STATIONARY AND DISCONTINUOUS WEAK SOLUTIONS OF THE NAVIER-STOKES EQUATIONS.

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ABSTRACT. We prove that there exists a nontrivial finite energy periodic stationary weak solution to the 3D Navier-Stokes equations (NSE). The construction relies on a convex integration scheme utilizing new stationary building blocks designed specifically for the NSE. The constructed family of approximate stationary solutions is also used to prove the existence of weak solutions of the NSE with energy profiles discontinuous on a dense set of positive Lebesgue measure.

1. INTRODUCTION

The 3D incompressible Navier-Stokes equations (NSE) on the torus $\mathbb{T}^3$ is the following systems of equations:

$$\begin{cases} \partial_t u - \Delta u + \operatorname{div}(u \otimes u) + \nabla p = 0 \\ \operatorname{div} u = 0, \end{cases} \tag{NSE}$$

where $u : \mathbb{T}^3 \times \mathbb{R} \to \mathbb{R}^3$ is the unknown velocity field and $p : \mathbb{T}^3 \times \mathbb{R} \to \mathbb{R}$ is the pressure.

Definition 1.1 (Weak solutions)**.** *A L^2-weakly continuous function $u \in C_w([0,T]; L^2(\mathbb{T}^3))$ with zero mean is a weak solution of* (NSE) *if $u(\cdot,t)$ is weakly divergence-free for all $t \in [0,T]$ and satisfies*

$$\int_{\mathbb{T}^3} u(x,0) \cdot \varphi(x,0)\, dx + \int_0^T \int_{\mathbb{T}^3} u \cdot (\partial_t \varphi + (u \cdot \nabla)\varphi + \Delta\varphi)\, dx d\tau = 0,$$

for any divergence-free zero-mean test function $\varphi \in C_c^\infty(\mathbb{T}^3 \times [0,T))$.

The vector field $u_0(\cdot) = u(\cdot, 0)$, which is also the weak L^2 limit of $u(\cdot,t)$ as $t \to 0^+$, is called the initial data. Often weak solutions with finite energy dissipation, i.e., $u \in L^2(0,T;H^1)$, are studied in the literature. Besides Definition 1.1, there are numerous equivalent ways to define such solutions, e.g., using alternative spaces of test functions (see [RRS16]).

Since the seminal work of Leray [Ler34] it has been known that any divergence-free initial data $u_0 \in L^2(\mathbb{T}^3)$ gives rise to a weak solution satisfying the following energy inequality:

$$\|u(t)\|_2^2 + 2\int_{t_0}^t \|\nabla u(s)\|_2^2\, ds \le \|u(t_0)\|_2^2, \tag{E.I.}$$

for any $t > 0$ and a.e. $t_0 \in [0,t)$ including 0. In the literature, such solutions are referred to as the Leray-Hopf weak solutions. There has been a long history of extensive studies of these solutions [Ler34, Hop51, Pro59, Ser62, Lad67, CF88, Tem01, ESv03], however, the global regularity and uniqueness of Leray-Hopf weak solutions remain among the most important unsolved questions in mathematical fluid dynamics. What is more related to the present work, is the validity of energy equality (also known as Onsager's conjecture in the case of the Euler equations [CET94]). In the recent groundbreaking work [BV19] Buckmaster and Vicol proved nonuniqueness and anomalous dissipation in the class of weak solutions, but this is still an open question for Leray-Hopf solutions. In fact, the continuity of the energy is not known either. If the energy has a jump discontinuity from the right, this immediately implies non-uniqueness since the solution can be restarted at that time to remove the jump. Moreover, infinitely many solutions can be obtained via interpolation [KV07].

The focus of this paper is to prove the existence of weak solutions to the (NSE) with very pathological energy behaviors. On one hand, we construct a finite energy stationary solution, which does not lose any energy even though its enstrophy is positive (in fact, infinite). These solutions exhibit what we call the *anomalous energy influx*, the backward energy cascade that precisely balances the energy dissipation at each scale. On the other hand, we construct weak solutions with energy profiles discontinuous on a dense set of positive Lebesgue measure. So the set of discontinuities of the energy can be very large at least in the class of weak solutions. Note that both results provide alternative proofs of the Buckmaster-Vicol nonuniqueness theorem [BV19] since there are Leray-Hopf

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solutions starting from the steady state or discontinuity points. The following theorems are direct consequences of our main results.

Theorem 1.2. *There exists a nontrivial stationary weak solution $u \in L^2(\mathbb{T}^3)$ to the 3D NSE.*

Theorem 1.3. *For any $\varepsilon, T > 0$, there exists a weak solution $u \in C_w([0,T]; L^2(\mathbb{T}^3))$ to the 3D NSE, which is discontinuous in L^2 on a set $E \subset [0,T]$, such that*

(1) E in dense in $[0,T]$.
(2) The Lebesgue measure of E^c is less than ε.

1.1. **Background.** Our work is based on the technique of convex integration. Although this method has been around since the work of Nash [Nas54], its application to fluid dynamics was brought to attention only in recent years by the pioneering work of De Lellis and Székelyhidi Jr. [DLS09]. Since [DLS09], it was developed over a series of works in the resolution of the Onsager's conjecture for the 3D Euler equations [DLS09, DLS13, DLS14, BDLIS15, BDLS16, Ise18, BLJV18]. Its extension to the NSE was done only very recently by Buckmaster-Vicol [BV19], where non-unique weak solutions of the Navier-Stokes equations in the sense of Definition 1.1 are constructed.

So far, the focus of the convex integration method has been to produce wild solutions that are as regular as possible. For instance, the regularity of wild solutions of the Euler equations was pushed to the critical Onsager's exponent $1/3$ by Isett [Ise18]. Also, the extension of [BV19] to the fractional NSE $(-\Delta)^\alpha$ setting for $1 \leq \alpha < \frac{5}{4}$ was done in [LT18]. Using the smoothing effect of the Stokes semigroup, Buckmaster-Colombo-Vicol [BCV18] were able to construct non-unique weak solutions whose singular sets have Hausdorff dimension less than 1. Nonuniqueness of Leray-Hopf solutions has also been obtained for ipodissipative NSE and Hall-MHD [CDLDR18, Dai18]. However, it is not clear whether a convex integration scheme could ever produce non-unique wild solutions in a class where the Leray structure theorem would hold[1], except perhaps one very specific scenario.

Finally, we mention another pathway in pursuing the possible nonuniqueness of the Leray-Hopf weak solutions aside from using convex integration. As pointed out by Jia and Šverák in [Jv14], one can also study the nonuniqueness issue via self-similar solutions for (-1)-homogeneous initial data. Indeed, in [Jv15] Jia and Šverák proved nonuniqueness of Leray-Hopf weak solutions under certain assumptions for the linearized Navier-Stokes operator. Even though a rigorous justification of those assumptions remains unavailable, very recently Guillod and Šverák provided numerical evidence indicating that the assumptions are likely to be true [Gv17].

1.2. **Motivations.** In contrast to the aforementioned results, we are focusing on the opposite direction, i.e. constructing more pathological solutions, especially solutions with anomalous energy behaviors.

The existence of a nontrivial stationary weak solution of d-dimensional NSE for $d \geq 4$ was recently proved by the second author in [Luo19], but the recaled Mikado flows used as building blocks had intermittency dimension $D = 1$, and hence could not be used for the 3D NSE. Nontrivial stationary solutions are also known to exist for the dyadic model of the NSE [BMR11], where one can precisely control the backward energy cascade to balance the energy dissipation, but the existence of such solutions has been an open question for the 3D NSE.

On the other hand, weak solutions (in the sense of Definition 1.1) are only lower semi-continuous in L^2. Therefore, it is natural to conjecture that there exist weak solutions that exhibit jumps in the energy. In fact, one can ask the following questions regarding the behavior of the energy:

Can energy $\|u(t)\|_2^2$ have jumps? Can it be discontinuous on a dense subset of $[0,T]$? Can it be discontinuous almost everywhere? Can it be discontinuous everywhere?

The answer to the last question is No. Indeed, the energy of a weak solution $\|u(t)\|_2^2$ is lower semi-continuous. Hence, by Baire's theorem, the energy is of the first Baire class and therefore the points of continuity are dense. Nevertheless, we believe that all the previous questions have positive answers. Theorem 1.3 is our first step in solving this conjecture.

1.3. **Main theorems.** We now state the main results of this paper. In particular, Theorem 1.2 and 1.3 are simpler versions of Theorem 1.4 and 1.6 accordingly.

The first theorem concerns the existence of stationary weak solutions for the 3D Navier-Stokes equations, which extends the previous work [Luo19] of the second author in dimension $d \geq 4$.

Theorem 1.4 (Finite energy stationary weak solution)**.** *Given any divergence-free $f \in C^\infty(\mathbb{T}^3)$ with zero mean, there is $M_f > 0$ such that for any $M \geq M_f$, there exists a weak solution $u \in L^2(\mathbb{T}^3)$ to (NSE) with forcing term f satisfying $\|u\|_2^2 = M$.*

[1]Note that the solutions in [CDLDR18, Dai18] do not obey the Leray structure theorem.

The next two theorems are about weak solutions with discontinuous energy profiles.

Theorem 1.5 (Energy with dense discontinuities)**.** *Let $\varepsilon, T > 0$ and $a \in C^\infty(\mathbb{T}^3 \times [0,T])$ be a smooth divergence-free vector field with zero mean for all $t \in [0,T]$. There exists a dense subset $E \subset [0,T]$ and a constant $M_a > 0$, such that for any $M \geq M_a$ there exists a weak solution $u \in C_w([0,T]; L^2(\mathbb{T}^3))$ to (NSE) so that the following holds:*

(1) The energy $\|u(t)\|_2^2$ is bounded by $2M$:

$$\|u(t)\|_2^2 \leq 2M \quad \textit{for any } t \in [0,T]\ , \tag{1.1}$$

and has jump discontinuities on set E:

$$\lim_{s\to t} \|u(s)\|_2^2 > \|u(t)\|_2^2 \quad \textit{for any } t \in E\ . \tag{1.2}$$

(2) $u(t)$ coincides with $a(t)$ at $t = 0, T$:

$$u(x,0) = a(x,0) \quad \textit{and} \quad u(x,T) = a(x,T), \tag{1.3}$$

but the energy jump is of size M:

$$\lim_{s\to 0+} \|u(s)\|_2^2 - \|u(0)\|_2^2 = \lim_{s\to T-} \|u(s)\|_2^2 - \|u(T)\|_2^2 = M. \tag{1.4}$$

(3) u is smooth on E:

$$u(t) \in C^\infty(\mathbb{T}^3) \quad \textit{for all } t \in E, \tag{1.5}$$

and uniformly ε-close to a in $W^{1,1}(\mathbb{T}^3)$:

$$\|u - a\|_{L^\infty_t W^{1,1}} < \varepsilon. \tag{1.6}$$

The set E in Theorem 1.5 is dense in $[0,T]$ and, in fact, countable. Using a gluing argument, we are also able to construct weak solutions whose energy discontinuities are dense and of positive measure.

Theorem 1.6 (Energy with dense and positive measure discontinuities)**.** *Let $\varepsilon > 0$ and $0 < \alpha \leq T$. There exist a set $E_\alpha \subset [0,T]$ with $E_\alpha = \mathcal{C}_\alpha \cup F_\alpha$ where $\mathcal{C}_\alpha$ is a fat Cantor set on $[0,T]$ such that $|[0,T] \setminus \mathcal{C}_\alpha| \leq \alpha$ and F_α is a countable dense subset of $[0,T]$, and a weak solution $u \in C_w([0,T]; L^2(\mathbb{T}^3))$ of (NSE) so that the following holds:*

(1) The energy profile $\|u(t)\|_2^2$ is discontinuous at every $t \in E_\alpha$. In fact,

$$\limsup_{s\to t} \|u(s)\|_2^2 > \|u(t)\|_2^2 \quad \textit{for all } t \in \mathcal{C}_\alpha, \tag{1.7}$$

and

$$\lim_{s\to t} \|u(s)\|_2^2 > \|u(t)\|_2^2 \quad \textit{for all } t \in F_\alpha. \tag{1.8}$$

(2) $u(t)$ is uniformly ε-small in $W^{1,1}(\mathbb{T}^3)$:

$$\|u\|_{L^\infty_t W^{1,1}} < \varepsilon, \tag{1.9}$$

smooth on F_α:

$$u(t) \in C^\infty(\mathbb{T}^3) \quad \textit{for all } t \in F_\alpha, \tag{1.10}$$

and vanishes on $\mathcal{C}_\alpha$:

$$u(t) = 0 \quad \textit{for all } t \in \mathcal{C}_\alpha. \tag{1.11}$$

1.4. **Some remarks on the results.**

Remark 1.7. *It is known that for any smooth force f (NSE) on torus $\mathbb{T}^3$ admits at least one smooth stationary solution* [CF88]*. Theorem 1.4 shows that there are infinitely many finite energy stationary weak solutions.*

Remark 1.8. *As our building blocks are compactly supported, it seems likely that there also exist finite energy stationary weak solutions in $\mathbb{R}^3$ with compact supports. We plan to address this problem in future works.*

Remark 1.9. *We note that weak solutions constructed in* [BV19, LT18, BCV18] *can not be stationary as the building blocks are time-dependent and their schemes rely on fast time oscillations.*

Remark 1.10. *The smoothness of the vector field a in Theorem 1.5 and the force f in Theorem 1.4 can definitely be lower, but we are not interested in this direction here. Also, Theorem 1.5 shows that any smooth initial data u_0 admits infinitely many weak solutions with discontinuous energy.*

Remark 1.11. *It is possible to construct a weak solution with discontinuous energy by gluing the solutions in* [BV19]*, see Appendix C. However, those discontinuities are not jumps. More importantly, such an argument can not generate dense discontinuities.*

Remark 1.12. *In view of the theory of Baire category, the set of discontinuities of a semi lower-continuous function is of Baire-1, which still can have full measure in* $[0, T]$*. At the moment, our method is not able to produce such examples.*

Remark 1.13. *Very recently, Luo and Titi* [LT18] *have extended the nonuniqueness result of* [BV19] *to factional NSE with* $(-\Delta)^\alpha$ *for any* $\alpha < \frac{5}{4}$*, which is sharp in view of Lion's wellposedness result* [Lio59, Lio69]*. Even though our method seems to work for factional NSE for some* $\alpha > 1$*, extensions to the full range of* $\alpha < \frac{5}{4}$ *are unavailable at this point.*

1.5. **Effect of intermittency.** The main technique used in the present paper is the convex integration that has been developed over the past decade for the incompressible Euler equations to tackle the famous Onsager's conjecture, see [DLS09, DLS13, DLS14, BDLIS15, BDLS16, Ise18, BLJV18], also inspired by the recent extension of this method to the Navier-Stokes equations [BV19, Luo19, BCV18].

The effect of intermittency on the regularity properties of solutions to the (NSE) and toy models has been also studied in the past decade [CF09, CS14a, CS14b]. Discontinuous weak solutions in the largest critical space and even supercritical spaces near L^2 were obtained in [CS10, CD14] using Beltrami type flows with the intermittency dimension $D = 0$. Such an extreme intermittency was achieved using Dirichlet kernels. Roughly speaking, in order for the d-dimensional Navier-Stokes equations to develop singularities, the intermittency dimension D of the flows should be less than $d - 2$, so that the Bernstein's inequality is highly saturated. So $D = 1$ is critical for the 3D NSE. It was also confirmed in [BV19, Luo19] that the main difficulty of conducting convex integration for the Navier-Stokes equations is the intermittency of the flow. Such a constraint, however, is not presented in the 3D Euler equations: Beltrami flows and Mikado flows used in the constructions of wild solutions for the 3D Euler equations are essentially homogeneous in space, namely the the intermittency dimension $D = 3$. This is also reflected in the difference between L^3 based norm in the best known energy conservation condition $L^3_t B^{\frac{1}{3}}_{3,c_0(\mathbb{N})}$ in [CCFS08] and L^∞ based norm of the counterexamples (CC^α for $\alpha < \frac{1}{3}$ in [Ise18]) for the 3D Euler equations [Ise18, BLJV18].

To resolve the issue of intermittency when applying convex integration, Buckmaster-Vicol introduced *intermittent Beltrami flows* in [BV19] and *intermittent jets* in [BCV18] as building blocks with arbitrary small intermittency dimension $D > 0$, allowing them to successfully implement convex integration scheme in the presence of the dissipative term Δu. This was done by introducing a Dirichlet type kernel to the classical Beltrami flows in [BV19] or using a space-time cutoff in [BCV18] respectively, rendering the linear term manageable. Even though such modifications produce unwanted interactions that are too large for the convex integration scheme to go through, they were handled with an additional "convex integration in time" with a help of very fast temporal oscillations. We note that even though it was possible to take advantage of all the interactions between Dirichlet kernels in [CS10, CD14], this is out of reach in the convex integration scheme at this point.

In this paper, we will design new building blocks specifically for the NSE. These vector fields, that we call *viscous eddies*, will be both stationary and compactly supported in $\mathbb{R}^3$. The construction is partly motivated by the geometric Lemma 3.1 used for the Mikado flows which were introduced in [DS17] and have been successfully used for the Euler equations on the torus $\mathbb{T}^n$ for $n \geq 3$. The Mikado flows can also be rescaled so that its intermittency dimension becomes $D = 1$ as demonstrated in [Luo19] (see also [MS18, MS19] for the setting in transport equation). This just misses the $D < 1$ requirement for the 3D NSE (see discussions in Section 2 of [Luo19] and heuristics in Section 2 of [CL20]).

In order to increase concentration that decreases the intermittency dimension, we start with a pipe flow in $\mathbb{R}^3$, use a lower order cutoff only in space along the direction of the flow, and add a correction profile to the existing one so that it will take advantage of the Laplacian to balance some of the unwanted interactions. This is possible due to the fact that the error introduced by the space cutoff along the major axis of the eddies is not a general stress term, but basically one-dimensional. By design, *viscous eddies* are divergence-free up to the leading term. Moreover, they are compactly supported approximate stationary solutions of the NSE (not the Euler equations). See Theorem 3.13 for a precise statement. Compared with the previously used building blocks for the NSE, such an approach mainly has two advantages. First, the new flows are time-independent and hence can be used to construct nontrivial stationary weak solutions, which was an open question for the 3D NSE. Second, they are compactly supported and can be used in the case of the whole space $\mathbb{R}^3$ in the future, whereas Beltrami flows, Mikado flows, intermittent Beltrami flows, and intermittent jets only exist on the torus $\mathbb{T}^d$.

1.6. **Energy pumping mechanism.** In order to produce discontinuous energy we introduce a new energy pumping mechanism that uses more energy than needed to cancel the stress error term in the convex integration scheme.

In previous works, there is a correspondence between the growth of the frequency and the decay of the energy so that the energy is not changed much along the iteration process. In other words, the high frequency part of the solution is very small uniformly in time. This is typical and desirable in order to improve the regularity of the wild solutions.

In contrast, to produce discontinuities in the energy, one can not adhere to such a uniformity in time in the scheme. We need to allow high frequencies to carry sizable energy on some time intervals, so that there is energy coming from/escaping to infinite wavenumber[2]. Consider the following toy model. Suppose $u(t)$ is a function with Fourier support in a shell of size $\lambda(t)$, and $\lambda(t) \to \infty$ as $t \to T$. Then the energy remains constant for $t < T$, but at $t = T$, the solution is zero, as all the energy has escaped to the infinite wavenumber. To reproduce this toy model in the convex integration scheme, one needs to construct an approximate sequence of solutions with temporal supports away from time T and sizable energy near T, such that the weak limit is 0 at $t = T$. Generalizing this example, one can construct a wild solution of the Navier-Stokes equations whose energy is constant on $(0, T)$ but vanishes at 0 and T.

However, if one uses solutions of such type with disjoint temporal support and glues them together, the resulting solution will only have finitely or countably many discontinuities. The next goal is to achieve the density of jumps. An exercise in real analysis shows that there exist unbounded L^2 functions that blow up on a dense subset of $[0, 1]$. Roughly speaking, we will construct solutions whose energy mimics the behavior of such functions. More precisely, there will be infinitely many blowing-up wavenumbers $\lambda(t)$ with smaller and smaller lifespan and energy. This is also consistent with the fact that the jumps decrease to zero along the iterations, which is anticipated as the energy, which we want to be bounded, needs some time to be transferred to lower/higher modes. We refer to Section 2 for more technical details in this regard.

1.7. **Organization of the paper.** The rest of the paper is organized as follows.

- In Section 2, we introduce the notations and the generalized Navier-Stokes system, for which we state the main proposition of the paper. Then using the main proposition, we prove Theorems 1.4, 1.5, and 1.6.
- In Section 3, we construct the building blocks for the convex integration, namely *viscous eddies*. We show that they are a family of approximate solutions of the stationary NSE. Several useful estimates are also derived.
- Section 4, Section 5 and Section 6 are devoted to proving the main proposition. Specifically, velocity perturbation is defined in Section 4, the new Reynolds stress is estimated in Section 5 and the energy behavior is proved in Section 6.
- In Appendix C, we show that one can use the solutions constructed by Buckmaster-Vicol to obtain discontinuities (but not jump-discontinuities) in the energy. Appendix D provides a proof of a technical tool, Proposition 4.7.

2. THE MAIN PROPOSITION

The main objective of this section is to prove Theorems 1.4, 1.5, and 1.6 using Proposition 2.1, which we will refer to as the main proposition.

2.1. **Notations.** Throughout the manuscript we use the following standard notations.

- $\|\cdot\|_p := \|\cdot\|_{L^p(\mathbb{T}^3)}$ is the Lebesgue norm (in space) for any $1 \leq p \leq \infty$ and $\|\cdot\|_{C^m} := \sum_{0 \leq i \leq m} \|\nabla^i \cdot\|_\infty$ for any m is the Hölder norm. For uniform in time bounds we will use standard notations $\|\cdot\|_{L^\infty_t L^p}$ and $\|\cdot\|_{L^\infty_t C^m}$.
- We say a function f is $\lambda^{-1}\mathbb{T}^3$-periodic if $f(x) = f(x+m)$ for any $m \in \lambda^{-1}\mathbb{Z}^3$. The space $C_0^\infty(\mathbb{T}^d)$ is the set of smooth functions with zero-mean on $\mathbb{T}^d$. $\fint_{\mathbb{T}^d} = \frac{1}{|\mathbb{T}|^d}\int_{\mathbb{T}^d}$ is the average integral any function $f \in L^1(\mathbb{T}^d)$.
- $x \lesssim y$ stands for the bound $x \leq Cy$ with some constant C which is independent of x and y but may change from line to line. Then $x \sim y$ means $x \lesssim y$ and $y \lesssim x$ at the same time. We use $x \ll y$ to indicate $x \leq cy$ for some small constant $0 < c < 1$.
- For vectors $a, b \in \mathbb{R}^d$, $a \otimes b$ is the matrix with $(a \otimes b)_{ij} = a_i b_j$. For matrix-value functions $f = f_{ij}$ and $g = g_{ij}$, $\operatorname{div} f = \partial_i f_{ij}$ and $f : g = f_{ij} g_{ij}$.
- The gradient ∇ always refers to differentiation in space only. Sometimes we use $\nabla_{t,x}$ to indicate that the differentiation is for space-time.
- Δ_q is the standard periodic Littlewood-Paley projections on to the dyadic frequency shell $2^{q-1} \leq |\xi| \leq 2^{q+1}$ for any $q \geq -1$ and $\Delta_{\leq q} = \sum_{r \leq q} \Delta_r$ and $\Delta_{\geq q} = \sum_{r \geq q} \Delta_r$.

