=1}^r \sum_{j<i} |k_i|^\alpha |k_j|^\alpha t e^{-|k_i|^{2\alpha}t} \sin((k_j' - k_i) \cdot x) v', \end{aligned}$$

where we used the fact that $\frac{1-e^{-x}}{x}$ is bounded for $x>0$. Hence, by (4.18) and (4.19) we infer that

$$\begin{aligned}\|u_{11}(\cdot,t)\|_\infty &\lesssim r^{-2\beta}\sum_{i=1}^{r}\sum_{j<i}|k_i|^\alpha|k_j|^\alpha te^{-|k_i|^{2\alpha}t}\\ &\lesssim r^{-2\beta}\sum_{i=1}^{r}|k_i|^{2\alpha}te^{-|k_i|^{2\alpha}t}\lesssim r^{-2\beta}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}u_{12}=&\frac{r^{-2\beta}}{2}\int_0^t\sum_{i=1}^{r}\sum_{j=1}^{r}|k_i|^\alpha|k_j|^\alpha e^{-(|k_i|^{2\alpha}+|k_j'|^{2\alpha})\tau}e^{-|k_i+k_j'|^{2\alpha}(t-\tau)}\sin((k_i+k_j')\cdot x)v'd\tau\\ =&\frac{r^{-2\beta}}{2}\sum_{i=1}^{r}\sum_{j=1}^{r}|k_i|^\alpha|k_j|^\alpha e^{-(|k_i|^{2\alpha}+|k_j'|^{2\alpha})t}\frac{1-e^{-(|k_i+k_j'|^{2\alpha}-|k_i|^{2\alpha}-|k_j'|^{2\alpha})t}}{|k_i+k_j'|^{2\alpha}-|k_i|^{2\alpha}-|k_j'|^{2\alpha}}\\ &\cdot\sin((k_i+k_j')\cdot x)v'\\ \sim&r^{-2\beta}\sum_{i=1}^{r}\sum_{j\le i}|k_i|^\alpha|k_j|^\alpha e^{-(|k_i|^{2\alpha}+|k_j'|^{2\alpha})t}t\sin((k_i+k_j')\cdot x)v'.\end{aligned}$$

Thus,

$$\begin{aligned}\|u_{12}(\cdot,t)\|_\infty &\lesssim r^{-2\beta}\sum_{i=1}^{r}\sum_{j\le i}|k_i|^\alpha|k_j|^\alpha te^{-|k_i|^{2\alpha}t}\\ &\lesssim r^{-2\beta}\sum_{i=1}^{r}|k_i|^{2\alpha}te^{-|k_i|^{2\alpha}t}\lesssim r^{-2\beta}.\end{aligned}$$

□

4.3. **Analysis of y.** In this section we analyze the part y of the solution. The idea is to control y using the estimate (2.7) of the bilinear operator $\mathcal{B}_\alpha$ in the space L^∞.

Recall from Subsection 3.1 that

$$y(t)=-\int_0^t e^{-(t-\tau)(-\Delta)^\alpha}[G_0(\tau)+G_1(\tau)+G_2(\tau)]d\tau. \tag{4.24}$$

Lemma 4.6. *Let $\alpha\ge 1$ and $\beta\in(0,1/2)$. Then*

$$\|y(t)\|_\infty\lesssim r^{1-3\beta}t^{\frac12-\frac{1}{2\alpha}}+r^{2-4\beta}t^{1-\frac{1}{2\alpha}},\qquad \forall t\in[0,T],$$

provided T is small and r is large enough.

Proof: It follows from (3.11) and (4.24) that

$$\begin{aligned}\|y(t)\|_\infty\lesssim&\|\mathcal{B}_\alpha(e^{-t(-\Delta)^\alpha}u_0,u_1)\|_\infty+\|\mathcal{B}_\alpha(u_1,u_1)\|_\infty\\ &+\|\mathcal{B}_\alpha(e^{-t(-\Delta)^\alpha}u_0,y)\|_\infty+\|\mathcal{B}_\alpha(u_1,y)\|_\infty+\|\mathcal{B}_\alpha(y,y)\|_\infty.\end{aligned}$$

Applying the bilinear estimate (2.7), Lemmas 4.3, 4.4, and 4.5 we infer

$$\begin{aligned}\|\mathcal{B}_\alpha(e^{-t(-\Delta)^\alpha}u_0, u_1)\|_\infty &\lesssim \int_0^t \frac{1}{(t-\tau)^{1/(2\alpha)}}\|e^{-\tau(-\Delta)^\alpha}u_0\|_\infty \|u_1(\tau)\|_\infty d\tau \\ &\lesssim r^{1-3\beta}\int_0^t (t-\tau)^{-1/(2\alpha)}\tau^{-1/2}d\tau \\ &\lesssim r^{1-3\beta}t^{\frac{1}{2}-\frac{1}{2\alpha}},\end{aligned}$$

where we used the boundedness of Beta function for $\alpha > 1/2$:

$$\int_0^t (t-\tau)^{-1/(2\alpha)}\tau^{-1/2}d\tau = t^{\frac{1}{2}-\frac{1}{2\alpha}}B(\frac{1}{2}, 1-\frac{1}{2\alpha}) \le Ct^{\frac{1}{2}-\frac{1}{2\alpha}}.$$

