On global attractors of the 3D Navier-Stokes equations

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Abstract

In view of the possibility that the 3D Navier-Stokes equations (NSE) might not always have regular solutions, we introduce an abstract framework for studying the asymptotic behavior of multi-valued dissipative evolutionary systems with respect to two topologies—weak and strong. Each such system possesses a global attractor in the weak topology, but not necessarily in the strong. In case the latter exists and is weakly closed, it coincides with the weak global attractor. We give a sufficient condition for the existence of the strong global attractor, which is verified for the 3D NSE when all solutions on the weak global attractor are strongly continuous. We also introduce and study a two-parameter family of models for the Navier-Stokes equations, with similar properties and open problems. These models always possess weak global attractors, but on some of them every solution blows up (in a norm stronger than the standard energy one) in finite time.

Keywords: Navier-Stokes equations, global attractor, blow-up in finite time.

1 Introduction

A remarkable feature of many dissipative partial differential equations (PDEs) is the existence of a global attractor to which all the solutions converge as time goes to infinity [21, 22]. The global attractor $\mathcal{A}$ is the minimal closed set in a phase space H (i.e., the functional space, usually a Banach space, in which the solutions exist) that uniformly attracts the trajectories starting from any a priori given bounded set in H. When the topology on H is referred as the strong (weak) topology of H, we will call $\mathcal{A}$ the strong (respectively weak) global attractor.

It is possible that a dissipative PDE does not have a strong global attractor. For instance, the 2D Navier-Stokes equations (NSE) on a bounded domain $\Omega \subset \mathbb{R}^2$, when supplemented with appropriate boundary conditions, possess a strong global attractor in H (a certain subspace of $L^2(\Omega)^2$) [15, 22], but it is not yet known whether this holds for the 3D NSE.

Nevertheless, even for the 3D NSE one can prove that there exists a weak global attractor [9]. When the strong global attractor is strongly compact in H (e.g., in the

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2D NSE), then it is also the weak global attractor. But, in any case, the weak global attractor is an appropriate generalization of the strong global attractor since it captures the long-time behavior of the solutions. In particular, the support of any time-average measure of the 3D NSE is included in the weak global attractor (see [8]). One should note that Sell [20] introduced a related notion of a trajectory attractor $\mathfrak{A}$ in the space of all trajectories, which was further studied in [4, 7, 20, 21]. The weak global attractor coincides with the set of values of all trajectories in $\mathfrak{A}$ at any fixed time t.

The aim of this study is to present a general abstract framework which is applicable to the 3D NSE even in the case where they do not possess a strong global attractor. This framework may be also useful in the study of other PDEs for which the existence of the strong global attractor is in limbo. This aim forces us to consider multi-valued evolutionary systems. A number of papers have been published concerning attractors of multi-valued semiflows. See [3] for a comparison of two canonical abstract frameworks by Melnik and Valero [17] and Ball [1]. The main difference between our evolutionary system $\mathcal{E}$ and Ball's generalized semiflow is that we do not include the hypotheses of concatenation and upper semicontinuity with respect to the initial data. This allows us to consider an evolutionary system whose trajectories are all Leray-Hopf weak solutions of the 3D NSE.

Our definition of the evolutionary system $\mathcal{E}$ already exploits the effect of dissipativity, namely the existence of an absorbing ball. In fact, the space X in which the trajectories of $\mathcal{E}$ live is, in applications, precisely such an absorbing ball. Since in most applications the phase space is a separable reflexive Banach space, both the strong and the weak topologies on X are metrizable. This is the motivation for us to define strong and weak topologies on X to be the ones induced by appropriate metrics.

We show that every evolutionary system always possesses a weak global attractor; moreover, if the strong global attractor exists and is weakly closed, then it has to coincide with the weak global attractor. Note that some classical definitions (see, e.g., [22]) require a global attractor to be an invariant set. We will see that under a condition, which is, for example, satisfied by the Leray-Hopf weak solutions of the 3D NSE, the weak global attractor is also the maximal bounded invariant set. We recall that those solutions are always weakly continuous in $L^2(\Omega)^3$.

It is known that if a weak global attractor for the 3D NSE is bounded in V, then it is in fact strong [9]. Moreover, Ball [1] showed that if a generalized semiflow for a dissipative evolutionary system is asymptotically compact, then a strong global attractor exists. This generalizes corresponding results for semiflows (see [11, 12, 16]) and implies that the strong global attractor for the 3D NSE exists under the condition that all weak solutions are strongly continuous from $(0, \infty)$ to $L^2(\Omega)^3$ (see [1]). In this paper we show that even without the assumptions of concatenation and upper semicontinuity with respect to the initial data, the asymptotic compactness implies that the weak global attractor is the minimal compact attracting set in the strong metric, i.e., the weak global attractor is in fact the strong compact global attractor (Theorem 2.16). Applied to the 3D NSE, this result implies the existence of a strong compact global attractor in the case when the solutions on the weak global attractor are continuous in $L^2(\Omega)^3$ (Theorem 3.30).

The convergence of Leray-Hopf weak solutions was also studied by Rosa [19], namely, he introduced an asymptotic regularity condition to insure the strong convergence of a weak solution towards its weak ω-limit. This condition requires the limit solutions to be strongly continuous in $L^2(\Omega)^3$, and implies that the weak global attractor is a strongly compact strongly attracting set if the weak solutions on the weak global attractor are strongly continuous in $L^2(\Omega)^3$. Moreover, since a trajectory of the 3D NSE that is not strongly continuous in $L^2(\Omega)^3$ also, obviously, is not relatively strongly compact in $L^2(\Omega)^3$, the strong continuity of weak solutions on the weak global attractor $\mathcal{A}_{\mathrm{w}}$ is also a necessary condition for $\mathcal{A}_{\mathrm{w}}$ to be strongly compact (see [19]).

Recall that we define a global attractor as the minimal closed (uniformly) attracting set in the corresponding metric, and hence, allow the possibility for the solutions on a strong global attractor to be discontinuous. We address this issue by studying the weak convergence of weak solutions towards a weak solution strongly continuous from the right in $L^2(\Omega)^3$. We show that the weak convergence is strong under the condition that the "energy jumps" of solutions converge to the "energy jumps" of the limit solution (Theorem 3.24). However, at this stage, no necessary condition for the strong convergence is known.

Finally, we provide an example of a dissipative evolutionary system for which all solutions on the weak global attractor blow-up in finite time. We introduce a two-parameter family of simple infinite-dimensional systems of differential equations. These systems, called *tridiagonal models for the NSE (TNS models)*, display basic features of the NSE. In particular, they are examples of dissipative systems that possess a weak global attractor, but the existence of a strong global attractor is not known.

TNS models have similar form and some similar properties to shell models, specifically dyadic models studied in [10, 14]. Moreover, a similar analysis of the dyadic models also results in a finite time blow-up (see [5]). In the TNS models though, the coefficients in the equations are chosen to yield NSE-like scaling properties. More precisely, we first mimic the Stokes operator in 3D via the choice a positive definite operator on l^2 whose eigenvalues grow with the same speed as the eigenvalues of the Stokes operator. Second, we mimic the nonlinear term of the NSE via the choice of a bilinear form, which scales like the Sobolev estimate for the NSE. Then we obtain the following system of differential equations:

$$\frac{d}{dt}u + \nu A u + B(u,u) = g, \tag{1}$$

where $u = (u_1, u_2, \dots)$,

$$(Au)_n = n^{\alpha} u_n,$$

and

$$(B(u,v))_n = -n^{\beta} u_{n-1} v_{n-1} + (n+1)^{\beta} u_n v_{n+1}, \qquad n = 1, 2, \dots,$$

with $u_0 = 0$. Here α and β are two positive parameters. Note that the orthogonality property in l^2 holds for B, which implies the existence of an absorbing ball and a weak global attractor. Moreover, when $\alpha = 2/3$, which corresponds to the speed with which the eigenvalues of the Stokes operator grow in three-dimensional space, and $\beta = 11/6$, we have the following sharp estimate:

$$|(B(u,u), Au)| \lesssim |Au|^{3/2} |A^{1/2}u|^{3/2},$$

where $|v|^2 = \sum v_n^2$. This estimate is exactly the same as the the estimate based on the Sobolev inequalities for the nonlinear term of the 3D NSE, with $|\cdot|$ being the L^2-norm. It is an open question whether solutions of (1) can lose regularity in the case

$(\alpha, \beta) = (2/3, 11/6)$. However, we show that in the nonviscous case $\nu = 0$, for every $\alpha > 0$, $\beta > 0$, $\gamma > 0$, and $g_n \geq 0$ for all $n \in \mathbb{N}$, the norm $|A^{(\beta+\gamma-1)/(3\alpha)}u|$ of every solution with $u_n(0) \geq 0$ and $u(0) \neq 0$ blows up (Theorem 4.7).

When the viscosity is not zero, the model always possesses a weak global attractor $\mathcal{A}_{\mathrm{w}}$. Moreover, we prove that if the force g is large enough, then all solutions on $\mathcal{A}_{\mathrm{w}}$ blow up in finite time in an appropriate norm when $2\beta > 3\alpha + 3$ (Remark 4.6). The question whether $\mathcal{A}_{\mathrm{w}}$ is a strong global attractor remains open in the case where $\beta > \alpha + 1$.

2 Evolutionary system and global attractors

Let $(X, \mathrm{d}_{\mathrm{s}}(\cdot, \cdot))$ be a metric space endowed with a metric d_{s}, which will be referred to as a strong metric. Let $\mathrm{d}_{\mathrm{w}}(\cdot, \cdot)$ be another metric on X satisfying the following conditions:

1. X is d_{w}-compact.
2. If $\mathrm{d}_{\mathrm{s}}(u_n, v_n) \to 0$ as $n \to \infty$ for some $u_n, v_n \in X$, then $\mathrm{d}_{\mathrm{w}}(u_n, v_n) \to 0$ as $n \to \infty$, that is, the identity map $(X, \mathrm{d}_{\mathrm{s}}) \mapsto (X, \mathrm{d}_{\mathrm{w}})$ is uniformly continuous.

Due to the latter property, d_{w} will be referred to as the weak metric on X. Note that any strongly compact (d_{s}-compact) set is weakly compact (d_{w}-compact), and any weakly closed set is strongly closed. Also it will be convenient to denote by $\overline{A}^{\bullet}$ the closure of the set $A \subset X$ in the topology generated by $\mathrm{d}_{\bullet}$; here and throughout $\bullet$ stands for either s or w.

To define an evolutionary system, first let

$$\mathcal{T} := \{I : \ I = [T, \infty) \text{ for some } T \in \mathbb{R}, \text{ or } I = (-\infty, \infty)\},$$

and for each $I \subset \mathcal{T}$ let $\mathcal{F}(I)$ denote the set of all X-valued functions on I. Now we define an evolutionary system $\mathcal{E}$ as follows.

Definition 2.1. *A map $\mathcal{E}$ that associates to each $I \in \mathcal{T}$ a subset $\mathcal{E}(I) \subset \mathcal{F}$ will be called an evolutionary system if the following conditions are satisfied:*

1. $\mathcal{E}([0, \infty)) \neq \emptyset$.
2. $\mathcal{E}(I + s) = \{u(\cdot) : \ u(\cdot - s) \in \mathcal{E}(I)\}$ *for all* $s \in \mathbb{R}$.
3. $\{u(\cdot)|_{I_2} : u(\cdot) \in \mathcal{E}(I_1)\} \subset \mathcal{E}(I_2)$ *for all pairs of* $I_1, I_2 \in \mathcal{T}$, *such that* $I_2 \subset I_1$.
4. $\mathcal{E}((-\infty, \infty)) = \{u(\cdot) : \ u(\cdot)|_{[T,\infty)} \in \mathcal{E}([T, \infty)) \ \forall T \in \mathbb{R}\}$.

We will refer to $\mathcal{E}(I)$ as the set of all trajectories (solutions) on the time interval I. Let $P(X)$ be the set of all subsets of X. For every $t \geq 0$ define a map

$$R(t) : P(X) \to P(X),$$
$$R(t)K := \{u(t) : u(0) \in K, u(\cdot) \in \mathcal{E}([0, \infty))\}, \qquad K \subset X.$$

Note that the assumptions on $\mathcal{E}$ imply that $R(t)$ enjoys the following property:

$$R(t + s)K \subset R(t)R(s)K, \qquad K \subset X, \quad t, s \geq 0. \tag{2}$$

Let us first point out that the trajectories are not required to be continuous (even with respect to d_{w}) nor are they uniquely determined by their starting points, i.e., it is possible to have two different trajectories $u, v \in \mathcal{E}([0,\infty))$, such that $u(0) = v(0)$. Second, there is no assumption of concatenation. If $u \in \mathcal{E}([0,\infty))$ and $v \in \mathcal{E}([T,\infty))$ for some $T > 0$, so that $u(T) = v(T)$, then the following function

$$w(t) = \begin{cases} u(t), & \text{if } t \in [0,T], \\ v(t), & \text{if } t \in (T,\infty). \end{cases}$$

need not be in $\mathcal{E}([0,\infty))$. We avoid the assumptions of the continuity, uniqueness, and concatenation in order to be able to consider an evolutionary system consisting of Leray-Hopf weak solutions to the 3D Navier-Stokes equations.

Often an evolution of a dynamical system can be described by a semigroup of continuous mappings acting on some metric space H:

$$S(t) : H \to H, \quad t \geq 0.$$

The semigroup properties are the following:

$$S(t+s) = S(t)S(s), \quad t, s \geq 0, \qquad S(0) = \text{Identity operator.} \tag{3}$$

Then, if $u(t) \in H$ represents a state of the dynamical system at time t, we have

$$u(t+s) = S(t)u(s), \qquad t, s \geq 0.$$

A ball $B \subset H$ is called an absorbing ball, if for any bounded set $K \subset H$ there exists t_0, such that

$$S(t)K \subset B, \qquad \forall t \geq t_0.$$

Hence, if we are interested in the long-time behavior of the dynamical system, it is enough to consider a restriction of the system to an absorbing ball. So, we let X be a closed absorbing ball, and define the map $\mathcal{E}$ in the following way:

$$\mathcal{E}(I) := \{u(\cdot) : u(t+s) = S(t)u(s) \text{ and } u(s) \in X \ \forall s \in I, \ t \geq 0\}.$$

Note that conditions 1–4 for the evolutionary system $\mathcal{E}$ automatically follow from the semigroup properties (3) of $S(t)$. In addition, let T be such that

$$S(t)X \subset X \qquad \forall t \geq T.$$

Then we have

$$R(t)K = S(t)K, \qquad \forall K \subset S(T)X, \ t \geq 0.$$

The 3D Navier-Stokes equations will serve as an instructive illustration and application of our consideration. As yet in our knlowledge of the 3D NSE, the time evolution of the 3D NSE cannot be described by a semigroup of maps. Therefore, for the 3D NSE we will have a more involved definition of $\mathcal{E}$ (see Section 3).