[2] Such possible scenarios are closely related to the energy balance equation for the Navier-Stokes equations. See for instance [CL20]

2.2. **Generalized Navier-Stokes system.** Let $a, f \in C^\infty(\mathbb{T}^3 \times [0,T])$ be smooth divergence-free vector fields with zero mean for all $t \in [0,T]$. We consider the following generalized Navier-Stokes system:

$$\begin{cases} \partial_t v + L_a v + \operatorname{div}(v \otimes v) + \nabla p = f \\ \operatorname{div} v = 0, \end{cases} \tag{gNSE}$$

where

$$L_a v = -\Delta v + \operatorname{div}(v \otimes a) + \operatorname{div}(a \otimes v).$$

The reason to consider such a generalization is as follows. Suppose v is a weak solution to (gNSE) with given vector field a and $f = -\partial_t a + \Delta a - \operatorname{div}(a \otimes a)$. Then $u := v + a$ solves (NSE). We note that the added terms are of lower order compared to the nonlinearity $\operatorname{div}(v \otimes v)$, and thus will not be of any trouble in the proof.

To construct weak solutions to (gNSE), let us consider the approximate equations

$$\begin{cases} \partial_t v + L_a v + \operatorname{div}(v \otimes v) + \nabla p = \operatorname{div} R + f \\ \operatorname{div} v = 0, \end{cases} \tag{gNSR}$$

where R is a symmetric traceless matrix. If (v, p, R, f) is a solution to (gNSR), then we say (v, R) is a solution to (gNSR) with data a and f. The above system is reminiscent to the so-called Navier-Stokes-Reynolds system used in the previous works [BCV18, BV19, Luo19]. Our main proposition is to construct weak solutions to (gNSE) using a sequence of solutions (v_n, R_n) of the approximate system (gNSR) so that the stress term $R_n \to 0$ as $n \to \infty$ in a suitable sense.

2.3. **Main proposition.** In this subsection, we will introduce the main proposition of the paper, which will enable us to prove all the main theorems listed in the introduction.

Throughout the paper we use the following notations. For any $r > 0$ and any finite set $F \subset [0,T]$, let

$$\begin{aligned} B_r(F) &= \{t \in [0,T] : \operatorname{dist}(t, F) < r\}, \\ I_r(F) &= [0,T] \setminus B_r(F). \end{aligned} \tag{2.1}$$

Proposition 2.1. *Let $c_0 = 10^{-2}$, $T > 0$.*[3] *Consider the system (gNSR) with given $a, f \in C^\infty(\mathbb{T}^3 \times [0,T])$ smooth vector fields with zero mean. There exists a small universal constant C such that the following holds.*

Let $\varepsilon, r > 0$, $0 < e_0 < e_1 < \infty$, and $\mathcal{F}_0, \mathcal{F}_1 \subset [0,T]$ be two finite sets such that $\mathcal{F}_0 \subset \mathcal{F}_1$. If (v_0, R_0) is a smooth solution to (gNSR) on $[0,T]$ with data a and f so that

(1) the energy $\|v_0(t)\|_2^2 \le e_0$ for all t, and is almost constant e_0 away from the set $\mathcal{F}_0$:

$$\big| \|v_0(t)\|_2^2 - e_0 \big| \le c_0(e_1 - e_0) \quad \text{for all } t \in I_r(\mathcal{F}_0)\ ,$$

(2) (v_0, R_0) is close to a solution of (gNSE) in the sense that

$$\delta_0 \le C(e_1 - e_0),$$

where $\delta_0 = \|R_0\|_{L^\infty_t L^1_x(\mathbb{T}^3 \times [0,T])}$,

then there is another smooth solution (v, R) to (gNSE) with data a and f such that

(1) The energy $\|v(t)\|_2^2 \le e_1$ for all t, and is almost constant e_1 away from the set $\mathcal{F}_1$:

$$\big| \|v(t)\|_2^2 - e_1 \big| \le \frac{c_0}{2}(e_1 - e_0) \quad \text{for all } t \in I_{4^{-1}r}(\mathcal{F}_1)\ .$$

(2) The new stress R verifies

$$\|R(t)\|_1 \le \begin{cases} \varepsilon & \text{for } t \in I_{4^{-1}r}(\mathcal{F}_1) \\ \delta_0 + \varepsilon & \text{for } t \in I_{4^{-2}r}(\mathcal{F}_1) \setminus I_{4^{-1}r}(\mathcal{F}_1) \\ \delta_0 & \text{for } t \in [0,T] \setminus I_{4^{-2}r}(\mathcal{F}_1). \end{cases} \tag{2.2}$$

.

Moreover, the velocity increment $w = v - v_0$ verifies

$$\operatorname{supp}_t w \subset I_{4^{-2}r}(\mathcal{F}_1) \quad \text{and} \quad \|w\|_{L^\infty_t W^{1,1}} \le \varepsilon, \tag{2.3}$$

and if $\mathcal{F}_0 = \mathcal{F}_1 = \emptyset$ and v_0 is stationary[4]*, i.e. $\partial_t v_0 = 0$, then w is also stationary: $\partial_t w = 0$.*

[3] Since we only use c_0 to measure the approximate level of the energy to a constant, the exact value of c_0 is not important.

[4] In this case, we of course require both a and f to be time-independent.

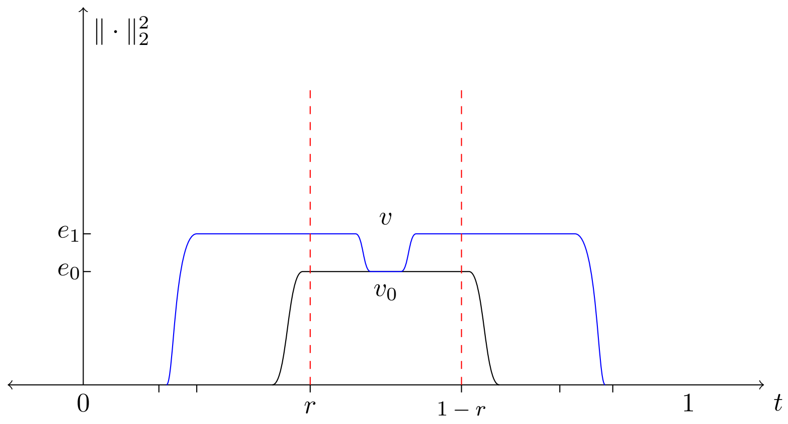

FIGURE 1. Construction of $v = v_0 + w$ in Proposition 2.1.

2.4. **Proof of main theorems.** We first prove Theorem 1.5, it suffices to prove the following result for (gNSE):

Theorem 2.2. *Let $\varepsilon > 0$ and $a \in C^\infty(\mathbb{T}^3 \times [0,T])$, $T > 0$ be a smooth divergence-free function with zero mean for all $t \in [0,T]$. Consider the associated generalized Navier-Stokes system (gNSE) with data a and $f = -\partial_t a + \Delta a - \operatorname{div}(a \otimes a)$. There exists a dense subset $E \subset [0,T]$, a constant $M_a > 0$ such that for any $M \geq M_a$ there exists weak solution $v \in C_w(0,T; L^2(\mathbb{T}^3))$ (NSE) so that the followings hold:*

(1) The energy $\|v(t)\|_2^2$ is bounded by M:
$$\|v(t)\|_2^2 \leq M \quad \textit{for any } t \in [0,T]\,, \tag{2.4}$$
and has jump discontinuities on set E:
$$\lim_{s\to t} \|v(s)\|_2^2 > \|v(t)\|_2^2 \quad \textit{for any } t \in E\,. \tag{2.5}$$
(2) $v(t)$ vanishes at $t = 0, T$:
$$v(x,0) = v(x,T) = 0, \tag{2.6}$$
but the energy jump is of size M:
$$\lim_{s\to 0+} \|v(s)\|_2^2 - \|v(0)\|_2^2 = \lim_{s\to T-} \|v(s)\|_2^2 - \|v(T)\|_2^2 = M. \tag{2.7}$$
(3) $v(x,t)$ is smooth on E:
$$v(t) \in C^\infty(\mathbb{T}^3) \quad \textit{for all } t \in E, \tag{2.8}$$
and is ε-small in $L_t^\infty W_x^{1,1}$:
$$\|v\|_{L_t^\infty W^{,1}} < \varepsilon. \tag{2.9}$$

The implication from Theorem 2.2 to Theorem 1.5 can be obtained simply by shifting $u = v + a$ since the vector field a is smooth. Now we prove Theorem 2.2 with the help of Proposition 2.1.

Proof of Theorem 2.2 assuming Proposition 2.1. We first construct the set E, then a sequence of approximate solution v_n such that v_n converges to the desire solution v in a suitable sense. Without loss of generality, we assume $T = 1$.

Step 1: Constructing the set E. Consider the binary representation of $x \in [0,1]$:
$$x = \sum_{j=0}^{\infty} x_j 2^{-j}.$$
Now let F_n be the collection of all real numbers in $[0,1]$ whose binary representation has at most n digits, namely $x \in F_n \subset [0,1]$ if and only if $x_j = 0$ for all $j > n$. Assuming $F_{-1} = \emptyset$, let also $E_n = F_{n+1} \setminus F_n$, $n \geq -1$. For instance, $E_{-1} = \{0,1\}$, $E_0 = \{1/2\}$, $E_1 = \{1/4, 3/4\}$. Let
$$E = \lim_{n\to\infty} F_n = \bigcup_{n \geq -1} E_n,$$
which is a dense subset of $[0,1]$.

Denoting $r_n = 4^{-n-1}$, let us show the following important property of the set E for later use:

$$\liminf_{n\to\infty} B_{r_n}(F_{n-1}) \subset E. \tag{2.10}$$

Suppose $t \in \liminf B_{r_n}(F_{n-1})$, which means that there exist N and $t_n \in F_{n-1}$ for every $n \geq N$, such that

$$|t - t_n| = \operatorname{dist}(t, F_{n-1}) < r_n. \tag{2.11}$$

We claim that $t_{n+1} = t_n$ for all $n \geq N$. Otherwise, for some $n \geq N$ there must be

$$|t - t_n| \geq |t_{n+1} - t_n| - |t - t_{n+1}| \geq 2^{-n} - r_{n+1} \geq 2^{-n-1},$$

which contradicts (2.11):

$$2^{-n-1} < r_n = 2^{-2n-2}.$$

Hence, it follows from (2.11) that $t = t_N \in F_{N-1}$ which implies that $t \in E$.

Step 2: Constructing approximate solutions v_n. Given smooth vector field a, we set $v_0 = 0$ and $R_0 = \mathcal{R}\big(\partial_t a - \Delta a + \operatorname{div}(a \otimes a)\big)$, where $\mathcal{R}$ is defined in Definition 5.1. Then (v_0, R_0) is a smooth solution of (gNSR) with data a and $f = -\partial_t a + \Delta a - \operatorname{div}(a \otimes a)$ on $[0,1]$. We choose

$$M_a = \frac{4}{C}\|R_0\|_{L^\infty_t L^1}, \tag{2.12}$$

where C is the constant in Proposition 2.1.

Let $r_n = 4^{-n-1}$ and $M \geq M_a$ and choose the energy level $e_n = (1 - 2^{-n})M$ for $n \in \mathbb{N}$. Note that the choice of e_n is admissible in view of (2.12).

Starting with (v_0, R_0), we apply Proposition 2.1 with data a and f on $[0,1]$ to obtain a sequence (v_n, R_n) of smooth solutions of (gNSR). More precisely, (v_{n+1}, R_{n+1}) is obtained by applying Proposition 2.1 to the previous solution (v_n, R_n) with parameters

$$(r, e_0, e_1, \varepsilon, \mathcal{F}_0, \mathcal{F}_1) := (r_n, e_n, e_{n+1}, \varepsilon_n, F_{n-1}, F_n),$$

where the small parameters ε_n are defined inductively by

$$\varepsilon_n = \frac{2^{-n-1}\varepsilon}{1 + \sum_{j\leq n-1} \sup_t \|w_j\|_\infty}, \tag{2.13}$$

and $w_j := v_j - v_{j-1}$ is the j-th velocity perturbation for $j \geq 1$.

Clearly, each (v_n, R_n) in the obtained sequence is a smooth solution of (gNSR) on $[0,1]$ with data a and $f = -\partial_t a + \Delta a - \operatorname{div}(a \otimes a)$, and by Proposition 2.1 we have the following properties:

(1) For any $n \in \mathbb{N}$

$$\begin{aligned} \Big| \|v_n(t)\|_2^2 - e_n \Big| &\leq c_0 2^{-n} M \\ \|R_n(t)\|_1 &\leq \varepsilon_n \end{aligned} \qquad \text{for all } t \in I_{r_n}(F_{n-1})\,, \tag{2.14}$$

and

$$\begin{aligned} \|v_n(t)\|_2^2 &\leq e_n \leq M, \\ \|R_n(t)\|_1 &\leq \|R_0\|_{L^\infty_t L^1} + \varepsilon. \end{aligned} \qquad \text{for all } t \in [0,1]. \tag{2.15}$$

(2) The velocity increment $w_n = v_n - v_{n-1}$ verifies that

$$\big\|w_n\big\|_{L^\infty_t W^{1,1}} \leq \varepsilon_n. \tag{2.16}$$

(3) If $t \in F_n$ for some $n \in \mathbb{N}$, then

$$v_k(t) = v_n(t) \quad \text{for all } k \geq n. \tag{2.17}$$

Step 3: L^2 convergence of v_n. The solution $v(t)$ is constructed as a strong L^2 limit of approximate smooth solutions $v_n(t)$,

$$v(t) = \lim_{n\to\infty} v_n(t) = \sum_{j=1}^\infty w_j, \qquad t \in [0,1].$$

We first prove that v is well-defined, i.e. v_n converges pointwise in L^2. Indeed, thanks to (2.13) and (2.16) the velocity perturbations w_k are almost orthogonal in L^2:

$$\sup_t |\langle w_j, w_k\rangle| \leq 2^{-j-1}\varepsilon \quad \text{for all } j > k. \tag{2.18}$$

As a result, due to (2.15)

$$\sum_{j=1}^{n}\|w_j\|_2^2 \le \|v_n\|_2^2 + 2\sum_{1\le j<k\le n}|\langle w_j, w_k\rangle| < M + 2\varepsilon \quad \text{for all } n.$$

So, for $0 \le n < m$ we have

$$\begin{aligned}\|v_m - v_n\|_2^2 &= \sum_{n<j\le m}\|w_j\|_2^2 + 2\sum_{n<j<k\le m}|\langle w_j, w_k\rangle| \\ &< \sum_{j>n}\|w_j\|_2^2 + 2^{-n+1}\varepsilon \to 0 \quad \text{as } n, m \to \infty,\end{aligned}$$

i.e., $v_n(t)$ is Cauchy in L^2 for every $t \in [0,1]$.

Next, we show that v is a weak solution of (gNSE). Let test function $\varphi \in C_c^\infty(\mathbb{T}^3 \times [0,1))$ be mean-free and divergence-free for all $t \in [0,1]$. Using the weak formulation for the solution $(v_n, R_n))$ of (gNSR) with data a and $f = -\partial_t a + \Delta a - \operatorname{div}(a \otimes a)$, we get

$$\begin{aligned}\int_{\mathbb{T}^3} v_n(\cdot,0)\cdot\varphi(\cdot,0) + &\int_{\mathbb{T}^3\times[0,1]} v_n\cdot\partial_t\varphi + v_n\cdot(v_n\cdot\nabla)\varphi + v_n\cdot\Delta\varphi \\ &+ \int_{\mathbb{T}^3\times[0,1]} a\cdot(v_n\cdot\nabla)\varphi + v_n\cdot(a\cdot\nabla)\varphi = \int_{\mathbb{T}^3\times[0,1]} R_n : \nabla\varphi + f\cdot\varphi.\end{aligned} \tag{2.19}$$

For simplicity of notation, let

$$I_n = \bigcap_{k\ge n} I_{r_k}(F_{k-1}).$$

Immediately

$$\big|[0,1]\setminus I_n\big| \lesssim 2^{-n}. \tag{2.20}$$

From (2.14) and (2.18) it follows that

$$\|v - v_n\|^2_{L^\infty L^2(\mathbb{T}^3\times I_n)} \le \sup_{I_n}\left(\|v(t)\|_2^2 - \|v_n(t)\|_2^2 - 2\langle v - v_n, v_n\rangle\right) \lesssim 2^{-n}, \tag{2.21}$$

and

$$\|R_n\|_{L_t^\infty L^1(\mathbb{T}^3\times I_n)} \lesssim 2^{-n}. \tag{2.22}$$

Using the bounds (2.20), (2.21), and (2.22) together with (2.15), it is easy to check the convergence of all the terms in (2.19) to their natural limits by splitting the domain of integrals into $\mathbb{T}^3 \times I_n$ and $\mathbb{T}^3 \times I_n^c$.

Next, let us show that as the pointwise L^2 limit of v_n, the solution v is weakly continuous. Let $\varphi \in L^2(\mathbb{T}^3)$ and $t_0 \in [0,1]$. Consider the following split:

$$\big|\langle v(t) - v(t_0), \varphi\rangle\big| \le \big|\langle v(t) - v_n(t), \varphi\rangle\big| + \big|\langle v_n(t) - v_n(t_0), \varphi\rangle\big| + \big|\langle v_n(t_0) - v(t_0), \varphi\rangle\big|.$$

The first and last terms go to zero as $n \to \infty$ by the uniform $W^{1,1}$ convergence of v_n. For the second term, since $v_n \in C_0^\infty(\mathbb{T}^3 \times [0,1])$, we get

$$\big|\langle v_n(t) - v_n(t_0), \varphi\rangle\big| \to 0 \quad \text{as } t \to t_0.$$

So we may conclude that $\langle v(t) - v(t_0), \varphi\rangle \to 0$ as $t \to t_0$.

Step 4: Verifying properties of v. Finally, we show that v is a weak solution satisfying all the properties (1), (2) and (3) stated in Theorem 2.2. First, $\|v(t)\|_2^2 \le M$ for all $t \in [0,1]$ due to (2.15). Therefore, to show (1), it remains to prove that E consists of jump discontinuities.

Indeed, given $t \in E$, there exists n such that $t \in E_n$, which implies $t \in I_{r_{n+1}}(F_n)$ and $v(t) = v_{n+1}(t)$. Using (2.14) we get

$$\begin{aligned}M - \|v(t)\|_2^2 &\ge M - e_{n+1} - c_0 M 2^{-n-1} \\ &\gtrsim M 2^{-n}.\end{aligned}$$

We will show that $\lim\limits_{s\to t}\|v(s)\|_2^2 = M$. To this end, let

$$I_\varepsilon = \{s \in [0,1] : t - \varepsilon < s < t \text{ or } t < s < t + \varepsilon\},$$

and

$$N_\varepsilon = \max\{j \in \mathbb{N} : I_\varepsilon \cap F_j = \emptyset\}.$$

By definitions of the sets F_n we have $N_\varepsilon > n$ provided $\varepsilon \leq 2^{-n-1}$, which implies that $\lim_{\varepsilon\to 0+} N_\varepsilon = \infty$. Moreover, from (2.10) it follows that

$$E^c = [0,1] \setminus E \subset \limsup I_{r_j}(F_{j-1}),$$

which by (2.14) and the pointwise L^2 convergence of v_n implies that

$$\|v(s)\|_2^2 = M \qquad \text{for all } s \in E^c.$$

Thus we only need to consider $s \in I_\epsilon \cap E$. In this case $s \notin F_{N_\varepsilon}$, however, $s \in E_m$ for some $m \geq N_\varepsilon$ and $v(s) = v_{m+1}(s)$. Then $s \in I_{r_{m+1}}(F_m)$, and therefore, (2.14) implies that

$$\big|\|v(s)\|_2^2 - M\big| \lesssim 2^{-N_\epsilon}.$$

Taking a limit $\epsilon \to 0$ we obtain $\lim_{s\to t} \|v(s)\|_2^2 = M$. Thus statement (1) is proved. As a special case of the jump discontinuities, statement (2) follows as well.

The smoothness of v on the set E and the uniform smallness of v in $W^{1,1}$ follow directly from (2.17) and (2.16) respectively. So, statement (3) has been obtained as well.

$\square$

Next, we use a gluing technique to glue pieces of weak solutions given by Theorem 1.5 to obtain Theorem 1.6.

Proof of Theorem 1.6. It is clear that Theorem 1.5 works for any interval $[t_0, t_1]$. Also, the energy level M_a depends only on the vector field a and M_a can be any positive number when $a = 0$. Without loss of generality, we assume $T = 1$.

Step 1: Constructing approximate sequence u_n. Let $\mathcal{C}_\alpha$ be a fat Cantor set on $[0,1]$ with measure $(1-\alpha)$ (each time remove the middle interval of length $(\frac{\alpha}{1+2\alpha})^n$). In other words,

$$\mathcal{C}_\alpha = [0,1] \setminus \bigcup_{n\geq 1} \bigcup_{1\leq j\leq 2^{n-1}} I^\alpha_{j,n},$$

where $I^\alpha_{j,n}$ are the open intervals removed from the fat Cantor set $\mathcal{C}_\alpha$ at step n.

Let us first construct a sequence of weak solutions of (NSE) that are supported on $\overline{I^\alpha_{j,n}}$. Applying Theorem 1.5 on each interval $I^\alpha_{j,n}$ with $(\varepsilon, a, M_a) := (\varepsilon 4^{-n}, 0, 1)$, we obtain a weak solution $u_{j,n}$, which we then extend trivially to the whole interval $[0,1]$. The resulting sequence of weak solutions $u_{j,n}$ satisfy

(1) $u_{j,n}$ is supported on $\overline{I^\alpha_{j,n}}$. Moreover,

$$u_{j,n}(t) = 0, \quad \text{for } t \notin I^\alpha_{j,n}.$$

(2) $u_{j,n}$ is small in $W^{1,1}$:

$$\|u_{j,n}\|_{L^\infty W^{1,1}} \leq \varepsilon 4^{-n}. \tag{2.23}$$

(3) $\|u_{j,n}\|_2^2$ is discontinuous on a dense subset $F^\alpha_{j,n} \subset \overline{I^\alpha_{j,n}}$.

Since $\overline{I^\alpha_{j,n}} \cap \overline{I^\alpha_{j',n'}} = \emptyset$ if $j \neq j'$ or $n \neq n'$, namely $u_{j,n}$ have disjoint temporal supports, we can construct another sequence of weak solutions of (NSE) by defining

$$u_n = \sum_{1\leq k\leq n} \sum_{1\leq j\leq 2^{n-1}} u_{j,k}.$$

As both summations are finite, u_n are weakly continuous in L^2 and are indeed weak solutions on $\mathbb{T}^3 \times [0,1]$.

Step 2: Convergence and weak continuity of u_n. We claim that $u_n(t)$ pointwise converges in L^2 and define

$$u(t) = \lim_{n\to\infty} u_n(t), \qquad t \in [0,1].$$

To prove this claim, consider two sub-cases.

(a) If $t \in \mathcal{C}_\alpha$, then $u_n(t) = \sum_{k\leq n}\sum_j u_{j,k}(t) = 0$ for all n. So, in particular, $u_n(t) \to 0$ in L^2.

(b) If $t \in [0,T] \setminus \mathcal{C}_\alpha$, then there exist $j, n \in \mathbb{N}$ such that $t \in I^\alpha_{j,n}$. Thus $u_m(t) = u_n(t)$ for any $m \geq n$, and consequently $u(t) = u_n(t)$.

Combining this with (2.23), it is also clear that statement (2) holds.