Similarly, using the estimates obtained in previous two subsections, we obtain

$$\|\mathcal{B}_\alpha(u_1, u_1)\|_\infty \lesssim \int_0^t \frac{1}{(t-\tau)^{1/(2\alpha)}}\|u_1(\tau)\|_\infty^2 d\tau \lesssim r^{2-4\beta}t^{1-\frac{1}{2\alpha}},$$

$$\begin{aligned}\|\mathcal{B}_\alpha(e^{-t(-\Delta)^\alpha}u_0, y)\|_\infty &\lesssim \int_0^t \frac{1}{(t-\tau)^{1/(2\alpha)}}\|e^{-\tau(-\Delta)^\alpha}u_0\|_\infty \|y(\tau)\|_\infty d\tau \\ &\lesssim r^{-\beta}\int_0^t (t-\tau)^{-1/(2\alpha)}\tau^{-1/2}d\tau \sup_{0<\tau<t}\|y(\tau)\|_\infty \\ &\lesssim r^{-\beta}t^{\frac{1}{2}-\frac{1}{2\alpha}}\sup_{0<\tau<t}\|y(\tau)\|_\infty,\end{aligned}$$

$$\begin{aligned}\|\mathcal{B}_\alpha(u_1, y)\|_\infty &\lesssim \int_0^t \frac{1}{(t-\tau)^{1/(2\alpha)}}\|u_1(\tau)\|_\infty \|y(\tau)\|_\infty d\tau \\ &\lesssim r^{1-2\beta}\int_0^t (t-\tau)^{-1/(2\alpha)}d\tau \sup_{0<\tau<t}\|y(\tau)\|_\infty \\ &\lesssim r^{1-2\beta}t^{1-\frac{1}{2\alpha}}\sup_{0<\tau<t}\|y(\tau)\|_\infty,\end{aligned}$$

$$\|\mathcal{B}_\alpha(y, y)\|_\infty \lesssim \int_0^t \frac{1}{(t-\tau)^{1/(2\alpha)}}\|y(\tau)\|_\infty^2 d\tau \lesssim t^{1-\frac{1}{2\alpha}}\left(\sup_{0<\tau<t}\|y(\tau)\|_\infty\right)^2.$$

Thus we have

$$\begin{aligned}\|y(t)\|_\infty &\lesssim r^{1-3\beta}t^{\frac{1}{2}-\frac{1}{2\alpha}} + r^{2-4\beta}t^{1-\frac{1}{2\alpha}} \\ &\quad + \left(r^{-\beta}t^{\frac{1}{2}-\frac{1}{2\alpha}} + r^{1-2\beta}t^{1-\frac{1}{2\alpha}} + t^{1-\frac{1}{2\alpha}}\sup_{0<\tau<t}\|y(\tau)\|_\infty\right)\sup_{0<\tau<t}\|y(\tau)\|_\infty.\end{aligned}$$

We choose large enough r and small enough $T > 0$, such that

$$A := r^{-\beta}t^{\frac{1}{2}-\frac{1}{2\alpha}} + r^{1-2\beta}t^{1-\frac{1}{2\alpha}} + t^{1-\frac{1}{2\alpha}}(r^{1-3\beta}t^{\frac{1}{2}-\frac{1}{2\alpha}} + r^{2-4\beta}t^{1-\frac{1}{2\alpha}}) \ll 1 \tag{4.25}$$

for $0 \le t \le T$. Indeed, note that the powers of t in A are all nonnegative for $\alpha \ge 1$. Thus,

$$A \le r^{-\beta}T^{\frac{1}{2}-\frac{1}{2\alpha}} + r^{1-2\beta}T^{1-\frac{1}{2\alpha}} + T^{1-\frac{1}{2\alpha}}(r^{1-3\beta}T^{\frac{1}{2}-\frac{1}{2\alpha}} + r^{2-4\beta}T^{1-\frac{1}{2\alpha}}).$$

Let $T = r^{-\gamma}$. It follows

$$A \le r^{-\beta-\gamma(\frac{1}{2}-\frac{1}{2\alpha})} + r^{1-2\beta-\gamma(1-\frac{1}{2\alpha})} + r^{1-3\beta-\gamma(\frac{3}{2}-\frac{1}{\alpha})} + r^{2-4\beta-\gamma(2-\frac{1}{\alpha})}. \tag{4.26}$$

We choose γ such that

$$\gamma > \frac{1-2\beta}{1-1/(2\alpha)}, \tag{4.27}$$

which guarantees all the powers of r are negative in (4.26). Hence (4.25) is satisfied for r large enough. Since $y(0) = 0$, we have the following bound by an absorbing argument:

$$\|y(t)\|_\infty \lesssim r^{1-3\beta} t^{\frac{1}{2}-\frac{1}{2\alpha}} + r^{2-4\beta} t^{1-\frac{1}{2\alpha}},$$

for all $0 < t \le T$.

