Boundary layer for the Navier-Stokes-alpha model of fluid turbulence ⋆

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Abstract

We study boundary layer turbulence using the Navier-Stokes-alpha model obtaining an extension of the Prandtl equations for the averaged flow in a turbulent boundary layer. In the case of a zero pressure gradient flow along a flat plate, we derive a nonlinear fifth-order ordinary differential equation, which is an extension of the Blasius equation. We study it analytically and prove the existence of a two-parameter family of solutions satisfying physical boundary conditions. Matching these parameters with the skin friction coefficient and the Reynolds number based on momentum thickness, we get an agreement of the solutions with experimental data in the laminar and transitional boundary layers, as well as in the turbulent boundary layer for moderately large Reynolds numbers.

1. Introduction

1.1. Boundary layer

Boundary-layer theory, first introduced by L. Prandtl in 1904, is now fundamental to many applications in fluid mechanics, especially in aerodynamics.

Prandtl argued that far away from a boundary the fluid behavior is practically inviscid, and the fluid velocity can be approximated by a solution of the Euler equations. Near the boundary though, in the region called the boundary layer, the viscosity can not be neglected. This layer is very thin for high Reynolds numbers, which allowed Prandtl to simplify the Navier-Stokes equations by neglecting some of the physical terms.

Consider the case of a two-dimensional steady incompressible viscous flow near a surface. Let x be the coordinate along the surface, y be the coordinate normal to the surface, and (u, v) be the corresponding velocity of the flow. For high

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Reynolds numbers, the boundary layer thickness δ is small. Neglecting the terms of the Navier-Stokes equations of high order in δ, one obtains the following Prandtl equations (see [30])

$$\begin{cases} u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u = \nu\frac{\partial^2}{\partial y^2}u - \frac{\partial}{\partial y}p \\ \frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0, \end{cases} \tag{1}$$

where ν is the kinematic viscosity, p is the pressure, and the density is chosen to be identically one.

In 1908 Blasius discovered that in the case of a zero pressure gradient flow along a flat plate, there exists a similarity variable $\xi = y/\sqrt{x}$, through which equations (1) can be reduced to the following ordinary differential equation:

$$h''' + \frac{1}{2}hh'' = 0 \tag{2}$$

with $h(0) = h'(0) = 0$, and $h'(\xi) \to 1$ as $\xi \to \infty$ (see [3]). H. Weyl [36] was the first to prove that there exists a unique solution h to (2) with such boundary conditions. For other proofs see [12], [17], [18], [21], [33], or [35]. See also Appendix, where it is proved that the Blasius profile $h'(\xi)$ has one inflection point in logarithmic coordinates. The Blasius equation with other boundary conditions was studied in [2], [12], [22], [33].

For the Blasius profile $h'(\xi)$ we have that

$$u(x,y) = u_{\mathrm{e}}h'\left(\frac{y}{\sqrt{l_{\mathrm{e}}x}}\right), \qquad v(x,y) = \frac{u_{\mathrm{e}}}{\sqrt{R_x}}h'\left(\frac{y}{\sqrt{l_{\mathrm{e}}x}}\right) \tag{3}$$

are solutions to (1) and they match experimental data in the laminar boundary layer. Here u_{e} is the horizontal velocity of the external flow, l_{e} is the external length scale $l_{\mathrm{e}} = \nu/u_{\mathrm{e}}$, and R_x is the local Reynolds number $R_x = x/l_{\mathrm{e}}$.

For high Reynolds numbers the flow becomes turbulent, and boundary layer equations for the averaged quantities $\bar{u}$, $\bar{v}$, $\bar{p}$ and fluctuating parts u', and v' can be written as

$$\begin{cases} \bar{u}\frac{\partial}{\partial x}\bar{u} + \bar{v}\frac{\partial}{\partial y}\bar{u} = \nu\frac{\partial^2}{\partial y^2}\bar{u} - \frac{\partial}{\partial y}\bar{p} - \frac{\partial}{\partial y}(\overline{u'v'}) \\ \frac{\partial}{\partial x}\bar{u} + \frac{\partial}{\partial y}\bar{v} = 0. \end{cases} \tag{4}$$

The above system, which is a boundary layer approximation of the Reynolds equations, is not closed. There are several models for the Reynolds shear stress term $-\overline{u'v'}$, e.g., the ones based on Prandtl's mixing length, eddy viscosity, or transport equation (see [34], [4], and references therein). In this paper, a summary of which was presented in [8], we derive a boundary layer approximation of the Navier-Stokes-alpha (NS-α) model of fluid turbulence, and use it, as an approximation of (4), to derive and study an extension of the Blasius equation for the averaged velocity.

1.2. Navier-Stokes-α model of fluid turbulence

The Euler-alpha model was first introduced in [19] as a generalization to n dimensions of the one-dimensional Camassa-Holm equation that describes shallow water waves. The Navier-Stokes-alpha (NS-α) model of fluid turbulence, also known as the viscous Camassa-Holm equations or LANS-α (Lagrangian averaged Navier-Stokes-alpha) model, is written as

$$\begin{cases} \frac{\partial}{\partial t}\mathbf{v} + (\mathbf{u}\cdot\nabla)\mathbf{v} + v_j\nabla u_j = \nu\Delta\mathbf{v} - \nabla q + f \\ \nabla\cdot\mathbf{u} = 0 \\ \mathbf{v} = \mathbf{u} - \frac{\partial}{\partial x_i}\left(\alpha^2\delta_{ij}\frac{\partial}{\partial x_j}\mathbf{u}\right), \end{cases} \tag{5}$$

where $\mathbf{u}$ represents the averaged physical velocity of the flow, q is a pressure analog, f is a force, and $\nu > 0$ is the viscosity. This model was proposed as a closed approximation to the Reynolds equations, and its solutions were compared with empirical data for turbulent flows in channels and pipes [5]-[7]. Analytical studies of the global existence, uniqueness and regularity of solutions to (5), as well as estimates of the dimension of the global attractor are done in [16]; the energy spectrum was studied in [15]. See also [13], [26], [27], [28] and references therein for other results and discussions.

In this paper, the filter length scale α, which represents the averaged size of the Lagrangian fluctuations (see [5]-[7]), will be considered as a parameter of the flow, changing along the streamlines in the boundary layer. More precisely, we propose an assumption that α should be proportional to the thickness of the boundary layer.

Our approach will be based on this particular model, among the large variety of models for turbulent phenomena (see, e.g., [1], [4], [11], [25], [34]).

1.3. Turbulent boundary layer

Similarly to Prandtl's derivation in the laminar case, we obtain a boundary layer approximation of the NS-α model, which extend the Prandtl equations for turbulent flows near boundaries. We then consider a classical problem of a zero pressure gradient boundary layer flow along a flat plate. Such boundary layers can occur on ships, lifting surfaces, airplane bodies, as well as on the blades of turbines and rotary compressors. In addition, the understanding of this problem is essential for the calculation of the skin friction drag for other body shapes, for which the separation does not occur [34].

In this case, using the Blasius similarity variable and assuming that the filter length scale α in the NS-α model is proportional to the boundary layer thickness $\delta(x)$, i.e., $\alpha(x) = \beta\delta(x)$, we reduce the boundary layer approximation of the NS-α model to the following ordinary differential equation:

$$m''' + \frac{1}{2}hm'' = 0, \tag{6}$$

where $m = h - \beta^2 h''$. The boundary conditions are $h(0) = h'(0) = 0$, and $h'(\xi) \to 1$ as $\xi \to \infty$. Our ansatz is that the averaged velocity of the flow (u, v) on a small interval $x_1 < x < x_2$ satisfies (3) for some h, a solution of (6).

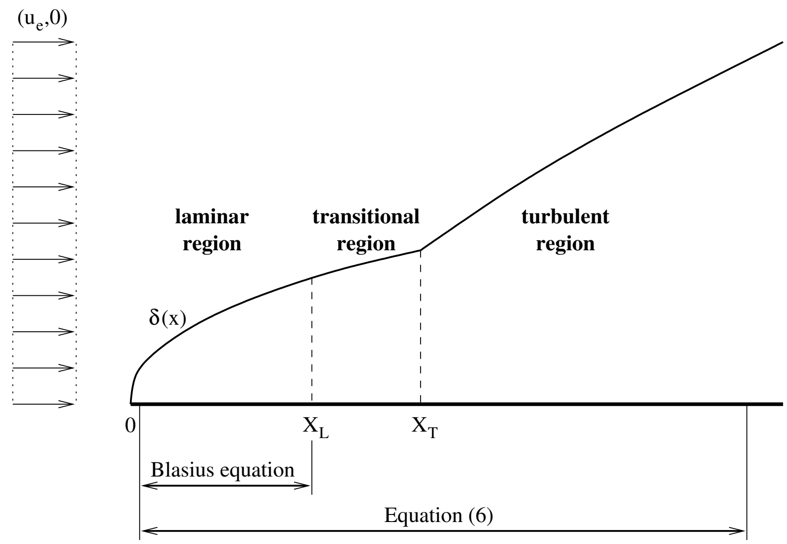

Fig. 1. Boundary layer along a flat plate

The dimensionless parameter β, a filter width scale defining a turbulent flow regime, is the ratio of the averaged size of the Lagrangian fluctuations to the boundary layer thickness. Observe that when β is zero, i.e., when there are no turbulent fluctuations, we have $m = h$ and equation (6) reduces to the Blasius equation. The Blasius profile h' has one inflection point in logarithmic coordinates (see Appendix 1), and it matches experimental data in the laminar region of the boundary layer. However, solutions of (6) match experimental data for a much larger interval of Reynolds numbers, containing the transitional region and a part of the turbulent region of the boundary layer (see Fig. 1 that illustrates these regions and the boundary layer thickness $\delta(x)$).