Having defined the evolutionary system $\mathcal{E}$, we proceed to define attracting sets and global attractors. For $A \subset X$ and $r > 0$, denote $B_\bullet(A, r) = \{u : \ \mathrm{d}_\bullet(u, A) < r\}$, where

$$\mathrm{d}_\bullet(u, A) := \inf_{x \in A} \mathrm{d}_\bullet(u, x).$$

Definition 2.2. *A set $A \subset X$ is a $\mathrm{d}_\bullet$-attracting set ($\bullet = \mathrm{s}, \mathrm{w}$) if it uniformly attracts X in $\mathrm{d}_\bullet$-metric, i.e., for any $\epsilon > 0$, there exists t_0, such that*

$$R(t)X \subset B_\bullet(A, \epsilon), \qquad \forall t \geq t_0.$$

Definition 2.3. *$\mathcal{A}_\bullet \subset X$ is a $\mathrm{d}_\bullet$-global attractor ($\bullet = \mathrm{s}, \mathrm{w}$) if $\mathcal{A}_\bullet$ is a minimal $\mathrm{d}_\bullet$-closed $\mathrm{d}_\bullet$-attracting set, i.e., $\mathcal{A}_\bullet$ is $\mathrm{d}_\bullet$-closed $\mathrm{d}_\bullet$-attracting and every subset $A \subset \mathcal{A}_\bullet$ that is also $\mathrm{d}_\bullet$-closed and $\mathrm{d}_\bullet$-attracting satisfies $A = \mathcal{A}_\bullet$.*

Note that the empty set is never an attracting set. Note also that since X is not strongly compact, the intersection of two d_s-closed d_s-attracting sets might not be d_s-attracting. Nevertheless, the uniqueness of a global attractor is a direct consequence of the following lemma.

Lemma 2.4. *If $\mathcal{A}_\bullet$ exists and A is a $\mathrm{d}_\bullet$-closed $\mathrm{d}_\bullet$-attracting set, then $\mathcal{A}_\bullet \subset A$ ($\bullet = \mathrm{s}, \mathrm{w}$).*

Proof. Let A be an arbitrary $\mathrm{d}_\bullet$-closed $\mathrm{d}_\bullet$-attracting set. Take any point $a \in \mathcal{A}_\bullet$. Let $\epsilon > 0$. If there exists $t_\epsilon > 0$, such that

$$R(t)X \cap B_\bullet(a, \epsilon) = \emptyset, \qquad \forall t \geq t_\epsilon,$$

then $\mathcal{A}_\bullet \setminus B_\bullet(a, \epsilon/2)$ is a $\mathrm{d}_\bullet$-closed $\mathrm{d}_\bullet$-attracting set contradicting the minimality of $\mathcal{A}_\bullet$. So, there exists a sequence $t_n \to \infty$ as $n \to \infty$, such that

$$R(t_n)X \cap B_\bullet(a, \epsilon) \neq \emptyset, \qquad \forall n.$$

On the other hand, since A is $\mathrm{d}_\bullet$-attracting, we infer that

$$R(t_n)X \subset B_\bullet(A, \epsilon),$$

for n large enough. It follows that

$$A \cap B_\bullet(a, 2\epsilon) \neq \emptyset.$$

Since A is $\mathrm{d}_\bullet$-closed, we have that $a \in A$. Thus, $\mathcal{A}_\bullet \subset A$. $\square$

As a direct consequence of this lemma we have the following.

Corollary 2.5. *If $\mathcal{A}_\bullet$ exists, then it is unique ($\bullet = \mathrm{s}, \mathrm{w}$).*

Assume now that the weak global attractor $\mathcal{A}_\mathrm{w}$ exists. If $\mathcal{A}_\mathrm{w}$ is a strongly attracting set, does it follow that the strong global attractor exists? In general, this may not be true. However, if the strong global attractor exists, then clearly it is a d_w-attracting set. Moreover, we have the following.

Theorem 2.6. *If $\mathcal{A}_\mathrm{s}$ exists, then $\mathcal{A}_\mathrm{w}$ exists and*

$$\mathcal{A}_\mathrm{w} = \overline{\mathcal{A}_\mathrm{s}}^{\mathrm{w}}.$$

Proof. If there exists a d_w-closed d_w-attracting set $A \subset \overline{\mathcal{A}_\mathrm{s}}^{\mathrm{w}}$ and $A \neq \overline{\mathcal{A}_\mathrm{s}}^{\mathrm{w}}$, then there exists $u_0 \in \mathcal{A}_\mathrm{s}$, such that

$$d = \mathrm{d}_\mathrm{w}(u_0, A) > 0.$$

By the definition of an attracting set, there exists a time $t_0 > 0$, such that

$$R(t)X \subset B_{\mathrm{w}}(A, d/2) \qquad \forall t \geq t_0. \tag{4}$$

Note that

$$\mathrm{d_w}(u_0, B_{\mathrm{w}}(A, d/2)) \geq d/2.$$

Therefore, by virtue of Property 2 in the definition of $\mathrm{d_w}$, there exists $\delta > 0$, such that

$$\mathrm{d_s}(u_0, B_{\mathrm{w}}(A, d/2)) > \delta,$$

whence,

$$B_{\mathrm{s}}(u_0, \delta) \cap B_{\mathrm{w}}(A, d/2) = \emptyset.$$

Now from (4) it follows that

$$B_{\mathrm{s}}(u_0, \delta) \cap R(t)X = \emptyset \qquad \forall t \geq t_0.$$

Consequently, $\mathcal{A}_{\mathrm{s}} \setminus B_{\mathrm{s}}(u_0, \delta/2)$ is a $\mathrm{d_s}$-closed $\mathrm{d_s}$-attracting set strictly included in $\mathcal{A}_{\mathrm{s}}$, a contradiction. Hence, $\overline{\mathcal{A}_{\mathrm{s}}}^{\mathrm{w}}$ is the weak global attractor. $\square$

The following are two simple examples of evolutionary systems that possess a weak global attractor $\mathcal{A}_{\mathrm{w}}$, but not a strong global attractor $\mathcal{A}_{\mathrm{s}}$.

Example 1. Let

$$X = \left\{ u \in L^2(-\infty, \infty) : \int_{-\infty}^{\infty} u(x)^2 \, dx \leq 1, u(x) = 0 \text{ for } x > 0 \right\},$$

and define on X the distances

$$\mathrm{d_s}(u, v) := \left(\int_{-\infty}^{\infty} (u(x) - v(x))^2 \, dx \right)^{1/2}, \qquad \mathrm{d_w}(u, v) = \int_{-\infty}^{\infty} \frac{1}{2^{|x|}} \frac{|u(x) - v(x)|}{1 + |u(x) - v(x)|} \, dx.$$

Consider the following partial differential equation:

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}.$$

The trajectories of the evolutionary system $\mathcal{E}$ will be solutions of this equation, i.e.,

$$\mathcal{E}([s, \infty)) = \{u \in \mathcal{F}([s, \infty)) : u(t) = u_0(\cdot + t - s), t \in [s, \infty), u_0 \in X\}, \qquad \forall s \in \mathbb{R},$$
$$\mathcal{E}((-\infty, \infty)) = \{0\}.$$

Then it is easy to show that $\mathcal{A}_{\mathrm{w}} = \{0\}$ (see also Theorem 2.14). However, no trajectory except the trivial one $u = 0$ strongly converges to 0 as $t \to \infty$.

Example 2. Take

$$X = \left\{ u \in l^2 : \sum_{n=1}^{\infty} u_n^2 \leq 1 \right\}.$$

where $u = (u_n)$, $u_n \in \mathbb{R}$ for all n. For $u, v \in X$, let

$$\mathrm{d_s}(u, v) := \left(\sum_{n=1}^{\infty} (u_n - v_n)^2 \right)^{1/2}, \qquad \mathrm{d_w}(u, v) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|u_n - v_n|}{1 + |u_n - v_n|}.$$

Consider the following differential equation:

$$\frac{d}{dt}u_n = -\frac{1}{n}u_n, \qquad n \in \mathbb{N}$$

The trajectories of the evolutionary system $\mathcal{E}$ will be solutions of this equation, i.e.,

$$\mathcal{E}([s,\infty)) = \{u \in \mathcal{F}([s,\infty)) : (u_n(t)) = (u_n^0 e^{(s-t)/n}), t \in [s,\infty), u^0 \in X\},$$

for $s \in \mathbb{R}$, and

$$\mathcal{E}((-\infty,\infty)) = \{0\}.$$

Take any $u^0 \in X$ and consider the trajectory $u \in \mathcal{E}([0,\infty))$ starting at u^0, i.e., $u(0) = u^0$. Then we have

$$\mathrm{d_s}(u(t),0)^2 = \sum_{n=1}^{\infty} u_n(t)^2 = \sum_{n=1}^{\infty} (u_n^0)^2 e^{-\frac{2t}{n}} \to 0 \qquad \text{as} \qquad t \to \infty.$$

However, the convergence is not uniform in the $\mathrm{d_s}$-metric, although it is uniform in the $\mathrm{d_w}$-metric. So again $\mathcal{A}_\mathrm{w} = \{0\}$, but $\mathcal{A}_\mathrm{s}$ does not exist.

Note that the nonexistence of $\mathcal{A}_\mathrm{s}$ in the two examples is due to two different behaviors of the trajectories. In the first example all the nontrivial trajectories converge to $\mathcal{A}_\mathrm{w}$ weakly, but not strongly. In the second example all the nontrivial trajectories converge to $\mathcal{A}_\mathrm{w}$ strongly, but not uniformly.

Definition 2.7. *The map $R(t)$ is uniformly $\mathrm{d}_\bullet$-compact ($\bullet = \mathrm{s},\mathrm{w}$) if there exists $t_0 \geq 0$, such that*

$$\bigcup_{t \geq t_0} R(t)X$$

is relatively $\mathrm{d}_\bullet$-compact.

Note that since X is $\mathrm{d_w}$-compact, $R(t)$ is automatically uniformly $\mathrm{d_w}$-compact.

Definition 2.8. *The $\omega_\bullet$-limit ($\bullet = \mathrm{s},\mathrm{w}$) of a set $K \subset X$ is*

$$\omega_\bullet(K) := \bigcap_{T \geq 0} \overline{\bigcup_{t \geq T} R(t)K}^{\bullet},$$

where the closure is taken in $\mathrm{d}_\bullet$-metric.

Lemma 2.9. *Let A be a $\mathrm{d}_\bullet$-closed $\mathrm{d}_\bullet$-attracting set. Then*

$$\omega_\bullet(X) \subset A.$$

Proof. Suppose that there exists $a \in \omega_\bullet(X) \setminus A$. Since A is $\mathrm{d}_\bullet$-closed, there exists $\epsilon > 0$, such that

$$A \cap B_\bullet(a,\epsilon) = \emptyset.$$

By the definition of the $\omega_\bullet$-limit, there exist a sequence $t_n \to \infty$ as $n \to \infty$ and a sequence $x_n \in R(t_n)X$, such that $\mathrm{d}_\bullet(x_n, a) \to 0$ as $n \to \infty$. Hence, there exists $N > 0$, such that

$$x_n \notin B_\bullet(A, \epsilon/2), \qquad \forall n \geq N.$$

This means that A is not $\mathrm{d}_\bullet$-attracting, a contradiction. $\square$

Lemma 2.10. *If the map $R(t)$ is uniformly $\mathrm{d}_\bullet$-compact ($\bullet = \mathrm{s}, \mathrm{w}$), then $\omega_\bullet(X)$ is a nonempty $\mathrm{d}_\bullet$-compact $\mathrm{d}_\bullet$-attracting set.*

Proof. By Definition 2.7 and the fact that $\mathcal{E}([0,\infty)) \neq \emptyset$, there exists t_0, such that

$$W(T) := \overline{\bigcup_{t \geq T} R(t)X}^{\bullet}$$

is a nonempty $\mathrm{d}_\bullet$-compact set for all $T \geq t_0$. In addition, $W(s) \subset W(t)$ for all $s \geq t \geq 0$. Thus,

$$\omega_\bullet(X) = \bigcap_{T \geq t_0} W(T)$$

is a nonempty $\mathrm{d}_\bullet$-compact set.

We will now prove that $\omega_\bullet(X)$ uniformly $\mathrm{d}_\bullet$-attracts X. Assume it does not. Then there exists $\epsilon > 0$, such that

$$V(t) := W(t) \cap (X \setminus B_\bullet(\omega_\bullet(X), \epsilon)) \neq \emptyset, \qquad \forall t \geq 0.$$

Since $V(t)$ is $\mathrm{d}_\bullet$-compact for $t \geq t_0$ and $V(s) \subset V(t)$ for $s \geq t \geq 0$, we have that there exists

$$x \in \bigcap_{t \geq t_0} V(t).$$

Hence, $x \in \omega_\bullet(X)$. However, this implies that $x \notin V(t)$, $t \geq 0$, a contradiction. $\square$

Theorem 2.11. *If the map $R(t)$ is uniformly $\mathrm{d}_\bullet$-compact ($\bullet = \mathrm{s}, \mathrm{w}$), then the $\mathrm{d}_\bullet$-global attractor exists and satisfies the following additional properties:*

(a) $\mathcal{A}_\bullet = \omega_\bullet(X)$.

(b) $\mathcal{A}_\bullet$ *is* $\mathrm{d}_\bullet$*-compact.*

Proof. By Lemma 2.10, $\omega_\bullet(X)$ is a nonempty $\mathrm{d}_\bullet$-compact $\mathrm{d}_\bullet$-attracting set. Moreover, $\omega_\bullet(X)$ is the minimal $\mathrm{d}_\bullet$-closed $\mathrm{d}_\bullet$-attracting set due to Lemma 2.9. Therefore, $\omega_\bullet(X)$ is the $\mathrm{d}_\bullet$-global attractor. $\square$

Note that since X is d_w-compact (see the definition of d_w), $R(t)$ is uniformly weakly compact. Hence we have the following.

Corollary 2.12. *The evolutionary system $\mathcal{E}$ always possesses a weak global attractor $\mathcal{A}_\mathrm{w}$.*

Our next goal is to investigate whether $\mathcal{A}_\mathrm{w}$ is an invariant set in the following sense.

Definition 2.13. *The set $A \subset X$ is invariant, if*

$$\{u(t) : u \in \mathcal{E}((-\infty,\infty)), u(0) \in A\} = A, \qquad \forall t \geq 0.$$

Assume that u_0 belongs to some invariant set. Then for all $t > 0$ we have that $u_0 \in R(t)X$. Hence, $u_0 \in \omega_{\mathrm{w}}(X) = \mathcal{A}_{\mathrm{w}}$. Therefore, $\mathcal{A}_{\mathrm{w}}$ contains every invariant set. Moreover, we will show that $\mathcal{A}_{\mathrm{w}}$ is invariant under some compactness property that is for instance satisfied by the family of all Leray-Hopf solutions of the 3D NSE (see Section 3).