Next, we show that $u \in C_w([0,1]; L^2)$, i.e., $u(t)$ is weakly continuous. Let $\varphi \in L^2(\mathbb{T}^3)$ and $t_0 \in [0,1]$. As usual, we consider the split

$$\big|\langle u(t) - u(t_0), \varphi\rangle\big| \leq \big|\langle u(t) - u_n(t), \varphi\rangle\big| + \big|\langle u_n(t) - u_n(t_0), \varphi\rangle\big| + \big|\langle u_n(t_0) - u(t_0), \varphi\rangle\big|. \tag{2.24}$$

Thanks to (2.23), for any $t \in [0,1]$ we have

$$\big|\langle u(t) - u_n(t), \varphi\rangle\big| \leq \big\|u - u_n\big\|_{L^\infty W^{1,1}}\|\varphi\|_\infty \leq \|\varphi\|_\infty \sum_{k>n}\sum_{1\leq j\leq 2^{n-1}} \big\|u_{j,k}\big\|_{L^\infty W^{1,1}} \leq \varepsilon 2^{-n}\|\varphi\|_\infty.$$

So the first and the last terms in (2.24) go to zero as $n \to \infty$, which together with the weak continuity of u_n implies the weak continuity of u in L^2.

Finally, we show that u is a weak solution of (NSE). Let test function $\varphi \in C_c^\infty(\mathbb{T}^3 \times [0,1))$ be mean-free and divergence-free for all $t \in [0,1]$. By the weak formulation of (NSE) for u_n we get

$$\int_{\mathbb{T}^3} u_n(x,0)\cdot\varphi(x,0)\,dx + \int_0^1\int_{\mathbb{T}^3} u_n\cdot\partial_t\varphi + u_n\cdot(u_n\cdot\nabla)\varphi + u_n\cdot\Delta\varphi\,dx d\tau = 0. \tag{2.25}$$

Since $u_n(0) = u(0) = 0$, the first term is zero. For the rest of the terms it suffices to show that

$$u_n \to u \quad \text{in } L^2_{t,x} \quad \text{as } n \to \infty.$$

Consider a remainder set

$$I_n = \bigcup_{m>n}\bigcup_{1\leq j\leq 2^{n-1}} I^\alpha_{j,m}.$$

Since $\operatorname{supp}_t u_{j,m} \subset I^\alpha_{j,m}$ we know that

$$u(t) = u_n(t) \quad \text{for all } t \in [0,1]\setminus I_n.$$

Moreover, the set I_n is small by direct computation:

$$|I_n| \lesssim \Big(\frac{2\alpha}{1+2\alpha}\Big)^n.$$

Thanks to the above, we have

$$\big\|u_n - u\big\|_{L^2_{t,x}(\mathbb{T}^3\times[0,1])} = \big\|u_n - u\big\|_{L^2_{t,x}(\mathbb{T}^3\times I_n)} \leq \big\|u_n - u\big\|_{L^\infty_t L^2_x}\big|I_n\big|^{\frac12} \to 0$$

as $n \to \infty$. So, we have proved that $u \in C_w(0,1;L^2)$ is a weak solution of (NSE) satisfying statement (2).

Step 3: Discontinuities of $\|u\|_2^2$ on E_α. We first define the countable set F_α:

$$F_\alpha = \bigcup_{j,m} F^\alpha_{j,m}$$

where recall that $F^\alpha_{j,m}$ is the set of jump discontinuities of $\|u_{j,m}\|_2^2$. From the definition of $F^\alpha_{j,m}$ it follows that $F_\alpha \cap \mathcal{C}_\alpha = \emptyset$. Moreover, it is clear that F_α is a dense subset of $[0,1]$.

Let us show the discontinuity on $E_\alpha = \mathcal{C}_\alpha \cup F_\alpha$. Suppose $t_0 \in F_\alpha$, then $t_0 \in I^\alpha_{j,m}$ for some j, m. Moreover, this implies that

$$u(s) = u_{j,m}(s) \quad \text{for all } s \in I^\alpha_{j,m}.$$

Since $u_{j,m}$ is a weak solution given by Theorem 1.5, $\|u\|_2^2$ is discontinuous at t_0:

$$\lim_{s\to t_0}\|u(s)\|_2^2 > \|u(t_0)\|_2^2\,. \tag{2.26}$$

Next, suppose $t_0 \in \mathcal{C}_\alpha$, then $\|u(t_0)\|_2^2 = 0$. Let t_k be a sequence such that $t_k \to t_0$ as $k \to \infty$ and each t_k is the endpoint of $I^\alpha_{j,k}$ for some $j = j(k)$. Then from Theorem 1.5 we get

$$\limsup_{s\to t_k}\|u(s)\|_2^2 \geq \limsup_{s\to t_k}\|u_k(s)\|_2^2 = 1.$$

So, for any $t_0 \in \mathcal{C}_\alpha$ we have

$$\limsup_{s\to t_0}\|u(s)\|_2^2 > \|u(t_0)\|_2^2.$$

Statement (1) is now proved.

□

We finish this section by proving Theorem 1.4.

Proof of Theorem 1.4 assuming Proposition 2.1. Given any smooth force term f, let $v_0 = 0$ and $R_0 = -\mathcal{R}f$. So (v_0, R_0) solves (gNSR) with data $a = 0$ and f. Then define

$$M_f = \frac{4}{C}\|R_0\|_{L^1}.$$

For any $M \geq M_f$ we can construct the solution as follows. Let the energy level $e_n = (1-2^{-n})M$ for $n \in \mathbb{N}$. Again, the choice of e_n is admissible due to $M \geq M_f$.

Starting with (v_0, R_0), we apply Proposition 2.1 to (v_n, R_n) with the same parameters as in the proof of Theorem 2.2:

$$(r, e_0, e_1, \varepsilon, \mathcal{F}_0, \mathcal{F}_1) = (4^{-n-1}, e_n, e_{n+1}, \varepsilon_n, \emptyset, \emptyset),$$

where ε_n is the same as (2.13). It should be noted that the value of r does not matter here as all v_n are stationary and $\mathcal{F}_0 = \mathcal{F}_1 = \emptyset$. Clearly, (v_n, R_n) are smooth solutions of (gNSR) with data $a = 0$ and f such that

$$\Big| \|v_n\|_2^2 - e_n \Big| \leq c_0 M 2^{-n-1},$$
$$\|R_n\|_1 \leq 2^{-n-1}\varepsilon.$$

Using the same argument as in the proof of Theorem 2.2, one can show that v_n converges to a stationary weak solution $v \in L^2$ of (gNSE) with data $a = 0$ and f such that $\|v\|_2^2 = M$. So v is a stationary weak solution of (NSE) with forcing term f. □

3. Stationary viscous eddies

In this section, the building blocks of the solution sequence are constructed. The entire construction is done in the whole space $\mathbb{R}^3$ not on torus $\mathbb{T}^3$. Recall the standard stationary Mikado flows can be rescaled so that the intermittency dimension $D = 1$ [Luo19], which is insufficiently intermittent to be the building blocks for the 3D Navier-Stokes equations. Being also stationary, our *viscous eddies* are in the intermittency regime $D < 1$, but the full range $0 < D < 1$ is unattainable.

There are two main major differences between our new building blocks and previous ones used for the NSE, *intermittent jets* in [BV19]. First, existing building blocks for the NSE are exact or approximate solutions of the Euler equations. As a result, the linear term is purely a useless error in those convex integration schemes. In contrast, *viscous eddies* are a family of approximate stationary solutions to the NSE, not Euler equations, see Theorem 3.13. The Laplacian is essential as it balances the leading term in the equations. Second, *viscous eddies* are time-independent, which enables us to obtain stationary weak solutions with time-independent (or zero) external force. In other words, our scheme does not require time oscillations, which might be of interest in improving the temporal regularity of wild solutions.

3.1. A geometric lemma.

We start with a geometric lemma that dates back to the work of Nash [Nas54]. A proof of the following version, which is essentially due to De Lellis and Székelyhidi Jr., can be found in [Sze13, Lemma 3.3]. This lemma allows us to reconstruct any stress tensor R in a compact subset of $\mathcal{S}^{3\times 3}_+$, the set of positive definite symmetric 3×3 matrices.

Lemma 3.1. *For any compact subset $\mathcal{N} \subset \mathcal{S}^{3\times 3}_+$, there exists $\lambda_0 \geq 1$ and smooth functions $\Gamma_k \in C^\infty(\mathcal{N}; [0,1])$ for any $k \in \mathbb{Z}^3$ with $|k| \leq \lambda_0$ such that*

$$R = \sum_{k \in \mathbb{Z}^3, |k| \leq \lambda_0} \Gamma_k^2(R) \frac{k}{|k|} \otimes \frac{k}{|k|} \qquad \textit{for all } R \in \mathcal{N}.$$

Lemma 3.1 is one of the reasons we choose to construct *viscous eddies*, which will be nonisotropic, closed to pipe flows, and divergence-free up to the leading order terms.

Fix a compact subset $\mathcal{N} \subset \mathcal{S}^{3\times 3}_+$ and let $\mathbb{K} \subset \mathbb{R}^3$ be the finite set of vectors given by Lemma 3.1[5], the directions of the major axis of *viscous eddies*. We can then choose a collection of points $p_k \in [0,1]^3$ for $k \in \mathbb{K}$ and a number $\mu_0 > 0$ such that

$$\bigcup_k B_{\mu_0^{-1}}(p_k) \subset [0,1]^3,$$

and

$$B_{2\mu_0^{-1}}(p_k) \cap B_{2\mu_0^{-1}}(p_{k'}) = \emptyset \quad \text{if } k \neq k'.$$

These points p_k will be the centers of our eddies and the balls $B_{\mu_0^{-1}}(p_k)$ will contain the supports of the eddies. Let

$$l_k := \{p_k + tk : t \in \mathbb{R}\} \subset \mathbb{R}^3$$

be the line passing through the point p_k in the k direction.

[5]For applications in this paper, the set $\mathcal{N} \subset \mathcal{S}^{3\times 3}_+$ is fixed. See Section 4.5.

3.2. **Velocity profiles.** Let $\psi \in C_c^\infty(\mathbb{R}^+)$ be a smooth non-negative non-increasing function so that $\operatorname{supp}\psi \subset [0,1]$. Then let

$$\phi(r) := -\frac{1}{r}\int_r^\infty \psi(s)s\,ds. \tag{3.1}$$

Note that $\phi \in C^\infty((0,\infty))$, $\phi(r) = 0$ for $r > 1$, and ϕ has a singularity r^{-1} near the origin due to the monotonicity of ψ.

At this time we also assume

$$\int_0^\infty (\psi^2 - \phi\psi')r\,dr = 0, \tag{3.2}$$

which will be verified in the next lemma.

Lemma 3.2. *There exits a smooth non-negative non-increasing $\psi \in C_c^\infty([0,1])$ such that (3.2) holds and $\psi' = 0$ in a neighborhood of 0.*

Proof. Integrating by parts we obtain

$$\begin{aligned}\int_0^\infty (\psi^2 - \phi\psi')\,r dr &= \int_0^\infty r\psi^2 dr + \int_0^\infty \int_r^\infty \psi(s)s\,ds\,\psi'(r)\,dr\\ &= 2\int_0^\infty r\psi^2 dr - \psi(0)\int_0^\infty r\psi\,dr.\end{aligned}$$

We first fix a non-negative non-increasing $\psi \in C_c^\infty([0,1])$ such that

$$\psi(r) = 1 \quad \text{for all } r \in [0,1/2] \quad \text{and} \quad 2\int_0^\infty r\psi^2 dr - \int_0^\infty r\psi dr > 0.$$

Note that the existence of such functions can be seen by taking mollification on the characteristic function $\chi_{[0,1]}$

Let us consider $\psi_a = \psi + a\psi(ar)$, $a \geq 1$ to be determined, for which we need the solve the equation

$$F(a) := 2\int_0^\infty r\psi_a^2\,dr - \psi_a(0)\int_0^\infty r\psi_a\,dr = 0.$$

It is clear that once a solution $F(a) = 0$ is found, the lemma is proven.

A direct computation yields that

$$F(a) = 4\left(\int_0^\infty r\psi^2\,dr + \int_0^\infty ra\psi(r)\psi(ar)\,dr\right) - (1+a)\left(\int_0^\infty r\psi\,dr + \int_0^\infty ra\psi(ar)\,dr\right). \tag{3.3}$$

In particular, our assumption on ψ implies

$$F(1) = 8\int_0^\infty r\psi^2\,dr - 4\int_0^\infty r\psi\,dr > 0.$$

As $a \to \infty$ we notice in (3.3) that

$$\int_0^\infty ra\psi(r)\psi(ar)dr \leq \int_0^\infty ra\psi(ar)dr = a^{-1}\int_0^\infty r\psi dr \to 0,$$

and thus there exist some $c_0, c_1 > 0$ depending of ψ such that

$$F(a) \leq c_0 - c_1(1+a) \quad \text{for all sufficiently large } a,$$

which implies that there exits $1 < a < \infty$ such that $F(a) = 0$. □

Throughout this section we will work in cylindrical coordinates to simply notations. Let

$$z_k = (x - p_k)\cdot\frac{k}{|k|}, \tag{3.4}$$

$$r_k = \operatorname{dist}(x, l_k) \tag{3.5}$$

be the cylindrical coordinates with respect to the basis$\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ centered at p_k, with $\mathbf{e}_z = \frac{k}{|k|}$.

It would also be convienet to introduce the following decomposition

$$\mathbb{R}^3 = \Omega_k \oplus l_k, \tag{3.6}$$

where $\Omega_k = \{x \in \mathbb{R}^3 : x\cdot k = 0\}$ is the plane orthogonal to l_k.

Finally, let us fix a smooth nontrivial function $\eta \in C_c^\infty(\mathbb{R})$ such that $\int \eta = 0$ and $\eta = 0$ for $|x| \geq 1$.

Definition 3.3 (Principle profiles ψ_k and η_k)**.** *For $k \in \mathbb{K}$ and $\mu \geq \tau \geq \mu_0$ let $\eta_k, \psi_k \in C^\infty(\mathbb{R}^3)$ and $\phi_k \in C^\infty(\mathbb{R}^3 \setminus l_k)$ be defined by*

$$\begin{aligned} \eta_k &= c\tau^{1/2}\eta(\tau z_k), \\ \psi_k &= \mu\psi(\mu r_k), \\ \phi_k &= \phi(\mu r_k), \end{aligned} \tag{3.7}$$

where c is a normalizing constant such that $\int_{\mathbb{R}^3} |\eta_k \psi_k|^2 dx = 1$.

Remark 3.4. *Note that η_k and ψ_k are smooth and compactly supported in Ω_k, but not ϕ_k which still has a compact support in Ω_k but also a singularity $1/r$ at the origin. We can use a mollification to smear out the singularity thanks to Proposition 3.9.*

Using cylindrical coordinates we can easily prove the following simple lemma regarding the profiles η_k and ψ_k.

Lemma 3.5. *For any $k \in \mathbb{K}$, the rescaled functions ψ_k and ϕ_k verify the identities*

$$\frac{\partial(r_k\phi_k)}{\partial r_k} = r_k\psi_k \quad \textit{and} \quad \int_0^\infty \left(\psi_k^2 - \phi_k \frac{\partial \psi_k}{\partial r_k}\right) r_k \, dr_k = 0. \tag{3.8}$$

For any $1 \leq p \leq \infty$, there hold

$$\begin{aligned} \|\eta_k\|_{L^p(l_k)} &\lesssim \tau^{1/2 - 1/p}, \\ \|\psi_k\|_{L^p(\Omega_k)} &\lesssim \mu^{1-2/p}, \end{aligned} \tag{3.9}$$

and

$$\|\phi_k\|_{L^p(\Omega_k)} \lesssim_p \mu^{-2/p} \quad \textit{if } 1 \leq p < 2. \tag{3.10}$$

Proof. The first two identities (3.8) follow from the rescalings (3.7), (3.1) as well as the zero-mean condition (3.2).

The first two estimates (3.9) follow from rescaling and the the fact that $\eta, \psi \in C_c^\infty(\mathbb{R}^+)$ while (3.10) follows from rescaling and the fact that $\phi \in L^p(r dr)$ for any $1 \leq p < 2$.

□

Next, we introduce another family of profiles that will be used to form the Laplacian corrector part of the eddies.

Thanks to the zero-mean condition (3.8) and the vanishing of ψ' near the origin obtained in Lemma 3.2, Lemma B.1 implies that there exists $h \in C^\infty(\mathbb{R}^+)$, such that $h(|\cdot|) \in C^\infty(\mathbb{R}^2) \cap W^{1,p}(\mathbb{R}^2)$ for $1 < p \leq \infty$, and

$$\Delta h(|x|) = (\psi(|x|))^2 - \phi(|x|)\psi'(|x|). \tag{3.11}$$

Then define $\Psi_k \in C^\infty(\mathbb{R}^3)$ by

$$\Psi_k := h(\mu r_k), \tag{3.12}$$

for which we have

$$\Delta\big(\Psi_k\big) = \psi_k^2 - \phi_k \frac{\partial \psi_k}{\partial r_k}. \tag{3.13}$$

.

Let us fix some nonnegative function $\varphi \in C_c^\infty(\mathbb{R}^+)$, such that $\phi(r) = 1$ for $r \leq 1$, $\operatorname{supp}\varphi \in [0, 2]$, and $\int_0^\infty \varphi \, r \, dr = 1$. This function will be used as a cutoff in Definition 3.6 below and a radial mollification in Definition 3.7.

Now we define another two profile functions, $\widetilde{\psi}_k$ and $\widetilde{\eta}_k$, which will constitute an important part of our eddies.

Definition 3.6 (Viscous profiles $\widetilde{\psi}_k$ and $\widetilde{\eta}_k$)**.** *For $k \in \mathbb{K}$ and $\mu \geq \tau \geq \mu_0$, define*

$$\widetilde{\psi}_k = \varphi(\tau r_k)\Psi_k,$$

and

$$\widetilde{\eta}_k = \frac{1}{2}\frac{\partial(\eta_k^2)}{\partial z_k}.$$

Note that the extra mild cutoff $\phi(\tau r_k)$ is to make sure the support of $\widetilde{\psi}_k$ is contained in a cylinder centered at the line l_k in $\mathbb{R}^3$ so that $\widetilde{\eta}_k \widetilde{\psi}_k$ is compactly supported.

3.3. **Vector fields $\mathbb{W}_k$ and $\mathbb{V}_k$.** Let us first introduce vector fields $\mathbb{W}_k$ and $\mathbb{V}_k$, which corresponds to the principle part and respectively the Laplacian correction part of the eddies.

Definition 3.7. *Let $\mathbb{K} \subset \mathbb{R}^3$ be a finite set and $\gamma > 0$ be a small constant. For each $k \in \mathbb{K}$ and $\mu \geq \tau \geq \mu_0$, the vector fields $\mathbb{W}_k : \mathbb{R}^3 \to \mathbb{R}^3$ and $\mathbb{V}_k : \mathbb{R}^3 \to \mathbb{R}^3$ are defined by*

$$\mathbb{W}_k = (W_z + W_r)_\gamma \quad \text{and} \quad \mathbb{V}_k = \widetilde{\eta}_k \widetilde{\psi}_k \mathbf{e}_z, \tag{3.14}$$

where the vector fields W_z and W_r are respectively defined by

$$W_z = \eta_k \psi_k \mathbf{e}_z, \qquad W_r = -\frac{\partial \eta_k}{\partial z_k} \phi_k \mathbf{e}_r. \tag{3.15}$$

Here $(\cdot)_\gamma := \varphi_\gamma *$ indicates a radial mollification at scale $\mu^{-1-\gamma}$ in the Ω_k-plane via the kernel

$$\varphi_\gamma = \tfrac{1}{2\pi} \mu^{2+2\gamma} \varphi(\mu^{1+\gamma} r_k).$$

In addition, let W_k be the non-smooth counterpart of $\mathbb{W}_k$ defined by

$$W_k = W_z + W_r. \tag{3.16}$$

The role of each parameter is as follows.

- μ^{-1} parametrizes the concentration level of eddies.
- τ^{-1} measures the closeness of eddies to the pipe flows
- γ is a small constant that we use to achieve the smoothness of the eddies.

We will choose the parameters so that $\|\mathbb{V}_k\|_2 \ll \|\mathbb{W}_k\|_2$ and $\|W_r\|_2 \ll \|W_z\|_2$. Hence, viscous eddies are quantitatively determined by W_z.

Note that $\mathbb{W}_k$ is divergence-free. Indeed, using standard vector calculus (see Appendix A) we compute

$$\begin{aligned} \operatorname{div}(\mathbb{W}_k) &= \operatorname{div}\left(\eta_k \psi_k \mathbf{e}_z - \frac{\partial \eta_k}{\partial z_k} \phi_k \mathbf{e}_r\right)_\gamma \\ &= \left(\frac{\partial \eta_k}{\partial z_k} \psi_k - \frac{\partial \eta_k}{\partial z_k} \frac{1}{r} \frac{\partial (r\phi_k)}{\partial r_k}\right)_\gamma \\ &= 0, \end{aligned}$$

thanks to (3.8).

Note that for $\mathbb{W}_k$ we can choose $\gamma \ll 1$ and $\tau \ll \mu$ so that it has any small intermittency $D > 0$:

$$\|\nabla^m \mathbb{W}_k\|_p \lesssim_m \mu^{m(1+\gamma)} \mu^{1-2/p} \tau^{1/2-1/p}, \tag{3.17}$$

however, besides being much smaller than $\mathbb{W}_k$, the viscous part $\mathbb{V}_k$ will impose other restrictions on admissible choices of τ, μ, as indicated by Proposition 3.11.

As a direct consequence of Definition 3.3 and 3.6 we obtain

Lemma 3.8 (Compact support of $\mathbb{W}_k$ and $\mathbb{V}_k$)**.** *For any $\mu \geq \tau \geq \mu_0$, the supports set of $\mathbb{W}_k$ and $\mathbb{V}_k$ verify*

$$\begin{aligned} &\operatorname{supp} \mathbb{W}_k \cup \operatorname{supp} \mathbb{V}_k \subset [0,1]^3 \quad \text{for any } k \in \mathbb{K}, \\ &\operatorname{supp} \mathbb{W}_k \cap \operatorname{supp} \mathbb{W}_{k'} = \emptyset \quad \text{and} \quad \operatorname{supp} \mathbb{V}_k \cap \operatorname{supp} \mathbb{V}_{k'} = \emptyset \quad \text{if } k \neq k', \end{aligned}$$

and the estimate

$$|\operatorname{supp} \mathbb{W}_k| \lesssim \tau^{-1} \mu^{-2}.$$

Moreover, the vector fields $\mathbb{W}_k$ have zero mean

$$\int_{\mathbb{R}^3} \mathbb{W}_k = 0. \tag{3.18}$$

Proof. The compactness and disjointness of the support follow from the definitions. The estimate of the support set follows from the fact that μ^{-1}-mollification only alter the diameter of the support set by μ^{-1} and $\tau \leq \mu$.

The zero-mean property (3.18) follows from integrating in cylindrical coordinates with basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ and the fact that the profile function $\eta \in C_c^\infty(\mathbb{R})$ used in (3.7) has zero mean. $\square$

3.4. **Definition of viscous eddies.** We will show that $\mathbb{W}_k$ and $\mathbb{V}_k$ can be used to form stationary solutions of the Navier-Stokes equations. The choice of $\mathbb{V}_k$ is inspired by the following results.

The first estimate shows that the leading order term in $\operatorname{div}(\mathbb{W}_k \otimes \mathbb{W}_k)$ is $\operatorname{div}(W_k \otimes W_z)$.