Unlike the Blasius equation, equation (6) has solutions with a large variety of behaviors. *A priori* it is not clear if it has physically acceptable solutions, i.e., solutions that satisfy the above physical boundary conditions. For example, it might happen that for every solution h to (6), h' is unbounded on $[0,\infty)$. In the other extreme case, $h'(\xi)$ may converge to one for any choice of free parameters $(h''(0), h'''(0), h''''(0), \beta)$, i.e., we may have existence of a four-parameter family of physically acceptable solutions. On the other hand, since we have four free parameters and one remaining boundary condition $h'(\xi) \to 1$ as $\xi \to \infty$, then by analogy with the Blasius equation (see Appendix 1), one might expect a three-parameter family of physically acceptable solutions.

However, we study equation (6) in Sections 3–7 and prove, among other things, that in a large open region of the space of boundary values, all physically acceptable solutions form exactly a *two*-parameter family. These two parameters are $a := \beta^3 h''(0)$, and $b := \beta^4 h'''(0)$, and they determine the velocity profile for each horizontal coordinate. They also correspond to two basic physical parameters of the turbulent boundary layer: the skin friction coefficient c_f, and the

Reynolds number based on momentum thickness R_θ. In Section 8 we find that the two-parameter family of velocity profiles $\{u_{R_\theta,c_f}\}$ match experimental data for Reynolds numbers based on momentum thickness $R_\theta < 3000$. It is remarkable that the case $a+b>0$ corresponds to the laminar boundary layer, and the case $a+b<0$ corresponds to the transitional and turbulent boundary layers (see Fig. 1). In addition, we would like to mention that for every physically acceptable solution h one automatically has

$$h^{(n)}(\xi) \to 0 \quad \text{as} \quad \xi \to 0, \qquad \forall n \geq 2.$$

There are still some fascinating open problems concerning equation (6). One of them is the question of whether a physically acceptable solution is unique for given $a>0$ and b in the turbulent region of the boundary layer, i.e., when $a+b<0$. The positive answer to this question is established in the case when $a+b \geq 0$. As we will see in Sections 6 and 7, solutions of (6) have very different behaviors in those cases (see Fig. 2–4 below). In addition, based on numerical computations, we have the following

Conjecture 1.1. There is an open region of the boundary values $\mathcal{T} \subset \{(a,b) : a+b<0\}$, for which velocity profiles have three inflection points in logarithmic coordinates. More precisely, if h is a physically acceptable solution of (6) corresponding to some $(a,b) \in \mathcal{T}$, then $F(z) := h'(e^z)$ has three inflection points.

Finally, we would like to mention that the equation (6) was recently derived from a different model, the so called Leray-α model of turbulence (see [9]).

1.4. Future studies

In this paper we concentrate on the case of the Blasius flow, which is important for understanding more complicated boundary layer flows. Nevertheless, the technique developed here can be also applied to other geometries. For example, in the case of a nonzero pressure gradient, using again the assumption, first introduced in [8], that α is proportional to the boundary layer thickness, one can study such boundary layer flows as two-dimensional flows past a wedge.

Another interesting example is a boundary layer along a three-dimensional axisymmetric body. When the body radius is much larger than the boundary layer thickness, the derived system (10), a two-dimensional boundary layer approximation of the NS-α model, can be used after applying the Mangler transformation (see [4]).

In addition, a modified version of system (10) and the assumption that α is proportional to the boundary layer thickness were used in [20] and [31] to study turbulent jets and wakes.

2. Derivation

We study a two-dimensional turbulent flow near a surface. Let x be the coordinate along the surface, y the coordinate normal to the surface, and $\mathbf{u} = (u,v)$ the velocity of the flow.

We will use the two dimensional Navier-Stokes-α model to study the averaged velocity in the boundary layer. This model is written as

$$\begin{cases} \frac{\partial}{\partial t}\mathbf{v} + (\mathbf{u}\cdot\nabla)\mathbf{v} + v_j \nabla u_j = \nu \Delta \mathbf{v} - \nabla q \\ \nabla \cdot \mathbf{u} = 0, \end{cases} \tag{7}$$

where $\mathbf{u} = (u, v)$ represents the averaged velocity, and

$$\mathbf{v} = \mathbf{u} - \frac{\partial}{\partial x_i}\left(\alpha^2 \delta_{ij} \frac{\partial}{\partial x_j}\mathbf{u}\right).$$

We supplement the system with no-slip boundary conditions $\mathbf{u}|_{y=0} = 0$, as well as

$$\lim_{y\to\infty} \mathbf{u}(x, y) = (U_{\mathrm{e}}(x), 0),$$

for all $x > 0$, where $(U_{\mathrm{e}}(x), 0)$ is an averaged external velocity of the flow. In addition, we assume that $\alpha(\cdot)$ is a function of x only. If the averaged velocity $\mathbf{u}$ is stationary in time, (7) becomes

$$\begin{cases} u(x,y)\frac{\partial}{\partial x}\gamma(x,y) + v\frac{\partial}{\partial y}\gamma + \gamma\frac{\partial}{\partial x}u + \tau\frac{\partial}{\partial x}v = \nu\frac{\partial^2}{\partial x^2}\gamma + \nu\frac{\partial^2}{\partial y^2}\gamma - \frac{\partial}{\partial x}q \\ u(x,y)\frac{\partial}{\partial x}\tau(x,y) + v\frac{\partial}{\partial y}\tau + \gamma\frac{\partial}{\partial y}u + \tau\frac{\partial}{\partial y}v = \nu\frac{\partial^2}{\partial x^2}\tau + \nu\frac{\partial^2}{\partial y^2}\tau - \frac{\partial}{\partial y}q \\ \frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0. \end{cases} \tag{8}$$

Let us fix l on the x-axis, and define $u_{\mathrm{e}} := U_{\mathrm{e}}(l)$ and

$$\epsilon(l) := \frac{1}{\sqrt{R_l}} = \sqrt{\frac{\nu}{u_{\mathrm{e}} l}}.$$

We change variables:

$$x_1 = \frac{x}{l}, \qquad y_1 = \frac{y}{\epsilon l}, \qquad u_1 = \frac{u}{u_{\mathrm{e}}}, \qquad v_1 = \frac{v}{\epsilon u_{\mathrm{e}}}, \qquad q_1 = \frac{q}{u_{\mathrm{e}}^2}, \qquad \alpha_1 = \frac{\alpha}{\epsilon l}.$$

Note that the new variables are dimensionless. Recall that $\alpha_1(\cdot)$ is a function of x. Then we obtain

$$\frac{1}{u_{\mathrm{e}}}\gamma(x,y) = u_1(x_1,y_1) - \epsilon^2\alpha_1^2\frac{\partial^2}{\partial x_1^2}u_1 - \alpha_1^2\frac{\partial^2}{\partial y_1^2}u_1 - \epsilon^2\frac{\partial}{\partial x_1}\alpha_1^2 \cdot \frac{\partial}{\partial x_1}u_1,$$
$$\frac{1}{u_{\mathrm{e}}}\tau(x,y) = \epsilon v_1(x_1,y_1) - \epsilon^3\alpha_1^2\frac{\partial^2}{\partial x_1^2}v_1 - \epsilon\alpha_1^2\frac{\partial^2}{\partial y_1^2}v_1 - \epsilon^3\frac{\partial}{\partial x_1}\alpha_1^2 \cdot \frac{\partial}{\partial x_1}v_1.$$

Neglecting the terms in equation (8) with high powers of ϵ, dropping subscripts and denoting

$$w = \left(1 - \alpha^2\frac{\partial^2}{\partial y^2}\right)u,$$

we arrive at

$$\begin{cases} u\frac{\partial}{\partial x}w + v\frac{\partial}{\partial y}w + w\frac{\partial}{\partial x}u = \frac{\partial^2}{\partial y^2}w - \frac{\partial}{\partial x}q \\ w\frac{\partial}{\partial y}u = -\frac{\partial}{\partial y}q \\ \frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0. \end{cases} \tag{9}$$

Now we introduce the expression for the averaged pressure p in the boundary layer in terms of the pressure analog q:

$$p(x,y) := q + \frac{1}{2}u^2 - \frac{1}{2}\alpha^2 \left(\frac{\partial}{\partial y}u\right)^2 .$$

Note that the second equation in (9) implies that $\frac{\partial}{\partial y}p = 0$, which is consistent with the fact that the variation of the averaged pressure in the vertical direction is negligible within the boundary-layer approximation (see [4]). Therefore, assuming Bernoulli's equation $U^2(x) + 2p(x) = const.$ for the rescaled external velocity $U(x) = U_e(x)/u_e$, we deduce that

$$\frac{\partial}{\partial x}p(x,y) = U(x)U'(x).$$

Finally, rewriting equations (9) in terms of p, we obtain the following boundary layer approximation of the NS-α model:

$$\begin{cases} u\frac{\partial}{\partial x}w + v\frac{\partial}{\partial y}w + z(u) = \frac{\partial^2}{\partial y^2}w - \frac{\partial}{\partial x}p \\ \frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0 \\ w = u - \alpha^2\frac{\partial^2}{\partial y^2}u, \end{cases} \tag{10}$$

where

$$z(u) = \alpha^2\left(\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}\right) + \frac{1}{2}\frac{\partial}{\partial x}\alpha^2 \cdot \left(\frac{\partial}{\partial y}u\right)^2 .$$

For ϵ small enough we have

$$u(x,y) \approx u_{\mathrm{e}}u_\infty\left(\frac{x}{l}, \frac{y}{\sqrt{l\cdot l_{\mathrm{e}}}}\right), \qquad v(x,y) \approx \frac{u_{\mathrm{e}}}{\sqrt{R_l}}v_\infty\left(\frac{x}{l}, \frac{y}{\sqrt{l\cdot l_{\mathrm{e}}}}\right),$$

where l_{e} is a length associated with the external flow $l_{\mathrm{e}} = \nu/u_{\mathrm{e}}$ and (u_∞, v_∞) is a solution of (10).