$\square$

4.4. **Finishing the proof.** Now we are ready to complete the proof of Theorem 1.1. Since u_0 is smooth and space-periodic, there exists $T^* > 0$ and a smooth space-periodic solution $u(t)$ to (1.1) on $[0, T^*)$ with $u(0) = u_0$, such that either $T^* = +\infty$ or

$$\limsup_{t \to T^*-} \|u(t)\|_\infty = +\infty.$$

Lemmas 4.4, 4.5, and 4.6 imply that $T^* > T$, where $T = r^{-\gamma}$ and γ is large enough so that (4.27) holds. Note that $T < 1$.

Now using (3.8), we combine the imbedding estimate (2.5), Lemmas 4.3, 4.4, 4.5 and 4.6 to obtain that, for $K^{-2\alpha} \le t \le T$,

$$\begin{aligned}
\|u(\cdot,t)\|_{\dot{B}^{-s}_{\infty,\infty}} \ge& \|u_{10}(\cdot,t)\|_{\dot{B}^{-s}_{\infty,\infty}} - \|u_{11}(\cdot,t)\|_\infty - \|u_{12}(\cdot,t)\|_\infty \\
&- \|e^{-t(-\Delta)^\alpha} u_0\|_\infty - \|y(\cdot,t)\|_\infty \\
\gtrsim& r^{1-2\beta}\left(1 - r^{-1} - r^{\beta-1} t^{-\frac{1}{2}} - r^{-\beta} t^{\frac{1}{2}-\frac{1}{2\alpha}} - r^{1-2\beta} t^{1-\frac{1}{2\alpha}}\right) \\
\gtrsim& r^{1-2\beta}\left(1 - r^{\beta-1} K^\alpha - r^{-\beta} T^{\frac{1}{2}-\frac{1}{2\alpha}} - r^{1-2\beta} T^{1-\frac{1}{2\alpha}}\right).
\end{aligned} \tag{4.28}$$

We will show that we can choose parameters so that

$$B := r^{\beta-1} K^\alpha + r^{-\beta} T^{\frac{1}{2}-\frac{1}{2\alpha}} + r^{1-2\beta} T^{1-\frac{1}{2\alpha}} \le 1/4, \qquad \text{for} \quad \alpha \ge 1. \tag{4.29}$$

Let $K = r^\zeta$ with positive ζ, and recall that $T = r^{-\gamma}$ as in Lemma 4.6. Then

$$B = r^{\beta-1+\zeta\alpha} + r^{-\beta-\gamma(\frac{1}{2}-\frac{1}{2\alpha})} + r^{1-2\beta-\gamma(1-\frac{1}{2\alpha})}.$$

For any $\beta \in (0, \frac{1}{2})$ we choose ζ, γ such that

$$0 < \zeta < \frac{1-\beta}{\alpha}, \qquad \frac{1-2\beta}{1-1/(2\alpha)} < \gamma < 2\alpha\zeta, \tag{4.30}$$

which can be done because $\alpha \ge 1$. This implies that all the powers of r in B are negative and hence (4.29) is satisfied for r large enough. Moreover, the condition $\gamma < 2\alpha\zeta$ guarantees that $K^{-2\alpha} < T$. Note that the conditions on γ in (4.27) and (4.30) coincide.

Given any $\delta > 0$ in Theorem 1.1, we now choose a suitable large r such that

$$r^{1-2\beta} \gtrsim \frac{1}{\delta}.$$

Therefore, it follows from (4.28) and (4.29) that

$$\|u(T)\|_{\dot{B}^{-s}_{\infty,\infty}} \gtrsim r^{1-2\beta} \gtrsim \frac{1}{\delta}.$$

Finally, Lemma 4.2 implies that the initial data u_0 satisfies

$$\|u_0\|_{\dot{B}^{-\alpha}_{\infty,p}} \lesssim r^{1/p-\beta} \lesssim r^{2\beta-1} \lesssim \delta,$$

as long as $3\beta \ge 1 + 1/p$, which holds for any $p > 2$ provided β is close enough to $1/2$. This competes the proof of Theorem 1.1.

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