Let $C([a,b];X_\bullet)$ be the space of $\mathrm{d}_\bullet$-continuous X-valued functions on $[a,b]$ endowed with the metric

$$\mathrm{d}_{C([a,b];X_\bullet)}(u,v) = \sup_{t\in[a,b]} \mathrm{d}_\bullet(u(t),v(t)).$$

Let also $C([a,\infty);X_\bullet)$ be the space of $\mathrm{d}_\bullet$-continuous X-valued functions on $[a,\infty)$ endowed with the metric

$$\mathrm{d}_{C([a,\infty);X_\bullet)}(u,v) = \sum_{T\in\mathbb{N}} \frac{1}{2^T} \frac{\sup\{\mathrm{d}_\bullet(u(t),v(t)) : a \le t \le a+T\}}{1+\sup\{\mathrm{d}_\bullet(u(t),v(t)) : a \le t \le a+T\}}.$$

Theorem 2.14. *If $\mathcal{E}([0,\infty))$ is compact in $C([0,\infty);X_{\mathrm{w}})$, then*

(a) $\mathcal{A}_{\mathrm{w}} = \mathcal{I} := \{u_0 :\ u_0 = u(0)$ *for some* $u \in \mathcal{E}((-\infty,\infty))\}$.

(b) $\mathcal{A}_{\mathrm{w}}$ *is the maximal invariant set.*

Proof. Since obviously $\mathcal{I}$ is the maximal invariant set, we have that $\mathcal{I} \subset \mathcal{A}_{\mathrm{w}}$. It remains to prove that $\mathcal{A}_{\mathrm{w}} \subset \mathcal{I}$. Take any $a \in \mathcal{A}_{\mathrm{w}}$. Since $\mathcal{A}_{\mathrm{w}} = \omega_{\mathrm{w}}(X)$, there exist $t_n \to \infty$, as $n \to \infty$ and $a_n \in R(t_n)X$, such that $a_n \to a$ weakly as $n \to \infty$. Using Property 2 in Definition 2.1, there exist $u_n \in \mathcal{E}([-t_n,\infty))$, such that $u_n(0) = a_n$. Also, Properties 2 and 3 in Definition 2.1 of $\mathcal{E}$ imply that $\mathcal{E}([-t_n,\infty))$ is compact in $C([-t_n,\infty);X_{\mathrm{w}})$ and

$$\{u|_{[-t_1,\infty)} : u \in \mathcal{E}([-t_n,\infty))\} \subset \mathcal{E}([-t_1,\infty)),$$

for every n. Now, passing to a subsequence and dropping a subindex, we can assume that $u_n|_{[-t_1,\infty)} \to u^1 \in \mathcal{E}([-t_1,\infty))$ in $C([-t_1,\infty);X_{\mathrm{w}})$ as $n \to \infty$. By a standard diagonalization process we obtain that there exist $u \in \mathcal{F}((-\infty,\infty))$ and a subsequence of u_n, still denoted by u_n, such that $u_n|_{[-T,\infty)} \to u_{[-T,\infty)}$ in $C([-T,\infty);X_{\mathrm{w}})$ for all $T > 0$. Thus, by the compactness we have that $u|_{[-T,\infty)} \in \mathcal{E}([-T,\infty))$ for all $T > 0$, and hence $u \in \mathcal{E}((-\infty,\infty))$. Finally, since $u(0) = a$. we have $a \in \mathcal{I}$. Hence, $\mathcal{A}_{\mathrm{w}} \subset \mathcal{I}$. $\square$

Now that we know that the weak global attractor always exists, we can weaken the condition on the existence of the strong global attractor.

Definition 2.15. *The map $R(t)$ is asymptotically $\mathrm{d}_\bullet$-compact ($\bullet = \mathrm{s},\mathrm{w}$) if for any $t_n \to \infty$ and any $x_n \in R(t_n)X$, the sequence $\{x_n\}$ is relatively $\mathrm{d}_\bullet$-compact.*

Theorem 2.16. *If the map $R(t)$ is asymptotically d_{s}-compact, then $\mathcal{A}_{\mathrm{w}}$ is d_{s}-compact strong global attractor.*

Proof. First note that $\omega_{\mathrm{s}}(X) \subset \omega_{\mathrm{w}}(X) = \mathcal{A}_{\mathrm{w}}$. On the other hand, let $a \in \mathcal{A}_{\mathrm{w}} = \omega_{\mathrm{w}}(X)$. By the definition of ω_{w}-limit, there exist $t_n \to \infty$ as $n \to \infty$ and $x_n \in R(t_n)X$, such that

$$\mathrm{d}_{\mathrm{w}}(x_n,a) \to 0 \qquad \text{as} \qquad n \to \infty.$$

Thanks to the asymptotic compactness of $R(t)$, this convergence is in fact strong. Therefore, $a \in \omega_{\mathrm{s}}(X)$. Hence, $\omega_{\mathrm{s}}(X) = \mathcal{A}_{\mathrm{w}}$.

Now let us show that $\omega_{\mathrm{s}}(X)$ is a $\mathrm{d_s}$-attracting set. Assume that it is not. Then there exist $\epsilon > 0$, $x_n \in X$, and $t_n \to \infty$ as $n \to \infty$, such that

$$x_n \in R(t_n)X \setminus B_{\mathrm{s}}(\omega_{\mathrm{s}}(X), \epsilon), \qquad \forall n \in \mathbb{N}.$$

Since $R(t)$ is asymptotically $\mathrm{d_s}$-compact, then $\{x_n\}$ is relatively $\mathrm{d_s}$-compact. Passing to a subsequence and dropping a subindex, we may assume that

$$x_n \to x \in X \qquad \text{strongly}, \qquad \text{as } n \to \infty.$$

Therefore, we have that $x \in \omega_{\mathrm{s}}(X)$, a contradiction.

Now note that $\omega_{\mathrm{s}}(X)$ is the minimal $\mathrm{d_s}$-closed $\mathrm{d_s}$-attracting set due to Lemma 2.9. Therefore, $\omega_{\mathrm{s}}(X)$ is the strong global attractor $\mathcal{A}_{\mathrm{s}}$. Finally, let us show that $\omega_{\mathrm{s}}(X)$ is strongly compact. Take any sequence $a_n \in \omega_{\mathrm{s}}(X)$. By the definition of ω_{s}-limit, there exist $t_n \to \infty$ and $x_n \in R(t_n)X$, such that

$$\mathrm{d_s}(x_n, a_n) \to 0 \qquad \text{as} \qquad n \to \infty.$$

Note that $\{x_n\}$ is relatively $\mathrm{d_s}$-compact due to the asymptotic compactness of $R(t)$. Hence, $\{a_n\}$ is relatively $\mathrm{d_s}$-compact, which concludes the proof.

$\square$

Finally, in the following example we show that the asymptotic compactness is a weaker condition than the uniform strong compactness.

Example. Take

$$X = \left\{ u \in l^2 : \sum_{n=1}^{\infty} u_n^2 \leq 1 \right\}.$$

where $u = (u_n)$, $u_n \in \mathbb{R}$ for all n. For $u, v \in X$, let

$$\mathrm{d_s}(u,v) := \left(\sum_{n=1}^{\infty} (u_n - v_n)^2 \right)^{1/2}, \qquad \mathrm{d_w}(u,v) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|u_n - v_n|}{1 + |u_n - v_n|}.$$

Consider the following differential equations:

$$\frac{d}{dt} u_1 = 0,$$

and

$$\frac{d}{dt} u_n = -u_n, \qquad n \in \mathbb{N}.$$

The trajectories of the evolutionary system $\mathcal{E}$ will be solutions of this equation, i.e.,

$$\begin{aligned} \mathcal{E}([s,\infty)) = \{ & u \in \mathcal{F}([s,\infty)) : u_1(t) = u_1^0, u_n(t) = u_n^0 e^{(s-t)} \text{ for } n \geq 2, \\ & t \in [s,\infty), u^0 \in X \}, \end{aligned}$$

for $s \in \mathbb{R}$, and

$$\mathcal{E}((-\infty,\infty)) = \{ u : u_1 \in [-1,1], u_n = 0, n \geq 2 \}.$$

Clearly, $R(t)$ is not uniformly strongly compact, but it is asymptotically compact. Hence, the strong global attractor exists and

$$\mathcal{A}_{\mathrm{s}} = \{u : u_1 \in [-1, 1], u_n = 0, n \geq 2\}.$$

Finally, Theorems 2.14 and 2.16 imply the following

Remark 2.17. *If $\mathcal{E}([0,\infty))$ is compact in $C([0,\infty); X_{\mathrm{w}})$ and $R(t)$ is asymptotically $\mathrm{d_s}$-compact, then the strong global attractor $\mathcal{A}_{\mathrm{s}}$ exists, and is the strongly compact maximal invariant set. Consequently, $\mathcal{A}_{\mathrm{s}}$ is a compact global attractor in the conventional sense.*

3 3D Navier-Stokes equations

Here we apply the results from the previous section to the space periodic 3D Navier-Stokes equations (NSE)

$$\begin{cases} \dfrac{d}{dt}u - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f, \\ \nabla\cdot u = 0, \\ u, p, f \text{ are periodic with period } L \text{ in each space variable,} \\ u, f \text{ are in } L^2_{\mathrm{loc}}(\mathbb{R}^3)^3, \\ u|_{t=0} = u_0, \end{cases} \tag{5}$$

where u, the velocity, and p, the pressure, are unknowns, f is a given driving force, and $\nu > 0$ is the kinematic viscosity coefficient of the fluid. By a Galilean change of variables, we can assume that the space average of u is zero, i.e.,

$$\int_\Omega u(x,t)\,dx = 0, \qquad \forall t,$$

where $\Omega = [0, L]^3$ is a periodic box.

In this section we will apply our general results from the previous section to the study of the assymptotical behaviour of weak solutions to (5).

3.1 Functional setting

First, let us introduce some notations and functional setting for (5). We denote by $(\cdot,\cdot)$ and $|\cdot|$ the $L^2(\Omega)^3$-inner product and the corresponding $L^2(\Omega)^3$-norm. Let $\mathcal{V}$ be the space of all $\mathbb{R}^3$ trigonometric polynomials of period L in each variable satisfying $\nabla\cdot u = 0$ and $\int_\Omega u(x)\,dx = 0$. Let H and V to be the closures of $\mathcal{V}$ in $L^2(\Omega)^3$ and $H^1(\Omega)^3$, respectively. Also, define the distances $\mathrm{d}_\bullet$ by

$$\mathrm{d_s}(u,v) := |u - v|, \qquad \mathrm{d_w}(u,v) = \sum_{\kappa\in\mathbb{Z}^3} \frac{1}{2^{|\kappa|}} \frac{|u_\kappa - v_\kappa|}{1 + |u_\kappa - v_\kappa|}, \qquad u, v \in H,$$

where u_κ and v_κ are Fourier coefficients of u and v respectively.

We denote by $P_\sigma : L^2(\Omega)^3 \to H$ the L^2-orthogonal projection, referred to as the Leray projector, and by $A = -P_\sigma\Delta = -\Delta$ the Stokes operator with the domain

$D(A) = (H^2(\Omega))^3 \cap V$. The Stokes operator is a self-adjoint positive operator with a compact inverse. Denote

$$\|u\| := |A^{1/2}u| = \left(\int_\Omega \sum_{i,j=1}^{3} \left| \frac{\partial u_i}{\partial x_j} \right|^2 \, dx \right)^{1/2}.$$

Note that $\|u\|$ is equivalent to the H^1-norm of u for $u \in D(A^{1/2})$.

For a rigorous mathematical study of the equation (5) we need a few more concepts from functional analysis. Namely, let V' be the set of all distributions of the form $v = \Delta u$, with $u \in V$. The V'-norm of this v is by definition $\|u\|$. Endowed with this norm, V' becomes the dual space of V, and if $v \in H$, then the value of v at a point $w \in H$ equals to the usual scalar product (v, w) in H. Now for u and v in V, let $B(u, v) := P_\sigma(u \cdot \nabla v)$, which is an element of V'. If $v \in \mathcal{D}(A)$, then $B(u, v) \in H$. Moreover,

$$\langle B(u, v), w \rangle = -\langle B(u, w), v \rangle, \qquad u, v, w \in V,$$

in particular, $\langle B(u, v), v \rangle = 0$ for all $u, v \in V$.

Equations (5) now can be condensed in the functional differential equation

$$\frac{d}{dt}u + \nu A u + B(u, u) = g \qquad \text{in} \qquad V', \tag{6}$$

where u is a V-valued function of time and $g = P_\sigma f$. Throughout, we will assume that g is time independent and $g \in H$.

Definition 3.1. *A weak solution of* (6) *on* $[T, \infty)$ *is an* H*-valued function* $u(t)$ *defined for* $t \in [T, \infty)$*, such that*

$$u(\cdot) \in C([T, \infty); H_{\mathrm{w}}) \cap L^2_{\mathrm{loc}}([T, \infty); V),$$

and

$$w(\cdot) := g - \nu A u(\cdot) - B(u(\cdot), u(\cdot)) \in L^1_{\mathrm{loc}}([T, \infty); V'), \tag{7}$$

$$\begin{aligned} (u(t) - u(T), v) &= \langle u(t) - u(T), v \rangle \\ &= \int_T^t \langle w(s), v \rangle \, ds \qquad \forall v \in V, t \geq T. \end{aligned} \tag{8}$$

The relations (7) and (8) imply that $\frac{d}{dt}u$ exists in V' a.e. in $[T, \infty)$. Therefore, often in the literature (8) and (7) are written as (6) and

$$\frac{d}{dt}u \in L^1_{\mathrm{loc}}([T, \infty); V'),$$

respectively.

The classical fundamental result concerning (6) is the following.

Theorem 3.2 (Leray, Hopf). *For every* $u_0 \in H$*, there exists a weak solution* $u(t)$ *of (6) on* $[T, \infty)$ *with* $u(T) = u_0$ *satisfying the following energy inequality:*

$$|u(t)|^2 + 2\nu \int_{t_0}^t \|u(s)\|^2 \, ds \leq |u(t_0)|^2 + 2 \int_{t_0}^t (g(s), u(s)) \, ds \tag{9}$$

for all $t \geq t_0$*,* t_0 *a.e. in* $[T, \infty)$*.*

See [13] for a hypothesis under which the energy equality holds. However, in general, the existence of weak solutions satisfying the energy equality is not known. Therefore, we introduce the following definition.