Proposition 3.9. *Suppose $\tau \leq \mu^{1-\gamma}$. Then the following estimate holds*

$$\big\|\mathbb{W}_k \otimes \mathbb{W}_k - W_k \otimes W_z\big\|_p \lesssim_p \mu^{-\gamma}\Big[\mu^{2-2/p}\tau^{1-1/p}\Big],$$

for all $1 \leq p < 2$.

The next two results show a precise structure of the error term $\operatorname{div}(W_k \otimes W_z)$. In particular, it has a fixed direction $\mathbf{e}_z$ and zero mean over the Ω_k-plane thanks to Lemma 3.5. Hence, it can be balanced by adding a Laplacian term.

Lemma 3.10. *There holds*

$$\operatorname{div}(W_k \otimes W_z) = \frac{1}{2}\frac{\partial(\eta_k^2)}{\partial z_k}\left(\psi_k^2 - \phi_k \frac{\partial \psi_k}{\partial r_k}\right)\mathbf{e}_z. \tag{3.19}$$

Proof. Since $W_k = W_z + W_r$ is divergence-free, by a direct computation using cylindrical coordinates (cf. Appendix A) we conclude

$$\begin{aligned}\operatorname{div}(W_k \otimes W_z) &= \big((W_z + W_r)\cdot\nabla\big)W_z\\ &= -\frac{\partial \eta_k}{\partial z_k}\phi_k\eta_k\frac{\partial\psi_k}{\partial r_k}\mathbf{e}_z + \eta_k\psi_k\frac{\partial\eta_k}{\partial z_k}\psi_k\mathbf{e}_z\\ &= \frac{1}{2}\frac{\partial(\eta_k^2)}{\partial z_k}\left(\psi_k^2 - \phi_k\frac{\partial\psi_k}{\partial r_k}\right)\mathbf{e}_z.\end{aligned}$$

□

Proposition 3.11. *Suppose $\tau \leq \mu$. Then the following important estimate holds:*

$$\big\|\operatorname{div}(W_k \otimes W_z) - \Delta\mathbb{V}_k\big\|_{L^p(\mathbb{R}^3)} \lesssim_p \tau^2\mu^{-1}\Big[\mu^{2-2/p}\tau^{1-1/p}\Big], \tag{3.20}$$

for all $1 < p \leq \infty$.

While Lemma 3.10 follows from a direct computation using cylindrical coordinates, we postpone the proofs of Proposition 3.9 and Proposition 3.11 to the end of this section. With these results at hand, it is natural to consider the following family of vector fields.

Definition 3.12 (Viscous eddies)**.** *Viscous eddies are vector fields of the form*

$$u = \sum_k a_k\mathbb{W}_k - a_k^2\mathbb{V}_k, \tag{3.21}$$

where coefficients $a_k \in \mathbb{R}$ for each $k \in \mathbb{K}$.

One of the advantages of *viscous eddies* is that they are approximate solutions of the stationary Navier-Stokes equations.

Theorem 3.13 (Approximate stationary solutions in $\mathbb{R}^3$)**.** *Let $\mathbb{K} \subset \mathbb{R}^3$ be finite and u be a viscous eddy:*

$$u = \sum_k a_k\mathbb{W}_k - a_k^2\mathbb{V}_k,$$

where constants $a_k \in \mathbb{R}$ for each $k \in \mathbb{K}$.

Then $u \in C_c^\infty(\mathbb{R}^3)$ is an approximate solution of the stationary Navier-Stokes equations in the following sense. There exist a stress $R \in C_c^\infty(\mathbb{R}^{3\times3})$ and a vector field $r \in C_c^\infty(\mathbb{R}^3)$ so that

$$\Delta u + \operatorname{div}(u \otimes u) = \operatorname{div} R + r.$$

Moreover, for any $\varepsilon > 0$, one can choose $\tau, \mu > 0$ such that

$$\|R\|_{L^1(\mathbb{R}^3)} + \|r\|_{L^1(\mathbb{R}^3)} \leq \varepsilon.$$

For simplicity of presentation we include the pressure in the stress term R and do not assume R is symmetric traceless. It might be possible to write the vector field r in the divergence form, gaining an additional one derivative. Such a method will require the use of inverse divergence operator on $\mathbb{R}^3$. However, the inverse divergence $\mathcal{R}$ in defined in 5.1 does not preserve compact support on $\mathbb{R}^3$.

As one can see, u is an approximate stationary solution to the NSE for an arbitrary direction k, whereas both intermittent jets in [BCV18] and Mikado flows in [Luo19] must have lattice directions to be periodic.

Proof of Theorem 3.13. Denote $u_1 = \sum_k a_k \mathbb{W}_k$ and $u_2 = -\sum_k a_k^2 \mathbb{V}_k$ then define the stress term R by

$$R = \nabla u_1 + u_1 \otimes u_2 + u_2 \otimes u_1 + u_2 \otimes u_2.$$

and the vector field r as

$$r = \Delta u_2 + \operatorname{div}(u_1 \otimes u_1).$$

Immediately, by direct computation

$$\Delta u + \operatorname{div}(u \otimes u) = \operatorname{div} R + r.$$

As a result,

$$\|R\|_{L^1(\mathbb{R}^3)} \lesssim \|\nabla u_1\|_1 + \|u_1\|_2 \|u_2\|_2 + \|u_2\|_2^2, \tag{3.22}$$

and

$$\|r\|_{L^p(\mathbb{R}^3)} \lesssim \sum_k \big\| \operatorname{div}(W_k \otimes W_z) - \Delta \mathbb{V}_k \big\|_{L^p(\mathbb{R}^3)} + \big\| \operatorname{div}(\mathbb{W}_k \otimes \mathbb{W}_k - W_k \otimes W_z) \big\|_{L^p(\mathbb{R}^3)}.$$

By Propositions 3.9 and 3.11, it is easy to choose $p > 1$ sufficiently close to 1 and τ, μ sufficiently large depending on a_k such that

$$\|R\|_{L^1(\mathbb{R}^3)} + \|r\|_{L^1(\mathbb{R}^3)} \le \|R\|_{L^1(\mathbb{R}^3)} + \|r\|_{L^p(\mathbb{R}^3)} \le \varepsilon.$$

□

3.5. Estimates for the *viscous eddies*.

Proposition 3.14. *For any $\tau \le \mu^{1-\gamma}$ and μ sufficiently large, the following estimates hold:*

$$\mu^{-m(1+\gamma)} \big\|\nabla^m \mathbb{W}_k\big\|_{L^p(\mathbb{R}^3)} \lesssim_m \mu^{1-2/p} \tau^{1/2-1/p}, \quad 1 \le p \le \infty,$$
$$\mu^{-m(1+\gamma)} \big\|\nabla^m \mathbb{V}_k\big\|_{L^p(\mathbb{R}^3)} \lesssim_{m,p} \mu^{-1} \tau^{3/2} \Big[\mu^{1-2/p} \tau^{1/2-1/p}\Big], \quad 1 < p \le \infty.$$

Proof. By a dimensional analysis and smoothness of $\mathbb{W}_k$ and $\mathbb{V}_k$, it suffices to prove the estimates for $m = 0$.

Let us first estimate $\mathbb{W}_k$. Definitions 3.3, 3.7 and Lemma 3.5 immediately imply that

$$\|W_z\|_{L^p} \lesssim \mu^{1-2/p} \tau^{1/2-1/p}, \qquad 1 \le p \le \infty, \tag{3.23}$$

and

$$\|W_r\|_{L^p} \lesssim_p \mu^{-2/p} \tau^{3/2-1/p}, \qquad 1 \le p < 2. \tag{3.24}$$

Note that $W_r \notin L^2$, and hence the implicit constant in (3.24) blows up as $p \to 2-$. Now we will show that the mollified radial component of the eddy satisfies

$$\|(W_r)_\gamma\|_{L^p} \lesssim \mu^{\gamma-2/p} \tau^{3/2-1/p}, \qquad 1 \le p \le \infty, \tag{3.25}$$

provided μ is large enough.

Indeed, due to Lemma 3.2, there exist constants $c_1 \in \mathbb{R}$ and $0 < \alpha_0 < 1$, such that $\psi(r) = c_1$ for all $r \le \alpha_0$. By definition (3.1), for all $r \le \alpha_0$ we have

$$\begin{aligned}
\phi(r) &= -\frac{1}{r} \int_r^\infty \psi(s) s \, ds \\
&= -\frac{1}{r_k} \left(\int_r^\alpha c_1 s \, ds + \int_\alpha^\infty \psi(s) s \, ds \right) \\
&= c_1 \frac{r}{2} + c_2 \frac{1}{r},
\end{aligned}$$

for some constant $c_2 \in \mathbb{R}$. Clearly there exists $\alpha \le \alpha_0$, such that $|\phi(r)|$ is decreasing for all $r \le \alpha$, and $|\phi(\alpha)| \ge |\phi(r)|$ for all $r \ge \alpha$. Therefore, $\big|(\phi_k)_\gamma\big|$ attains a global maximum at $r_k = 0$, provided $2\mu^{-\gamma} \le \alpha$. A direct computation shows that

$$\begin{aligned}
\big|(\phi_k)_\gamma(0)\big| &= \left| \int_0^{\mu^{-1-\gamma}} \varphi_\gamma(r) \phi_k(r) r \, dr \right| \\
&= \left| \int_0^{\mu^{-1-\gamma}} \mu^{2+2\gamma} \varphi(\mu^{1+\gamma} r) \frac{1}{\mu r} r \, dr \right| \\
&\lesssim \mu^\gamma.
\end{aligned}$$

Now using the fact that $|\operatorname{supp}(W_r)_\gamma| \lesssim \mu^{-2}\tau^{-1}$, we can conclude that

$$\begin{aligned}\|(W_r)_\gamma\|_{L^p} &\lesssim \mu^{-2/p}\tau^{-1/p}\|(W_r)_\gamma\|_{L^\infty}\\ &\lesssim \mu^{-2/p}\tau^{-1/p}\left\|\frac{\partial \eta_k}{\partial z_k}\right\|_{L^\infty}\|(\phi_k)_\gamma\|_{L^\infty}\\ &\lesssim \mu^{-2/p}\tau^{-1/p}\tau^{3/2}\mu^{\gamma},\end{aligned}$$

provided μ is large enough (so that $2\mu^{-\gamma} \leq \alpha$).

Now we can easily estimate viscous eddies using (3.23) and (3.25):

$$\|\mathbb{W}_k\|_{L^p} \leq \|(W_z)_\gamma\|_{L^p} + \|(W_r)_\gamma\|_{L^p} \lesssim (1+\tau\mu^{\gamma-1})\Big[\mu^{1-2/p}\tau^{1/2-1/p}\Big] \lesssim \Big[\mu^{1-2/p}\tau^{1/2-1/p}\Big],$$

due to the assumption $\tau \leq \mu^{1-\gamma}$.

Next, we estimate $\|\mathbb{V}_k\|_{L^p}$ in cylindrical coordinates. Since $\mathbb{V}_k$ is axisymetric, using the decomposition $\mathbb{R}^3 = \Omega_k \oplus l_k$, we obtain

$$\|\mathbb{V}_k\|_{L^p(\mathbb{R}^3)} \lesssim \|\widetilde{\eta}_k\|_{L^p(l_k)}\|\widetilde{\psi}_k\|_{L^p(\Omega_k)}.$$

By Definitions 3.3 snd 3.6,

$$\|\widetilde{\eta}_k\|_{L^p(l_k)} \lesssim \left\|\frac{\partial(\eta_k^2)}{\partial z_k}\right\|_{L^p(l_k)} \lesssim \tau^{2-\frac{1}{p}}. \tag{3.26}$$

Then for $p>1$ we have

$$\begin{aligned}\|\widetilde{\psi}_k\|_{L^p(\Omega_k)} &\leq \|\varphi\|_{L^\infty(\Omega_k)}\big\|\Psi_k\big\|_{L^p(\Omega_k)}\\ &\lesssim_p \Big(\int \big|h(\mu r_k)\big|^p r_k dr_k\Big)^{\frac{1}{p}}\\ &\lesssim \mu^{-2/p},\end{aligned} \tag{3.27}$$

where in the last estimate we have used the fact that $h \in L^p(\mathbb{R}^2)$ for any $1<p\leq\infty$.

Putting together (3.26) and (3.27) we obtain the desired estimate

$$\|\mathbb{V}_k\|_{L^p} \lesssim_p \tau^{3/2}\mu^{-1}\Big[\mu^{1-2/p}\tau^{1/2-1/p}\Big] \quad \text{for any } 1<p\leq\infty.$$

□

Using the above estimates, we prove Proposition 3.9 and Proposition 3.11.

Proof of Proposition 3.9. We start with the decomposition

$$\mathbb{W}_k \otimes \mathbb{W}_k = W_k \otimes W_z + \big(\mathbb{W}_k - W_k\big)\otimes W_z + \mathbb{W}_k \otimes \big((W_z)_\gamma - W_z\big) + \mathbb{W}_k \otimes (W_r)_\gamma.$$

So by Hölder's inequality we will focus on the following

$$\begin{aligned}\big\|\mathbb{W}_k \otimes \mathbb{W}_k - W_k\otimes W_z\big\|_p &\lesssim \big\|\mathbb{W}_k - W_k\big\|_p\|W_z\|_\infty + \|\mathbb{W}_k\|_\infty\big\|(W_z)_\gamma - W_z\big\|_p + \|\mathbb{W}_k\|_\infty\big\|(W_r)_\gamma\big\|_p\\ &\lesssim X_1 + X_2 + X_3\end{aligned} \tag{3.28}$$

Let us first estimate X_1. We start with the definition of $\mathbb{W}_k$ and obtain

$$\begin{aligned}X_1 &\lesssim \big(\|W_z - (W_z)_\gamma\|_p + \|W_r - (W_r)_\gamma\|_p\big)\|W_z\|_{L^\infty}\\ &\lesssim \big(\|W_z - (W_z)_\gamma\|_p + \|W_r\|_p\big)\|W_z\|_{L^\infty}.\end{aligned} \tag{3.29}$$

To estimate the above terms, we first notice that by a standard approach to mollification,

$$\|W_z - (W_z)_\gamma\|_p \lesssim \|W_z\|_{W^{1,p}}\mu^{-1-\gamma}. \tag{3.30}$$

Moreover, by Lemma 3.5 (cf. (3.23) and (3.24)), we have

$$\|W_z\|_{W^{1,p}} \lesssim \mu\mu^{1-2/p}\tau^{1/2-1/p}, \qquad \|W_z\|_{L^\infty} \lesssim \mu\tau^{1/2}, \tag{3.31}$$

and, since $1 \leq p < 2$,

$$\|W_r\|_{L^p} \lesssim_p \mu^{-1}\tau\mu^{1-2/p}\tau^{1/2-1/p}. \tag{3.32}$$

Substituting bounds (3.30), (3.31), and (3.32) into (3.29) gives

$$X_1 \lesssim (\mu^{-\gamma} + \mu^{-1}\tau)\big[\mu^{2-2/p}\tau^{1-1/p}\big], \tag{3.33}$$

which is the desired estimate since $\tau \leq \mu^{1-\gamma}$.

Next, we estimate X_2. By Proposition 3.14 we have

$$\big\|\mathbb{W}_k\big\|_\infty \lesssim \mu\tau^{1/2}, \tag{3.34}$$

which together with (3.30) and (3.31) implies that

$$X_2 \lesssim \mu^{-\gamma}\big[\mu^{2-2/p}\tau^{1-1/p}\big]. \tag{3.35}$$

Finally, we need to bound X_3. All the estimates for X_3 have been obtained before. In particular, since $1 \le p < 2$, (3.32) and (3.34) imply

$$\begin{aligned} X_3 &\lesssim \big\|\mathbb{W}_k\big\|_\infty \big\|W_r\big\|_p \\ &\lesssim \mu^{-1}\tau\big[\mu^{2-2/p}\tau^{1-1/p}\big], \end{aligned}$$

which is what we need due to the assumption $\tau \le \mu^{1-\gamma}$.

□

Proof of Proposition 3.11. By a direct computation,

$$\Delta\mathbb{V}_k = \Delta(\widetilde{\eta}_k\varphi)\Psi_k\mathbf{e}_z + 2\nabla(\widetilde{\eta}_k\varphi)\nabla\Psi_k\mathbf{e}_z + \widetilde{\eta}_k\varphi\Delta\Psi_k\mathbf{e}_z, \tag{3.36}$$

where we write $\varphi = \varphi(\tau r_k)$ for short. Recall from (3.13) that

$$\Delta\big(\Psi_k\big) = \psi_k^2 - \phi_k\frac{\partial\psi_k}{\partial r_k},$$

and, in particular, $\Delta\Psi_k = 0$ for $r_k \ge \mu^{-1}$. Since $\tau \le \mu$, we have that $\varphi(\tau r_k) = 1$ on $\operatorname{supp}\Delta\Psi_k$. Then using Definition 3.6 and Lemma 3.10, we obtain

$$\begin{aligned} \widetilde{\eta}_k\varphi\Delta\Psi_k\mathbf{e}_z &= \frac{1}{2}\frac{\partial(\eta_k^2)}{\partial z_k}\Big(\psi_k^2 - \phi_k\frac{\partial\psi_k}{\partial r_k}\Big)\mathbf{e}_z \\ &= \operatorname{div}(W_k\otimes W_z). \end{aligned}$$

Combining this with (3.36), we get

$$\big\|\operatorname{div}(W_k\otimes W_z) - \Delta\mathbb{V}_k\big\|_{L^p(\mathbb{R}^3)} \lesssim \big\|\Delta(\widetilde{\eta}_k\varphi)\Psi_k\big\|_{L^p(\mathbb{R}^3)} + \big\|\nabla(\widetilde{\eta}_k\varphi)\nabla\Psi_k\big\|_{L^p(\mathbb{R}^3)}. \tag{3.37}$$

Since $\tau \le \mu$, it suffices to bound the second term in (3.37). By Definition 3.6, we have a pointwise bound

$$\big|\nabla(\widetilde{\eta}_k\varphi)\big| \lesssim \big|\nabla^2(\eta_k^2)\big| + \tau\big|\nabla(\eta_k^2)\big|.$$

Thus for the second term in (3.37) we have

$$\big\|\nabla(\widetilde{\eta}_k\varphi)\nabla\Psi_k\mathbf{e}_z\big\|_{L^p(\mathbb{R}^3)} \lesssim \big\|\nabla^2(\eta_k^2)\big\|_{L^p(l_k)}\big\|\nabla\Psi_k\big\|_{L^p(\Omega_k)} + \tau\big\|\nabla(\eta_k^2)\big\|_{L^p(l_k)}\big\|\nabla\Psi_k\big\|_{L^p(\Omega_k)}. \tag{3.38}$$

Now by rescaling (3.12) and Definition 3.3,

$$\big\|\nabla\Psi_k\big\|_{L^p(\Omega_k)} \lesssim_p \mu^{1-2/p} \quad \text{for } 1 < p \le \infty, \quad \text{and} \quad \big\|\nabla^n(\eta_k^2)\big\|_{L^p(l_k)} \lesssim_n \tau^n\tau^{1-1/p} \quad \text{for } 1 \le p \le \infty,$$

so we get

$$\big\|\nabla(\widetilde{\eta}_k\varphi)\nabla\Psi_k\mathbf{e}_z\big\|_{L^p(\mathbb{R}^3)} \lesssim_p \tau^2\mu^{-1}\Big[\mu^{2-2/p}\tau^{1-1/p}\Big],$$

which implies the desired bound:

$$\big\|\operatorname{div}(W_k\otimes W_z) - \Delta\mathbb{V}_k\big\|_{L^p(\mathbb{R}^3)} \lesssim_p \tau^2\mu^{-1}\Big[\mu^{2-2/p}\tau^{1-1/p}\Big] \quad \text{for } 1 < p \le \infty. \tag{3.39}$$

□

4. PROOF OF MAIN PROPOSITION: VELOCITY PERTURBATION

In this section, we start proving Proposition 2.1. The main objective of the section is to define and estimate the velocity perturbation. More specifically, we will carefully design the velocity perturbation w so that the new solution $v = v_0 + w$ has the desired properties listed in Proposition 2.1. The key is to reduce the size of the stress error term and make sure w carries a precise amount of energy on the intervals $I_{4^{-1}r}(\mathcal{F}_1)$ at the same time.

The rest of this section is organized as follows. We first give a general introduction of the proof, and then introduce all the necessary preparation work to define w, namely, fix constants τ and μ appeared in the *viscous eddies*, choose suitable cutoff functions in space and time, and introduce the Leray projection and a fast oscillation operator $\mathbf{P}_\sigma$. Finally, we define the velocity perturbation w and derive various estimates needed in the next two sections.

4.1. **General introduction.** To better illustrate the idea, we provide some heuristics and try to outline the general idea of the proof here. To the leading order, the velocity perturbation w consists of finitely many highly oscillating *viscous eddies*:

$$w = \sum_k a_k \mathbf{P}_\sigma \mathbb{W}_k + a_k^2 \mathbf{P}_\sigma \mathbb{V}_k := w^{(p)} + w^{(l)},$$

where coefficients a_k are determined by the old Reynolds stress R_0, and $\mathbf{P}_\sigma$ is a fast oscillation operator (see Definition 4.4).

On one hand, we nee to control the new stress term, which, according to (gNSR), is implicitly defined by

$$\operatorname{div} R = \partial_t w + L_a w + \operatorname{div}(w \otimes v_0 + v_0 \otimes w) + \operatorname{div}(R_0 + w \otimes w) - \nabla p_1.$$

The old Reynolds stress R_0 will be canceled by the interaction $w^{(p)} \otimes w^{(p)}$ together with $w^{(l)}$. More precisely,

$$\operatorname{div}(w^{(p)} \otimes w^{(p)}) + \operatorname{div} R_0 + \Delta w^{(l)} = \text{High frequency errors} + \text{Lower order terms}.$$

On the left hand side, R_0 will be canceled by the high-high interaction of $w^{(p)} \otimes w^{(p)}$, and $\Delta w^{(l)}$ will balance the error essentially introduced by the unwanted $\operatorname{div}(\mathbb{W}_k \otimes \mathbb{W}_k)$ as shown in Theorem 3.13. On the right hand side, lower order terms are automatically small, but high frequency errors will gain a factor of σ^{-1} after inverting the divergence. This will be shown in Lemma 5.8, Section 5.

On the other hand, we need to make sure the new solution v has the desired energy profile. This is in fact mostly compatible with the above effort of controlling the new stress error. Heuristically, to balance the stress term R_0, one must spend the energy of size at least $\sim \|R_0\|_1$. In other words,

$$\|w(t)\|_2^2 \gtrsim \|R_0(t)\|_1 \quad \text{for all } t.$$

There is a lot of flexibility in choosing the size of w though, as one can use more energy than needed to balance the old stress term R_0. In our scheme, the size of $\|w\|_2$ is determined by the given energy levels e_0 and e_1 on the intervals $I_{4^{-1}r}(\mathcal{F}_1)$, where the old stress error term is already quite small (the second condition for (v_0, R_0) in Proposition 2.1). This makes control of the stress and pumping of the energy compatibility. See (4.3) and Section 6 for more details.

4.2. **Setup of constants.** First, we set up the constants appeared in the definition of the vector fields $\mathbb{W}_k^{\tau,\mu}$ and the *viscous eddies*.

The major parameter λ, the (spacial) frequency of the perturbation, will be a sufficiently large. The parameters μ and τ in the *viscous eddies* are defined explicitly as powers of λ while γ is taken to be small. Moreover, we also define an integer σ to parametrize the oscillations of the eddies.