When $\alpha = 0$, the system (10) reduces to the Prandtl equations (1) modulo rescaling. Our aim is to solve this system in the case of a zero pressure gradient flow along a flat plate. We will study the flow near some fixed point x_0 on the plate. Let us choose the origin on the plate so that the point x_0 has the coordinates $(l,0)$, and consider l is to be a parameter of the boundary layer at point x_0. Now, we assume that α is proportional to $\sqrt{x}$, i.e.,

$$\alpha = \sqrt{x}\beta,$$

where β is another parameter of the boundary layer defining a turbulent flow regime, which represents the ratio of the averaged size of the Lagrangian fluctuations to the boundary layer thickness. In addition, we will study the solutions (u_∞, v_∞) of (10) such that

$$u_\infty = f(\xi), \qquad v_\infty = \frac{1}{\sqrt{x}}g(\xi), \qquad \xi = \frac{y}{\sqrt{x}} \tag{11}$$

for some functions f and g on an adequate interval $l-\epsilon < x < l+\epsilon$. It is remarkable that the self-similarity assumption (11) yields $z(u_\infty) \equiv 0$.

Substituting the expressions (11) for u and v into (10), and setting $\frac{\partial}{\partial x} p = 0$, we obtain the following equations for f and g:

$$\begin{cases} -\frac{1}{2} f f' \xi + \beta^2 f(\frac{1}{2} f''' \xi + f'') - \beta^2 f f'' + g f' - \beta^2 f''' g = f'' - \beta^2 f'''' \\ g' - \frac{1}{2}\xi f' = 0. \end{cases}$$

Let

$$h(\xi) = \int_0^\xi f(\eta)\, d\eta.$$

Then $g = \frac{1}{2}\xi h' - \frac{1}{2} h$, and we have the following equation for h:

$$h''' + \frac{1}{2} h h'' - \beta^2 \left(h''''' + \frac{1}{2} h'''' \right) = 0. \tag{12}$$

The boundary condition $u|_{y=0} = 0$ requires $f(0) = 0$ and thus $h(0) = h'(0) = 0$. In addition, the physical interpretation of $\nu \frac{\partial}{\partial y} u$ for $y = 0$ as the shear stress on the wall imposes the condition $f'(0) > 0$, that is, $h''(0) > 0$. Moreover, $u(x, y) \to u_{\mathrm{e}}$ as $y \to \infty$ requires that $h'(\xi) \to 1$ as $\xi \to \infty$.

Note that if $\hat{h}(\xi)$ is a solution of (12), then $h(x) := \beta \hat{h}(\beta x)$ is a solution of

$$-h''''' - \frac{1}{2} h h'''' + h''' + \frac{1}{2} h h'' = 0. \tag{13}$$

This equation can be also written as

$$\begin{cases} m''' + \frac{1}{2} h m'' = 0 \\ m = h - h''. \end{cases} \tag{14}$$

The physical boundary conditions are $h(0) = h'(0) = 0$, $h''(0) > 0$, and $h'(x) \to \beta^2$ as $x \to \infty$.

Note that the last boundary condition is equivalent to the condition that $\lim_{x\to\infty} h'(x)$ exists and satisfies

$$0 < \lim_{x\to\infty} h'(x) < \infty. \tag{15}$$

In this case h is a solution to our boundary value problem with β defined as

$$\beta := \left(\lim_{x\to\infty} h'(x) \right)^{1/2}.$$

Recall that $\beta = \alpha(x)/\sqrt{x}$, and *a priori* we do not know how to choose this physical parameter. By finding a solution h subjected to the condition (15), we will also find the value of the parameter β.

As we already discussed in Introduction, we have a fifth-order nonlinear ordinary differential equation, and since $h(0) = h'(0) = 0$, we have to specify three boundary conditions $(h''(0), h'''(0), h''''(0))$ in order to solve it. In Sections 3–7 we will prove that given a pair $(h''(0), h'''(0))$ in a large adequate region, there is a right choice of $h''''(0)$ that guarantees (15), i.e., there is a two-parameter family of physically acceptable solutions. More precisely, we prove the following

Theorem 2.1. *There exists a continuous function $b_0 : (0, \infty) \to \mathbb{R}$ such that $b_0(a) < -a$, and for each $a > 0$ and $b \in (b_0(a), \infty)$ we have that $\lim_{x\to\infty} h'(x)$ exists and satisfies*

$$0 < \lim_{x\to\infty} h'(x) < \infty,$$

where $h(x)$ is a solution to (13) with $h(0) = h'(0) = 0$, $h''(0) = a$, $h'''(0) = b$, and $h''''(0) = C(a, b)$. The function $C(a, b)$ is defined in Section 5.

We also show that the condition $b \in (b_0(a), \infty)$ cannot be removed, i.e., there are no physically acceptable solutions for some choices of $(h''(0), h'''(0))$ in the region where $h''(0) + h'''(0) < 0$ and $|h''(0) + h'''(0)|$ is large enough (see Corollary 7.6).

Here is a brief outline of the strategy to prove Theorem 2.1. In Section 3 we classify solutions of (13) and characterize their properties (see Theorem 3.1). In Section 4 we show how the different types of solutions depend upon the boundary conditions. In Section 5 we define a function $C(a, b)$, which we call the connecting function, and study its properties. This function connects the boundary values of the physically acceptable solutions:

$$h''''(0) = C(h''(0), h'''(0)), \tag{16}$$

as we show in Sections 6 and 7. In Section 6 we prove that in the case $h''(0) + h'''(0) \geq 0$, h is physically acceptable if and only if (16) holds (Theorem 6.3). In Section 7 we show that in the case $h''(0) + h'''(0) < 0$, when $|h''(0) + h'''(0)|$ is not large, (16) implies that h is physically acceptable (Theorem 7.3). When $|h''(0) + h'''(0)|$ is large enough, physically acceptable solutions might not exists (Corollary 7.6).

3. Classification of the solutions of (13)

Let $h(x)$ be a solution of (13). The first equation in (14) implies that

$$m''(x) = m''(0)e^{-\frac{1}{2}\int_0^x h(y)\,dy}. \tag{17}$$

Since m'' has a constant sign, m' is monotonic and therefore the integral

$$I(h) := \int_0^{x_*} e^{-x} m'(x)\,dx,$$

where $[0, x_*)$ is the maximal existence interval of the solution, makes sense provided the values $\pm\infty$ are also accepted.

The following is a characterization of the behavior of the solutions for $x \to \infty$.

Theorem 3.1. *Let $h(x)$ be a solution of (13) with $h(0) = 0$, $h'(0) = 0$, $h''(0) > 0$. Then only the following four cases are possible:*

1. $x_* = \infty,\ I(h) < h''(0),\ m'(x) \to \gamma \in \mathbb{R},$ *and* $h'(x) \to \infty$ *as* $x \to \infty$.
2. $x_* = \infty,\ I(h) = h''(0),\ m'(x) \to \gamma \geq 0,$ *and* $h'(x) \to \gamma$ *as* $x \to \infty$.
3. $x_* = \infty,\ I(h) > h''(0),\ m''(0) = 0,$ *and* $h'(x) \to -\infty$ *as* $x \to \infty$.

4. $I(h) > h''(0),\ m''(0) > 0,\ m'(x) \to \infty,$ *and* $h'(x) \to -\infty$ *as* $x \to x_*$.

Proof. It is useful to consider first the particular case when $m''(0) = 0$. In this case $m'(x) = const =: \gamma$, and elementary computations show that one of the cases 1, 2, or 3 occurs. From now on we will suppose that $m''(0) \neq 0$.

Consider

$$h' - h''' = m'.$$

A general solution h' has the following integral representation:

$$h'(x) = \frac{1}{2}\left(e^x - e^{-x}\right)h''(0) - \frac{1}{2}\int_0^x \left(e^{x-y} - e^{-x+y}\right)m'(y)\,dy. \tag{18}$$

Note that if $I(h)$ is finite, then (18) implies that

$$\begin{aligned} &2h'(x) - e^x\left(h''(0) - I(h)\right) \\ &\qquad = -e^{-x}h''(0) + \int_x^{x_*} e^{x-y}m'(y)\,dy + \int_0^x e^{-x+y}m'(y)\,dy. \end{aligned} \tag{19}$$

First, assume $m'(x) \to \gamma \in \mathbb{R}$ as $x \to x_*$. Then $I(h)$ is finite. Suppose, $x_* < \infty$. Then (18) implies that h' is bounded on $[0, x_*)$. Thus h is also bounded. Since $m' = h' - h'''$, it follows that h''' is bounded. Thus h'' is bounded. Also, (17) implies that m'' is bounded. However, $m'' = h'' - h''''$. Therefore h'''' is bounded. This contradicts the definition of x_*. Hence $x_* = \infty$. Now (19) implies that

$$2h'(x) - e^x\left(h''(0) - I(h)\right) \to 2\gamma \quad \text{as} \quad x \to \infty.$$

If $h''(0) < I(h)$, then $h'(x) \to -\infty$ as $x \to \infty$ and from (17) it follows that $|m''(x)| \to \infty$ as $x \to \infty$. This contradicts the assumption that $m'(x) \to \gamma \in \mathbb{R}$ as $x \to \infty$. Thus we have concluded that if this latter condition holds, then either case 1 or case 2 occurs.