Definition 3.3. *A Leray-Hopf solution of the* (6) *on the interval* $[T, \infty)$ *is a weak solution on* $[T, \infty)$ *satisfying the energy inequality (9) for all* $T \leq t_0 \leq t$*,* t_0 *a.e. in* $[T, \infty)$*. The set* Ex *of measure* 0 *of points* t_0 *for which the energy inequality does not hold will be called the exceptional set (of the solution). In addition, the solution* $u(t)$ *will be called regular on an interval* $(\alpha, \beta) \subset [T, \infty)$ *if* $u(t) \in V$ *and* $\|u(t)\|$ *is continuous on* (α, β)*.*

Note that the uniqueness of Leray-Hopf solutions of the Initial Value Problem is not known.

Theorem 3.4 (Leray). *For every* $u_0 \in V$*, there exists a strong solution* $u(t)$ *of* (6) *on some interval* $[0, T)$*,* $T > 0$*, with* $u(0) = u_0$*.*

Theorem 3.5 (Leray). *Let* $u(t)$ *be a Leray-Hopf solution of* (6) *on* $[T, \infty)$*. Then there are at most countably many distinct open intervals* I_j*, such that*

$$[T, \infty) = \bigcup_j \bar{I}_j,$$

$u(t)$ *is regular on each* I_j*, and the measure of* $[T, \infty) \setminus \cup_j I_j$ *is zero.*

Theorem 3.6. *Let* $u(t)$ *be a regular solution on* $[0, T)$*. Then every Leray-Hopf solution* $v(t)$ *on* $[0, \infty)$ *with* $v(0) = u(0)$ *coincides with* $u(t)$ *in* $[0, T)$*.*

Finally, we recall several well-known supplementary facts.

Remark 3.7. *The complement of the exceptional set* Ex *coincides with the set of points of strong continuity from the right.*

Theorem 3.8. *Let* $u(t)$*,* $u_n(t)$ *be Leray-Hopf solutions on the interval* $[0, \infty)$*, such that*

$$u_n \to u \qquad \textit{in} \qquad C([0, T]; H_{\mathrm{w}}),$$

as $n \to \infty$*, for some* $T > 0$*. Let* $(\alpha, \beta) \subset (0, T)$ *be an interval of regularity of* $u(t)$*. Then for every* $0 < \delta < (\beta - \alpha)/2$*,*

$$\|u_n(t) - u(t)\| \to 0 \qquad \textit{uniformly on} \qquad [\alpha + \delta, \beta - \delta],$$

as $n \to \infty$*.*

Remark 3.9. *Let* $u(t)$ *be a Leray-Hopf solution. As a* V*-valued function,* $u(t)$ *is analytic in time on every interval of regularity.*

3.2 The weak global attractor for the 3D NSE

A ball $B \subset H$ is called an absorbing ball for the equation (6) if for any bounded set $K \subset H$, there exists t_0, such that

$$u(t) \in B, \qquad \forall t \geq t_0,$$

for all Leray-Hopf solutions $u(t)$ of (6) on $[0, \infty)$ with $u(0) \in K$. It is well known that there exists an absorbing ball in H for the 3D NSE. In fact, one has the following.

Proposition 3.10. *The 3D Navier-Stokes equations possess an absorbing ball*

$$B = B_{\mathrm{s}}(0, R),$$

where R is any number larger than $|g|\nu^{-1}L/(2\pi)$ (see, e.g., [6]).

Fix $R > |g|\nu^{-1}L/(2\pi)$ and let X be the closed absorbing ball

$$X = \{u \in H : |u| \le R\},$$

which is, clearly, weakly compact. Then for any bounded set $K \subset H$, there exists a time t_0, such that

$$u(t) \in X, \qquad \forall t \ge t_0,$$

for every Leray-Hopf solution $u(t)$ with the initial data $u(0) \in K$. Classical NSE estimates (see [6]) imply that for any sequence of Leray–Hopf solutions $u_n(t)$ (not only for the ones guaranteed by Theorem 3.2), the following result holds.

Lemma 3.11. *Let $u_n(t)$ be a sequence of Leray-Hopf solutions, such that $u_n(t) \in X$ for all $t \ge t_0$. Then*

$$\begin{aligned} u_n \ \textit{ is bounded in } \ & L^2([t_0, T]; V), \\ \frac{d}{dt}u_n \ \textit{ is bounded in } \ & L^{4/3}([t_0, T]; V'), \end{aligned}$$

for all $T > t_0$. Moreover, there exists a subsequence u_{n_j} of u_n, which converges in $C([t_0, T]; H_{\mathrm{w}})$ to some Leray-Hopf solution $u(t)$, i.e.,

$$(u_{n_j}, v) \to (u, v) \qquad \textit{uniformly on} \qquad [t_0, T],$$

as $n_j \to \infty$, for all $v \in H$.

Consider an evolutionary system for which a family of all trajectories consists of all Leray-Hopf solutions of the 3D Navier-Stokes equations in X. More precisely, define

$$\begin{aligned} \mathcal{E}([T, \infty)) := \{u(\cdot) : u(\cdot) &\text{ is a Leray-Hopf solution on } [T, \infty) \\ &\text{and } u(t) \in X \ \forall t \in [T, \infty)\}, \qquad T \in \mathbb{R}, \end{aligned}$$

$$\begin{aligned} \mathcal{E}((\infty, \infty)) := \{u(\cdot) : u(\cdot) &\text{ is a Leray-Hopf solution on } (-\infty, \infty) \\ &\text{and } u(t) \in X \ \forall t \in (-\infty, \infty)\}. \end{aligned}$$

Since X is weakly compact, the existence of the weak global attractor is a direct consequence of Theorem 2.14. Moreover, we have the following.

Lemma 3.12. *$\mathcal{E}([0, \infty))$ is compact in $C([0, \infty); H_{\mathrm{w}})$.*

Proof. Take any sequence $u_n \in \mathcal{E}([0, \infty))$, $n \in \mathbb{N}$. Thanks to Lemma 3.11, there exists a subsequence, still denoted by u_n, that converges to some $u^1 \in \mathcal{E}([0, \infty))$ in $C([0, 1]; H_{\mathrm{w}})$ as $n \to \infty$. Now, passing to a subsequence and dropping a subindex, we obtain that there exists $u^2 \in \mathcal{E}([0, \infty))$, such that $u_n \to u^2$ in $C([0, 2]; H_{\mathrm{w}})$ as $n \to \infty$. Note that $u^1(t) = u^2(t)$ on $[0, 1]$. Continuing this diagonalization process, we obtain a subsequence u_{n_j} of u_n that converges to some $u \in \mathcal{E}([0, \infty))$ in $C([0, \infty); H_{\mathrm{w}})$ as $n_j \to \infty$, which concludes the proof. □

Now Theorem 2.14 yields the following.

Theorem 3.13. *The weak global attractor $\mathcal{A}_{\mathrm{w}}$ for the 3D Navier-Stokes equations exists and satisfies*

(a) $\mathcal{A}_{\mathrm{w}} = \{u(0) : u \in \mathcal{E}((-\infty, \infty))\}$.

(b) $\mathcal{A}_{\mathrm{w}}$ *is the maximal invariant set.*

Lemma 3.14. *If $u(t)$, a Leray-Hopf solution of the 3D NSE, satisfies*

$$\limsup_{t\to\infty} \|u(t)\| < \infty,$$

then $u(t)$ converges strongly in H to the weak global attractor $\mathcal{A}_{\mathrm{w}}$.

Proof. Suppose that $u(t)$ does not converge strongly in H to $\mathcal{A}_{\mathrm{w}}$. Then there exist $M > 0$ and a sequence $t_n \to \infty$ as $n \to \infty$, such that

$$\mathrm{d_s}(u(t_n), \mathcal{A}_{\mathrm{w}}) > M, \qquad n \in \mathbb{N}. \tag{10}$$

Note that there exists a time $T > 0$, such that $\{u(t) : t \geq T\}$ is relatively compact in H. Therefore, passing to a subsequence, we may assume that $u(t_n)$ converges strongly (and hence weakly) in H to some $a \in H$. Therefore, $a \in \mathcal{A}_{\mathrm{w}}$, which contradicts (10). □

3.3 Strong convergence of Leray-Hopf solutions

The aim of this subsection is to give sufficient conditions for a sequence of Leray-Hopf solutions on $[T, \infty)$ to converge in $C([T, \infty); H)$, provided it converges in $C([T, \infty); H_{\mathrm{w}})$. We start with preliminary properties, some of which may have an intrinsic interest.

Theorem 3.15. *Let $u(t)$ be a Leray-Hopf solution of* (6) *on $[T, \infty)$. Let Ex be the exceptional set for this solution (see Def. 3.3). Then for any time $t_0 > T$, there exist A_- and A_+, such that*

(a) For every sequence $\{t_n\} \subset [T, \infty) \setminus Ex$, such that $t_n \to t_0$, $t_n < t_0$, it follows that $|u(t_n)| \to A_-$ as $n \to \infty$.

(b) For every sequence $\{t_n\} \subset [T, \infty) \setminus Ex$, such that $t_n \to t_0$, $t_n > t_0$, it follows that $|u(t_n)| \to A_+$ as $n \to \infty$.

For these A_- and A_+ we will use the following notations:

$$\widetilde{\lim_{t\to t_0-}} |u(t)| := A_- \qquad \textit{and} \qquad \widetilde{\lim_{t\to t_0+}} |u(t)| := A_+.$$

Proof. For $\{t_n\}$ as in (a), the energy inequality (9) on $[t_n, t_{n+k}]$ is

$$|u(t_{n+k})|^2 + 2\nu \int_{t_n}^{t_{n+k}} \|u(s)\|^2 \, ds \leq |u(t_n)|^2 + 2 \int_{t_n}^{t_{n+k}} (g, u(s)) \, ds,$$

provided $t_{n+k} \geq t_n$. Taking the upper limit as $k \to \infty$, we obtain

$$\limsup_{n\to\infty} |u(t_n)|^2 + 2\nu \int_{t_n}^{t_0} \|u(s)\|^2 \, ds \leq |u(t_n)|^2 + 2 \int_{t_n}^{t_0} (g, u(s)) \, ds.$$

Taking the lower limit as $n \to \infty$, we arrive at

$$\limsup_{n\to\infty} |u(t_n)|^2 \le \liminf_{n\to\infty} |u(t_n)|^2,$$

i.e., $\lim_{n\to\infty} |u(t_n)|$ exists. Since the limit exists for any sequence t_n, it does not depend on the choice of a sequence.

For $\{t_n\}$ as in (b), the energy inequality (9) on $[t_{n+k}, t_n]$ is

$$|u(t_n)|^2 + 2\nu \int_{t_{n+k}}^{t_n} \|u(s)\|^2\, ds \le |u(t_{n+k})|^2 + 2 \int_{t_{n+k}}^{t_n} (g, u(s))\, ds,$$

provided $t_{n+k} \le t_n$. Taking the lower limit as $k \to \infty$, we obtain

$$|u(t_n)|^2 + 2\nu \int_{t_0}^{t_n} \|u(s)\|^2\, ds \le \liminf_{n\to\infty} |u(t_n)|^2 + 2 \int_{t_0}^{t_n} (g, u(s))\, ds.$$

Finally, taking the upper limit as $n \to \infty$, we arrive at

$$\limsup_{n\to\infty} |u(t_n)|^2 \le \liminf_{n\to\infty} |u(t_n)|^2,$$

i.e., $\lim_{n\to\infty} |u(t_n)|$ exists. Since the limit exists for any sequence t_n, it does not depend on the choice of a sequence. □

Lemma 3.16. *Let $u(t)$ be a Leray-Hopf solution of* (6) *on* $[T, \infty)$. *Then*

$$\widetilde{\lim_{t\to t_0-}} |u(t)| = \limsup_{t\to t_0-} |u(t)|,$$
$$\widetilde{\lim_{t\to t_0+}} |u(t)| = \limsup_{t\to t_0+} |u(t)|,$$

and,

$$\widetilde{\lim_{t\to t_0-}} |u(t)| \ge \widetilde{\lim_{t\to t_0+}} |u(t)| \ge |u(t_0)|. \tag{11}$$

for all $t_0 > T$.

Proof. Take any $t_0 > T$. Obviously, we have

$$\widetilde{\lim_{t\to t_0-}} |u(t)| \le \limsup_{t\to t_0-} |u(t)|,$$
$$\widetilde{\lim_{t\to t_0+}} |u(t)| \le \limsup_{t\to t_0+} |u(t)|.$$

To show the opposite inequalities, note that for any $t_1 \in [T, \infty) \setminus Ex$ and $t_2 > t_1$, the energy inequality (9) on $[t_1, t_2]$ is

$$|u(t_2)|^2 + 2\nu \int_{t_1}^{t_2} \|u(s)\|^2\, ds \le |u(t_1)|^2 + 2 \int_{t_1}^{t_2} (g, u(s))\, ds.$$

First, we fix $t_1 < t_0$ and take the upper limit as $t_2 \to t_0-$, obtaining

$$\limsup_{t\to t_0-} |u(t)|^2 + 2\nu \int_{t_1}^{t_0} \|u(s)\|^2\, ds \le |u(t_1)|^2 + 2 \int_{t_1}^{t_0} (g, u(s))\, ds.$$

Now we take the limit as $t_1 \to t_0-$ avoiding the exceptional set (see Theorem 3.15). We get

$$\limsup_{t \to t_0-} |u(t)|^2 \leq \widetilde{\lim_{t \to t_0-}} |u(t)|.$$

Second, we fix $t_2 > t_0$ and take the limit as $t_1 \to t_0+$ avoiding the exceptional set. We arrive at

$$|u(t_2)|^2 + 2\nu \int_{t_0}^{t_2} \|u(s)\|^2 \, ds \leq \widetilde{\lim_{t \to t_0+}} |u(t)|^2 + 2 \int_{t_0}^{t_2} (g, u(s)) \, ds.$$

Taking the upper limit as $t_2 \to t_0+$, we get

$$\limsup_{t \to t_0+} |u(t)|^2 \leq \widetilde{\lim_{t \to t_0+}} |u(t)|.$$

Third, we fix $t_2 > t_0$ and take the limit as $t_1 \to t_0-$ avoiding the exceptional set. We obtain

$$|u(t_2)|^2 + 2\nu \int_{t_0}^{t_2} \|u(s)\|^2 \, ds \leq \widetilde{\lim_{t \to t_0-}} |u(t)|^2 + 2 \int_{t_0}^{t_2} (g, u(s)) \, ds.$$

Taking the limit as $t_2 \to t_0+$ avoiding the exceptional set, we get

$$\widetilde{\lim_{t \to t_0+}} |u(t)|^2 \leq \widetilde{\lim_{t \to t_0-}} |u(t)|.$$

Finally the weak continuity of $u(t)$ yields

$$\widetilde{\lim_{t \to t_0+}} |u(t)| \geq |u(t_0)|,$$

which concludes the proof. $\square$

Remark 3.17. *We can now rewrite the energy inequality for a Leray-Hopf solution $u(t)$ in the following form:*

$$|u(t)|^2 + 2\nu \int_{t_0}^{t} \|u(s)\|^2 \, ds \leq \widetilde{\lim_{t \to t_0+}} |u(t)|^2 + 2 \int_{t_0}^{t} (g, u(s)) \, ds, \tag{12}$$

for all $0 \leq t_0 \leq t$.