In the sequel, we fix

$$\begin{cases} \sigma = \lambda^{1/30} \\ \mu = \lambda^{14/15} \\ \tau = \lambda^{2/5} \\ \gamma = \frac{1}{28} \end{cases} \tag{4.1}$$

Clearly, it holds that $\mu^\gamma = \sigma$ and $\sigma\mu^{1+\gamma} = \lambda$. We also have the following hierarchy of constants:

$$\sigma \ll \tau \ll \mu \ll \lambda.$$

For periodicity, we also require σ to be an integer. Let us briefly discuss the scales involved in the definition of w. In essence, the choice of parameters ensures that by raising the value of λ, the new stress term R_0 introduced by w on $I_{4^{-1}r}(\mathcal{F}_1)$ can be as small as we want, and, at the same time, the energy of new solution $\|v(t)\|_2^2$ can be controlled precisely.

There are mainly four constraints in choosing the scales:

- The first constraint is due to the small intermittency requirement. Since λ is the frequency of w which consists of oscillation σ and concentration τ and μ, then for w to be small in $W^{1,1}$ it requires (see (3.9))
$$\lambda \tau^{-\frac{1}{2}} \mu^{-1} \ll 1.$$
- The second constraint is needed to achieve the correct energy level. Since $\|w^{(p)}\|_2$ controls the energy level of the new solution v, we need $\|w^{(l)}\|_2 \ll \|w^{(p)}\|_2$ and $\|w^{(c)}\|_2 \ll \|w^{(p)}\|_2$. According to definitions of $w^{(l)}$ and $w^{(c)}$, i.e. (4.12) and (4.13), this implies
$$\tau^{\frac{3}{2}} \ll \mu.$$
- The previous two constraints are due to the viscous part $w^{(l)}$. There is a new error introduced by Δ, namely R_{low} in Lemma 5.8. To make sure R_{low} is small, we need
$$\tau^2 \ll \mu.$$

- We use a mollification in the scale $\mu^{-1-\gamma}$ to remove $1/r$ singularity of a viscous eddy in the radial direction. This singularity is needed so that we can take advantage of the Laplacian. In order to control norms of the viscous eddy, we need an upper bound on γ. More precisely, as we have seen in the previous section, we need the following condition:
$$\tau \leq \mu^{1-\gamma}.$$

It is easy to verify that our choice of constants (4.1) satisfies all the above constraints.

Next, we introduce a constant M, whose role is to limit the order of the derivative that we will be taking so that the implicit constants stay bounded.

Definition 4.1 (The constant M)**.** *Let $N = 300$ and $\theta = 1/2$. We define M to be the constant obtained from applying Proposition 4.7 with such θ and N.*

4.3. **Cut-offs in space and time.** Let $\chi : \mathbb{R}^{3\times 3} \to \mathbb{R}^+$ be a positive smooth function so that it is monotone increasing with respect to $|x|$ and
$$\chi^2(x) = \begin{cases} 1, & 0 \leq |x| \leq 1 \\ |x|, & |x| \geq 2 \end{cases} \tag{4.2}$$
where $|\cdot|$ denotes the Euclidean matrix norm. Note that by definition
$$\|\nabla^m \chi\|_\infty \lesssim_m 1.$$
Now we choose a proper threshold $\rho_0(t)$ to control how much energy is added. Given an solution (v_0, R_0) and energy level e_1 as in the statement of Proposition 2.1, let
$$\rho_0(t) = \frac{1}{12}(\widetilde{e}_1 - \|v_0(t)\|_2^2), \tag{4.3}$$
where $\widetilde{e}_1 = e_1 - 10^{-6}(e_1 - e_0)$ is to leave room for future corrections. Note that ρ_0 is bounded from below:
$$\rho_0(t) \gtrsim e_1 - e_0 \gtrsim C^{-1}\delta_0, \tag{4.4}$$
due to the assumptions (1) and (2) in Proposition 2.1, where $\delta_0 = \|R_0\|_{L^\infty_t L^1_x(\mathbb{T}^3\times[0,T])}$ and the universal constant C in Proposition 2.1 will be specified in Section 6.

To deal with the issue of the Reynolds stress R_0 having large magnitudes, we introduce a divisor as follows. Define $\rho : \mathbb{T}^3 \times [0,T] \to \mathbb{R}^+$ to be
$$\rho(x,t) = 4\rho_0 \chi^2(\rho_0^{-1} R_0). \tag{4.5}$$
It follows from the above definitions that
$$\frac{|R_0|}{\rho} = \frac{|R_0|}{4\rho_0\chi^2(\rho_0^{-1}R_0)} \leq 1/2 \quad \text{for all } (x,t) \in \mathbb{T}^3 \times [0,T].$$

Next, we introduce a cutoff in time so that the energy profile of the new solution satisfies all the required properties. For the exceptional set $\mathcal{F}_1$ (cf. (2.1)), let $\theta : \mathbb{R} \to \mathbb{R}^+$ be a smooth cut-off function such that
$$\theta(t) = \begin{cases} 1, & t \in I_{4^{-1}r}(\mathcal{F}_1) \\ 0, & t \notin I_{4^{-2}r}(\mathcal{F}_1), \end{cases} \tag{4.6}$$
and
$$\|\theta^{(n)}\|_\infty \lesssim_n r^{-n} \quad \text{for all } n \in \mathbb{N}. \tag{4.7}$$

Remark 4.2. *When $\mathcal{F}_1 = \emptyset$, we take $\theta = 1$, so there is no cutoff in time. This will ensure that if $\mathcal{F}_0 = \mathcal{F}_1 = \emptyset$ and the solution v_0 is stationary, then the velocity perturbation w is also stationary.*

4.4. **Leray projection and fast periodization operator.** To define the velocity perturbation, we recall the definition of Leray projection.

Definition 4.3 (Leray projection)**.** *Let $v \in C^\infty(\mathbb{T}^3, \mathbb{R}^3)$ be a smooth vector field. Define the operator $\mathcal{Q}$ as*
$$\mathcal{Q}v := \nabla f + \fint_{\mathbb{T}^3} v,$$
where $f \in C^\infty(\mathbb{T}^3)$ is the smooth zero-mean solution of
$$\Delta f = \operatorname{div} v, \qquad x \in \mathbb{T}^3.$$
Furthermore, let $\mathcal{P} = \mathrm{Id} - \mathcal{Q}$ be the Leray projection onto divergence-free vector fields with zero mean.

To avoid potential abuse of notation, we will utilize the following fast periodization operator $\mathbf{P}_\sigma$ for functions whose support sets are contained in $[0,1]^3$. We will apply $\mathbf{P}_\sigma$ to the *viscous eddies* so that they oscillate at a frequency much higher than that of the solution (v_0, R_0).

Definition 4.4 (Fast periodization operator $\mathbf{P}_\sigma$)**.** *Let $\sigma \in \mathbb{N}$. Suppose $f \in C_c^\infty(\mathbb{R}^3)$ and $\operatorname{supp} f \subset [0,1]^3$, define the fast periodization operator $\mathbf{P}_\sigma$ by*

$$\mathbf{P}_\sigma f(x) = \sum_{m \in \mathbb{Z}^3} f(\sigma x + m). \tag{4.8}$$

By definition $\mathbf{P}_\sigma f$ is $\sigma^{-1}\mathbb{T}^3$-periodic, and for any differentiation ∇^n, we have

$$\nabla^n \mathbf{P}_\sigma f = \sigma^n \mathbf{P}_\sigma \nabla^n f \tag{4.9}$$

which will be used without mentioning in the future.

4.5. **Definitions of the perturbation.** With all the preparations in hand, we can define the velocity perturbation w.

We first apply Lemma 3.1 for $\mathcal{B} = \{R \in \mathcal{S}^{3\times 3}_+ : |\operatorname{Id} - R| \leq 1/2\}$ to obtain smooth functions $\Gamma_k : \mathcal{B} \to \mathbb{R}$ for $k \in \mathbb{Z}^3$, $|k| \leq \lambda_0$. Then the coefficients for the *viscous eddies* are defined by

$$a_k(x,t) = \rho^{1/2}(x,t)\Gamma_k\Big(\operatorname{Id} - \frac{R_0}{\rho}\Big) \quad \text{for } k \in \mathbb{Z}^3, |k| \leq \lambda_0. \tag{4.10}$$

In view of Theorem 3.13, define vector fields

$$w^{(p)} = \theta \sum_k a_k \mathbf{P}_\sigma \mathbb{W}_k = w_z^{(p)} + w_r^{(p)}, \tag{4.11}$$

where

$$w_z^{(p)} = \theta \sum_k a_k \mathbf{P}_\sigma (W_z)_\gamma, \qquad \text{and} \qquad w_r^{(p)} = \theta \sum_k a_k \mathbf{P}_\sigma (W_r)_\gamma,$$

and

$$w^{(l)} = -\theta^2 \sigma^{-1} \sum_k a_k^2 \mathbf{P}_\sigma \mathbb{V}_k. \tag{4.12}$$

Also define a divergence-free correction term

$$w^{(c)} = -\mathcal{Q} w^{(p)} - \mathcal{Q} w^{(l)}. \tag{4.13}$$

Finally, the velocity increment w is defined by

$$w = \theta \sum_k a_k \mathbf{P}_\sigma \mathbb{W}_k - \theta^2 \sigma^{-1} \sum_k a_k^2 \mathbf{P}_\sigma \mathbb{V}_k + w^{(c)}. \tag{4.14}$$

which also reads

$$w = w^{(p)} + w^{(l)} + w^{(c)}. \tag{4.15}$$

Thanks to Lemma 3.8, $\mathbf{P}_\sigma$ may be applied and w is well-defined. It is clear that w is periodic due to the periodicity of coefficients a_k and the periodization operator $\mathbf{P}_\sigma$. By design w is divergence-free. Also since the operator $\mathcal{P}$ removes the mean, w has zero mean as well.

Next, we show the smoothness of w, for which it suffices to show the following simple result for the coefficients a_k.

Lemma 4.5 (Properties of coefficients a_k)**.** *The coefficients a_k defined by (4.10) are smooth on $\mathbb{T}^3 \times [0,T]$. There exist a number $\kappa = \kappa(e_1, v_0, R_0) \geq r^{-1}$ such that*

$$\max_k \|a_k\|_{C^m_{t,x}} \leq \kappa^{m+1}, \quad \textit{for any integer} \quad 0 \leq m \leq 4M;$$

the following bounds hold

$$\begin{aligned} \|\rho(t)\|_{L^1} &\lesssim \rho_0(t), \\ \|a_k(t)\|_{L^2} &\lesssim \rho_0(t)^{1/2}; \end{aligned} \tag{4.16}$$

and we have the identity

$$\sum_k a_k^2 \fint_{\mathbb{T}^3} \mathbf{P}_\sigma (W_k \otimes W_z) = \rho \operatorname{Id} - R_0. \tag{4.17}$$

Proof. Recall that

$$a_k = 2\rho_0^{1/2}\chi(\rho_0^{-1}R_0)\Gamma_k\Big(\operatorname{Id}-\frac{R_0}{\rho}\Big). \tag{4.18}$$

To show that a_k has bounded space-time Hölder norms of order $4M$, it suffices to check that each factor above is smooth as the domain $\mathbb{T}^3\times[0,T]$ is compact. Since

$$\rho_0^{1/2} = \frac{1}{2\sqrt{3}}(\widetilde{e}_1 - \|v_0(t)\|_2^2)^{1/2},$$

which is bounded from below by (4.4), the function $\rho_0^{1/2}$ is smooth on $[0,T]$. By the same argument and the definition of χ in (4.2), we may also conclude that $\chi(\rho_0^{-1}R_0)\in C^\infty_{x,t}(\mathbb{T}^3\times[0,T])$. Since $\Gamma_k\in C^\infty(\mathcal{B})$, the last term in (4.18) is also in $C^\infty_{t,x}$.

Next, let us prove (4.16). Since $0\le\theta\le 1$, by definition of ρ in (4.5), we have

$$\begin{aligned}\|\rho(t)\|_{L^1} &\le \int_{|R_0|\le\rho_0}\rho(x,t)dx + \int_{|R_0|\ge\rho_0}\rho(x,t)dx\\ &\lesssim \rho_0\Big(\int_{|R_0|\le\rho_0}1dx + \int_{|R_0|\ge\rho_0}|R_0|dx\Big)\lesssim\rho_0,\end{aligned}$$

where we have used $\|R_0\|_{L^\infty_tL^1}=\delta_0\lesssim\rho_0$ due to (4.4).

For the second bound in (4.16), we can directly compute to obtain:

$$\|a_k(t)\|_2^2 \lesssim \rho_0\theta^2\int_{\mathbb{T}^3}\chi^2(\rho_0^{-1}R_0)dx\lesssim\rho_0\theta^2.$$

To show the last identity, thanks to Lemma 3.1, it suffices to show

$$\fint_{\mathbb{T}^3}\mathbf{P}_\sigma\left(W_k\otimes W_z\right) = \frac{k}{|k|}\otimes\frac{k}{|k|}.$$

Since

$$W_r\otimes W_z = \frac{\partial\eta_k}{\partial z_k}\phi_k\eta_k\psi_k\,\mathbf{e}_r\otimes\mathbf{e}_z.$$

where profile function $\frac{\partial\eta_k}{\partial z_k}\phi_k\eta_k\psi_k$ is axisymmetric, we have

$$\fint_{\mathbb{T}^3}\mathbf{P}_\sigma\left(W_r\otimes W_z\right)=0.$$

Then by Definitions 4.4 and 3.3,

$$\begin{aligned}\fint_{\mathbb{T}^3}\mathbf{P}_\sigma\left(W_k\otimes W_z\right) = \fint_{\mathbb{T}^3}\mathbf{P}_\sigma\left(W_z\otimes W_z\right) &= \int_{\mathbb{R}^3}W_z\otimes W_z\\ &=\int_{\mathbb{R}^3}|\eta_k\psi_k|^2\mathbf{e}_z\otimes\mathbf{e}_z\\ &=\frac{k}{|k|}\otimes\frac{k}{|k|}.\end{aligned}$$

Hence, the identity (4.17) follows from (4.10) and Lemma 3.1. □

4.6. **Estimates for the perturbations.** This subsection is devoted to various estimates for the perturbation w. We start with decomposing the corrector $w^{(c)}$ using standard vector calculus. Here the inverse Laplacian Δ^{-1} on torus $\mathbb{T}^3$ is defined via a multiplier with symbol $-|k|^{-2}$ for $k\neq 0$ and 0 for $k=0$.

Lemma 4.6 (Structure of the corrector)**.** *The corrector $w^{(c)}$ verifies*

$$w^{(c)} = w^{(cp)} + w^{(cl)}$$

where $w^{(cp)}$ and $w^{(cl)}$ are respectively

$$w^{(cp)} = \theta\sum_k\nabla\Delta^{-1}\Big(\nabla a_k\cdot\mathbf{P}_\sigma\mathbb{W}_k\Big) - \fint_{\mathbb{T}^3}w^{(p)},$$

and

$$w^{(cl)} = \theta^2\sigma^{-1}\mathcal{Q}\Big(\sum_k a_k^2\mathbf{P}_\sigma\mathbb{V}_k\Big).$$

Proof. Noticing that $\operatorname{div}\mathbb{W}_k=0$, these formulae immediately follow from Definition 4.3. □

We recall the following improved Hölder's inequality for functions with fast oscillation proven in [Luo19], which is crucial in obtaining the L^2 decay of the perturbation w. For convenience we include a proof in Appendix D.

Proposition 4.7. *For any small $\theta > 0$ and any large $N > 0$ there exist $M \in \mathbb{N}$ and $\lambda_0 \in \mathbb{N}$ so that for any $\mu > 0$, $\sigma \in \mathbb{N}$ satisfying $\lambda_0 \leq \sigma$ and $\mu \leq \sigma^{1-\theta}$ the following holds. Suppose $a \in C^\infty(\mathbb{T}^3)$ and let $C_a > 0$ be such that*

$$\|\nabla^i a\|_\infty \leq C_a \mu^i \quad \text{for any } 0 \leq i \leq M.$$

Then for any $\sigma^{-1}\mathbb{T}^3$ periodic function $f \in L^p(\mathbb{T}^3)$, $1 < p < \infty$, the following estimates are satisfied.

- *If $p \geq 2$ is even, then*
$$\|af\|_p \lesssim_{p,\theta,N} \|a\|_p \|f\|_p + C_a \|f\|_p \sigma^{-N}. \tag{4.19}$$
- *If $\fint_{\mathbb{T}^d} f = 0$ then for $0 \leq s \leq 1$*
$$\big\||\nabla|^{-1}(af)\big\|_p \lesssim_{p,s,\theta,N} \sigma^{-1+s} \big\||\nabla|^{-s}(af)\big\|_p + C_a \|f\|_p \sigma^{-N}. \tag{4.20}$$

All the implicit constants appeared in the statement are independent of a, μ and σ.

Remark 4.8. *Throughout the paper, we will always apply Proposition 4.7 for $\theta = \frac{1}{2}$ and $N = 300$. These two fixed constants determine the constant M.*

With the help of Proposition 4.7, we are in the position to derive useful estimates for the velocity perturbation w.

Proposition 4.9 (Spacial frequency estimates)**.** *For any λ sufficiently large and integer $0 \leq m \leq M$ the following estimates hold:*

$$\lambda^{-m} \|\nabla^m w^{(p)}(t)\|_p \lesssim \rho_0^{1/2}(t) \Big[\mu^{1-2/p} \tau^{1/2-1/p}\Big], \qquad 1 \leq p \leq 2, \tag{4.21}$$
$$\lambda^{-m} \|\nabla^m w_r^{(p)}(t)\|_p \lesssim \rho_0^{1/2}(t) \tau \mu^{2\gamma-1} \Big[\mu^{1-2/p} \tau^{1/2-1/p}\Big], \qquad 1 \leq p \leq 2, \tag{4.22}$$
$$\lambda^{-m} \|\nabla^m w^{(l)}(t)\|_p \lesssim_p \tau^{3/2} \mu^{-1} \Big[\mu^{1-2/p} \tau^{1/2-1/p}\Big], \qquad 1 < p \leq 2, \tag{4.23}$$
$$\lambda^{-m} \|\nabla^m w^{(c)}(t)\|_p \lesssim \sigma^{-1} \Big[\mu^{1-2/p} \tau^{1/2-1/p}\Big], \qquad 1 \leq p \leq 2. \tag{4.24}$$

Proof. *Bounds for $w^{(p)}$:*

Since by Lemma 3.8

$$\big|\mathbb{T}^3 \cap \operatorname{supp} \mathbf{P}_\sigma \mathbb{W}_k\big| \lesssim \tau^{-1} \mu^{-2}, \tag{4.25}$$

it suffices to show (4.21) for $p = 2$.

By product rule,

$$\big|\nabla^m w^{(p)}\big| \lesssim_m \sum_k \sum_{0 \leq i \leq m} \sigma^{m-i} \big|\nabla^i a_k\big| \big|\nabla^{m-i} \mathbf{P}_\sigma \mathbb{W}_k\big|. \tag{4.26}$$

As $\mathbf{P}_\sigma \mathbb{W}_k$ is $\sigma^{-1}\mathbb{T}^3$-periodic and, thanks to Lemma 4.5,

$$\|\nabla^i a_k\|_{C^m_x} \leq \|a_k\|_{C^{m+i}_x} \leq \kappa^{i+1+m} \quad \text{for all } 0 \leq m \leq M.$$

Since for large enough λ we have $\kappa^2 < \sigma \in \mathbb{N}$, we can apply Proposition 4.7 with $\theta = \frac{1}{2}$, $N = 300$, and $C_a = \kappa^{i+1}$ (cf. Definition 4.1) to obtain that

$$\big\||\nabla^i a_k| |\mathbf{P}_\sigma \nabla^{m-i} \mathbb{W}_k|\big\|_2 \lesssim \|\nabla^i a_k\|_2 \|\mathbf{P}_\sigma \nabla^{m-i} \mathbb{W}_k\|_2 + \kappa^{i+1} \|\mathbf{P}_\sigma \nabla^{m-i} \mathbb{W}_k\|_2 \sigma^{-N}. \tag{4.27}$$

Let us consider two sub-cases: $m = 0$ and $m \geq 1$. When $m = 0$, it follows that

$$\|a_k \mathbf{P}_\sigma \mathbb{W}_k\|_2 \lesssim \rho_0^{1/2} + \kappa \sigma^{-N}.$$

As $\sigma^{-N} = \lambda^{-10}$ and $\rho_0 \gtrsim e_1 - e_0 > 0$, we can make sure for any sufficiently large $\lambda(e_0, e_1, \kappa)$ that

$$\|a_k \mathbf{P}_\sigma \mathbb{W}_k\|_2 \lesssim \rho_0^{1/2},$$

from which we immediately get

$$\|w^{(p)}(t)\|_2 \lesssim \rho_0^{1/2}.$$

When $m \geq 1$, we consider the split:

$$\sum_{0 \leq i \leq m} \sigma^{m-i} \big\||\nabla^i a_k| |\mathbf{P}_\sigma \nabla^{m-i} \mathbb{W}_k|\big\|_2 \leq \sigma^m \|a_k \mathbf{P}_\sigma \nabla^m \mathbb{W}_k\|_2 + \sum_{1 \leq i \leq m} \sigma^{m-i} \big\||\nabla^i a_k| |\mathbf{P}_\sigma \nabla^{m-i} \mathbb{W}_k|\big\|_2. \tag{4.28}$$

We will bound these two terms separately. For the first term in (4.28), we use (4.27), Lemma 4.5, and Proposition 3.14 to obtain

$$\begin{aligned}\sigma^m \|a_k \mathbf{P}_\sigma \nabla^m \mathbb{W}_k\|_2 &\lesssim \sigma^m \left(\rho_0^{1/2} \|\mathbf{P}_\sigma \nabla^m \mathbb{W}_k\|_2 + \sigma^{-N}\kappa \|\mathbf{P}_\sigma \nabla^m \mathbb{W}_k\|_2\right)\\ &\lesssim \sigma^m \mu^{m(1+\gamma)} \left(\rho_0^{1/2} + \sigma^{-N}\kappa\right).\end{aligned}$$

Since $\sigma^{-N} = \lambda^{-10}$, $\sigma\mu^{1+\gamma} = \lambda$, and $\rho_0 \gtrsim e_1 - e_0$, for λ sufficiently large we get

$$\sigma^m \|a_k \mathbf{P}_\sigma \nabla^m \mathbb{W}_k\|_2 \lesssim \rho_0^{1/2} \lambda^m. \tag{4.29}$$

For the second term in (4.28), we simply use Hölder's inequality, Lemma 4.5, and Proposition 3.14 to obtain

$$\begin{aligned}\sum_{1\le i\le m} \sigma^{m-i} \big\| |\nabla^i a_k| |\mathbf{P}_\sigma \nabla^{m-i} \mathbb{W}_k| \big\|_2 &\le \sum_{1\le i\le m} \sigma^{m-i} \|\nabla^i a_k\|_{L^\infty_{x,t}} \|\mathbf{P}_\sigma \nabla^{m-i} \mathbb{W}_k\|_2\\ &\lesssim \sum_{1\le i\le m} \sigma^{m-i}\kappa^{i+1}\mu^{(m-i)(1+\gamma)} \lesssim \kappa^2 \sigma^{m-1}\mu^{(m-1)(1+\gamma)},\end{aligned}$$

where we have also used $\kappa \ll \mu$ in the last inequality. Then again, for λ sufficiently large, we get

$$\sum_{1\le i\le m} \sigma^{m-i} \big\| |\nabla^i a_k| |\mathbf{P}_\sigma \nabla^{m-i} \mathbb{W}_k| \big\|_2 \lesssim \rho_0^{1/2} \lambda^m. \tag{4.30}$$

So for $\lambda(\rho_0, \kappa, e_1, e_0)$ sufficiently large, putting together (4.29) and (4.30), we can bound (4.28) as

$$\sum_{0\le i\le m} \sigma^{m-i} \big\| |\nabla^i a_k| |\mathbf{P}_\sigma \nabla^{m-i} \mathbb{W}_k| \big\|_2 \lesssim \rho_0^{1/2} \lambda^m,$$

which implies that

$$\|\nabla^m w^{(p)}(t)\|_2 \lesssim \rho_0^{1/2}\lambda^m, \quad \text{for any } 1 \le m \le M\,.$$

Since for any integer $0 \le m \le M$ the desire estimate holds for $p = 2$, by Hölder's inequality and (4.25), for $1 \le p \le 2$ we have

$$\lambda^{-m} \|\nabla^m w^{(p)}(t)\|_p \lesssim \rho_0^{1/2} \mu^{1-2/p} \tau^{1/2-1/p}.$$

Bounds for $w_r^{(p)}$:

In light of estimate (3.25), the above argument also gives the desired bound for $w_r^{(p)}$. In particular, for $m = 0$, thanks to Proposition 4.7 we have

$$\begin{aligned}\|a_k \mathbf{P}_\sigma (W_r)_\sigma\|_2 &\lesssim \|\nabla^i a_k\|_2 \|\mathbf{P}_\sigma (W_r)_\sigma\|_2 + \kappa \|\mathbf{P}_\sigma (W_r)_\sigma\|_2 \sigma^{-N}\\ &\lesssim (\rho_0^{1/2} + \kappa\sigma^{-N})\tau\mu^{2\gamma-1}\\ &\lesssim \rho_0^{1/2}\tau\mu^{2\gamma-1}.\end{aligned}$$

Bounds for $w^{(l)}$:

Without loss of generality, we prove this bound for $m = 0$ as well, since general cases for $0 \le m \le M$ follow from applying an additional product rule, which can be seen in the estimates for $w^{(p)}$.