Second, assume that $m'(x) \to -\infty$ as $x \to x_*$. If $x_* < \infty$, then (18) implies that $h'(x)$ is bounded from below on $[0, x_*)$. However, since $\liminf_{x\to x_*} m''(x) = -\infty$, from (17) we conclude that $\liminf_{x\to x_*} \int_0^x h(y)\,dy = -\infty$. This implies that h' is not bounded from below, a contradiction. Therefore $x_* = \infty$ and since $h''(0) > 0$, (18) implies that $h'(x) \to \infty$. Thus, by (17), $m''(x)$ goes to zero faster than an exponential function, so that $m'(x)$ is bounded on $[0, \infty)$, a contradiction. Hence $m'(x)$ has to be bounded from below on $[0, x_*)$.

Third, assume that $m'(x) \to \infty$ as $x \to x_*$ and $x_* < \infty$. Since we have that $\limsup_{x\to x_*} m''(x) = \infty$, we get $\liminf_{x\to x_*} \int_0^x h(y)\,dy = -\infty$ (see (17)), which implies that $h'(x)$ is not bounded from below. If

$$I(h) \leq h''(0),$$

then, in particular, $I(h)$ is finite and consequently (19) implies that

$$2h'(x) \geq -e^{-x}h''(0) + \int_x^{x_*} e^{x-y}m'(y)\,dy + \int_0^x e^{-x+y}m'(y)\,dy.$$

Thus h' is bounded from below, a contradiction. Hence

$$I(h) > h''(0).$$

Since m'' has a constant sign (see (17)), we have $m''(0) > 0$ and $m''(x) > 0$ on $[0, x_*)$. It follows that $\int_0^x \left(e^{x-y} - e^{-x+y}\right) m''(y)\, dy$ is nonnegative on $[0, x_*)$. Note that the general solution of $h'' - h'''' = m''$ is

$$2h''(x) = \left(e^x - e^{-x}\right) h'''(0) + \left(e^x + e^{-x}\right) h''(0) - \int_0^x \left(e^{x-y} - e^{-x+y}\right) m''(y)\, dy.$$

Thus $h''(x)$ is bounded from above on $[0, x_*)$. This and the fact that h' is unbounded from below force $h'(x) \to -\infty$ as $x \to x_*$. Therefore case 4 occurs.

The last case to be considered is when $m'(x) \to \infty$ as $x \to \infty$. If

$$I(h) \leq h''(0),$$

then, in particular, $I(h)$ is finite and consequently (19) implies that

$$2h'(x) \geq -e^{-x} h''(0) + \int_x^\infty e^{x-y} m'(y)\, dy + \int_0^x e^{-x+y} m'(y)\, dy \to \infty$$

as $x \to \infty$. But in this case from (17) it follows that m' has to be bounded, a contradiction. Hence

$$I(h) > h''(0).$$

Let $x_0 := \inf\{x : x > 0, m'(x) > 0\}$. Then for each $x \geq 2x_0$ we have the following inequality:

$$\begin{aligned}
\int_{x_0}^x \left(e^{-y} - e^{-2x+y}\right) m'(y)\, dy &\geq \int_{x_0}^{x/2} \left(e^{-y} - e^{-2x+y}\right) m'(y)\, dy \\
&\geq \left(1 - e^{-x}\right) \int_{x_0}^{x/2} e^{-y} m'(y)\, dy.
\end{aligned}$$

Therefore from (18) we obtain that for $x \geq 2x_0$

$$\begin{aligned}
2h'(x)e^{-x} &\leq h''(0) - \int_0^x \left(e^{-y} - e^{-2x+y}\right) m'(y)\, dy \\
&\leq h''(0) - \int_0^{x_0} e^{-y} m'(y)\, dy - \left(1 - e^{-x}\right) \int_{x_0}^{x/2} e^{-y} m'(y)\, dy \\
&\to h''(0) - I(h) < 0
\end{aligned}$$

as $x \to \infty$. Hence $h'(x) \to -\infty$ as $x \to \infty$, which corresponds to case 4. $\square$

Fix $a > 0$ and b. For a given c let h_c be a solution of (13) with $h_c(0) = h_c'(0) = 0$, $h_c''(0) = a$, $h_c'''(0) = b$, and $h_c''''(0) = c$. Let also $m_c = h_c - h_c''$.

Recall that the physically acceptable solutions have to satisfy the following boundary condition at infinity:

$$h_c'(x) \to \beta^2 \quad \text{as} \quad x \to \infty, \tag{20}$$

for some $\beta > 0$. By Theorem 3.1 such a solution has to fall in the second case, that is, $I(h_c) = a$. Conversely, any solution from the second case has to satisfy (20) for some $\beta > 0$ except the non-generic case when the limit of h'_c is zero.

We would like to know whether for given a and b there exists $\tilde{c}$ such that $h_{\tilde{c}}$ is physically acceptable. To this aim we will study in more detail the four cases of the Theorem 3.1 in the next section.

4. Classification criteria

Lemma 4.1. *If $c > a$, then h_c is in one of the cases 1 or 2 in Theorem 3.1, i.e.,*

$$I(h_c) \le a.$$

Proof. Indeed, since $c > a$, the second equation in (14) implies that $m''(0) < 0$ and therefore cases 3 and 4 of the Theorem 3.1 are excluded. $\square$

We can supplement this lemma with the following

Lemma 4.2. *For each $a > 0$ and $b \in \mathbb{R}$ there exists $c_1(a, b)$ such that h_c is in one of the cases 3 or 4 in Theorem 3.1, i.e,*

$$I(h_c) > a$$

for all $c < c_1(a, b)$.

Proof. Take any $c < a$. Assume that h_c exists for all $x \ge 0$. Note that since $m''_c(0) = a - c > 0$, we have $m'_c(x) > m'_c(0)$ for each $x > 0$. Therefore

$$\begin{aligned}\int_0^x \left(e^{x-y} - e^{-x+y}\right) m'_c(y)\,dy \;&\ge\; \left(e^x + e^{-x} - 2\right) m'_c(0)\\ &=\; -\left(e^x + e^{-x} - 2\right) b\\ &=:\; \psi_1(b, x)\end{aligned}$$

for each $x \ge 0$. Hence, due to the integral representation (18) of h'_c, we obtain

$$\begin{aligned}2h'_c(x) \;&\le\; \left(e^x - e^{-x}\right) h''_c(0) - \psi_1(b, x)\\ &=\; \left(e^x - e^{-x}\right) a - \psi_1(b, x).\\ &=:\; \psi_2(a, b, x)\end{aligned}$$

for each $x \ge 0$. Consequently

$$2h_c(x) = 2\int_0^x h'_c(y)\,dy \le \int_0^x \psi_2(a, b, y)\,dy =: \psi_3(a, b, x), \qquad x \ge 0.$$

Thus, since $m''(0) = a - c > 0$, we have that

$$m''_c(x) = m''_c(0) e^{-\frac{1}{2}\int_0^x h_c(y)\,dy} \ge m''_c(0) e^{-\frac{1}{4}\int_0^x \psi_3(a,b,y)\,dy} =: m''_c(0)\psi_4(a, b, x)$$

for each $x \geq 0$. Note that $\psi_4(a,b,x)$ does not depend on c, which is an essential part of this proof. Now we have the following inequality for $m_c'(x)$:

$$m_c'(x) \geq m_c'(0) + m_c''(0) \int_0^x \psi_4(a,b,y)\,dy, \qquad x \geq 0.$$

Thus

$$\begin{aligned} I(h_c) &= \int_0^\infty e^{-x} m_c'(x)\,dx \\ &\geq \int_0^\infty e^{-x} \left(m_c'(0) + m_c''(0) \int_0^x \psi_4(a,b,y)\,dy \right) dx \\ &= -b + (a-c)\Theta_{a,b}\ , \end{aligned} \tag{21}$$

where

$$\Theta_{a,b} := \int_0^\infty dx\, e^{-x} \int_0^x dy\, \psi_4(a,b,y).$$

Since $\Theta_{a,b} \in (0,\infty]$, we can define

$$c_\mathrm{l}(a,b) = \min\left\{a, a - (a+b)/\Theta_{a,b}\right\}.$$

Take any $c < c_\mathrm{l}(a,b)$. If the maximal existence interval of h_c is finite ($x_* < \infty$), then Theorem 3.1 implies that $I(h_c) > a$. If h_c exists for all $x \geq 0$, then $I(h_c) > a$ by (21). □

In addition, we have the following result:

Lemma 4.3. *For any $a > 0$ and $b < -a$ there exists $c_\mathrm{u}(a,b)$ such that $h_c'(x) > 0$ for all $x > 0$, and $h_c'(x) \to \infty$ as $x \to \infty$ for all $c > c_\mathrm{u}(a,b)$.*

In particular, h_c is in the case 1 in Theorem 3.1, i.e.,

$$I(h_c) < a$$

for all $c > c_\mathrm{u}(a,b)$.

Proof. Differentiating (18), we obtain

$$2h_c''(x) = \left(e^x + e^{-x}\right) a - \int_0^x \left(e^{x-y} + e^{-x+y}\right) m'(y)\,dy. \tag{22}$$

For $c > a$ we have $m_c''(x) < 0$ for all $x \geq 0$ (see (17)). This implies that $m_c'(x) \leq -b$ for all $x \geq 0$. Therefore (22) implies that

$$\begin{aligned} 2h_c''(x) &\geq \left(e^x + e^{-x}\right) a + b \int_0^y \left(e^{x-y} + e^{-x+y}\right)\,dy \\ &= \left(e^x + e^{-x}\right) a + \left(e^x - e^{-x}\right) b \end{aligned}$$

for all $x \geq 0$. Since $a > 0$, there exists x_0, independent of c, such that $h_c''(x) > 0$ for all $c > a$ and $0 \leq x \leq x_0$.