Recall that if the energy norm $|u(t)|$ of a Leray-Hopf solution is continuous from the right at some $t = t_0$, then t_0 does not belong to the exceptional set for $u(t)$, i.e. the energy inequality holds for t_0 (see Remark 3.7).

Lemma 3.18. *Let $u(t)$ be a Leray-Hopf solution of* (6) *on $[T, \infty)$. Then $|u(t)|$ is continuous from the right at $t = t_0 \geq T$ if and only if*

$$\widetilde{\lim_{t \to t_0+}} |u(t)| = |u(t_0)|.$$

Proof. If $|u(t)|$ is continuous from the right at $t = t_0 \geq T$, then, thanks to Lemma 3.16, we have that

$$\widetilde{\lim_{t \to t_0+}} |u(t)| = \limsup_{t \to t_0+} |u(t)| = |u(t_0)|. \tag{13}$$

Assume now that (13) holds. Due to the weak continuity of $u(t)$, we have

$$\liminf_{t \to t_0+} |u(t)|^2 \geq |u(t_0)|.$$

Hence,

$$\lim_{t \to t_0+} |u(t)|^2 = |u(t_0)|.$$

□

Now we will show that the strong continuity of a Leray-Hopf solution is equivalent to the strong continuity from the left (avoiding the exceptional set).

Lemma 3.19. *Let $u(t)$ be a Leray-Hopf solution of* (6) *on $[T, \infty)$. Then $|u(t)|$ is continuous at $t = t_0 > T$ if and only if*

$$\widetilde{\lim_{t \to t_0-}} |u(t)| = |u(t_0)|.$$

Proof. Clearly, if $|u(t)|$ is continuous at $t = t_0 > T$, then

$$\widetilde{\lim_{t \to t_0-}} |u(t)| = \limsup_{t \to t_0-} |u(t)| = |u(t_0)|. \tag{14}$$

Assume now that (14) holds. Then due to the weak continuity of $u(t)$, we have

$$\lim_{t \to t_0-} |u(t)| = |u(t_0)|.$$

In addition, Lemma 3.16 (equation (11)) implies that

$$\widetilde{\lim_{t \to t_0+}} |u(t)| = |u(t_0)|.$$

Finally, thanks to Lemma 3.18, we have

$$\lim_{t \to t_0+} |u(t)| = |u(t_0)|.$$

Therefore, $|u(t)|$ is continuous at $t = t_0$. □

We will now study a weak convergence of Leray-Hopf solutions. Our goal is to obtain sufficient conditions for a strong convergence.

Lemma 3.20. *Let $\{u_n(t)\}$, $u(t)$ be Leray-Hopf solutions of* (6) *on $[T_1, \infty)$. If $u_n \to u$ in $C([T_1, T_2]; H_{\mathrm{w}})$, then*

$$\limsup_{n \to \infty} \widetilde{\lim_{t \to t_0-}} |u_n(t)| \leq \widetilde{\lim_{t \to t_0-}} |u(t)|,$$

for all $t_0 \in (T_1, T_2]$.

Proof. Suppose this is not true for some $t_0 \in (T_1, T_2]$. Then passing to a subsequence and dropping the subindexes, we can assume that

$$\widetilde{\lim_{t\to t_0-}}|u_n(t)| - \widetilde{\lim_{t\to t_0-}}|u(t)| \geq \delta > 0, \qquad \forall n$$

and $|u_n(t)| \to |u(t)|$ on $[T_1, T_2] \setminus S$, where S is a set of zero measure, which includes the exceptional set for $u(t)$.

Let $S' := S \cup (\bigcup_n Ex_n)$, where Ex_n is the exceptional set for $u_n(t)$. The energy inequality for $u_n(t)$ implies

$$\widetilde{\lim_{\tau\to t_0-}}|u_n(\tau)|^2 \leq |u_n(t)|^2 + 2\int_t^{t_0} (g, u_n(s))\ ds,$$

for all $t \in [T_1, t_0] \setminus S'$. Taking the upper limit as $n \to \infty$ and using the strong convergence of $u_n(t)$ to $u(t)$ on $[T_1, T_2] \setminus S'$, we obtain

$$\limsup_{n\to\infty} \widetilde{\lim_{\tau\to t_0-}}|u_n(\tau)|^2 \leq |u(t)|^2 + 2\int_t^{t_0} (g, u(s))\ ds, \qquad t \in [T_1, t_0] \setminus S'.$$

Finally, letting $t \to t_0-$, we get

$$\limsup_{n\to\infty} \widetilde{\lim_{t\to t_0-}}|u_n(t)|^2 \leq \widetilde{\lim_{t\to t_0-}}|u(t)|^2,$$

which is in contradiction with the definition of δ. □

Lemma 3.21. *Let $\{u_n(t)\}$, $u(t)$ be Leray-Hopf solutions of* (6) *on $[T_1, \infty)$. If $u_n \to u$ in $C([T_1, T_2]; H_{\mathrm{w}})$, then*

$$\liminf_{n\to\infty} \widetilde{\lim_{t\to t_0+}}|u_n(t)| \geq \widetilde{\lim_{t\to t_0+}}|u(t)|,$$

for all $t_0 \in [T_1, T_2)$.

Proof. Suppose this is not true for some $t_0 \in [T_1, T_2)$. Then passing to a subsequence and dropping the subindexes, we can assume that

$$\widetilde{\lim_{t\to t_0+}}|u_n(t)| - \widetilde{\lim_{t\to t_0+}}|u(t)| \leq \delta < 0, \qquad \forall n$$

and $|u_n(t)| \to |u(t)|$ on $[T_1, T_2] \setminus S$, where S is a zero measure set, which includes the exceptional set for $u(t)$. The energy inequality (12) for $u_n(t)$ implies

$$|u_n(t)|^2 \leq \widetilde{\lim_{\tau\to t_0+}}|u_n(\tau)|^2 + 2\int_{t_0}^{t} (g, u_n(s))\ ds,$$

for all $t \in [t_0, T_2]$. Taking the lower limit as $n \to \infty$ and using the strong convergence of $u_n(t)$ to $u(t)$ on $[T_1, T_2] \setminus S$, we obtain

$$|u(t)|^2 \leq \liminf_{n\to\infty} \widetilde{\lim_{\tau\to t_0+}}|u_n(\tau)|^2 + 2\int_{t_0}^{t} (g, u(s))\ ds, \qquad t \in [t_0, T_2] \setminus S.$$

Finally, letting $t \to t_0+$, we get

$$\widetilde{\lim_{t\to t_0+}}|u(t)|^2 \leq \liminf_{n\to\infty} \widetilde{\lim_{t\to t_0+}}|u_n(t)|^2,$$

which is in contradiction with the definition of δ. □

Definition 3.22. *For $u(t)$, a Leray-Hopf solution of* (6), *denote*

$$[u(t_0)] := \widetilde{\lim_{t\to t_0-}}|u(t)| - \widetilde{\lim_{t\to t_0+}}|u(t)|,$$

which we will call the energy norm jump (loss) at $t = t_0$.

Note that due to (11), the energy norm jumps of the Leray-Hopf solutions are never negative, i.e.,

$$[u(t)] \geq 0, \qquad \forall t,$$

for any Leray-Hopf solution $u(t)$. Moreover, we have the following result.

Theorem 3.23. *Let $\{u_n(t)\}$, $u(t)$ be Leray-Hopf solutions of* (6) *on $[T_1, \infty)$. If $u_n \to u$ in $C([T_1, T_2], H_{\mathrm{w}})$, then*

$$\limsup_{n\to\infty}[u_n(t)] \leq [u(t)],$$

for all $t \in (T_1, T_2)$.

Proof. Indeed, Lemmas 3.20 and 3.21 yield

$$\begin{aligned}
\limsup_{n\to\infty}[u_n(t_0)] &= \limsup_{n\to\infty}\left(\widetilde{\lim_{t\to t_0-}}|u_n(t)| - \widetilde{\lim_{t\to t_0+}}|u_n(t)|\right)\\
&\leq \limsup_{n\to\infty}\widetilde{\lim_{t\to t_0-}}|u_n(t)| - \liminf_{n\to\infty}\widetilde{\lim_{t\to t_0+}}|u_n(t)|\\
&\leq \widetilde{\lim_{t\to t_0-}}|u(t)| - \widetilde{\lim_{t\to t_0+}}|u(t)|\\
&= [u(t_0)].
\end{aligned}$$

□

Assume now that Leray-Hopf solutions converge weakly to a strongly continuous from the right in H Leray-Hopf solution. We will show that the weak convergence is strong if the energy jumps of solutions converge to the energy jumps of the limit solution.

Theorem 3.24. *Let $\{u_n(t)\}$, $u(t)$ be Leray-Hopf solutions of* (6) *on $[T_1, \infty)$. If $u_n \to u$ in $C([T_1, T_2]; H_{\mathrm{w}})$,*

$$\widetilde{\lim_{t\to t_0+}}|u(t)| = |u(t_0)|, \qquad \textit{and} \qquad \liminf_{n\to\infty}[u_n(t_0)] \geq [u(t_0)],$$

for some $t_0 \in (T_1, T_2)$, then

$$\lim_{n\to\infty}|u_n(t_0)| = |u(t_0)|.$$

Proof. Thanks to (11) and Lemma 3.20, we have

$$\begin{aligned}
\limsup_{n\to\infty} |u_n(t_0)| &\leq \limsup_{n\to\infty} \widetilde{\lim_{t\to t_0+}} |u_n(t)| \\
&\leq \liminf_{n\to\infty} \widetilde{\lim_{t\to t_0-}} |u_n(t)| - \liminf_{n\to\infty} [u_n(t_0)] \\
&\leq \limsup_{n\to\infty} \widetilde{\lim_{t\to t_0-}} |u_n(t)| - [u(t_0)] \\
&\leq \widetilde{\lim_{t\to t_0-}} |u(t)| - [u(t_0)] \\
&= \widetilde{\lim_{t\to t_0+}} |u(t)| \\
&= |u(t_0)|,
\end{aligned}$$

which concludes the proof. □

Note that if a Leray-Hopf solution $u(t)$ is continuous at $t = t_0$, then $[u(t_0)] = 0$. Therefore, in particular, Theorem 3.24 immediately implies the following.

Corollary 3.25. *Let $\{u_n\}$ be a sequence of Leray-Hopf solutions of* (6) *on $[T_1, \infty)$. If $u_n \to u$ in $C([T_1, T_2]; H_{\mathrm{w}})$, and $|u(t)|$ is continuous at some $t = t_0 \in (T_1, T_2)$, then $u_n(t_0) \to u(t_0)$ strongly in H.*

So, if a solution $u(t)$ on the weak global atractor is continuous in H, then it attracts its basin strongly. Note that this is exactly Rosa's asymptotic regularity condition (see [19]).

3.4 The strong global attractor for the 3D NSE

In this subsection we will see that if all solutions on the weak global attractor $\mathcal{A}_{\mathrm{w}}$ are continuous in H, then $\mathcal{A}_{\mathrm{w}}$ is the strong global attractor. First, we will show that if a solution belongs to $\mathcal{A}_{\mathrm{w}}$ on some open time-interval I, then on any closed subinterval of I it has to coincide with a solution that stays on $\mathcal{A}_{\mathrm{w}}$ for all time.

Lemma 3.26. *Let $u \in \mathcal{E}([T, \infty))$ and $\tilde{u} \in \mathcal{E}([0, \infty))$, such that $u(T) = \tilde{u}(T)$ and*

$$\widetilde{\lim_{t\to T-}} |\tilde{u}(t)| \geq \widetilde{\lim_{t\to T+}} |u(t)|,$$

for some $T > 0$. Let

$$v(t) = \begin{cases} \tilde{u}(t), & 0 \leq t \leq T, \\ u(t), & t > T. \end{cases}$$

Then $v \in \mathcal{E}([0, \infty))$.

Proof. Obviously, $v(t)$ is a weak solution of the 3D NSE. To show that it satisfies the energy inequality, take any $t \geq T$ and $t_0 \in (0, T)$, $t_0 \notin Ex$, where Ex is the exceptional set for $\tilde{u}$. Then we have

$$|v(t)|^2 + 2\nu \int_T^t \|v(s)\|^2 \, ds \leq \widetilde{\lim_{s\to T+}} |v(s)|^2 + 2 \int_T^t (g, v(s)) \, ds,$$

and

$$\widetilde{\lim_{s\to T-}}|v(s)|^2 + 2\nu\int_{t_0}^{T}\|v(s)\|^2\,ds \le |v(t_0)|^2 + 2\int_{t_0}^{T}(g,v(s))\,ds.$$

Adding these inequalities, we obtain

$$|v(t)|^2 + 2\nu\int_{t_0}^{t}\|v(s)\|^2\,ds \le |v(t_0)|^2 + 2\int_{t_0}^{t}(g,v(s))\,ds,$$

which concludes the proof. □

Corollary 3.27. *Let $u \in \mathcal{E}([T,\infty))$ and $\tilde{u} \in \mathcal{E}((-\infty,\infty))$, such that $u(T) = \tilde{u}(T)$ and $u(t)$ is strongly continuous from the right at $t = T$. Let*

$$v(t) = \begin{cases} \tilde{u}(t), & t \le T, \\ u(t), & t > T. \end{cases}$$

Then $v \in \mathcal{E}((-\infty,\infty))$.