Recall the definition (4.12) that

$$w^{(l)} = -\sigma^{-1}\theta^2 \sum_k a_k^2 \mathbf{P}_\sigma \mathbb{V}_k.$$

By Hölder's inequality, Lemma 4.5, and Proposition 3.14, we have

$$\begin{aligned}\|w^{(l)}\|_p &\lesssim \sigma^{-1} \sum_k \|a_k^2\|_{L^\infty_{t,x}} \|\mathbf{P}_\sigma \mathbb{V}_k\|_p\\ &\lesssim \kappa^2 \sigma^{-1} \tau \mu^{-2} \tau^{1-1/p} \mu^{2-2/p}.\end{aligned}$$

Therefore, for sufficiently large $\lambda(\kappa)$, we can use σ^{-1} to absorb the factor with κ to obtain

$$\|w^{(l)}\|_p \lesssim \tau^{3/2}\mu^{-1}\left[\mu^{1-2/p}\tau^{1/2-1/p}\right]. \tag{4.31}$$

Bounds for $w^{(c)}$:

Again, we only prove the bound for $m = 0$.Thanks to Lemma 4.6, we need to estimate $\|w^{(cp)}\|_p$ and $\|w^{(cl)}\|_p$. It suffices to estimate the following term:

$$\Big\| \sum_k \nabla\Delta^{-1}\Big(\nabla a_k \cdot \mathbf{P}_\sigma \mathbb{W}_k\Big)\Big\|_p = \Big\| \sum_k \mathcal{R}|\nabla|^{-1}\Big(\nabla a_k \cdot \mathbf{P}_\sigma \mathbb{W}_k\Big)\Big\|_p$$
$$\lesssim \Big\| \sum_k |\nabla|^{-1}\Big(\nabla a_k \cdot \mathbf{P}_\sigma \mathbb{W}_k\Big)\Big\|_p,$$

where $\mathcal{R}$ is the Riesz transform. $\mathcal{R}$ and $|\nabla|^{-1}$ are defined via multipliers with symbols $-i\frac{k}{|k|}$ and $|k|^{-1}$ respectively for $k \neq 0$, and zero for $k = 0$. Recall that $\mathbf{P}_\sigma \mathbb{W}_k$ is $\sigma^{-1}\mathbb{T}^3$-periodic and of zero mean. Moreover, due to Lemma 4.5,

$$\|\nabla a_k\|_{C^m_x} \leq \|a_k\|_{C^{m+1}_x} \leq \kappa^{m+2} \quad \text{for all } 0 \leq m \leq M.$$

Once again we can apply Proposition 4.7 with $C_a = \kappa^2$ to obtain the bound

$$\Big\||\nabla|^{-1}\Big(\nabla a_k \cdot \mathbf{P}_\sigma \mathbb{W}_k\Big)\Big\|_p \lesssim \sigma^{-1}\Big\|\Big(\nabla a_k \cdot \mathbf{P}_\sigma \mathbb{W}_k\Big)\Big\|_p + \kappa^2\Big\|\Big(\mathbf{P}_\sigma \mathbb{W}_k\Big)\Big\|_p \sigma^{-N}$$
$$\lesssim \big(\sigma^{-1}\|\nabla a_k\|_\infty + \kappa^2\sigma^{-N}\big)\,\|\mathbf{P}_\sigma \mathbb{W}\|_p$$
$$\lesssim (\sigma^{-1}\kappa^2 + \kappa^2\sigma^{-300})\mu^{1-2/p}\tau^{1/2-1/p}$$
$$\lesssim \sigma^{-1}\kappa^2\big[\mu^{1-2/p}\tau^{1/2-1/p}\big].$$

Finally, since

$$\Big|\fint_{\mathbb{T}^3} |w^{(p)}|\Big| \lesssim \rho_0^{1/2}\lambda^{-17/15},$$

we have

$$\|w^{(cp)}\|_p \lesssim \sigma^{-1}\big[\mu^{1-2/p}\tau^{1/2-1/p}\big],$$

provided $\lambda(\kappa, e_1)$ is large enough.

To estimate the term $w^{(cl)}$, let us introduce $p_\varepsilon = p + \varepsilon$, for $\varepsilon \geq 0$, such that $1 < p_\varepsilon \leq 2$ and

$$\tau^{3/2}\mu^{-1}\big[\mu^{1-2/p_\varepsilon}\tau^{1/2-1/p_\varepsilon}\big] \leq \sigma^{-1}\big[\mu^{1-2/p}\tau^{1/2-1/p}\big].$$

Note that the operator $\mathcal{Q}$ is bounded on $L^{p_\varepsilon}(\mathbb{T}^3)$, and hence we have

$$\|w^{(cl)}\|_p \leq \|w^{(cl)}\|_{p_\varepsilon} \lesssim \|w^{(l)}\|_p \lesssim \tau^{3/2}\mu^{-1}\big[\mu^{1-2/p_\varepsilon}\tau^{1/2-1/p_\varepsilon}\big] \leq \sigma^{-1}\big[\mu^{1-2/p}\tau^{1/2-1/p}\big], \tag{4.32}$$

due to the choice of constants (4.1).

□

Using the choice of constants (4.1) and the established bounds (4.21), (4.24), and (4.23), we get the next useful corollary.

Corollary 4.10 (Estimates with explicit exponents)**.** *For any λ sufficiently large we have*

$$\|w^{(p)}\|_p + \lambda^{-1}\|\nabla w^{(p)}\|_p \lesssim \rho_0^{1/2}\lambda^{\frac{17}{15}(1-\frac{2}{p})}, \qquad 1 \leq p \leq 2$$
$$\|w_r^{(p)}\|_p + \lambda^{-1}\|\nabla w_r^{(p)}\|_p \lesssim \rho_0^{1/2}\lambda^{-\frac{7}{15}}\lambda^{\frac{17}{15}(1-\frac{2}{p})}, \qquad 1 \leq p \leq 2$$
$$\|w^{(l)}\|_p + \lambda^{-1}\|\nabla w^{(l)}\|_p \lesssim_p \lambda^{-\frac{1}{3}}\lambda^{\frac{17}{15}(1-\frac{2}{p})}, \qquad 1 < p \leq 2,$$
$$\|w^{(c)}\|_p + \lambda^{-1}\|\nabla w^{(c)}\|_p \lesssim \lambda^{-\frac{1}{30}}\lambda^{\frac{17}{15}(1-\frac{2}{p})}, \qquad 1 \leq p \leq 2,$$

and consequently

$$\|w\|_p + \lambda^{-1}\|\nabla w\|_p \lesssim \rho_0^{1/2}\lambda^{\frac{17}{15}(1-\frac{2}{p})}, \qquad 1 \leq p \leq 2. \tag{4.33}$$

In particular, given any $\varepsilon > 0$, for λ sufficiently large,

$$\|w\|_{L^\infty_t W^{1,1}_x} \leq \varepsilon. \tag{4.34}$$

The last estimate concerns the time derivative of the perturbation w. Since the velocity profiles in $\mathbb{W}_k$ and $\mathbb{V}_k$ are stationary, time derivative only falls on the slow variables a_k and θ.

Proposition 4.11 (Temporal frequency estimates)**.** *For any λ sufficiently large, $1 \leq p \leq 2$, and integer $0 \leq m \leq M$, the following estimate holds:*

$$\kappa^{-m-1}\|\partial_t^m w\|_{L^\infty_t L^p_x} \lesssim \mu^{1-2/p}\tau^{1/2-1/p}. \tag{4.35}$$

Moreover, if (v_0, R_0) is stationary and $\mathcal{F}_0 = \mathcal{F}_1 = \emptyset$, then $v = v_0 + w$ is also stationary.

Proof. The last statement follows from (4.6) and the definition of a_k, namely (4.10). Let us show (4.35). In view of Lemma 4.5, it suffices to prove the bound for $m = 1$. Thanks to Lemma 4.6, we can use the decomposition

$$\partial_t w = \partial_t w^{(p)} + \partial_t w^{(cp)} + \partial_t w^{(cl)} + \partial_t \mathcal{P} w^{(l)}.$$

We first bound the term $\partial_t w^{(p)}$. By its definition, Lemma 4.5, Hölder's inequality and Proposition 3.14 we have that

$$\begin{aligned}\left\|\partial_t w^{(p)}\right\|_p &\lesssim \sum_k \left\|\theta a_k\right\|_{C^1_{t,x}} \left\|\mathbf{P}_\sigma \mathbb{W}_k\right\|_p \\ &\lesssim \kappa^2 \left[\mu^{1-2/p} \tau^{1/2-1/p}\right],\end{aligned}$$

which is exactly the bound that we need.

Next, we show the same estimate holds for the term $\partial_t \mathcal{P} w^{(l)}$. As done in the proof of Proposition 4.9, let $p_\varepsilon = p + \varepsilon$ with $\varepsilon \geq 0$ chosen small enough such that $1 < p_\epsilon \leq 2$ and

$$\mu^{1-2/p_\varepsilon} \tau^{1/2-1/p_\varepsilon} \leq \mu^{1-2/p} \tau^{1/2-1/p} \sigma^{1/2},$$

which is possible thanks to (4.1). Then, using the L^{p_ε} boundedness of the Leray projection, Hölder's inequality, Proposition 3.14 and the above choice of p_ε, for any $1 \leq p \leq 2$ it follows that

$$\begin{aligned}\left\|\partial_t \mathcal{P} w^{(l)}\right\|_p \leq \left\|\mathcal{P} \partial_t w^{(l)}\right\|_{p_\varepsilon} &\lesssim \left\|\partial_t w^{(l)}\right\|_{p_\varepsilon} \lesssim \sigma^{-1} \sum_k \left\|\theta^2 a_k^2\right\|_{C^1_{t,x}} \left\|\mathbf{P}_\sigma \mathbb{V}_k\right\|_{p_\varepsilon} \\ &\lesssim \kappa^3 \sigma^{-1} \tau^{3/2} \mu^{-1} \mu^{1-2/p_\varepsilon} \tau^{1/2-1/p_\varepsilon} \lesssim \kappa^3 \sigma^{-1/2} \tau^{3/2} \mu^{-1} \left[\mu^{1-2/p} \tau^{1/2-1/p}\right].\end{aligned}$$

Due to our choice of constants, (4.1), for any sufficiently large $\lambda(\kappa)$ we have $\kappa^3 \sigma^{-1/2} \tau^{3/2} \mu^{-1} \leq \kappa^2$ and hence

$$\left\|\partial_t \mathcal{P} w^{(l)}\right\|_p \lesssim \kappa^2 \left[\mu^{1-2/p} \tau^{1/2-1/p}\right].$$

Finally, it remains to bound the terms $\partial_t w^{(cp)}$ and $\partial_t w^{(cl)}$. As in the proof of Proposition 4.9, we have the following estimates:

$$\begin{aligned}\left\|\sum_k \partial_t(\theta a_k) \nabla \Delta^{-1} \left(\nabla a_k \cdot \mathbf{P}_\sigma \mathbb{W}_k\right)\right\|_p &\lesssim \left\|\theta a_k\right\|_{C^1_{t,x}} \left\|\nabla \Delta^{-1} \left(\nabla a_k \cdot \mathbf{P}_\sigma \mathbb{W}_k\right)\right\|_p \\ &\lesssim \kappa^2 \sigma^{-1} \left[\mu^{1-2/p} \tau^{1/2-1/p}\right] \\ &\lesssim \kappa^2 \left[\mu^{1-2/p} \tau^{1/2-1/p}\right],\end{aligned}$$

which is the desired bound. □

5. PROOF OF ITERATION LEMMA: NEW REYNOLDS STRESS

In this section, we construct a new Reynolds stress R such that (2.2) holds. The majority of this section is devoted to obtaining bounds on the new Reynolds stress R using the established estimates for the velocity perturbations in Section 4. We split R into four parts and then estimate them separately.

To do this, one needs to obtain a symmetric traceless matrix R as the new stress term. Since the underdetermined system (gNSR) only provides an implicit definition of R, i.e. its divergence, the divergence has to be "inverted". This is a standard technique in elliptic PDEs. Here, we follow the one used in [BLJV18].

Definition 5.1 (Inverse divergence). *Let $f \in C^\infty(\mathbb{T}^3)$ be a smooth vector field. The inverse divergence operator $\mathcal{R} : C^\infty(\mathbb{T}^3, \mathbb{R}^3) \to \mathbb{R}^{3\times 3}$ is defined by*

$$\begin{aligned}&(\mathcal{R} f)_{ij} = \mathcal{R}_{ijk} f_k, \\ &\mathcal{R}_{ijk} = -\frac{1}{2} \Delta^{-2} \partial_i \partial_j \partial_k - \frac{1}{2} \Delta^{-1} \partial_k \delta_{ij} + \Delta^{-1} \partial_i \delta_{jk} + \Delta^{-1} \partial_j \delta_{ik}.\end{aligned} \tag{5.1}$$

Remark 5.2. *We note that in the definition, the inverse Laplacian Δ^{-1} is defined on $\mathbb{T}^3$ and gives functions with zero mean. So $\mathcal{R} f$ is always well-defined and mean free.*

With the above definition, a simple exercise leads to the following.

Lemma 5.3. *The operator $\mathcal{R}$ defined by (5.1) has the following properties. For any vector field $f \in C^\infty(\mathbb{T}^3)$ the matrix $\mathcal{R} f$ is symmetric trace-free, and*

$$\operatorname{div} \mathcal{R} f = f. \tag{5.2}$$

If additionally $\operatorname{div} f = 0$, *then*

$$\mathcal{R} \Delta f = \nabla f + (\nabla f)^T. \tag{5.3}$$

With this inverse divergence operator, we are ready to give the definition of the new Reynolds stress.

Definition 5.4 (New Reynolds stress R)**.** *Define the new Reynolds stress by*

$$R = \mathcal{R}\Big(\partial_t w + L_a w + \operatorname{div}(w \otimes v_0 + v_0 \otimes w) + \operatorname{div}(\theta^2 R_0 + w \otimes w) - \nabla p_1\Big) + (1-\theta^2) R_0 \tag{5.4}$$

where the pressure term $p_1 = \theta^2 \rho$ and ρ is defined in (4.5)*.*

It is immediate that the new Reynold stress R verifies the following equation thanks to Lemma 5.3

$$\operatorname{div} R = \partial_t w + L_a w + \operatorname{div}(w \otimes v_0 + v_0 \otimes w) + \operatorname{div} R_0 + \operatorname{div}(w \otimes w) - \nabla p_1.$$

Consequently, since (v_0, R_0) is a solution of (gNSR), there exists a uniquely determined zero-mean pressure P such that the new solution $v = v_0 + w$ verifies

$$\partial_t v + L_a v + \operatorname{div}(v \otimes v) + \nabla P = \operatorname{div} R.$$

In view of $w = w^{(p)} + w^{(l)} + w^{(c)}$, the new Reynolds stress can be rewritten as

$$R = R_{\text{lin}} + R_{\text{cor}} + R_{\text{osc}} + R_{\text{rem}}, \tag{5.5}$$

where the linear part R_{lin}, the correction part R_{cor}, oscillation part R_{osc} and the reminder part R_{rem} are respectively defined by

$$\begin{aligned}
R_{\text{lin}} &= \mathcal{R}\big(\partial_t w + L_a w - \Delta w^{(l)} + \operatorname{div}(w \otimes v_0 + v_0 \otimes w)\big), \\
R_{\text{cor}} &= \mathcal{R}\big(\operatorname{div}\big((w^{(c)} + w^{(l)}) \otimes w + w^{(p)} \otimes (w^{(c)} + w^{(l)})\big)\big), \\
R_{\text{osc}} &= \mathcal{R}\big(\operatorname{div}(\theta^2 R_0 + w^{(p)} \otimes w^{(p)}) + \Delta w^{(l)} - \nabla p_1\big), \\
R_{\text{rem}} &= (1-\theta^2) R_0.
\end{aligned}$$

In the remainder of this section, we will estimate R via the decomposition $\|R\|_1 \leq \|R_{\text{lin}}\|_1 + \|R_{\text{cor}}\|_1 + \|R_{\text{osc}}\|_1 + \|R_{\text{rem}}\|_1$ and show the following.

Lemma 5.5 (Estimates for R)**.** *The new Reynolds stress R obeys the estimates:*

$$\|R(t)\|_1 \leq \begin{cases} \varepsilon & \text{for } t \in I_{4^{-1}r}(\mathcal{F}_1) \\ \delta_0 + \varepsilon & \text{for } t \in I_{4^{-2}r}(\mathcal{F}_1) \setminus I_{4^{-1}r}(\mathcal{F}_1) \\ \delta_0 & \text{for } t \in [0,T] \setminus I_{4^{-2}r}(\mathcal{F}_1). \end{cases} \tag{5.6}$$

Since $\operatorname{supp}_t w \subset I_{4^{-2}r}(\mathcal{F}_1)$, it is sufficient to show that

$$\big\|R_{\text{lin}}\big\|_{L^\infty_t L^1_x} + \big\|R_{\text{cor}}\big\|_{L^\infty_t L^1_x} + \|R_{\text{osc}}\|_{L^\infty_t L^1_x} \leq \varepsilon.$$

We first estimate the linear part. For this term, the smallness of the intermittency plays a key role.

Lemma 5.6 (Linear error)**.** *For any λ sufficiently large,*

$$\big\|R_{\text{lin}}\big\|_{L^\infty_t L^1_x} \leq \frac{\varepsilon}{4}. \tag{5.7}$$

Proof. Considering the fact that

$$\|\mathcal{R}\|_{L^p(\mathbb{T}^3) \to L^p(\mathbb{T}^3)} \lesssim 1 \quad \text{for any } 1 < p < \infty \tag{5.8}$$

due to the Hardy-Littlewood-Sobolev inequality, and that

$$\|\mathcal{R} \operatorname{div} \cdot\|_{L^p(\mathbb{T}^3) \to L^p(\mathbb{T}^3)} \lesssim 1 \quad \text{for any } 1 < p < \infty \tag{5.9}$$

due to the boundedness of the Riesz transform, throughout the proof we fix $p > 1$ close to 1 such that

$$\mu^{1-2/p} \tau^{1/2-1/p} = \lambda^{\frac{17}{15}(1-2/p)} \leq \lambda^{-16/15}. \tag{5.10}$$

Split the linear error $R_{\text{lin}} = R_t + R_d$, where the first part R_t is the error caused by time derivative $R_t = \mathcal{R}\partial_t w$, and the second part R_d consists of the dissipative and drifts errors

$$R_d = \mathcal{R}\Delta(w^{(p)} + w^{(c)}) + \mathcal{R} \operatorname{div}\big(w \otimes (a + v_0)\big) + \mathcal{R} \operatorname{div}\big((a + v_0) \otimes w\big).$$

For the liner error caused by time derivative, by (5.8) and Proposition 4.11 we have

$$\big\|R_t\big\|_1 \leq \big\|\mathcal{R}\partial_t w\big\|_p \lesssim \big\|\partial_t w\big\|_p \lesssim \kappa^2 \mu^{1-2/p} \tau^{1/2-1/p} \leq \kappa^2 \lambda^{-\frac{16}{15}}. \tag{5.11}$$

We turn to estimate the liner error caused by drifts and the Laplacian. So using Lemma 5.3, (5.9) and Hölder's inequality we get

$$\begin{aligned}
\big\|R_d\big\|_1 &\leq \|\mathcal{R}\Delta(w^{(p)} + w^{(c)})\|_1 + \big\|\mathcal{R} \operatorname{div}\big(w \otimes (a + v_0)\big)\big\|_p + \big\|\mathcal{R} \operatorname{div}\big((a + v_0) \otimes w\big)\big\|_p \\
&\lesssim \|\nabla(w^{(p)} + w^{(c)})\|_1 + \|w\|_p \big[\|a\|_\infty + \|v_0\|_\infty\big].
\end{aligned} \tag{5.12}$$

By Corollary 4.10 and using (5.10) we have

$$\|\nabla(w^{(p)} + w^{(c)})\|_1 \lesssim \left[\rho_0^{1/2} + \lambda^{-1/3}\right]\lambda^{-2/15}$$
$$\|w\|_p \lesssim \rho_0^{1/2}\lambda^{-16/15}.$$

It follows from the above and (5.12) that

$$\left\|R_d\right\|_1 \lesssim \rho_0^{1/2}\lambda^{-2/15} + \rho_0^{1/2}\lambda^{-16/15}\left(\|a\|_\infty + \|v_0\|_\infty\right). \tag{5.13}$$

Combining (5.11) and (5.13), for any sufficiently large $\lambda(a, \varepsilon, e_1, \kappa, v_0)$ it holds

$$\left\|R_{\text{lin}}\right\|_1 \leq \left\|R_t\right\|_1 + \left\|R_d\right\|_1 \leq \frac{\varepsilon}{4}. \tag{5.14}$$

□

Next, we turn to estimating the correction part of the new Reynolds stress R. This part is essentially caused by $w^{(c)}$ and $w^{(l)}$ which are both much smaller than $w^{(p)}$.