The next step is to prove that there exists c_{u} such that

$$m_c'(x_0) < 0, \qquad \forall c > c_{\mathrm{u}}.$$

Choose $c_{\mathrm{u}} > a$ such that

$$-b + (a-c)\int_0^{x_0} e^{-\frac{1}{4}(e^x - e^{-x} - 2x)a}\, dx < 0, \qquad \forall c > c_{\mathrm{u}}.$$

Choose any $c > c_{\mathrm{u}}$. Suppose that $m_c'(x) \geq 0$ for all $x \in [0, x_0]$. Then

$$\int_0^x \left(e^{x-y} - e^{-x+y}\right) m_c'(y)\, dy \geq 0, \qquad x \in [0, x_0].$$

Therefore (18) forces

$$2h_c'(x) \leq \left(e^x - e^{-x}\right) a, \qquad x \in [0, x_0],$$

which implies that

$$2h_c(x) \leq \left(e^x + e^{-x} - 2\right) a, \qquad x \in [0, x_0].$$

Thus

$$m_c''(x) = (a-c)e^{-\frac{1}{2}\int_0^x h_c(y)\, dy} \leq (a-c)e^{-\frac{1}{4}(e^x - e^{-x} - 2x)a}, \qquad x \in [0, x_0].$$

Finally,

$$m_c'(x_0) \leq -b + (a-c)\int_0^{x_0} e^{-\frac{1}{4}(e^x - e^{-x} - 2x)a}\, dx < 0,$$

a contradiction. Hence $m_c'(x_0) < 0$.

Consequently, since $m_c'(x)$ is decreasing, we have $m_c'(x) < 0$ for all $x \geq x_0$. Note that $h_c'''(x_0) = h_c'(x_0) - m_c'(x_0) > 0$. We can show now that $h_c'''(x) > 0$ for all $x \geq x_0$. Indeed, suppose that there exists $x_1 > x_0$ such that $h_c'''(x) > 0$ on $[x_0, x_1)$ and $h_c'''(x_1) = 0$. Recall that $h_c''(x) > 0$ for all $x \in [0, x_0]$. Then $h_c'(x_1) > 0$, and, consequently, $m'(x_1) = h_c'(x_1) - h_c'''(x_1) > 0$, a contradiction.

Hence $h_c''(x) > 0$ and $h_c'(x) > 0$ for all $x > 0$. In addition, noting that $\lim_{x\to\infty} m_c'(x) < 0$, we conclude from Theorem 3.1 that $h_c'(x) \to \infty$ as $x \to \infty$. $\square$

Lemma 4.4. *If $I(h_{\tilde{c}}) \leq a$ for some $\tilde{c} \leq a$, then $I(h_c) < a$ for all $\tilde{c} < c \leq a$. If $I(h_{\tilde{c}}) \geq a$ for some $\tilde{c} \leq a$, then $I(h_c) > a$ for all $c < \tilde{c}$.*

Proof. Let $c_1 < c_2 \leq a$. Since $h_{c_1}''''(0) < h_{c_2}''''(0)$, then $h_{c_1}'(x) < h_{c_2}'(x)$ for $x > 0$ small enough. Suppose there exists x_0 that belongs to the existence intervals of both h_{c_1} and h_{c_2} such that $h_{c_1}'(x_0) = h_{c_2}'(x_0)$ and $h_{c_1}'(x) < h_{c_2}'(x)$ for $0 < x < x_0$. Since $m_{c_1}''(0) > m_{c_2}''(0) \geq 0$, (17) implies that $m_{c_1}''(x) > m_{c_2}''(x)$ for $0 < x < x_0$. Therefore $m_{c_1}'(x) > m_{c_2}'(x)$ for $0 < x < x_0$. Comparing the integral representations (18) of h_{c_1}' and h_{c_2}', we obtain $h_{c_1}'(x_0) < h_{c_2}'(x_0)$, a contradiction.

Hence $h_{c_1}'(x) < h_{c_2}'(x)$ and consequently $m_{c_1}'(x) > m_{c_2}'(x)$ for each x that belongs to the existence intervals of both h_{c_1} and h_{c_2}. This, together with the definition of $I(h)$ and Theorem 3.1, completes the proof. $\square$

5. Connecting function

The aim of this section is to introduce and study several regions in the space of the boundary values a, b, and c, which are useful in the study of the existence of a physically acceptable solution. The following result will play an essential role in this study.

Denote by $h_{a,b,c}$ the solution of (13) with $h_{a,b,c}(0) = h'_{a,b,c}(0) = 0$, $h''_{a,b,c}(0) = a > 0$, $h'''_{a,b,c}(0) = b$, and $h''''_{a,b,c}(0) = c$. Whenever a and b are fixed, we will use for simplicity our previous notation $h_c = h_{a,b,c}$.

Proposition 5.1. *Let*

$$\Omega := \{(a, b, c) : a > 0, I(h_{a,b,c}) < a\},$$

$$\Omega_0 := \Omega \cap \{(a, b, c) : h'_{a,b,c}(x) > 0 \ \forall x > 0\}.$$

Then Ω and Ω_0 are open in $\mathbb{R}^3$.

Proof. Choose any $(a, b, c) \in \Omega$. Let $\hat{h} = h_{a,b,c}$ and $\widehat{m} := \hat{h} - \hat{h}''$. Write

$$\delta := a - I(\hat{h}) > 0.$$

By Theorem 3.1, the maximal existence interval of $\hat{h}$ is infinite. Moreover, $\widehat{m}'(x) \to \gamma$ as $x \to \infty$ for some $\gamma \in \mathbb{R}$. Since

$$\int_x^\infty e^{x-y} \widehat{m}'(y)\, dy + \int_0^x e^{-x+y} \widehat{m}'(y)\, dy \to 2\gamma$$

as $x \to \infty$, we can choose $\hat{x}$ so that

$$\int_x^\infty e^{x-y} \widehat{m}'(y)\, dy + \int_0^x e^{-x+y} \widehat{m}'(y)\, dy > 2\gamma - 1, \qquad \forall x \geq \hat{x}. \tag{23}$$

Since $\hat{h}'(x) \to \infty$ (see Theorem 3.1), we have that $\hat{h}(x) \to \infty$, and, consequently,

$$\lim_{x\to\infty} \frac{\widehat{m}''(x)}{\hat{h}(x)} = 0.$$

Therefore we can choose $x_0 > \hat{x}$, such that $\hat{h}(x_0) > 0$ and $\hat{h}'(x) > 0$ for all $x \geq x_0$, as well as

$$\frac{|\widehat{m}''(x_0)|}{\hat{h}(x_0)} < \frac{\delta}{32}, \qquad \frac{\delta}{2} e^{x_0} > a - 2\gamma + 2. \tag{24}$$

Continuous dependence on the initial data implies that there exists $\epsilon_0 \in (0, \delta/4)$ such that any solution h of (13) with the boundary values satisfying

$$|h^{(n)}(0) - \hat{h}^{(n)}(0)| < \epsilon_0, \qquad n = 0, ..., 4,$$

exists on $[0, x_0]$ and satisfies the following conditions:

$$h(x_0) > 0, \qquad h'(x_0) > 0, \qquad \frac{|m''(x_0)|}{h(x_0)} < \frac{\delta}{32},$$

and

$$|m'(x) - \widehat{m}'(x)| < \frac{\delta}{8}, \qquad \forall x \in [0, x_0].$$

If there exists $x \in (0, x_0]$ such that $\hat{h}'(x) \leq 0$, define $\epsilon = \epsilon_0$. In the opposite case when $\hat{h}'(x) > 0$ for all $x > 0$, choose $\epsilon \in (0, \epsilon_0)$ small enough that $h'(x) > 0$ on $(0, x_0]$ for all solutions h with boundary values satisfying

$$|h^{(n)}(0) - \hat{h}^{(n)}(0)| < \epsilon, \qquad n = 0, ..., 4. \tag{25}$$

This can be done because $\hat{h}''(0) = a > 0$.

We now fix any solution h satisfying (25). Let $[0, x_*)$ be the maximal existence interval of h. Define

$$x_1 := \sup\{\tilde{x} : \tilde{x} \leq x_*, h'(x) > 0 \text{ on } [x_0, \tilde{x})\}$$

Then (17) implies that for any $x \in [x_0, x_1)$ we have

$$|m''(x)| \leq |m''(x_0)| e^{-\frac{1}{2}(x - x_0)h(x_0)}.$$

Therefore

$$\begin{aligned} |m'(x) - m'(x_0)| &\leq 2\frac{|m''(x_0)|}{h(x_0)} \left(1 - e^{-\frac{1}{2}(x-x_0)h(x_0)}\right) \\ &\leq 2\frac{|m''(x_0)|}{h(x_0)} \\ &< \frac{\delta}{16} \end{aligned}$$

for all $x \in [x_0, x_1)$. Similarly, $|\widehat{m}'(x) - \widehat{m}'(x_0)| < \delta/16$ for all $x \in [x_0, x_1)$. Thus we have

$$|m'(x) - \widehat{m}'(x)| < |m'(x_0) - \widehat{m}'(x_0)| + \frac{\delta}{8} < \frac{\delta}{4}, \qquad x \in [x_0, x_1).$$

Hence (see (18))

$$\begin{aligned} 2|h'(x) - \hat{h}'(x)| &\leq \left(e^x - e^{-x}\right)\epsilon + \int_0^x \left(e^{x-y} - e^{-x+y}\right)|m'(y) - \widehat{m}'(y)|\, dy \\ &< \frac{\delta}{4}e^x + \frac{\delta}{4}\left(e^x + e^{-x} - 2\right) \\ &< \frac{\delta}{2}e^x \end{aligned} \tag{26}$$

for all $x \in [x_0, x_1)$. On the other hand, from (19) and (23) it follows that

$$\begin{aligned} 2\hat{h}'(x) &= e^x\left(a - I(\hat{h})\right) - e^{-x}a + \int_x^\infty e^{x-y}\widehat{m}'(y)\, dy + \int_0^x e^{-x+y}\widehat{m}'(y)\, dy \\ &> \delta e^x - a + 2\gamma - 1 \end{aligned}$$

for all $x \in [x_0, x_1)$. This, together with (26) and (24), allows us to conclude that

$$2h'(x) > \frac{\delta}{2} e^x - a + 2\gamma - 1 > 1, \qquad x \in [x_0, x_1). \tag{27}$$

Assume now that $x_1 < x_*$. Since h' is continuous, by the definition of x_1 we have $h'(x_1) = 0$. On the other hand, (27) implies that $h'(x_1) \geq 1$, a contradiction. Thus $x_1 = x_*$. Therefore $h'(x)$ does not converge to $-\infty$ when $x \nearrow x_*$, and, consequently, from Theorem 3.1 we must have $x_* = \infty$. Then (27) yields

$$\lim_{x \to \infty} h'(x) = \infty,$$

i.e., $I(h) < a$, according to Theorem 3.1. Moreover, $h'(x) > 0$ on the whole interval $[x_0, \infty)$.