Lemma 3.28. *Let $u(t)$ be a Leray-Hopf solution $u \in \mathcal{E}([T_1,\infty))$, such that $u(t) \in \mathcal{A}_{\mathrm{w}}$ for all $t \in (T_1,T_2)$. Then for any $T_0 \in (T_1,T_2)$, there exists $v \in \mathcal{E}((-\infty,\infty))$, such that $u(t) = v(t)$ on $[T_0,\infty)$. In particular, $u(t) \in \mathcal{A}_{\mathrm{w}}$ for all $t > T_1$.*

Proof. First note that (T_1,T_0) contains some interval of regularity of $u(t)$ and take T in the interior of this interval. Since $u(T) \in \mathcal{A}_{\mathrm{w}}$, there exists a solution $\tilde{u} \in \mathcal{E}((-\infty,\infty))$, such that $\tilde{u}(T) = u(T)$. We will now glue them at point $t = T$, obtaining

$$v(t) = \begin{cases} \tilde{u}(t), & t \le T, \\ u(t), & t > T. \end{cases}$$

Since $u(t)$ is strongly continuous at $t = T$, Corolary 3.27 implies that $v \in \mathcal{E}((-\infty,\infty))$. □

Assume now that all solutions that stay on the weak global attractor are strongly continuous in H. We will prove that in this case the weak global attractor is strong. This generalizes a well-known result in [9]. Indeed, if $\mathcal{A}_{\mathrm{w}}$ is bounded in V, then all the solutions, as long as they stay on the weak global attractor, are regular, and are therefore continuous in H. This also weakens the condition of Ball [1] – the continuity of all Leray-Hopf solutions from $(0,\infty)$ to H.

Lemma 3.29. *If all solutions on the weak global attractor are strongly continuous in H, i.e., if $\mathcal{E}((-\infty,\infty)) \subset C((-\infty,\infty);H)$, then $R(t)$ is asymptotically compact.*

Proof. Take any $\{t_n\}$, such that $t_n \to \infty$ as $n \to \infty$, and $x_n \in R(t_n)X$. Then there exists a sequence of solutions $v_n \in \mathcal{E}([0,\infty))$, such that $v_n(t_n) = x_n$. We will show that $\{x_n\}$ has a convergent subsequence. Without loss of generality, there exists $T > 0$, such that $t_n \ge 2T$ for all n. Consider a sequence $u_n(t) = v_n(t + t_n - T)$, where $t \ge 0$. Due to Lemma 3.12, $\mathcal{E}([0,\infty))$ is compact in $C([0,\infty);H_{\mathrm{w}})$. Hence, passing to a subsequence and dropping a subindex, we can assume that u_n coverges to some $u \in \mathcal{E}([0,\infty))$ in $C([0,\infty);H_{\mathrm{w}})$ as $n \to \infty$. By the definition of the weak global attractor, $u(t) \in \mathcal{A}_{\mathrm{w}}$ for all $t \in [0,\infty)$. Applying Lemma 3.28 with $(T_1,T_2) =$

$(0, T)$, we obtain that there exists a complete trajectory $v \in \mathcal{E}((-\infty, \infty))$, such that $u(t) = v(t)$ on $[T/2, \infty)$. Therefore,

$$u \in C([T/2, \infty); H).$$

Hence, Corollary 3.25 yields that $u_n(T) \to u(T)$ strongly in H, i.e., $x_n \to u(T)$ strongly in H. $\square$

We can now conclude with the two main results of this section. First, as a direct consequence of Theorem 2.16 and Lemma 3.29 we obtain

Theorem 3.30. *If all solutions on the weak global attractor are strongly continuous in H, then the strong global attractor $\mathcal{A}_{\mathrm{s}}$ exists, is strongly compact, and coincides with $\mathcal{A}_{\mathrm{w}}$.*

Proof. Due to Lemma 3.29 we have that $R(t)$ is asymptotically compact. Then Theorem 2.16 implies that $\mathcal{A}_{\mathrm{s}}$ exists, is strongly compact, and coincides with $\mathcal{A}_{\mathrm{w}}$. $\square$

Second, we prove that the condition $\mathcal{E}((-\infty, \infty)) \subset C((-\infty, \infty); H)$ is equivalent to a condition that all the "energy jumps" uniformly converge to zero as time goes to infinity. More precisely, let

$$[R(t)X] := \sup\{[u(t)] : u \in \mathcal{E}([0, \infty))\}.$$

Then we have the following.

Theorem 3.31. $\mathcal{E}((-\infty, \infty)) \subset C((-\infty, \infty); H)$ *if and only if* $[R(t)X] \to 0$ *as* $t \to \infty$.

Proof. It is obvious that if $[R(t)X] \to 0$ as $t \to \infty$, then we have $\mathcal{E}((-\infty, \infty)) \subset C((-\infty, \infty); H)$.

Assume now that $\mathcal{E}((-\infty, \infty)) \subset C((-\infty, \infty); H)$, but $[R(t)X]$ does not converge to 0 as $t \to \infty$. Then there exist a sequence of Leray-Hopf solutions $v_n \in \mathcal{E}([0, \infty))$ and a time sequence $t_n \to \infty$ as $n \to \infty$, such that

$$\limsup_{n \to \infty} [v_n(t_n)] > 0. \tag{15}$$

Proceeding as in the proof of Lemma 3.29, we can now assume that there exists some $T > 0$, such that the sequence $u_n(t) = v_n(t + t_n - T)$ (where $t \geq 0$) converges in $C([0, \infty); H_{\mathrm{w}})$ to the restriction $u = v|_{[0,\infty)}$ of some complete trajectory $v \in \mathcal{E}((-\infty, \infty))$. Since $v(t)$ is continuous in H, by Theorem 3.23 we must have

$$\lim_{n \to \infty} [v_n(t_n)] = \lim_{n \to \infty} [u_n(T)] = [u(T)] = [v(T)] = 0,$$

a contradiction.

$\square$

3.5 Regular part of the global attractor

We define the regular part of the weak global attractor, first introduced is [9], as follows.

$$\begin{aligned}\mathcal{A}_{\mathrm{reg}} := \{\, & u_0 : \exists \tau > 0, u \in \mathcal{E}((-\infty,\infty)) \text{ with } u(0) = u_0, \text{ such that } u(t) \text{ is regular} \\ & \text{on } (-\tau,\tau), \text{ and for each } \tilde{u} \in \mathcal{E}((-\infty,\infty)) \text{ with } \tilde{u}(0) = u(0) \text{ we have} \\ & u(t) = \tilde{u}(t)\ \forall t \in (-\tau,\tau) \,\}.\end{aligned}$$

The following result was proven in [9]:

Theorem 3.32. *The regular part of the global attractor satisfies the following properties:*

(a) $\mathcal{A}_{\mathrm{reg}}$ *is weakly open in* $\mathcal{A}_{\mathrm{w}}$*,*

(b) $\mathcal{A}_{\mathrm{reg}}$ *is weakly dense in* $\mathcal{A}_{\mathrm{w}}$*,*

(c) If all solutions in $\mathcal{A}_{\mathrm{w}}$ *are regular, then* $\mathcal{A}_{\mathrm{w}}$ *is bounded in* V *(hence,* $\mathcal{A}_{\mathrm{reg}} = \mathcal{A}_{\mathrm{w}}$*).*

Part (c) was misstated in [9] as

(c') If $\mathcal{A}_{\mathrm{w}} \subset V$*, then* $\mathcal{A}_{\mathrm{reg}} = \mathcal{A}_{\mathrm{w}}$*.*

However, the proof provided in [9] yields only (c). As yet it is not known whether (c') is true.

Now we will further study the case when all weak solutions on the weak global attractor of the 3D NSE are continuous in H. Under this assumption, Theorem 3.30 implies that the strong compact global attractor $\mathcal{A}_{\mathrm{s}}$ exists and $\mathcal{A}_{\mathrm{s}} = \mathcal{A}_{\mathrm{w}}$. Moreover,

Theorem 3.33. *If* $\mathcal{E}((-\infty,\infty)) \subset C((-\infty,\infty);H)$*, then the regular part of the global attractor* $\mathcal{A}_{\mathrm{reg}}$ *is strongly dense in* $\mathcal{A}_{\mathrm{s}}$*.*

Proof. Due to Theorem 3.30 $\mathcal{A}_{\mathrm{s}}$ exists and is strongly compact. Therefore, weak and strong topologies are equivalent on $\mathcal{A}_{\mathrm{s}}$. Then since $\mathcal{A}_{\mathrm{reg}}$ is weakly dense in $\mathcal{A}_{\mathrm{w}}$, it is also strongly dense in $\mathcal{A}_{\mathrm{s}} = \mathcal{A}_{\mathrm{w}}$. □

We conclude this section with the following remark. In the case when the cubic box $\Omega = [0,L]^3$ is replaced by the cuboid $\Omega = [0,L]^2 \times [0,l]$ with $0 < l \ll L$, there exists a function $\alpha(g) > 0$, such that if

$$\frac{l}{L} \le \alpha(g), \tag{16}$$

then $\mathcal{A}_{\mathrm{w}}$ is the strong global attractor and is regular (see [18, 23]). It would be interesting to show that for some values of l/L larger than $\alpha(g)$, all the Leray-Hopf solutions on $\mathcal{A}_{\mathrm{w}}$ are strongly continuous, i.e., $\mathcal{A}_{\mathrm{w}}$ is also the strong global attractor.

4 Tridiagonal models for the Navier-Stokes equations

In this section we introduce a two-parameter family of new simple models for the Navier-Stokes equations with a nonlinear term enjoying the same basic properties as the nonlinear term $B(u,u)$ in the NSE (6).

The role of the space H will be played by l^2 with the usual inner product and norm:

$$(u,v)=\sum_{n=1}^{\infty} u_n v_n, \qquad |u|=\sqrt{(u,u)}.$$

The norm $|u|$ will be called the energy norm. Let $A : D(A) \to H$ be the Laplace operator defined by

$$(Au)_n = n^{\alpha} u_n, \qquad n \geq 1,$$

for some $\alpha > 0$. The domain $D(A)$ of this operator is

$$\left\{ u : \sum_{n=1}^{\infty} n^{2\alpha} u_n^2 < \infty \right\}.$$

Clearly, $D(A)$ is dense in H and A is a positive definite operator whose eigenvalues are

$$1, 2^{\alpha}, 3^{\alpha}, \ldots$$

Let $V = A^{-1/2}H$ endowed with the following inner product and norm:

$$((u,v)) = \sum_{n=1}^{\infty} n^{\alpha} u_n v_n, \qquad \|u\| = \sqrt{((u,u))}.$$

Here $\|u\|$ is an analog of H^1-norm of u and we will call it the enstrophy norm. Let also

$$\|u\|_{\gamma} = \left(\sum_{n=1}^{\infty} n^{\gamma} u_n^2 \right)^{1/2},$$

which is an analog of $H^{\gamma/\alpha}$-norm of u.

Our models for the NSE are given by the following equations:

$$\begin{cases} \dfrac{d}{dt} u_n + \nu n^{\alpha} u_n - n^{\beta} u_{n-1}^2 + (n+1)^{\beta} u_n u_{n+1} = g_n, \qquad n = 1, 2, 3 \ldots \\ u_0 = 0. \end{cases} \tag{17}$$

Here, $\nu > 0$, $\alpha > 0$, and $\beta > 1$. Note that the value of $\frac{d}{dt} u_n$ is determined only by the values of u_{n-1}, u_n, and u_{n+1}. Therefore, we will refer to the equations (17) as the tridiagonal model for the Navier-Stokes equations or shortly TNS equations. For $u = (u_1, u_2, \ldots)$ they can be written in a more condensed form as

$$\frac{d}{dt} u + \nu A u + B(u,u) = g, \tag{18}$$

where

$$(B(u,v))_n = -n^{\beta} u_{n-1} v_{n-1} + (n+1)^{\beta} u_n v_{n+1},$$

and $u_0 = 0$. Note that the orthogonality property holds for B:

$$\begin{aligned} (B(u,v),v) &= \sum_{n=1}^{\infty} \left(-n^{\beta} u_{n-1} v_{n-1} v_n + (n+1)^{\beta} u_n v_{n+1} v_n \right) \\ &= \sum_{n=1}^{\infty} \left(-n^{\beta} u_{n-1} v_{n-1} v_n + n^{\beta} u_{n-1} v_n v_{n-1} \right) \\ &= 0. \end{aligned}$$

In the case of TNS equations, a weak solution on $[T, \infty)$ (or $(-\infty, \infty)$, if $T = -\infty$) of (17) is actually a locally bounded H-valued function $u(t)$ on $[T, \infty)$, such that $u_n \in C^1([T, \infty))$ and $u_n(t)$ satisfies (17) for all n. From now on weak solutions will be called just solutions.

A solution $u(t)$ is strong (or regular) on some interval $[T_1, T_2]$, if $\|u(t)\|$ is bounded on $[T_1, T_2]$. A solution is strong on $[T_1, \infty)$, if it is strong on every interval $[T_1, T_2]$, $T_2 \geq 0$.

A Leray-Hopf solution of (17) on the interval $[T, \infty)$ is a solution of (17) on $[T, \infty)$ satisfying the energy inequality

$$|u(t)|^2 + 2\nu \int_{t_0}^{t} \|u(\tau)\|^2 \, d\tau \leq |u(t_0)|^2 + 2 \int_{t_0}^{t} (g, u(\tau)) \, d\tau,$$

for all $T \leq t_0 \leq t$, t_0 a.e. in $[T, \infty)$. The set Ex of those t_0 for which the energy inequality does not hold will be called the exceptional set.

4.1 A priori estimates and the existence of strong solutions

Taking a limit of the Galerkin approximation, the existence of Leray-Hopf solutions follows in exactly the same way as for the 3D NSE. In this paper we will show some *a priori* estimates, which can be obtained rigurously for the Leray-Hopf solutions. For simplicity, we assume that g is independent of time, $g \in H$, and $g_n \geq 0$ for all n.

Energy estimates. Formally taking a scalar product of (17) with u, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|u|^2 &\leq -\nu\|u\|^2 + |g||u| \\ &\leq -\nu|u|^2 + \frac{\nu}{2}|u|^2 + \frac{|g|^2}{2\nu} \\ &= -\frac{\nu}{2}|u|^2 + \frac{|g|^2}{2\nu}.\end{aligned}$$

Using Gronwall's inequality, we conclude that

$$|u(t)|^2 \leq e^{-\nu t}|u(0)|^2 + \frac{|g|^2}{\nu^2}(1 - e^{-\nu t}). \tag{19}$$

Hence $B = \{u \in H : \ |u| \leq R\}$ is an absorbing ball for the Leray-Hopf solutions, where R is any number larger that $|g|/\nu$.