Lemma 5.7 (Correction error)**.** *For any λ sufficiently large,*

$$\left\|R_{\text{cor}}\right\|_{L_t^\infty L_x^1} \leq \frac{\varepsilon}{8}. \tag{5.15}$$

Proof. In view of Corollary 4.10, fix a $p > 1$ close to 1 such that

$$\|w^{(c)}\|_{\frac{2p}{p-2}} \lesssim \lambda^{-\frac{1}{30}},$$
$$\|w^{(l)}\|_{\frac{2p}{p-2}} \lesssim \lambda^{-\frac{1}{30}}.$$

By the L^p boundedness of $\mathcal{R}\operatorname{div}$ and Hölder's inequality, we have

$$\|R_{\text{cor}}\|_1 \lesssim \|R_{\text{cor}}\|_p \lesssim_p \|\big((w^{(c)} + w^{(l)}) \otimes w\|_p + \|w^{(p)} \otimes (w^{(c)} + w^{(l)})\|_p \tag{5.16}$$
$$\lesssim \big(\|w^{(c)}\|_{\frac{2p}{p-2}} + \|w^{(l)}\|_{\frac{2p}{p-2}}\big)\|w\|_2 \tag{5.17}$$
$$\lesssim \lambda^{-\frac{1}{30}}\big(\rho_0^{1/2} + \lambda^{-\frac{1}{3}} + \lambda^{-\frac{1}{30}}\big). \tag{5.18}$$

Due to the negative exponent in λ on the right hand side, for any sufficiently large $\lambda(\varepsilon, e_0, e_1, \kappa)$ we have

$$\|R_{\text{cor}}\|_1 \leq \frac{\varepsilon}{8}.$$

□

Finally, we turn to estimating the oscillation error R_{osc}, where we will utilize the fact that *viscous eddies* are approximate stationary solutions of the NSE.

Lemma 5.8 (Decomposition of R_{osc})**.** *The oscillation error R_{osc} can be decomposed into two parts:*

$$R_{\text{osc}} = R_{\text{high}} + R_{\text{low}} + R_{\text{err}}, \tag{5.19}$$

where R_{high} is the high frequency part

$$R_{\text{high}} = \theta^2\mathcal{R}\sum_k \nabla(a_k)^2\mathbb{P}_{\neq 0}\mathbf{P}_\sigma\big(W_k \otimes W_z\big), \tag{5.20}$$

R_{low} consists of lower order terms

$$\begin{aligned} R_{\text{low}} =& \sigma\theta^2\mathcal{R}\sum_k a_k^2\mathbf{P}_\sigma\big(\operatorname{div}(W_k \otimes W_z) - \Delta\mathbb{V}_k\big) \\ &- \sigma^{-1}\theta^2\mathcal{R}\sum_k\Big[\Delta a_k^2\mathbf{P}_\sigma\mathbb{V}_k + 2\nabla a_k^2 \cdot \mathbf{P}_\sigma\nabla\mathbb{V}_k\Big], \end{aligned} \tag{5.21}$$

and R_{err} is the symmetry breaking error

$$R_{\text{err}} = \theta^2\mathcal{R}\operatorname{div}\sum_k a_k^2\Big(\mathbf{P}_\sigma\big(\mathbb{W}_k \otimes \mathbb{W}_k\big) - \mathbf{P}_\sigma\big(W_k \otimes W_z\big)\Big).$$

Proof. Since $\mathbb{W}_k$ has disjoint support in space, we have

$$w^{(p)} \otimes w^{(p)} = \theta^2 \sum_k (a_k)^2 \mathbf{P}_\sigma \big(\mathbb{W}_k \otimes \mathbb{W}_k \big),$$

which in view of Lemma 4.5 gives

$$\begin{aligned} w^{(p)} \otimes w^{(p)} - \theta^2 \sum_k a_k^2 \Big(\mathbf{P}_\sigma \big(\mathbb{W}_k \otimes \mathbb{W}_k \big) - \mathbf{P}_\sigma \big(W_k \otimes W_z \big) \Big) & \\ = R_{\mathrm{err}} + \theta^2(t) \sum_k a_k^2 \fint_{\mathbb{T}^3} \mathbf{P}_\sigma \big(W_k \otimes W_z \big) + \theta^2 \sum_k a_k^2 \Big(\mathbf{P}_\sigma \big(W_k \otimes W_z \big) - \fint_{\mathbb{T}^3} \mathbf{P}_\sigma \big(W_k \otimes W_z \big) \Big) & \\ = \theta^2 \rho \operatorname{Id} - \theta^2 R_0 + \theta^2 \sum_k (a_k)^2 \mathbb{P}_{\neq 0} \mathbf{P}_\sigma \Big(W_k \otimes W_z \Big). & \qquad (5.22) \end{aligned}$$

Upon taking the divergence on both sides of (5.22) we have for the oscillation error

$$\begin{aligned} R_{\mathrm{osc}} &= R_{\mathrm{err}} + \mathcal{R} \big(\operatorname{div} \theta^2 R_0 + \operatorname{div}(w^{(p)} \otimes w^{(p)}) - \nabla p_1 + \Delta w^{(l)} \big) \\ &= R_{\mathrm{err}} + \mathcal{R} \Big(\theta^2 \operatorname{div} \sum_k (a_k)^2 \mathbb{P}_{\neq 0} \mathbf{P}_\sigma \big(W_k \otimes W_z \big) + \Delta w^{(l)} \Big). \end{aligned}$$

By the product rule we may obtain

$$R_{\mathrm{osc}} = R_{\mathrm{err}} + R_{\mathrm{high}} + \mathcal{R} \Big(\sigma \theta^2 \sum_k a_k^2 \mathbf{P}_\sigma \operatorname{div} \big(W_k \otimes W_z \big) + \Delta w^{(l)} \Big). \tag{5.23}$$

It remains to compute the second term in (5.23). Using the definition of $w^{(l)}$, a routine computation gives

$$\Delta w^{(l)} = -\sigma \theta^2 \sum_k a_k^2 \mathbf{P}_\sigma \Delta \mathbb{V}_k - \theta^2 \sum_k \Big[\sigma^{-1} \Delta a_k^2 \mathbf{P}_\sigma \mathbb{V}_k + 2 \nabla a_k^2 \mathbf{P}_\sigma \nabla \mathbb{V}_k \Big],$$

which implies exactly

$$\mathcal{R} \Big(\sigma \theta^2 \sum_k a_k^2 \mathbf{P}_\sigma \operatorname{div} \big(W_k \otimes W_z \big) + \Delta w^{(l)} \Big) = R_{\mathrm{low}}.$$

Hence the oscillation error verifies the identity $R_{\mathrm{osc}} = R_{\mathrm{high}} + R_{\mathrm{low}} + R_{\mathrm{err}}$.

□

Remark 5.9. *The term R_{high} is typical in convex integration, where the derivative falls on "slow variable" a_k and the term $\mathbb{P}_{\neq 0} \mathbf{P}_\sigma \big(\mathbb{W}_k \otimes \mathbb{W}_k \big)$ has fast oscillation and zero mean. The presence of R_{low} and R_{err} is one the fundamental differences between our scheme and previous ones.*

We are ready to estimate the oscillation error. The term R_{high} will be able to gain a factor of σ^{-1} via the inverse divergence $\mathcal{R}$, while the term R_{low} is already quite small thanks to the inverse Laplacian. In other words, R_{high} is of high frequency, while R_{low} is not of high frequency but instead lower order.

Lemma 5.10 (Oscillation error: R_{high})**.** *For any λ sufficiently large,*

$$\big\| R_{\mathrm{high}} \big\|_{L^\infty_t L^1_x} \leq \frac{\varepsilon}{4}. \tag{5.24}$$

Proof. Throughout the proof, let us fix two parameters $0 < \alpha < 1$ and $1 < p < 2$, such that the Sobolev embedding $W^{\alpha,1}(\mathbb{T}^3) \hookrightarrow L^p(\mathbb{T}^3)$ holds.

It follows from the L^p boundedness of the Riezs transform that

$$\| R_{\mathrm{high}} \|_{L^1(\mathbb{T}^3)} \leq \| R_{\mathrm{high}} \|_{L^p(\mathbb{T}^3)} \lesssim \sum_k \Big\| |\nabla|^{-1} \big(\nabla (a_k^2) \mathbb{P}_{\neq 0} \mathbf{P}_\sigma \big(W_k \otimes W_z \big) \big) \Big\|_p. \tag{5.25}$$

Obviously $\mathbb{P}_{\neq 0} \mathbf{P}_\sigma \big(W_k \otimes W_z \big)$ is $\sigma^{-1} \mathbb{T}^3$-periodic and has zero mean, and by Lemma 4.5

$$\big\| \nabla a_k^2 \big\|_{C^m_x} \leq \big\| a_k^2 \big\|_{C^{m+1}_x} \leq \kappa^{m+3} \quad \text{for all } 0 \leq m \leq M.$$

Thus we may apply Proposition 4.7 with $C_a = \kappa^3$ to obtain that

$$\begin{aligned} \Big\| |\nabla|^{-1} \big(\nabla (a_k^2) \mathbb{P}_{\neq 0} \mathbf{P}_\sigma \big(W_k \otimes W_z \big) \big) \Big\|_p \lesssim & \sigma^{-1+\alpha} \Big\| |\nabla|^{-\alpha} \big(\nabla (a_k)^2 \mathbb{P}_{\neq 0} \mathbf{P}_\sigma \big(W_k \otimes W_z \big) \big) \Big\|_p \\ & + \kappa^3 \sigma^{-N} \big\| \mathbf{P}_\sigma \big(W_k \otimes W_z \big) \big\|_p. \end{aligned} \tag{5.26}$$

The first term in (5.26) can be estimated by the Sobolev embedding $W^{\alpha,1}(\mathbb{T}^3) \hookrightarrow L^p(\mathbb{T}^3)$, and Lemma 4.5 as follows:

$$\begin{aligned} \sigma^{-1+\alpha}\left\| |\nabla|^{-\alpha}\big(\nabla(a_k)^2\mathbb{P}_{\neq 0}\mathbf{P}_\sigma\big(W_k \otimes W_z\big)\big)\right\|_p &\lesssim \sigma^{-1+\alpha}\left\|a_k^2\right\|_{C^1_{t,x}}\left\|\mathbf{P}_\sigma(W_k \otimes W_z)\right\|_1 \\ &\lesssim \sigma^{-1+\alpha}\kappa^4\left\|\mathbf{P}_\sigma\big((W_z+W_r)\otimes W_z\big)\right\|_1. \end{aligned} \tag{5.27}$$

Now recall that $W_r \notin L^2$ due to the $1/r$ singularity on the Ω_k-plane, but $W_r \in L^p$ since $1 \leq p < 2$ (see (3.24)). Hence, Hölder's inequality, (3.23), and (3.24) imply

$$\begin{aligned} \left\|\mathbf{P}_\sigma\big((W_z+W_r)\otimes W_z\big)\right\|_1 &\lesssim \big(\|W_z\|_{L^p(\mathbb{R}^3)} + \|W_r\|_{L^p(\mathbb{R}^3)}\big)\|W_z\|_{L^{1-1/p}(\mathbb{R}^3)} \\ &\lesssim_p \big(\mu^{1-2/p}\tau^{1/2-1/p} + \mu^{-2/p}\tau^{3/2-1/p}\big)\mu^{-1+2/p}\tau^{-1/2+1/p} \\ &= \mu^0\tau^0 + \mu^{-1}\tau^1 \\ &\lesssim 1. \end{aligned} \tag{5.28}$$

The second term in (5.26) can be handled easily using Proposition 3.14 and $N = 300$,

$$\kappa^3\sigma^{-N}\left\|\mathbf{P}_\sigma\big(W_k \otimes W_z\big)\right\|_p \lesssim \kappa^3\lambda^{-10}\big(\|W_z\|_{L^p(\mathbb{R}^3)} + \|W_r\|_{L^p(\mathbb{R}^3)}\big)\|W_z\|_{L^\infty(\mathbb{R}^3)} \lesssim \kappa^3\lambda^{-1}. \tag{5.29}$$

Collecting (5.25), (5.26), (5.27), (5.28), and (5.29) we arrive at

$$\left\|R_{\text{high}}\right\|_1 \lesssim (\kappa^3+\kappa^4)\sigma^{-1+\alpha}.$$

As $0<\alpha<1$, for all $\lambda(\varepsilon,\kappa)$ sufficiently large we can conclude that

$$\left\|R_{\text{high}}\right\|_{L^\infty_t L^1_x} \leq \frac{\varepsilon}{8}.$$

□

Lemma 5.11 (Oscillation error: R_{low})**.** *For any λ sufficiently large*

$$\left\|R_{\text{low}}\right\|_{L^\infty_t L^1_x} \leq \frac{\varepsilon}{8}. \tag{5.30}$$

Proof. Let us fix $p>1$ such that

$$\sigma\tau^2\mu^{-1}\big(\tau^{1-1/p}\mu^{2-2/p}\big) \leq \lambda^{-\frac{1}{30}}. \tag{5.31}$$

So by the boundedness of $\mathcal{R}$ on L^p and Hölder's inequality, we have

$$\begin{aligned} \left\|R_{\text{low}}\right\|_{L^1(\mathbb{T}^3)} \leq \left\|R_{\text{low}}\right\|_{L^p(\mathbb{T}^3)} &\lesssim \sum_k \sigma\left\|a_k^2\right\|_{L^\infty_{t,x}}\left\|\mathbf{P}_\sigma\big(\operatorname{div}(W_k\otimes W_z) - \Delta\mathbb{V}_k\big)\right\|_p \\ &\quad + \sigma^{-1}\left\|a_k^2\right\|_{C^2_{t,x}}\left\|\mathbf{P}_\sigma\mathbb{V}_k\right\|_p + \sigma^{-1}\left\|a_k^2\right\|_{C^1_{t,x}}\left\|\mathbf{P}_\sigma\nabla\mathbb{V}_k\right\|_p \end{aligned}$$

Thanks to Proposition 3.11,

$$\left\|\mathbf{P}_\sigma\big(\operatorname{div}(W_k\otimes W_z) - \Delta\mathbb{V}_k\big)\right\|_p \lesssim \tau^2\mu^{-1}\big(\tau^{1-1/p}\mu^{2-2/p}\big).$$

Combining this with the estimates in Proposition 3.14 and Lemma 4.5, it follows that

$$\begin{aligned} \left\|R_{\text{low}}\right\|_1 &\lesssim \big(\kappa^2\sigma\tau^2\mu^{-1} + \kappa^4\sigma^{-1}\tau\mu^{-2} + \kappa^3\sigma^{-1}\tau\mu^{\gamma-1}\big)\big(\tau^{1-1/p}\mu^{2-2/p}\big) \\ &\lesssim \big(\kappa^2+\kappa^3+\kappa^4\big)\sigma\tau^2\mu^{-1}\big(\tau^{1-1/p}\mu^{2-2/p}\big), \end{aligned} \tag{5.32}$$

where we used $\mu^\gamma = \sigma \leq \sigma^2\tau$ for the third term.

Using (5.31) and taking $\lambda(\kappa,\varepsilon)$ sufficiently large, the desired bound follows:

$$\left\|R_{\text{low}}\right\|_1 \leq \frac{\varepsilon}{8}.$$

□

Lemma 5.12 (Symmetry breaking error: R_{err})**.**

$$\left\|R_{\text{err}}\right\|_{L^\infty_t L^1_x} \leq \frac{\varepsilon}{8}.$$

Proof. We fix $1<p<2$ so that $\mu^{-\gamma}\mu^{2-2/p}\tau^{1-1/p} \leq \mu^{-\gamma/2}$. Recall that $\mathcal{R}\operatorname{div}$ is bounded on L^p. Then

$$\left\|R_{\text{err}}\right\|_{L^1(\mathbb{T}^3)} \leq \left\|R_{\text{err}}\right\|_{L^p(\mathbb{T}^3)} \lesssim \sum_k \left\|a_k^2\right\|_{L^\infty_{t,x}}\left\|\mathbf{P}_\sigma\big(\mathbb{W}_k\otimes\mathbb{W}_k\big) - \mathbf{P}_\sigma\big(W_k\otimes W_z\big)\right\|_p.$$

Now using Lemma 4.5 and Proposition 3.9, we obtain

$$\begin{aligned}\left\|R_{\text{err}}\right\|_1 &\lesssim \kappa^2\mu^{-\gamma}\mu^{2-2/p}\tau^{1-1/p}\\ &\lesssim \kappa^2\mu^{-\gamma/2}\\ &\leq \frac{\varepsilon}{8},\end{aligned}$$

for $\lambda(\varepsilon,\kappa)$ large enough. □

Note that Lemma 5.5 is proved, as it follows directly from Lemma 5.6, 5.7, 5.10, 5.11, and 5.12.

6. Proof of iteration lemma: energy level

In this section, we prove properties related to the energy in the main proposition. To show the correct energy level of the solution v, let us first show that the energy in the perturbation w is dominated by $w_z^{(p)}$, which is anticipated in view of the estimates in Proposition 4.9.

Lemma 6.1. *For any λ sufficiently large*

$$\left|\|v(t)\|_2^2 - \|v_0(t)\|_2^2 - \|w_z^{(p)}(t)\|_2^2\right| \leq 10^{-7}(e_1-e_0) \quad \text{for all } t\in[0,T]. \tag{6.1}$$

Proof. Since $w = w_z^{(p)} + w_r^{(p)} + w^{(l)} + w^{(c)}$, we have

$$\|v(t)\|_2^2 - \|v_0(t)\|_2^2 - \|w_z^{(p)}(t)\|_2^2 = E_{\text{error}}$$

where the error term E_{error} is

$$E_{\text{error}} = 2\langle w, v_0\rangle + 2\langle w_z^{(p)}, w_r^{(p)} + w^{(c)} + w^{(l)}\rangle + \|w_r^{(p)} + w^{(c)} + w^{(l)}\|_2^2.$$

Fix any $1<p<2$. By Hölder's inequality, we have

$$\left|E_{\text{error}}\right| \lesssim \|w(t)\|_p\|v_0(t)\|_{\frac{p}{p-1}} + \left(\|w_r^{(p)}\|_2 + \|w^{(c)}\|_2 + \|w^{(l)}\|_2\right)\|w_z^{(p)}\|_2 + \|w_r^{(p)}\|_2^2 + \|w^{(c)}\|_2^2 + \|w^{(l)}\|_2^2.$$

Thanks to Corollary 4.10, for any sufficiently large $\lambda(e_1,\kappa,v_0)$ we have

$$\begin{aligned}\|w^{(c)}\|_2^2 + \|w^{(l)}\|_2^2 &\lesssim \lambda^{-\frac{3}{10}},\\ \|w_r^{(p)}\|_2 &\lesssim \rho_0^{1/2}\lambda^{-\frac{7}{15}}\lambda^{\frac{17}{15}(1-\frac{2}{p})},\\ \|w_z^{(p)}\|_2 &\lesssim \|w^{(p)}\|_2 + \|w_r^{(p)}\|_2 \lesssim \rho_0^{1/2},\\ \|w\|_p &\lesssim (\rho_0^{1/2} + \lambda^{-\frac{1}{30}})\lambda^{\frac{17}{15}(1-\frac{2}{p})}.\end{aligned}$$

Since $\rho_0(t)\lesssim e_1$, for any sufficiently large $\lambda(e_1,e_0,\kappa,v_0)$, we can make sure that

$$\left|E_{\text{error}}\right| \leq 10^{-7}(e_1-e_0).$$

□

Next, we estimate the energy of $w^{(p)}$ more precisely than Proposition 4.9. Note that the choice of ρ_0, namely (4.3), is crucial in the proof. Recall that $\widetilde{e}_1 = e_1 - 10^{-6}(e_1-e_0)$.

Lemma 6.2. *Suppose that the constant C in the statement of Proposition 2.1 is small enough. For any λ sufficiently large, the energy of $w^{(p)}$ verifies*

$$\left|\|w_z^{(p)}\|_2^2 - \theta^2(\widetilde{e}_1 - \|v_0\|_2^2)\right| \leq 10^{-7}(e_1-e_0) \quad \text{for all } t\in[0,T].$$

Proof. First, note that as in (3.30) and (3.31),

$$\|W_z - (W_z)_\gamma\|_2 \lesssim \mu^{-1-\gamma}\|W_z\|_{H^1} \lesssim \mu^{-2-\gamma}\tau = \lambda^{-1/3}.$$

Hence, thanks to Lemma 4.5,

$$\left|\|w_z^{(p)}\|_2 - \Big\|\theta a_k\sum_k \mathbf{P}_\sigma W_z\Big\|_2\right| \lesssim \Big\|\theta a_k\sum_k \mathbf{P}_\sigma(W_z - (W_z)_\gamma)\Big\|_2 \lesssim \kappa\lambda^{-1/3},$$

and consequently

$$\left|\|w_z^{(p)}\|_2 - \Big\|\theta a_k\sum_k \mathbf{P}_\sigma W_z\Big\|_2\right| \leq 10^{-8}(e_1-e_0) \tag{6.2}$$

for $\lambda(e_0,e_1,\kappa)$ large enough. Now recall that

$$\fint_{\mathbb{T}^3}\mathbf{P}_\sigma(W_k\otimes W_z) = \fint_{\mathbb{T}^3}\mathbf{P}_\sigma(W_z\otimes W_z).$$

Thus, similarly to (5.22), we obtain

$$\theta^2 \sum_k a_k^2 \fint_{\mathbb{T}^3} \mathbf{P}_\sigma(W_z \otimes W_z) = \theta^2 \sum_k a_k^2 \fint_{\mathbb{T}^3} \mathbf{P}_\sigma(W_z \otimes W_z) + \sum_k a_k^2 \Big(\mathbf{P}_\sigma(W_z \otimes W_z) - \fint_{\mathbb{T}^3} \mathbf{P}_\sigma(W_z \otimes W_z) \Big)$$
$$= \theta^2 \rho \operatorname{Id} - \theta^2 R_0 + \theta^2 \sum_k (a_k)^2 \mathbb{P}_{\neq 0} \mathbf{P}_\sigma \Big(W_z \otimes W_z \Big).$$

Upon taking the trace and integrating in space, it follows that

$$\Big\| \theta a_k \sum_k \mathbf{P}_\sigma W_z \Big\|_2^2 = 3\theta^2 \int_{\mathbb{T}^3} \rho(x,t) + \theta^2 \sum_k \int_{\mathbb{T}^3} (a_k)^2 \mathbb{P}_{\neq 0} \mathbf{P}_\sigma \operatorname{Tr} \big(W_z \otimes W_z \big),$$

Using the definition of ρ_0 (4.3), we can consider the split

$$\Big\| \theta a_k \sum_k \mathbf{P}_\sigma W_z \Big\|_2^2 - \theta^2 (\widetilde{e}_1 - \|v_0\|_2^2) = X_l + X_h, \tag{6.3}$$

where X_l is the low frequency error term

$$X_l = 3\theta^2 \int_{\mathbb{T}^3} \rho(x,t) - \theta^2 (\widetilde{e}_1 - \|v_0\|_2^2), \tag{6.4}$$

and X_h is the high frequency error term

$$X_h = \theta^2 \int_{\mathbb{T}^3} (a_k)^2 \mathbb{P}_{\neq 0} \mathbf{P}_\sigma \operatorname{Tr} \big(W_z \otimes W_z \big). \tag{6.5}$$