The second part of the proposition follows from the fact that in the case under consideration $h'(x) > 0$ on $(0, x_0]$. $\square$

Among all the solutions h_c consider the special solution h_a. From (18) it follows that

$$2h_a'(x) = (a+b)e^x - (a-b)e^{-x} - 2b, \qquad I(h_a) = -b. \tag{28}$$

We have two major cases. If $a + b \geq 0$, then we have $I(h_a) \leq a$. If $a + b < 0$, then $I(h_a) > a$. In addition, from Lemma 4.4 we deduce that

$$I(h_c) > a \quad \text{for} \quad b < -a < 0, \;\; c \leq a. \tag{29}$$

Let

$$\Lambda := \{(a, b) : \exists c \text{ such that } (a, b, c) \in \Omega_0\}.$$

Define the connecting function $C : \Lambda \to \mathbb{R}$ by

$$C(a, b) := \inf\{c : (a, b, c) \in \Omega_0\}. \tag{30}$$

For convenience $\{(a, b, C(a, b)\}$ will be called the connecting surface.

Lemma 5.2. *We have*

$$\Lambda = (0, \infty) \times (-\infty, \infty).$$

In addition, $(a, -a, c) \in \Omega_0$ *for all* $c > a > 0$*, and*

$$\begin{aligned} a < C(a, b) < \infty, &\quad \textit{if} \quad a + b < 0, \\ C(a, b) = a, &\quad \textit{if} \quad a + b = 0, \\ -\infty < C(a, b) < a, &\quad \textit{if} \quad a + b > 0. \end{aligned}$$

Proof. First, if $a+b<0$, then Lemma 4.3 implies that there exists c large enough that $h_c'(x)>0$ for all $x>0$ and $h_c'(x)\to\infty$ as $x\to\infty$. Therefore $(a,b)\in\Lambda$ and $C(a,b)<\infty$. In addition, (29) implies that $C(a,b)\geq a$. Moreover, since $h_a'(x)$ is negative for some $x>0$ (see (28)), we have $C(a,b)>a$ by virtue of continuous dependence on the initial data.

Second, if $a+b>0$, then $h_a'(x)>0$ for all $x>0$, and $h_a'(x)\to\infty$ as $x\to\infty$, i.e., $(a,b,a)\in\Omega_0$. Therefore $(a,b)\in\Lambda$ and $C(a,b)\leq a$. Moreover, since Ω_0 is open, we obtain $(a,b,C(a,b))\notin\Omega_0$, and consequently $C(a,b)<a$. In addition, from Lemma 4.2 we infer that $C(a,b)>-\infty$.

Now note that (17) yields $m_c''(x)<0$ for all $c>a$, all $x>0$. Therefore

$$m_c'(x)<m_c'(0)=-b=m_a'(x),\qquad c>a,\ x>0. \tag{31}$$

Hence,

$$I(h_c)=\int_0^\infty e^{-y}m_c'(y)\,dy<m_c'(0)=-b,\qquad c>a.$$

Thus, if $a+b=0$, we get that $I(h_c)<a$ for all $c>a$, and consequently by Theorem 3.1 we have $h_c'(x)\to\infty$ as $x\to\infty$ for all $c>a$. In addition, we get $h_c'(x)>h_a'(x)>0$ for all $c>a$ and $x>0$ due to (18) and (31). Therefore, $(a,-a,c)\in\Omega_0$ for all $c>a$. Hence $(a,-a)\in\Lambda$, and $C(a,-a)\leq a$. Since $I(h_a)=a$ we obtain $C(a,-a)=a$ due to Lemma 4.4. $\square$

Lemma 5.3. *The function $C(a,b)$ is upper semi-continuous for all $a>0$ and $b\in\mathbb{R}$.*

Proof. Notice that the set Ω_0 is open due to Theorem 5.1. Thus the set

$$\{(a,b,c): a>0, c>C(a,b)\}=\bigcup_{p>0}(\Omega_0+(0,0,p))$$

is open. Therefore $C(a,b)$ is upper semi-continuous. $\square$

Figures 2–4 show numerical solutions h_c' for some values of c. The physical solutions represented by solid lines correspond to $c=C(a,b)$. The solutions h_c' for which $c\neq C(a,b)$, represented by dotted lines, with the special solution h_a' pinpointed out, were numerically found to be unbounded, i.e., they do not satisfy the physical boundary condition $0<\lim_{x\to\infty}h_c'(x)<\infty$ (see (15)). Note that in the case $a+b<0$ with $|a+b|$ large, the solution $h_{C(a,b)}'$ is not bounded either (see Figure 4).

However, when $a+b\geq 0$, or $a+b<0$ but $|a+b|$ is not large, $h_{C(a,b)}'(x)$ has a finite positive limit, i.e. it is physically acceptable (see Figures 2 and 3). Therefore, for given a and b, the solution $h_{C(a,b)}'$ is a candidate for a physically acceptable solution. In the next two sections we will give rigorous results confirming these numerical facts.

The parameters a and b depend on the skin friction coefficient and Reynolds number based on momentum thickness. However, this dependence is quite involved, and will be discussed in Section 8. There we will show how to choose a and b, and compare $h_{C(a,b)}'$ with experimental data.

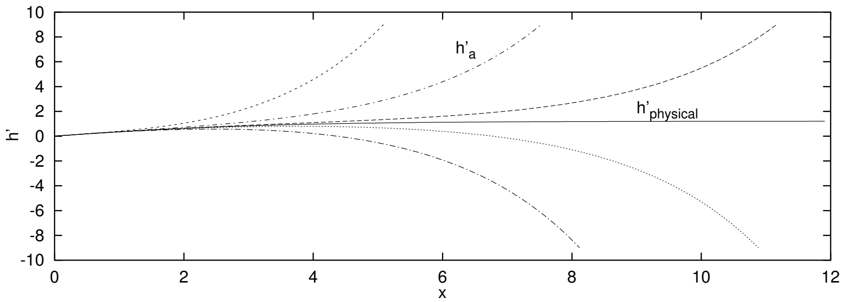

Fig. 2. $a + b > 0$

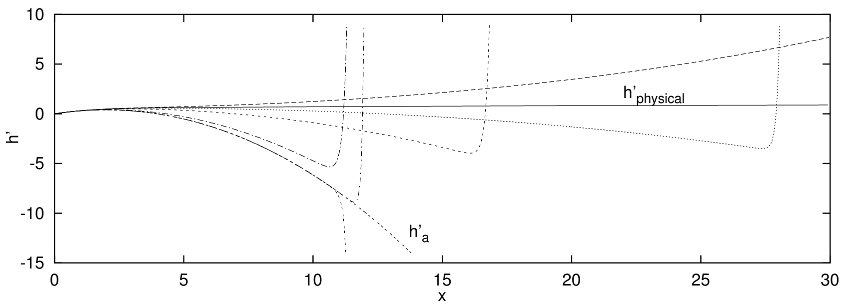

Fig. 3. $a + b < 0$

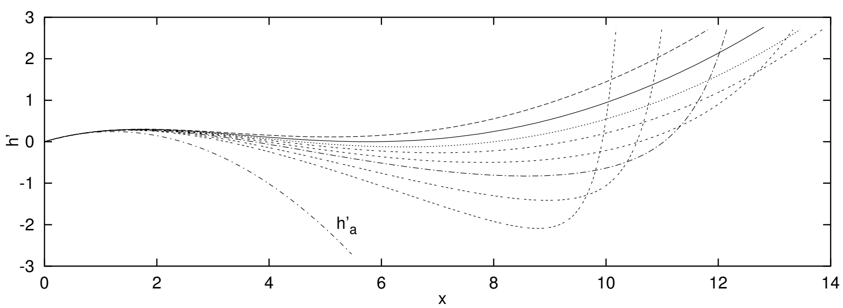

Fig. 4. $a + b < 0$, $|a + b|$ large

6. Existence of a physically acceptable solution in the case $a + b \geq 0$

We fix $a > 0$ and $b \geq -a$. For a given c, let h_c be the solution of (13) with $h_c(0) = h_c'(0) = 0$, $h_c''(0) = a$, $h_c'''(0) = b$, and $h_c''''(0) = c$. Let also $m_c = h_c - h_c''$.