Enstrophy estimates. Let $v = A^{1/2}u$ and

$$c_{\mathrm{b}} := \begin{cases} \alpha 2^{\beta}, & 0 < \alpha \leq 1, \\ \alpha 2^{\alpha+\beta-1}, & \alpha > 1. \end{cases}$$

Using Hölder's inequality, we obtain the following estimate for the nonlinear term:

$$
\begin{aligned}
|(B(u,u),Au)| &\le \left|\sum_{n=1}^{\infty} [(n+1)^{\alpha} - n^{\alpha}]\,(n+1)^{\beta} u_n^2 u_{n+1}\right| \\
&\le c_{\mathrm{b}} \sum_{n=1}^{\infty} n^{\beta-\alpha/2-1} |v_n|^2 |v_{n+1}| \\
&\le c_{\mathrm{b}} (\max_n |v_n|) \sum_{n=1}^{\infty} n^{\beta-\alpha/2-1} v_n^2 \\
&\le c_{\mathrm{b}} |v| |A^{1/2} v|^{2\beta/\alpha - 2/\alpha - 1} |v|^{-2\beta/\alpha + 2/\alpha + 3} \\
&= c_{\mathrm{b}} |A^{1/2} v|^{2\beta/\alpha - 2/\alpha - 1} |v|^{-2\beta/\alpha + 2/\alpha + 4} \\
&= c_{\mathrm{b}} |Au|^{2\beta/\alpha - 2/\alpha - 1} \|u\|^{-2\beta/\alpha + 2/\alpha + 4},
\end{aligned}
$$

whenever $\beta \in [\alpha/2+1, 3\alpha/2+1]$. Choosing u to have only two consecutive nonzero terms, it is easy to check that this estimate is sharp. Moreover, when $\alpha = 2/3$ and $\beta = 11/6$, we have

$$
|(B(u,u),Au)| \le c_{\mathrm{b}} |Au|^{3/2} \|u\|^{3/2},
$$

which is exactly what Sobolev estimates give for the 3D NSE. Therefore, formally taking a scalar product of (17) with Au, we obtain

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u\|^2 &\le -\nu |Au|^2 + c_{\mathrm{b}} |Au|^{3/2} \|u\|^{3/2} + (g, Au) \\
&\le -\nu|Au| + \frac{\nu}{3}|Au|^2 + \frac{3^6 c_{\mathrm{b}}^4}{2^8 \nu^3}\|u\|^6 + \frac{3}{4\nu}|g|^2 + \frac{\nu}{3}|Au|^2 \\
&\le -\frac{\nu}{3}|Au|^2 + \frac{3^6 c_{\mathrm{b}}^4}{2^8 \nu^3}\|u\|^6 + \frac{3}{4\nu}|g|^2,
\end{aligned}
$$

a Riccati-type equation for $\|u\|^2$. Hence, the model has the same enstrophy estimates as the 3D NSE, similar properties, and the same open question concerning the regularity of the solutions in the case $(\alpha, \beta) = (2/3, 11/6)$. In particular, we have a global existence of Leray-Hopf weak solutions (see Theorem 3.2), local existence of strong solutions (see Theorem 3.4), Leray's structure theorem (see Theorem 3.5), uniqueness of strong solutions in the class of Leray-Hopf solutions (see Theorem 3.6), and existence of a weak global attractor (see Theorem 3.13).

In the case $(\alpha, \beta) = (1/2, 7/4)$, we have

$$
|(B(u,u),Au)| \le c_{\mathrm{b}} |Au|^2 \|u\|,
$$

which corresponds to the 4D Navier-Stokes equations. In the case $(\alpha, \beta) = (2/5, 17/10)$, we have

$$
|(B(u,u),Au)| \le c_{\mathrm{b}} |Au|^{5/2} \|u\|^{1/2},
$$

which corresponds to the 5D Navier-Stokes equations. In general, the choice

$$
\alpha = \frac{2}{d}, \qquad \beta = \frac{3}{2} + \frac{1}{d}
$$

would correspond to the d-dimensional Navier-Stokes equations.

The similarity of the TNS equations (17) with the NSE holds also for values of $d \neq 3$. Indeed, when $2\beta < 3\alpha + 2$, the enstrophy estimate implies a local existence of strong solutions, i.e., solutions whose enstrophy norms are continuous. More precisely, if $2\beta < 3\alpha + 2$, then for any initial data $u_0 \in V$, there exists a strong solution $u(t)$ with $u(0) = u_0$ on some interval $[0, T]$. In terms of the dimension, a sufficient condition for the local existence of strong solutions is $d < 4$. In the case when $\beta \leq \alpha + 1$, the enstrophy estimate implies a global regularity. In terms of the dimension, a sufficient condition for the global existence of strong solutions is $d \leq 2$.

Now we will concentrate on the solutions with initial data $u_n(0) \geq 0$ for all n.

Theorem 4.1. *Let $u(t)$ be a solution of* (17) *with $u_n(0) \geq 0$. Then $u_n(t) \geq 0$ for all $t > 0$, and $u(t)$ satisfies the energy inequality*

$$|u(t)|^2 + 2\nu \int_{t_0}^{t} \|u(\tau)\|^2 \, d\tau \leq |u(t_0)|^2 + 2 \int_{t_0}^{t} (g, u(\tau)) \, d\tau \tag{20}$$

for all $0 \leq t_0 \leq t$.

Proof. A general solution for $u_n(t)$ can be written as

$$\begin{aligned} u_n(t) = u_n(0) \exp\left(- \int_0^t \left[\nu n^\alpha + (n+1)^\beta u_{n+1}(\tau) \right] \, d\tau \right) \\ + \int_0^t (g_n + n^\beta u_{n-1}^2(s)) \exp\left(- \int_s^t \left[\nu n^\alpha + (n+1)^\beta u_{n+1}(\tau) \right] \, d\tau \right) \, ds. \end{aligned}$$

Since $u_n(0) \geq 0$ for all n, then $u_n(t) \geq 0$ for all n, $t > 0$. Hence, multiplying (17) by u_n, taking a sum from 1 to N, and integrating between t_0 and t, we obtain

$$\begin{aligned} \sum_{n=1}^{N} u_n(t)^2 - \sum_{n=1}^{N} u_n(t_0)^2 + 2\nu \int_{t_0}^{t} \sum_{n=1}^{N} n^\alpha u_n^2 \, d\tau \\ = -2 \int_{t_0}^{t} (N+1)^\beta u_N^2 u_{N+1} \, d\tau + 2 \int_{t_0}^{t} \sum_{n=1}^{N} g_n u_n \, d\tau \\ \leq 2 \int_{t_0}^{t} \sum_{n=1}^{N} g_n u_n \, d\tau. \end{aligned}$$

Taking the limit as $N \to \infty$, we obtain (20).

□

4.2 Blow-up in finite time

Here, when $2\beta - 3\alpha - 3 > 0$ and $g_1 > 0$ is large enough, we will show that every solution $u(t)$ of (17) with $u_n(0) \geq 0$ blows up in finite time in an appropriate norm. First, we need the following two lemmas.

Lemma 4.2. *Let $u(t)$ be a solution to* (17) *on $[0, \infty)$ with $u_n(0) \geq 0$ for all n. Assume that $\|u(t)\|_{2(\beta+\gamma-1)/3} \in L^3_{\text{loc}}([0, \infty); \mathbb{R})$. Then*

$$\int_{t_0}^{t} \sum_{n=1}^{\infty} n^{\beta+\gamma-1} u_n^2 u_{n+1} \, d\tau < \infty, \qquad \int_{t_0}^{t} \sum_{n=1}^{\infty} n^{\beta+\gamma-1} u_n^3 \, d\tau < \infty, \tag{21}$$

and

$$\|u(t)\|_\gamma^2 - \|u(t_0)\|_\gamma^2 + 2\nu \int_{t_0}^t \|u\|_{\alpha+\gamma}^2 \, d\tau \geq 2\gamma \int_{t_0}^t \sum_{n=1}^\infty (n+1)^{\beta+\gamma-1} u_n^2 u_{n+1} \, d\tau \quad (22)$$

for all $0 \leq t_0 \leq t$, $0 < \gamma \leq 1$.

Proof. Thanks to Theorem 4.1, $u_n(t) \geq 0$ for all $n, t > 0$. Since $\|u(t)\|_{2(\beta+\gamma-1)/3}^3$ is integrable on $[t_0, t]$ for all $0 \leq t_0 \leq t$, we obtain

$$\begin{aligned}
\int_{t_0}^t \sum_{n=1}^\infty n^{\beta+\gamma-1} u_n^2 u_{n+1} \, d\tau &\leq 2 \int_{t_0}^t \sum_{n=1}^\infty n^{\beta+\gamma-1} u_n^3 \, d\tau \\
&\leq 2 \int_{t_0}^t \left(\sum_{n=1}^\infty n^{\frac{2}{3}(\beta+\gamma-1)} u_n^2 \right)^{3/2} d\tau \\
&= 2 \int_{t_0}^t \|u\|_{2(\beta+\gamma-1)/3}^3 \, d\tau \\
&< \infty.
\end{aligned}$$

Hence, the relations in (21) hold. In particular,

$$\liminf_{n\to\infty} \int_{t_0}^t n^{\beta+\gamma} u_n^2 u_{n+1} \, d\tau = 0. \quad (23)$$

Now multiplying (17) by $n^\gamma u_n$, taking a sum from 1 to N, and integrating from t_0 to t, we obtain

$$\begin{aligned}
&\sum_{n=1}^N n^\gamma u_n(t)^2 - \sum_{n=1}^N n^\gamma u_n(t_0)^2 + 2\nu \int_{t_0}^t \sum_{n=1}^N n^{\alpha+\gamma} u_n^2 \, d\tau \\
&\quad = 2 \int_{t_0}^t \sum_{n=1}^{N-1} (n+1)^\beta ((n+1)^\gamma - n^\gamma) u_n^2 u_{n+1} \, d\tau \\
&\qquad - 2 \int_{t_0}^t (N+1)^\beta N^\gamma u_N^2 u_{N+1} \, d\tau + 2 \int_{t_0}^t \sum_{n=1}^N n^\gamma g_n u_n \, d\tau \\
&\quad \geq 2\gamma \int_{t_0}^t \sum_{n=1}^{N-1} (n+1)^{\beta+\gamma-1} u_n^2 u_{n+1} \, d\tau - 2 \int_{t_0}^t (N+1)^\beta N^\gamma u_N^2 u_{N+1} \, d\tau
\end{aligned}$$

Thanks to (23), taking the lower limit as $N \to \infty$, we get (22).

$\square$

Lemma 4.3. *For any* $c > 0$, *there exists* $g_1 > 0$, *such that*

$$\int_t^{t+1} |u|^2 \, d\tau > c, \qquad \forall t \geq 0,$$

for all solutions $u(t)$ *with* $g = (g_1, g_2, \dots)$.

Proof. Take any $c > 0$. Thanks to Theorem 4.1, $u_n(t) \geq 0$ for all $n, t > 0$. Therefore,

$$u_1(t + 1/2) \geq u_1(t) - \nu \int_t^{t+1/2} u_1 \, d\tau - 2^\beta \int_t^{t+1/2} u_1 u_2 \, d\tau + g_1.$$

Now integrating this inequality over $[t, t + 1/2]$, we obtain

$$\int_t^{t+1} u_1 \, d\tau + \nu \int_t^{t+1} u_1 \, d\tau + 2^{\beta-1} \int_t^{t+1} u_1^2 \, d\tau + 2^{\beta-1} \int_t^{t+1} u_2^2 \, d\tau \geq \frac{g_1}{2}.$$

Hence, using Cauchy-Schwarz inequality, we get

$$(\nu + 1) \left(\int_t^{t+1} (u_1^2 + u_2^2) \, d\tau \right)^{1/2} + 2^{\beta-1} \int_t^{t+1} (u_1^2 + u_2^2) \, d\tau \geq \frac{g_1}{2}.$$

Obviously, for g_1 large enough, we have that

$$\int_t^{t+1} |u|^2 \, d\tau \geq \int_t^{t+1} (u_1^2 + u_2^2) \, d\tau > c.$$

□

Now we proceed to our main result in this section.

Theorem 4.4. *For every solution $u(t)$ to equation (17) with $u_n(0) \geq 0$, $\nu > 0$, $2\beta - 3\alpha - 3 > 0$, and g_1 large enough, $\|u(t)\|^3_{2(\beta+\gamma-1)/3}$ is not locally integrable for all $\gamma > 0$.*

Proof. Thanks to Theorem 4.1, $u_n(t) \geq 0$ for all $n, t > 0$. Assume that there exist

$$\gamma \in (0, \min\{2\beta - 3\alpha - 3, 1\})$$

and a solution $u(t)$ to (17), such that $\|u(t)\|_{2(\beta+\gamma-1)/3} \in L^3_{\mathrm{loc}}([0, \infty); \mathbb{R})$. Then Lemma 4.2 implies that

$$\int_0^T \sum_{n=1}^\infty n^{\beta+\gamma-1} u_n^2 u_{n+1} \, d\tau < \infty \qquad \text{and} \qquad \int_0^T \sum_{n=1}^\infty n^{\beta+\gamma-1} u_n^3 \, d\tau < \infty,$$

for all $T > 0$. Moreover, since $\alpha + \gamma < 2(\beta + \gamma - 1)/3$, we have that $\|u(t)\|^2_{\alpha+\gamma}$ is locally integrable on $[0, \infty)$.