The goal is to show that $|X_l| + |X_h| \le 10^{-7}(e_1 - e_0)$. Let us first estimate the term X_h. Using a standard integration by parts argument, we have[6]

$$\big| X_h \big| \lesssim \sum_k \|a_k^2\|_{C^M_{t,x}} \big\| |\nabla|^{-M} \mathbb{P}_{\neq 0} \mathbf{P}_\sigma \operatorname{Tr} \big(W_z \otimes W_z \big) \big\|_2, \tag{6.6}$$

where M is as defined in Definition 4.1. Since $\mathbb{P}_{\neq 0} \mathbf{P}_\sigma \operatorname{Tr} \big(\mathbb{W}_k \otimes \mathbb{W}_k \big)$ is $\sigma^{-1}\mathbb{T}$-periodic and of zero mean, we have

$$\begin{aligned} \big\| |\nabla|^{-M} \mathbb{P}_{\neq 0} \mathbf{P}_\sigma \operatorname{Tr} \big(W_z \otimes W_z \big) \big\|_2 &\lesssim \sigma^{-M+3} \big\| |\nabla|^{-3} \mathbb{P}_{\neq 0} \mathbf{P}_\sigma \operatorname{Tr} \big(W_z \otimes W_z \big) \big\|_2 \\ &\lesssim \sigma^{-M+3} \big\| \mathbb{P}_{\neq 0} \mathbf{P}_\sigma \operatorname{Tr} \big(W_z \otimes W_z \big) \big\|_1 \\ &\lesssim \sigma^{-M+3} \big\| W_z \big\|_{L^2(\mathbb{R}^3)}^2 \\ &\lesssim \sigma^{-M+3}, \end{aligned}$$

where the second inequality follows from the Sobolev embedding $H^{-3}(\mathbb{T}^3) \hookrightarrow L^1(\mathbb{T}^3)$, and the last inequality follows from Proposition 3.14. Combining this with (6.6) and using Lemma 4.5, we get

$$\big| X_h \big| \lesssim \|a_k^2\|_{C^M_{t,x}} \sigma^{-M+3} \lesssim \kappa^{M+2} \sigma^{-M+3}. \tag{6.7}$$

Hence for sufficiently large $\lambda(e_0, e_1, \kappa)$, we can ensure that

$$\big| X_h \big| \le 10^{-8}(e_1 - e_0). \tag{6.8}$$

On the other hand, for the term X_l using the definitions of ρ and ρ_0 (namely (4.5) and (4.3)) we get

$$X_l = -12\theta^2 \rho_0 \left(1 - \int \chi^2(\rho_0^{-1} R_0) \right)$$

First, Let us split the integral

$$\int \chi^2(\rho_0^{-1} R_0) = \left(\int_{|R_0| \le \rho_0} + \int_{|R_0| \ge \rho_0} \right) \chi^2(\rho_0^{-1} R_0).$$

Next, by the above split we have

$$\big| X_l \big| \lesssim \rho_0 \left| 1 - \int_{|R_0| \le \rho_0} \chi^2(\rho_0^{-1} R_0) \right| + \rho_0 \left| \int_{|R_0| \ge \rho_0} \chi^2(\rho_0^{-1} R_0) \right|. \tag{6.9}$$

Since $\delta_0 = \|R_0\|_{L^\infty_t L^1_x}$, thanks to the Chebyshev inequality we have

$$\big| \{ x \in \mathbb{T}^3 : |R_0| \ge \rho_0 \} \big| \le \frac{\delta_0}{\rho_0},$$

[6] Recall that $\|a_k\|_{C^m_{t,x}} \le \kappa^{m+1}$ is only valid for $0 \le m \le 4M$.

which together with the definition of χ in (4.2) and the fact that $|\mathbb{T}^3| = 1$ implies that

$$
\begin{aligned}
|X_l| &\lesssim \rho_0 \left| 1 - \int_{|R_0|\le \rho_0} 1 dx \right| + \rho_0 \int_{|R_0|\ge \rho_0} \rho_0^{-1} |R_0| \\
&\lesssim \rho_0 \left| \int_{|R_0|>\rho_0} 1 dx \right| + \int_{|R_0|\ge \rho_0} |R_0| \\
&\lesssim \delta_0 .
\end{aligned}
$$

Note that in the estimates for X_l, all implicit constants are universal. In view of the assumption $\delta_0 \le C(e_1 - e_0)$ in the statement of Proposition 2.1, we may choose the constant C small enough such that

$$
|X_l| \le 10^{-8}(e_1 - e_0). \tag{6.10}
$$

Combining (6.2), (6.8), and (6.10) with (6.3) we obtain

$$
\left| \|w^{(p)}\|_2^2 - \theta^2(\widetilde{e}_1 - \|v_0\|_2^2) \right| \le 10^{-7}(e_1 - e_0). \tag{6.11}
$$

□

With the help of Lemma 6.1 and 6.2, we obtain the desire energy level of the new solution v as a corollary.

Corollary 6.3. *Suppose that the constant C in the statement of Proposition 2.1 is small enough. For any λ sufficiently large, the energy of new solution $v(t)$ verifies*

$$
\sup_t \|v(t)\|_2^2 \le e_1,
$$

and

$$
\left| \|v(t)\|_2^2 - e_1 \right| \le \frac{c_0}{2}(e_1 - e_0) \quad \textit{for all } t \in I_{4^{-1}r}(\mathcal{F}_1).
$$

Proof. Both bounds immediately follow from Lemma 6.1, 6.2 and the facts that $\widetilde{e}_1 = e_1 - 10^{-6}(e_1 - e_0)$ and $\theta = 1$ on $I_{4^{-1}r}(\mathcal{F}_1)$. □

Appendix A. Del formulae in cylindrical coordinates

In this appendix, we collect some useful vector calculus identities concerning the cylindrical coordinates (see for example [Ach90]).

Let f be a scaler function. The gradient of f

$$
\nabla f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_r + \frac{\partial f}{\partial z}\mathbf{e}_z. \tag{A.1}
$$

For vector field $A = A_r\mathbf{e}_r + A_\theta\mathbf{e}_\theta + A_z\mathbf{e}_z$, its divergence

$$
\operatorname{div} A = \frac{1}{r}\frac{\partial(rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}, \tag{A.2}
$$

and curl

$$
\begin{aligned}
\nabla \times A = &\Big(\frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}\Big)\mathbf{e}_r \\
&+ \Big(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\Big)\mathbf{e}_\theta \\
&+ \frac{1}{r}\Big(\frac{\partial(rA_r)}{\partial r} - \frac{\partial A_r}{\partial \theta}\Big)\mathbf{e}_z.
\end{aligned} \tag{A.3}
$$

For two vector field A and B, the material derivative

$$
\begin{aligned}
(A\cdot\nabla)B = &\Big(A_r\frac{\partial B_r}{\partial r} + \frac{A_\theta}{r}\frac{\partial B_r}{\partial \theta} + A_z\frac{\partial B_r}{\partial z} - \frac{A_\theta B_\theta}{r}\Big)\mathbf{e}_r \\
&+ \Big(A_r\frac{\partial B_\theta}{\partial r} + \frac{A_\theta}{r}\frac{\partial B_\theta}{\partial \theta} + A_z\frac{\partial B_\theta}{\partial z} + \frac{A_\theta B_r}{r}\Big)\mathbf{e}_\theta \\
&+ \Big(A_r\frac{\partial B_z}{\partial r} + \frac{A_\theta}{r}\frac{\partial B_z}{\partial \theta} + A_z\frac{\partial B_z}{\partial z}\Big)\mathbf{e}_z.
\end{aligned} \tag{A.4}
$$

APPENDIX B. DECAY ESTIMATES FOR THE POSSION EQUATION

Here we derive some decay estimates for solutions of the planar Poisson equation. Let $f \in C_c^\infty(\mathbb{R}^2)$ be radially symmetric with zero mean

$$\int_{\mathbb{R}^2} f\,dx = 0. \tag{B.1}$$

We show that

Lemma B.1. *Let h be the solution of*

$$\Delta h = f \quad \text{on } \mathbb{R}^2, \tag{B.2}$$

such that $|h| \to 0$ as $x \to \infty$. Then h is radially symmetric and $h \in W^{1,p}(\mathbb{R}^2)$ for $1 < p \le \infty$.

Proof. Since the solution h is given explicitly by the Newton potential

$$h(x) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \ln(|x-y|)f(y)\,dy, \tag{B.3}$$

we only need to verify the decay estimates.

The first decay $|h| \to 0$ as $x \to \infty$ follows from removing the mean

$$h(x) = -\frac{1}{2\pi}\int (\ln(|x-y|) - \ln(x))f(y)\,dy,$$

and the Mean Value Theorem.

To show that $h \in W^{1,p}(\mathbb{R}^2)$ for $1 < p \le \infty$, let us consider the Taylor expansion of $\ln(|x-y|)$

$$\ln(|x-y|) = \ln(|x|) - \frac{x\cdot y}{|x|^2} + \sum_{|\beta|=2} R_\beta(x,y)y^\beta, \tag{B.4}$$

where the remainder is given by

$$R_\beta(x,y) = \int_0^1 (1-t)D^\beta g(x-ty)\,dt, \tag{B.5}$$

with $g(x) = \ln(|x|)$ and $|\nabla^2 g| \lesssim \frac{1}{|x|^2}$.

Let us show that $h \in L^p$ for $1 < p \le \infty$. Since f has zero mean and zero first moment due to radial symmetry, combining (B.4) and (B.3) we have

$$h(x) = -\frac{1}{2\pi}\sum_{\beta=2}\int R_\beta(x,y)y^\beta f(y)\,dy. \tag{B.6}$$

Then by Minkowski's inequality, we have

$$\|h\|_{L^p(\mathbb{R}^2)} \lesssim \sum_{\beta=2}\int \left(\int \Big|R_\beta(x,y)\Big|^p dx\right)^{\frac{1}{p}} |f(y)||y|^2 dy. \tag{B.7}$$

To estimate $R_\beta(x,y)$, we use Minkowski's inequality once again

$$\int \Big|R_\beta(x,y)\Big|^p dx \lesssim \left[\int_0^1 \left(\int \Big|D^\beta g(x-ty)\Big|^p dx\right)^{\frac{1}{p}} dt\right]^p.$$

Note that from definition,

$$\Big|D^\beta g(x-ty)\Big| \lesssim_\beta \frac{1}{|x-ty|^2} \tag{B.8}$$

and we get

$$\int \Big|R_\beta(x,y)\Big|^p dx \lesssim_p 1, \quad \text{for } 1 < p \le \infty,$$

which implies

$$\|h\|_{L^p(\mathbb{R}^2)} < \infty, \quad \text{for } 1 < p \le \infty.$$

The claim that $\nabla h \in L^p$ for $1 < p \le \infty$ is easier since differentiating (B.3) already gives a decay of $1/|x|$ in the kernel, and in this case just removing the mean is sufficient. $\square$

Appendix C. Essential discontinuities by Buckmaster-Vicol solutions

In this section, we show that it is possible to use the weak solution constructed in [BV19] to obtained essential discontinuities of positive measure in the energy profile. First, recall

Theorem C.1 (Theorem 1.2 of [BV19])**.** *There exists $\beta > 0$, such that for any nonnegative smooth function $e(t) : [0,T] \to \mathbb{R}^+$, there exists $v \in C([0,T]; H^\beta(\mathbb{T}^3))$ a weak solution of the Navier-Stokes equations, such that $\int_{\mathbb{T}^3} |v(x,t)|^2 dx = e(t)$ for all $t \in [0,T]$.*

Let $e(t)$ be a nonnegative bump function supported on $(1/2,1)$ such that $\max_t e(t) = 1$. Consider a weak solution $u \in C((0,1]; L^2(\mathbb{T}^3))$ such that on each interval $[2^{-n-1}, 2^{-n}]$, $u(t)$ is the Buckmaster-Vicol solution with energy profile $e(2^n t)$. As a consequence, we have

$$\liminf_{t\to 0-} \|u(t)\|_2^2 = 0, \qquad \limsup_{t\to 0-} \|u(t)\|_2^2 = 1.$$

Such an example does not extend to the whole interval $[0,1]$ as Theorem C.1 on its own does not guarantee the existence of the weak limit as $t \to 0+$ since there are no other available bounds as opposed to in the proof of Theorem 1.6 where we used (2.24).

However, we can modify this construction in the following way. Consider a Buckmaster-Vicol solution $u_n(t)$ on $[1/2,1]$ with the energy profile $e_n(t) = 2^{-2n} e(t)$ and define (on $\mathbb{T}^3$)

$$u(t) = \sum_{n=0}^{\infty} 2^n u_n(2^n x, 2^{2n} t).$$

Then $u(t)$ is weakly continuous at $t = 0$, as the weak limit is zero. And it is a weak solution on $[0,1]$ with energy bounded by 1. Moreover,

$$\liminf_{t\to 0+} \|u(t)\|_2 = 0, \qquad \limsup_{t\to 0+} \|u(t)\|_2 = 1.$$

Using a similar argument in the proof of Theorem 1.6, one can also use Buckmaster-Vicol solutions to obtain weak solutions whose discontinuities have positive measure in time. Note that this method does not produce jump discontinuities nor density of the set of discontinuities since the resulting solution is "intermittent" on the time interval.

Appendix D. Proof of Proposition 4.7

We include a proof of Proposition 4.7 . Let us recall the following result on the Hölder norms of composition of functions. A proof using the multivariable chain rule can be found in [DLS14].

Proposition D.1. *Let $F : \Omega \to \mathbb{R}$ be a smooth function with $\Omega \subset \mathbb{R}^d$. For any smooth function $u : \mathbb{R}^d \to \Omega$ and any $1 \le m \in \mathbb{N}$ we have*

$$\|\nabla^m (F \circ u)\|_\infty \lesssim \|\nabla^m u\|_\infty \sum_{1 \le i \le m} \|\nabla^i F\|_\infty \|u\|_\infty^{i-1} \tag{D.1}$$

where the implicit constant depends on m, d.

Proof of Proposition 4.7. We present a proof in the d-dimensional case. By considering $\widetilde{a} := \frac{1}{C_a} a$ it suffices to prove both of the results for $C_a = 1$. Notice that since $p \ge 2$ is even, the function $|a|^p$, which is a composition of $a : \mathbb{T}^d \to [-1,1]$ and x^p, is smooth. Therefore, applying Proposition D.1 we see that

$$\begin{aligned} \|\nabla^m |a|^p\|_\infty &\lesssim_p \|\nabla^m a\|_\infty + \sum_{i \le m} \|\nabla a\|_\infty^{i-1} \\ &\lesssim_p \mu^m \end{aligned} \qquad \text{for any } m \in \mathbb{N}.$$

We can now introduce the split:

$$\|af\|_p^p = \int_{\mathbb{T}^d} (a^p - \overline{|a|^p})(|f|^p - \overline{|f|^p})\, dx + \|a\|_p^p \|f\|_p^p,$$

where $\overline{\cdot}$ denotes the integral over $\mathbb{T}^d$. By Parseval's theorem, we get[7]

$$\|af\|_p^p \le \left| \int_{\mathbb{T}^d} |\nabla|^M (a^p - \overline{|a|^p}) |\nabla|^{-M} (|f|^p - \overline{|f|^p}) dx \right| + \|a\|_p^p \|f\|_p^p.$$

[7] The nonlocal operators $|\nabla|^s$ and $|\nabla|^{-s}$ are defined respectively by multipliers with symbols $|k|^s$ and $|k|^{-s}$ for $k \ne 0$ and zero for $k = 0$.

We need show the first term is very small. By Hölder's inequality:

$$\left|\int_{\mathbb{T}^d} |\nabla|^M (a^p - \overline{|a|^p})|\nabla|^{-M}(|f|^p - \overline{|f|^p})dx\right| \lesssim \big\||\nabla|^M a^p\big\|_2 \big\||\nabla|^{-M}(|f|^p - \overline{|f|^p})\big\|_2. \tag{D.2}$$

By the L^2 boundedness of Riesz transform we can replace the nonlocal $|\nabla|^M$ by ∇^M to obtain

$$\begin{aligned}\big\||\nabla|^M a^p\big\|_2 &\lesssim \big\|\nabla^M a^p\big\|_2 \\ &\leq \big\|\nabla^M a^p\big\|_\infty \\ &\lesssim \mu^M.\end{aligned} \tag{D.3}$$

We turn to estimate the second factor in (D.2). Considering the fact that the function $(|f|^p - \overline{|f|^p})$ is zero-mean and $\sigma^{-1}\mathbb{T}^d$-periodic we have

$$\begin{aligned}\Big\||\nabla|^{-M}(|f|^p - \overline{|f|^p})\Big\|_2 &\lesssim \sigma^{-M+d}\Big\||\nabla|^{-d}(|f|^p - \overline{|f|^p})\Big\|_2 \\ &\lesssim \sigma^{-M+d}\Big\|(|f|^p - \overline{|f|^p})\Big\|_1 \\ &\lesssim \sigma^{-M+d}\|f\|_p^p,\end{aligned}$$

where the first inequality is a direct consequence of the Littlewood-Paley theory and the second inequality follows from the Sobolev embedding $L^1(\mathbb{T}^d) \hookrightarrow H^d(\mathbb{T}^d)$.

Combining this with estimates (D.2) and (D.3) we find that

$$\left|\int_{\mathbb{T}^d} |\nabla|^M (a^p - \overline{|a|^p})|\nabla|^{-M}(|f|^p - \overline{|f|^p})\,dx\right| \lesssim \sigma^{-M+d}\mu^M\|f\|_p^p.$$

By the assumption $\mu \leq \sigma^{1-\theta}$, there exists a number $M_{\theta,p,N,d} \in \mathbb{N}$ sufficiently large so that

$$\sigma^{-M+d}\mu^M \leq \sigma^{-Np}. \tag{D.4}$$

Then we have

$$\left|\int_{\mathbb{T}^d} |\nabla|^M (a^p - \overline{|a|^p})|\nabla|^{-M}(|f|^p - \overline{|f|^p})\,dx\right| \lesssim \sigma^{-Np}\|f\|_p^p,$$

which finishes the proof of (4.19) due to the elementary inequality $(a^p + b^p) \leq (a+b)^p$.

To prove (4.20) let us first recall the wavenumber projection. For any $\lambda \in \mathbb{R}$ define $\mathbb{P}_{\leq\lambda} = \sum_{q:2^q\leq\lambda}\Delta_q$ and $\mathbb{P}_{\geq\lambda} = \mathrm{Id} - \mathbb{P}_{\leq\lambda}$, where Δ_q is the Littlewood-Paley projection. Consider the following decomposition:

$$\begin{aligned}|\nabla|^{-1}(af) &= |\nabla|^{-1+s}|\nabla|^{-s}\big(\mathbb{P}_{\leq 2^{-4}\sigma}a\big)f + |\nabla|^{-1+s}|\nabla|^{-s}\big(\mathbb{P}_{\geq 2^{-4}\sigma}a\big)f \\ &:= |\nabla|^{-1+s}A_1 + |\nabla|^{-1+s}A_2\end{aligned}$$

For the term A_1, since f is $\sigma^{-1}\mathbb{T}^d$-periodic and zero-mean, it follows that

$$\mathbb{P}_{\geq 2^{-1}\sigma}f = f$$

and then by the support of Fourier modes of $\big(\mathbb{P}_{\leq 2^{-4}\sigma}a\big)f$ we have

$$\mathbb{P}_{\leq 2^{-2}\sigma}\Big[\mathbb{P}_{\leq 2^{-4}\sigma}af\Big] = 0 \quad \text{and} \quad \fint_{\mathbb{T}^d}\mathbb{P}_{\leq 2^{-4}\sigma}af = 0$$

which implies that

$$|\nabla|^{-1+s}A_1 = |\nabla|^{-1+s}\mathbb{P}_{\geq 2^{-2}\sigma}A_1.$$

By the Littlewood-Paley theory, we have

$$\Big\||\nabla|^{-1+s}\mathbb{P}_{\geq 2^{-2}\sigma}\Big\|_{L^p\to L^p} \lesssim_p \sigma^{-1+s}, \qquad 1 < p < \infty.$$

So, we have

$$\big\||\nabla|^{-1+s}A_1\big\|_p \lesssim_p \sigma^{-1+s}\Big\||\nabla|^{-s}\big(\mathbb{P}_{\leq 2^{-4}\sigma}af\big)\Big\|_p.$$

To get the exact form of the estimate, noticing that $|\nabla|^{-s}$ is bounded on L^p, $1 < p < \infty$, we conclude that

$$\begin{aligned}\big\||\nabla|^{-1+s}A_1\big\|_p &\leq \sigma^{-1+s}\big\||\nabla|^{-s}(af)\big\|_p + \sigma^{-1+s}\big\||\nabla|^{-s}\big(\mathbb{P}_{\geq 2^{-4}\sigma}af\big)\big\|_p \\ &\lesssim \sigma^{-1+s}\big\||\nabla|^{-s}(af)\big\|_p + \sigma^{-1+s}\big\|\mathbb{P}_{\geq 2^{-4}\sigma}a\big\|_\infty\|f\|_p.\end{aligned} \tag{D.5}$$

Similarly for A_2, since $|\nabla|^{-1}$ is bounded on L^p, we have

$$\big\||\nabla|^{-1+s}A_2\big\|_p = \big\||\nabla|^{-1}\big(\mathbb{P}_{\geq 2^{-4}\sigma}a\big)f\big\|_p \lesssim \big\|\mathbb{P}_{\geq 2^{-4}\sigma}af\big\|_p \leq \big\|\mathbb{P}_{\geq 2^{-4}\sigma}a\big\|_\infty\|f\|_p.$$

So it suffices to show $\|\Delta_q a\|_\infty \lesssim 2^{-Nq}$ for all $2^q \geq 2^{-4}\sigma$. Recall from the definition of the periodic Littlewood-Paley projection that

$$\Delta_q a = \int_{\mathbb{T}^d} \varphi_q(x-y)a(y)dy,$$

where the frequency cutoffs satisfy

$$\big\||\nabla|^{-M}\varphi_q\big\|_2 \lesssim 2^{-qM}\|\varphi_q\|_2 \lesssim 2^{-qM+qd}. \tag{D.6}$$

By Parseval's theorem and Young's inequality,

$$\begin{aligned}\|\Delta_q a\|_\infty &= \left\|\int_{\mathbb{T}^d} |\nabla|^{-M}\varphi_q(\cdot - y)|\nabla|^M a(y)\,dy\right\|_\infty \\ &\leq \big\||\nabla|^{-M}\varphi_q\big\|_2 \big\||\nabla|^M a\big\|_2.\end{aligned}$$

From L^2 boundedness of Riesz transform and the assumption on a it follows

$$\||\nabla|^M a\|_2 \lesssim \|\nabla^M a\|_2 \lesssim \|\nabla^M a\|_\infty \leq \mu^M, \tag{D.7}$$

where we used $C_a = 1$. Thus, combining estimates (D.7) and (D.6) we find

$$\begin{aligned}\|\Delta_q a\|_\infty &\lesssim 2^{qd}\mu^M 2^{-qM} \\ &\leq 2^{qd}\sigma^{(1-\theta)M}2^{-qM},\end{aligned}$$

where we used the fact that $\mu \leq \sigma^{1-\theta}$. Now choosing λ_0 large enough so that $\sigma^{\theta/2} \geq 2^{4(1-\theta/2)}$ for all $\sigma \geq \lambda_0$, we obtain

$$\begin{aligned}\|\Delta_q a\|_\infty &\leq 2^{qd}\sigma^{(1-\theta/2)M}2^{-4(1-\theta/2)}2^{-qM} \\ &\leq 2^{qd}2^{-q\theta M/2},\end{aligned} \tag{D.8}$$

provided $2^q \geq 2^{-4}\sigma$. Choosing any $M \geq 2(N-d)/\theta$, in view of (D.8), we have

$$\|\Delta_q a\|_\infty \lesssim 2^{-Nq} \quad \text{for all } 2^q \geq 2^{-4}\sigma.$$

After taking a summation in q for $2^q \geq 2^{-4}\sigma$ we obtain

$$\|\mathbb{P}_{\geq 2^{-4}\sigma} a\|_\infty \lesssim \sigma^{-N}.$$

Then collecting all the estimates, we have

$$\begin{aligned}\big\||\nabla|^{-1+s}(af)\big\|_p &\leq \big\||\nabla|^{-1+s}A_1\big\|_p + \big\||\nabla|^{-1+s}A_2\big\|_p \\ &\lesssim \sigma^{-1+s}\||\nabla|^{-s}(af)\|_p + \sigma^{-N}\|f\|_p.\end{aligned}$$

□

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