Lemma 6.1. *If $I(h_c) \leq a$ for some $c \leq a$, then $h_c''(x) \geq 0$ for all $x \geq 0$.*

Proof. Assume that $h_c''(x_1) < 0$ for some $x_1 > 0$. According to Theorem 3.1, $\lim_{x\to\infty} h_c'(x) \geq 0$ and therefore $\limsup_{x\to\infty} h_c''(x) \geq 0$. Hence there exists

$x_2 > x_1$ such that $h_c''(x_2) > h_c''(x_1)$. Consequently, since

$$h_c''(0) = a > 0 > h_c''(x_1),$$

a local minimum of h_c'' is attained at some $x_0 \in (0, x_2)$. Then $h_c''(x_0) < 0$ and $h_c''''(x_0) \geq 0$. This implies that $m_c''(x_0) < 0$. However, since $c \leq a$, we have that $m_c''(x) \geq 0$ for all $x > 0$, a contradiction. Therefore $h_c''(x) \geq 0$ for all $x > 0$. □

Lemma 6.2. *The set*

$$(\Omega \setminus \Omega_0) \cap \{(a, b, c) : a + b \geq 0, c \leq a\}$$

is empty.

Proof. Assume that $(a, b, c) \in \Omega$, $a + b \geq 0$, and $c \leq a$. Since $I(h_c) < a$ and $c \leq a$, Lemma 6.1 implies that $h_c''(x) \geq 0$ for all $x \geq 0$. Since $h_c'(0) = 0$ and $h_c''(0) = a > 0$, we have that $h_c'(x) > 0$ for all $x > 0$. Therefore $(a, b, c) \in \Omega_0$. □

Theorem 6.3. *If $a > 0$ and $a + b \geq 0$, then $h_{C(a,b)}$ is physically acceptable, i.e.,*

$$\lim_{x \to \infty} h_{C(a,b)}'(x) = \gamma \tag{32}$$

for some $\gamma = \gamma(a, b) \in (0, \infty)$. In addition, $h_{C(a,b)}''(x) \geq 0$ for all $x > 0$, $C(a, b)$ is continuous and strictly less than a for $a + b > 0$, and $C(a, -a) = a$, Moreover, the physically acceptable solution $h_{C(a,b)}$ is unique, i.e.,

$$I(h_c) \neq a, \qquad \forall c \neq C(a, b).$$

Proof. First, if $a + b = 0$, then from (28) it follows that $I(h_a) = a$ and $h_a''(x) \geq 0$ for all $x \geq 0$. Since $C(a, -a) = a$ (see Lemma 5.2), (32) holds due to Theorem 3.1.

Assume now that $a + b > 0$. Let $C = C(a, b)$. Note that $C < a$ by virtue of Lemma 5.2. Since Ω_0 is open, $(a, b, C) \notin \Omega_0$. Therefore Lemma 6.2 yields

$$(a, b, C) \notin \Omega$$

Hence $I(h_C) \geq a$ by the definition of Ω. Assume that $I(h_C) > a$. Then Theorem 3.1 yields that

$$(a, b, C) \in \Theta := \{(a, b, c) : a > 0, a + b > 0, c < a, h_c'(x) < 0 \text{ for some } x > 0\},$$

which is an open set by virtue of continuous dependence on the initial data. Now recall that the connecting surface belongs to the closure of Ω_0. On the other hand, since $\Omega_0 \cap \Theta = \emptyset$, we obtain that (a, b, C) does not belong to the closure of Ω_0, a contradiction. Therefore $I(h_C) = a$.

In addition, Lemma 4.4 implies that $I(h_c) > a$ for all $c < C(a, b)$, and $I(h_c) < a$ for all $C(a, b) < c \leq a$. Therefore, to show uniqueness of the physically acceptable solution we have to prove that $I(h_c) \neq a$ for $c > a$. Indeed, due to (17), $m_c''(x) < 0$ for all $c > a$, all $x \geq 0$. Consequently, by the definition of

$I(h)$, we have $I(h_c) < m'_c(0) = -b \le a$ for $c > a$. The following is a summary of what was proved so far:

$$\begin{aligned} I(h_c) < a \ \text{ for } \ c > C(a,b), \\ I(h_c) = a \ \text{ for } \ c = C(a,b), \\ I(h_c) > a \ \text{ for } \ c < C(a,b). \end{aligned}$$

In particular, by Theorem 3.1 and Lemma 6.1 we have

$$\{(a,b,c) : a > 0, a+b > 0, c < C(a,b)\} = \Theta,$$

which is an open set, as we argued above. So, $C(a,b)$ is lower semi-continuous for $a > 0$, $a+b > 0$. Combining this with Theorem 5.3 we obtain that $C(a,b)$ is a continuous function for $a > 0$ and $a+b > 0$. □

7. Existence of a physically acceptable solution in the case $a+b<0$

For given $a > 0$, $b < -a$ and $c \in \mathbb{R}$, let $h_{a,b,c}$ be the solution of (13) with $h_{a,b,c}(0) = h'_{a,b,c}(0) = 0$, $h''_{a,b,c}(0) = a$, $h'''_{a,b,c}(0) = b$, $h''''_{a,b,c}(0) = c$. Let also $m_{a,b,c} = h_{a,b,c} - h''_{a,b,c}$.

Due to (29), $I(h_{a,b,c}) > a$ for all $b < -a$, $c \le a$. So, in order to obtain a physically acceptable solution, we should concentrate on the case $c > a$, i.e., when $m''_{a,b,c}(0) < 0$ (see (17)). Lemma 4.1 yields that $I(h_{a,b,c}) \le a$ in this case. In particular, this implies that $h_{a,b,c}$ is defined on $[0,\infty)$ (see Theorem 3.1).

Let

$$\Gamma = \{(a,b,c) : a > 0, c = C(a,b)\},$$

i.e., Γ is the connecting surface (see Section 5). We start with the following

Lemma 7.1. *Define $\gamma : \Gamma \to \mathbb{R}$ by*

$$\gamma(a,b,c) = \lim_{x\to\infty} m'_{a,b,c}(x).$$

Then γ is continuous at any point $(a,b,c) \in \Gamma$ such that $\gamma(a,b,c) > 0$.

Proof. The definition of Γ and continuous dependence on the initial data imply that

$$h'_{a,b,c}(x) \ge 0, \qquad \forall (a,b,c) \in \Gamma, \ \ \forall x > 0. \tag{33}$$

Take any $(\hat{a},\hat{b},\hat{c}) \in \Gamma$ such that $\gamma(\hat{a},\hat{b},\hat{c}) > 0$. Let $\hat{h} := h_{\hat{a},\hat{b},\hat{c}}$ and $\hat{m} := m_{\hat{a},\hat{b},\hat{c}}$. Suppose that $\lim_{x\to\infty} \hat{h}'(x) = 0$. Then, according to Theorem 3.1, $\gamma(a,b,c) = 0$, a contradiction. Therefore, by (33), we obtain $\lim_{x\to\infty} \hat{h}'(x) > 0$. Thus we have (see (17))

$$\hat{h}(x) \to \infty, \qquad \widehat{m}''(x) \to 0 \qquad \text{as} \qquad x \to \infty.$$

Given $\epsilon > 0$, choose x_0 large enough that

$$\hat{h}(x) > 1, \qquad |\widehat{m}''(x)| < \frac{\epsilon}{8}, \qquad \forall x \ge x_0.$$

The continuous dependence on the initial data implies that there exists a neighborhood N of $(\hat{a}, \hat{b}, \hat{c})$ in Γ such that for any $h = h_{a,b,c}$ and $m = m_{a,b,c}$, with the initial data $(a, b, c) \in N$, we have

$$h(x_0) > 1, \qquad |m''(x_0)| < \frac{\epsilon}{8}, \qquad |m'(x_0) - \widehat{m}'(x_0)| < \frac{\epsilon}{2}.$$

In addition, by virtue of (33), we obtain

$$h(x) > 1, \qquad x \geq x_0.$$

Hence, using (17), we derive that

$$|m''(x)| = |m''(x_0)| e^{-\frac{1}{2}\int_{x_0}^{x} h(y)\, dy} < |m''(x_0)| e^{-\frac{1}{2}(x-x_0)} < \frac{\epsilon}{8} e^{-\frac{1}{2}(x-x_0)}$$

for all $x \geq x_0$. Similarly,

$$|\widehat{m}''(x)| < \frac{\epsilon}{8} e^{-\frac{1}{2}(x-x_0)}, \qquad x \geq x_0.$$

Thus

$$|m'(x) - \widehat{m}'(x)| < |m'(x_0) - \widehat{m}'(x_0)| + 2\int_{x_0}^{x} \frac{\epsilon}{8} e^{-\frac{1}{2}(x-x_0)}\, dx < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for all $x \geq x_0$. Consequently,

$$|\gamma(a, b, c) - \gamma(\hat{a}, \hat{b}, \hat{c})| \leq \epsilon.$$

This concludes the proof of the lemma. $\square$

Lemma 7.2. *If $\gamma(a, b, c) \geq 0$ for some $(a, b, c) \in \Gamma$ such that $a + b < 0$, then*

$$I(h_{a,b,c}) = a,$$

and

$$\lim_{x\to\infty} h'_{a,b,c}(x) = \gamma(a, b, c).$$

Proof. Take any $(a, b, c) \in \Gamma$ such that $a + b < 0$. Lemma 5.2 implies that $c > a$ and, consequently, $I(h_{a,b,c}) \leq a$ due to Lemma 4.1. Assume that $I(h_{a,b,c}) < a$, i.e., $h'_{a,b,c}(x) \to \infty$ as $x \to \infty$ (see Theorem 3.1). Since Ω_0 is open, $\Gamma \cap \Omega_0 = \emptyset$, and we have that $(a, b, c) \notin \Omega_0$. Thus, by the definition of Ω_0, $h'_{a,b,c}(x_0) = 0$ for some $x_0 > 0$. Note that we also have $h'_{a,b,c}(0) = 0$ and $h''_{a,b,c}(0) = a > 0$. Thus the minimum of $h'_{a,b,c}$ is attained for some $x_1 > 0$. Then we have $h'_{a,b,c}(x_1) \leq 0$ and $h'''_{a,b,c}(x_1) \geq 0$. Consequently, we obtain that

$$m'_{a,b,c}(x_1) = h'_{a,b,c}(x_1) - h'''_{a,b,c}(x_1) \leq 0.$$

Since $c > a$, we have that $m''_{a,b,c}(x) < 0$ for all $x \geq 0$ (see (17). Hence $m'_{a,b,c}(x)$ is strictly decreasing in x for $x \geq 0$. Therefore $\gamma(a, b, c) < 0$, which allows us to conclude that if $\gamma(a, b, c) \geq 0$, then $I(h_{a,b,c}) = a$. $\square$

Now we are ready to proceed to the main result in this section.