Now note that if $u_{n+1} \geq 2u_n$, then $u_n u_{n+1}^2 \leq \frac{1}{2} u_{n+1}^3$. Otherwise, $u_n u_{n+1}^2 \leq 2u_n^2 u_{n+1}$. Hence,

$$u_n u_{n+1}^2 \leq \tfrac{1}{2} u_{n+1}^3 + 2u_n^2 u_{n+1}, \qquad n \in \mathbb{N}. \tag{24}$$

This also implies

$$\begin{aligned} u_n u_{n+1} u_{n+2} &\leq \tfrac{1}{2} u_n^2 u_{n+1} + \tfrac{1}{2} u_{n+1} u_{n+2}^2 \\ &\leq \tfrac{1}{2} u_n^2 u_{n+1} + \tfrac{1}{4} u_{n+2}^3 + u_{n+1}^2 u_{n+2}, \end{aligned} \tag{25}$$

for all $n \in \mathbb{N}$. From (17) we have

$$\frac{d}{dt}(u_n u_{n+1}) = -\nu(n^\alpha + (n+1)^\alpha) u_n u_{n+1} + n^\beta u_{n-1}^2 u_{n+1} - (n+1)^\beta u_n u_{n+1}^2 \\ + (n+1)^\beta u_n^3 - (n+2)^\beta u_n u_{n+1} u_{n+2} + u_n g_{n+1} + u_{n+1} g_n.$$

From this, using inequalities (24) and (25), we obtain

$$\sum_{n=1}^{\infty}(n+1)^{\gamma-1}(u_n u_{n+1})(t+1) - \sum_{n=1}^{\infty}(n+1)^{\gamma-1}(u_n u_{n+1})(t) \\ + 2\nu \int_t^{t+1} \sum_{n=1}^{\infty}(n+1)^{\alpha+\gamma-1} u_n u_{n+1}\, d\tau \\ + (3 + (3/2)^\beta) \int_t^{t+1} \sum_{n=1}^{\infty}(n+1)^{\beta+\gamma-1} u_n^2 u_{n+1}\, d\tau \\ \geq \frac{1}{4} \int_t^{t+1} \sum_{n=1}^{\infty} n^{\beta+\gamma-1} u_n^3\, d\tau, \quad (26)$$

for all $t > 0$. On the other hand, Lemma 4.2 yields

$$\|u(t+1)\|_\gamma^2 - \|u(t)\|_\gamma^2 + 2\nu \int_t^{t+1} \|u\|_{\alpha+\gamma}^2\, d\tau \geq 2\gamma \int_t^{t+1} \sum_{n=1}^{\infty}(n+1)^{\beta+\gamma-1} u_n^2 u_{n+1}\, d\tau \quad (27)$$

for all $t > 0$. Denote

$$\Theta(t) = \int_t^{t+1} \|u(\tau)\|_\gamma^2\, d\tau + \frac{2\gamma}{3 + (3/2)^\beta} \int_t^{t+1} \sum_{n=1}^{\infty}(n+1)^{\gamma-1}(u_n u_{n+1})(\tau)\, d\tau.$$

Note that $\Theta(t)$ is absolutely continuous on $[0, \infty)$. We will show that $\Theta(t)$ is a Lyapunov function for the equation, i.e., $\Theta(t)$ is always increasing. Indeed, multiplying the inequality (26) by $2\gamma/(3 + (3/2)^\beta)$ and adding (27), we obtain

$$\frac{d}{dt}\Theta(t) \geq -2\nu \int_t^{t+1} \|u(\tau)\|_{\alpha+\gamma}^2\, d\tau - \frac{4\gamma\nu}{3 + (3/2)^\beta} \int_t^{t+1} \sum_{n=1}^{\infty}(n+1)^{\alpha+\gamma-1} u_n u_{n+1}\, d\tau \\ + \frac{\gamma}{6 + 2(3/2)^\beta} \int_t^{t+1} \sum_{n=1}^{\infty} n^{\beta+\gamma-1} u_n^3\, d\tau,$$

a.e. on $(0, \infty)$. Since γ is such that

$$\epsilon := 2\beta - 3\alpha - \gamma - 3 > 0,$$

let

$$A := \left(\sum_{n=1}^{\infty} n^{-1-\epsilon}\right)^{-1/2}.$$

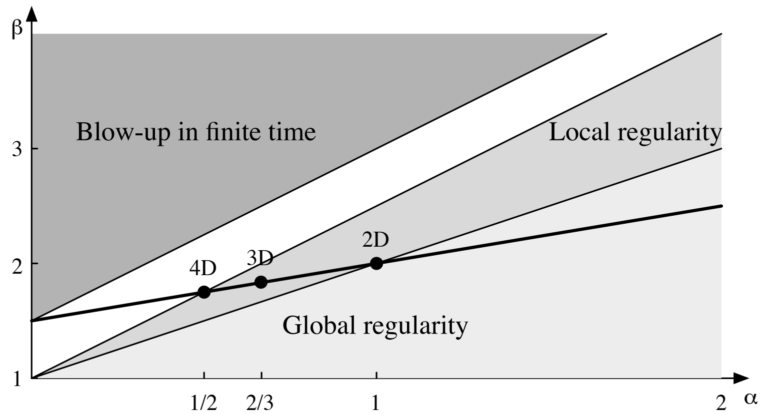

Figure 1: Regions of local regularity, global regularity, and blow-up.

Now Hölder's inequality yields

$$\int_t^{t+1} \sum_{n=1}^{\infty} n^{\alpha+\gamma} u_n^2 \, d\tau \leq \left(\int_t^{t+1} \sum_{n=1}^{\infty} n^{-1-\epsilon} \, d\tau \right)^{1/3} \left(\int_t^{t+1} \sum_{n=1}^{\infty} n^{\beta+\gamma-1} u_n^3 \, d\tau \right)^{2/3}$$
$$= A^{-2/3} \left(\int_t^{t+1} \sum_{n=1}^{\infty} n^{\beta+\gamma-1} u_n^3 \, d\tau \right)^{2/3}.$$

Hence,

$$\int_t^{t+1} \sum_{n=1}^{\infty} n^{\beta+\gamma-1} u_n^3 \, d\tau \geq A \left(\int_t^{t+1} \sum_{n=1}^{\infty} n^{\alpha+\gamma} u_n^2 \, d\tau \right)^{3/2}.$$

Finally, we obtain

$$\frac{d}{dt}\Theta(t) \geq -2\nu \left(1 + 2\gamma \frac{(3/2)^{\alpha+\gamma}}{3+(3/2)^{\beta}} \right) \int_t^{t+1} \|u(\tau)\|_{\alpha+\gamma}^2 \, d\tau$$
$$+ \frac{\gamma A}{6+2(3/2)^{\beta}} \left(\int_t^{t+1} \|u(\tau)\|_{\alpha+\gamma}^2 \, d\tau \right)^{3/2},$$

a.e. on $(0,\infty)$. Due to Lemma 4.3, if g_1 is large enough, then there exists a positive constant c, such that

$$\frac{d}{dt}\Theta(t) \geq c\Theta(t)^{3/2}, \qquad \text{a.e. on } (0,\infty).$$

This is a Riccati-type equation. Hence, $\Theta(t)$ blows up in finite time, which contradicts the fact that it is continuous on $[0,\infty)$. $\square$

Figure 1 shows three regions, the ones where we were able to prove local regularity, global regularity, and and blow-up in finite time. The labels $2D$, $3D$, and $4D$ show the dimensions of the Navier-Stokes systems corresponding to the models at those points.

4.3 Non-regular weak global attractor

As in Subsection 3.2, we can define an evolutionary system $\mathcal{E}$ whose trajectories are all Leray-Hopf solutions of the TNS equations. The weak global attractor for this system is

$$\begin{aligned}\mathcal{A}_{\mathrm{w}} = \{u_0 \in H : \ &\text{there exists a Leray-Hopf solution } u(t) \text{ on } (-\infty,\infty),\\ &\text{such that } u(0)=u_0 \text{ and } |u(t)| \text{ is bounded on } (-\infty,\infty)\}.\end{aligned}$$

Recall that $g_n \geq 0$ for all $n \in \mathbb{N}$. Obviously, if $g=0$, then $\mathcal{A}_{\mathrm{w}} = \{0\}$. Henceforth we will assume that $g \neq 0$.

Theorem 4.5. *If $g_n = 0$ for all $n \geq N_g$, then every $u = (u_1, u_2, \dots) \in \mathcal{A}_{\mathrm{w}}$ satisfies*

$$u_n \geq 0, \qquad n = 1, 2, \dots$$

Proof. A general solution for $u_n(t)$ can be written as

$$\begin{aligned}u_n(t) &= u_n(t_0)\exp\left(-\int_{t_0}^t \nu n^\alpha + (n+1)^\beta u_{n+1}(\tau)\,d\tau\right)\\ &\quad + \int_{t_0}^t \exp\left(-\int_s^t \nu n^\alpha + (n+1)^\beta u_{n+1}(\tau)\,d\tau\right)(g_n + n^\beta u_{n-1}^2(s))\,ds. \qquad (28)\end{aligned}$$

Clearly, this implies the following facts.

(a) If $u_n(t_0) \geq 0$ for some n and t_0, then $u_n(t) \geq 0$ for all $t \geq t_0$.

(b) If $|u(t)|$ is bounded for all $t \in \mathbb{R}$, then $u_n(t) \geq 0$ for all $t \in \mathbb{R}$, whenever $u_{n+1}(t) \geq 0$ for all $t \in \mathbb{R}$.

Now assume that there exists $u^0 \in \mathcal{A}_{\mathrm{w}}$, such that $u_N^0 < 0$ for some $N \geq N_g$. Then there exists a Leray-Hopf solution $u(t)$, such that $u(0) = u^0$ and $|u(t)|$ is bounded on $(-\infty, \infty)$. For such a solution we have $u_N(t) < 0$ for all $t \leq 0$. In addition, from the energy inequality for $u(t)$ we deduce that

$$\begin{aligned}\sum_{n=N}^\infty u_n(t_1)^2 - \sum_{n=N}^\infty u_n(t_0)^2 &\leq 2\int_{t_0}^{t_1}\left[N^\beta u_{N-1}(\tau)^2 u_N(\tau) - \nu\sum_{n=N}^\infty n^\alpha u_n(\tau)^2\right]d\tau\\ &\leq -2\nu\int_{t_0}^{t_1}\sum_{n=N}^\infty u_n(\tau)^2\,d\tau,\end{aligned}$$

for all $t_0 \leq t_1 \leq 0$. Hence,

$$\sum_{n=N}^\infty u_n(t_0)^2 \geq e^{-2\nu t_0}\sum_{n=N}^\infty u_n(0)^2,$$

for all $t_0 \leq 0$. This implies that $|u(t)|^2$ is not bounded backwards in time, a contradiction.

Now take any $u^0 \in \mathcal{A}_{\mathrm{w}}$. There exists a Leray-Hopf solution $u(t)$, such that $u(0) = u^0$ and $|u(t)|$ is bounded on $(-\infty, \infty)$. For such a solution we proved that

$$u_n(t) \geq 0, \qquad \forall t \in \mathbb{R}, n \geq N_g.$$

Now note that due to the remark (b) above, if $u_{n+1}(t) \geq 0$ for all $t \in \mathbb{R}$, then $u_n(t) \geq 0$ for all $t \in \mathbb{R}$, which concludes the proof. $\square$

Now Theorem 4.5 allows us to apply all the results in Subsection 4.2 to the solutions on the weak global attractor $\mathcal{A}_{\mathrm{w}}$. Therefore, we have the following.

Remark 4.6. *Let $2\beta > 3\alpha + 3$ and g_1 be large enough. Then $\|u(t)\|_{2(\beta+\gamma-1)/3}$ blows up in finite time for every solution $u(t)$ on $\mathcal{A}_{\mathrm{w}}$, i.e., $\mathcal{A}_{\mathrm{w}}$ is not bounded in $H^{2(\beta+\gamma-1)/(3\alpha)}$ for any $\gamma > 0$.*

However, this does not mean that the weak global attractor cannot be strong. The question whether $\mathcal{A}_{\mathrm{w}}$ is the strong global attractor remains open in the case $\beta > \alpha+1$.

4.4 Tridiagonal models for the Euler equations

In this section we consider the tridiagonal models for the Euler equations (TE), the equations (17) with $\nu = 0$. First, let us show the global existence of the weak solutions to the TE equations. Take a sequence $\nu_j \to 0$ as $j \to \infty$. Given $u^0 \in H$, let $u^j(t)$ be a solution of (17) with $\nu = \nu_j$ and $u^j(0) = u^0$. It is easy to show that the sequence $\{u^j\}$ is weakly equicontinuous. Therefore, thanks to Ascoli-Arzela theorem, passing to a subsequence and dropping a subindex, we obtain that there exists a function $u : [0,\infty) \to H$, such that $u^j \to u$ in $C([0,\infty); H_{\mathrm{w}})$ as $j \to \infty$. Clearly, $u(t)$ is a solution of the TE equations, in the sense that it is a locally bounded H-valued function on $[T,\infty)$, such that $u_n \in C^1([0,\infty))$ and $u_n(t)$ satisfies (17) for all n.

Now let us show that in the nonviscous case $\nu = 0$, for every solution $u(t)$ of (17), the norm $\|u(t)\|_{2(\beta+\gamma-1)/3}$ blows up for any $\alpha > 0$, $\beta > 1$, and $\gamma > 0$, reflecting the fact that there is no backward energy transfer for this model.

Theorem 4.7. *Let $u(t)$ be a solution of* (17) *on $[0,\infty)$ with $\nu = 0$, $g_n \geq 0$, $u_n(0) \geq 0$ for all n, and $u(0) \neq 0$. Then $\|u(t)\|_{2(\beta+\gamma-1)/3}$ is not bounded on $[0,\infty)$ for every $\gamma > 0$.*

Proof. Clearly, it is enough to prove the theorem in the case where $0 < \gamma < \min\{1, 2(\beta-1)\}$. Assume that $\|u(t)\|_{2(\beta+\gamma-1)/3}$ is bounded on $[0,\infty)$. Then Lemma 4.2 implies that

$$\|u(t)\|_{\gamma}^2 - \|u(t_0)\|_{\gamma}^2 \geq 2\gamma \int_{t_0}^{t} \sum_{n=1}^{\infty} (n+1)^{\beta+\gamma-1} u_n^2 u_{n+1}\, d\tau \geq 0, \tag{29}$$

for all $0 \leq t_0 \leq t$. Thus, $\|u(t)\|_{\gamma}^2$ is non-decreasing. Since $\gamma < 2(\beta+\gamma-1)/3$, $\|u(t)\|_{\gamma}$ is bounded on $[0,\infty)$. Then there exists $E_0 > 0$ such that

$$\lim_{t\to\infty} \|u(t)\|_{\gamma}^2 = E_0.$$

Then (29) implies that

$$\lim_{t\to\infty} \int_t^{\infty} u_n(\tau)^2 u_{n+1}(\tau)\, d\tau = 0, \qquad n \in \mathbb{N}. \tag{30}$$

Hence,

$$\begin{aligned} u_n(t)^2 - u_n(0)^2 &= 2n^{\beta} \int_0^t u_{n-1}^2 u_n\, d\tau - 2(n+1)^{\beta} \int_0^t u_n^2 u_{n+1}\, d\tau + 2\int_0^t g_n u_n\, d\tau \\ &\to 2n^{\beta} \int_0^{\infty} u_{n-1}^2 u_n\, d\tau - 2(n+1)^{\beta} \int_0^{\infty} u_n^2 u_{n+1}\, d\tau + 2\int_0^{\infty} g_n u_n\, d\tau, \end{aligned}$$

as $t \to \infty$. Hence $u_n(\infty) := \lim_{t\to\infty} u_n(t)$ exists for all n. Now (30) implies that $u_n(\infty)u_{n+1}(\infty) = 0$ for all n. Suppose that $u_k(\infty) \neq 0$ for some k. Then $u_{k+1}(\infty) = 0$ and there exists $t_0 > 0$, such that

$$(k+2)^\beta u_{k+1}(t)u_{k+2}(t) \le \tfrac{1}{3}(k+1)^\beta u_k(\infty)^2 \qquad \text{and} \qquad u_k^2(t) \ge \tfrac{2}{3}u_k(\infty)^2,$$

for all $t \ge t_0$. Thus,

$$\begin{aligned}\frac{d}{dt}u_{k+1} &= (k+1)^\beta u_k(t)^2 - (k+2)^\beta u_{k+1}(t)u_{k+2}(t) + g_{k+1} \\ &\ge \tfrac{1}{3}(k+1)^\beta u_k(\infty)^2,\end{aligned}$$

for all $n \ge t_0$. Therefore,

$$\lim_{t\to\infty} u_{k+1}(t) = \infty,$$

a contradiction. □

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