Theorem 7.3. *There exists a continuous function $b_0 : (0,\infty) \to \mathbb{R}$ such that $b_0(a) < -a$, and for each $a > 0$ and $b \in (b_0(a), -a)$ we have*

$$h'_{a,b,C(a,b)}(x) \to \gamma(a,b,C(a,b)) > 0 \quad \textit{as} \quad x \to \infty,$$

i.e., $h_{a,b,C(a,b)}$ is physically acceptable for all $b \in (b_0(a), -a)$.

Proof. First, Lemma 5.2 yields that $(a,-a,a) \in \Gamma$ for each $a > 0$,

$$(a,-a,c) \in \Omega_0, \qquad \forall c > a > 0, \tag{34}$$

and

$$C(a,b) > a, \qquad \forall b < -a < 0. \tag{35}$$

Take any $\hat{a} > 0$. Recall that Ω_0 is open and that $\Gamma \cap \Omega_0 = \emptyset$. Therefore, using (34), we infer that

$$\limsup_{(a,b)\to(\hat{a},-\hat{a})} C(a,b) \leq \hat{a}.$$

This, together with (35) and the fact that $C(\hat{a},-\hat{a}) = \hat{a}$, implies that $C(a,b)$ is continuous at $(a,b) = (\hat{a},-\hat{a})$.

Since $\gamma(\hat{a},-\hat{a},\hat{a}) = \lim_{x\to\infty} m'_{\hat{a},-\hat{a},\hat{a}}(x) = \hat{a} > 0$, Lemma 7.1 implies that there exists a neighborhood N of $(\hat{a},-\hat{a},\hat{a})$ such that $\gamma(a,b,c) > 0$ for all $(a,b,c) \in N \cap \Gamma$. Moreover, the continuity of $C(a,b)$ at $(\hat{a},-\hat{a})$ implies that there exists $\delta > 0$ such that for $|a-\hat{a}| < \delta$ and $-\delta < b + \hat{a} < 0$ we have

$$(a,b,C(a,b)) \in N.$$

Therefore Lemma 7.2 implies that $I(h_{a,b,C(a,b)}) = a$ for all $|a-\hat{a}| < \delta$ and $-\delta < b+\hat{a} < 0$. Then Theorem 3.1 yields

$$h'_{a,b,C(a,b)}(x) \to \gamma(a,b,C(a,b)) \quad \text{as} \quad x \to \infty \tag{36}$$

for $|a-\hat{a}| < \delta$ and $-\delta < b+\hat{a} < 0$.

Using a partition of unity we see that there exists a continuous function $b_0 : (0,\infty) \to \mathbb{R}$ such that (36) holds for all $a > 0$ and $b_0(a) < b < -a$. $\square$

We note that Theorem 7.3 was stated without the condition "$b > b_0(a)$" in [8] (see Theorem 2.1, [8]). To rigorously show that the theorem does not hold without such an assumption, let h be a solution of (13) with $h(0) = h'(0) = 0$, $h''(0) = a > 0$, $h'''(0) = b < -a$, $h''''(0) = c$. Assume that this h above is physically acceptable, i.e., that $I(h) = a$. We will show that this assumption puts some constraints on a and b.

Lemma 7.4. *If $b < -a < 0$ and $I(h) = a$, then the following inequalities hold:*

1. *$h'(x) > 0$ for all $x > 0$,*
2. *$h''(x) < a$ for all $x > 0$,*
3. *$h'''(x) > b$ for all $x > 0$.*

Proof. First, according to (29) we have $c > a$. Therefore (17) yields that

$$m''(x) < 0, \qquad x \geq 0. \tag{37}$$

Since $I(h) = a$, from Theorem 3.1 it follows that

$$0 \leq \lim_{x\to\infty} h'(x) = \lim_{x\to\infty} m'(x) < \infty. \tag{38}$$

Hence

$$h'''(x) = h'(x) - m'(x) \to 0 \quad \text{as} \quad x \to \infty. \tag{39}$$

Relations (37) and (38) imply that $m'(x) > 0$ for all $x \geq 0$. Moreover, relations (38) and (39) imply that

$$\lim_{x\to\infty} h''(x) = \lim_{x\to\infty} h'''(x) = 0. \tag{40}$$

Inequalities (1) – (3) obviously hold for x small enough. First, suppose $h'(x_0) = 0$ for some $x_0 > 0$. Since $\lim_{x\to\infty} h'(x) \geq 0$, there exists $x_1 > 0$ such that $h'(x_1) \leq 0$ and $h'''(x_1) \geq 0$. Then $m'(x_1) \leq 0$, a contradiction. Therefore $h'(x) > 0$ for all $x > 0$. In particular, this yields that $h(x) > 0$ for all $x > 0$ and that

$$m'''(x) = \frac{1}{2}(c-a)h(x)e^{-\frac{1}{2}\int_0^x h(y)\,dy} > 0, \qquad x > 0. \tag{41}$$

Second, suppose $h''(x_0) = a$ for some $x_0 > 0$. Since $\lim_{x\to\infty} h''(x) = 0$, there exists $x_1 > 0$ such that $h''(x_1) \geq a > 0$ and $h''''(x_1) \leq 0$. Then $m''(x_1) > 0$, contradicting (37). Therefore $h''(x) < a$ for all $x > 0$.

Third, suppose $h'''(x_0) = b$ for some $x_0 > 0$. Since $\lim_{x\to\infty} h'''(x) = 0$, there exists $x_1 > 0$ such that $h'''(x_1) \leq b < 0$ and $h'''''(x_1) \geq 0$. Then $m'''(x_1) < 0$, contradicting (41). Therefore $h'''(x) > b$ for all $x > 0$. $\square$

Theorem 7.5. *If* $b < -a < 0$ *and* $I(h) = a$, *then*

$$|b| \leq c \leq a + k\sqrt[3]{a}|b|,$$

where

$$k = \left(\int_0^\infty e^{-\frac{1}{12}y^3}\,dy\right)^{-1} \approx 2.0.$$

In particular, we have

$$|b| \leq C(a,b) \leq a + k\sqrt[3]{a}|b|, \qquad \forall b \in (b_0(a), -a).$$

Proof. Recall that $h''' - h''''' = m'''$. Since $m'''(x) > 0$ for all $x \geq 0$ (see (41)), we obtain

$$\begin{aligned} 2h'''(x) &= \left(e^x - e^{-x}\right)c + \left(e^x + e^{-x}\right)b - \int_0^x \left(e^{x-y} - e^{-x+y}\right)m'''(y)\,dy. \\ &< \left(e^x - e^{-x}\right)c + \left(e^x + e^{-x}\right)b \end{aligned}$$

for all $x > 0$. Since $\lim_{x\to\infty} h'''(x) = 0$ (see 40), we obtain $c + b \geq 0$. To prove the other inequality, notice that, due to Lemma 7.4, $h''(x) < a$ for all $x > 0$. Therefore

$$\int_0^x h(y)\,dy < \frac{1}{6}ax^3, \qquad x > 0.$$

According to (29), $c > a$. Then it follows that

$$m''(x) < (a-c)e^{-\frac{1}{12}ax^3}, \qquad x > 0,$$

and

$$m'(x) < -b + (a-c)\int_0^x e^{-\frac{1}{12}ay^3}\,dy, \qquad x > 0.$$

Due to (38), we obtain

$$\int_0^\infty e^{-\frac{1}{12}ay^3}\,dy \leq \frac{-b}{c-a}.$$

Define

$$k := \left(\int_0^\infty e^{-\frac{1}{12}y^3}\,dy\right)^{-1}.$$

Then we have

$$c \leq a - k\sqrt[3]{a}b,$$

and, finally,

$$|b| \leq c \leq a + k\sqrt[3]{a}|b|.$$

□

This theorem yields the following

Corollary 7.6. *If $0 < a < 1/k^3$ and $b < -a/(1 - k\sqrt[3]{a})$, then there is no physically acceptable solution h with $h''(0) = a$ and $h'''(0) = b$. Here k is the universal constant from Theorem 7.5.*

8. Comparison with experimental data

It is common to use the wall coordinates

$$y^+ = \frac{u_\tau y}{\nu}, \qquad u^+ = \frac{u}{u_\tau}$$

in the turbulent boundary layer, where

$$u_\tau = \sqrt{\frac{1}{\rho}\tau} = \sqrt{\nu\frac{\partial u}{\partial y}\Big|_{y=0}},$$

and τ is the shear stress at the wall.

Fix x_0 on the horizontal axis and denote

$$l_{\mathrm{e}} = \frac{\nu}{u_{\mathrm{e}}}, \qquad R_l = \frac{l}{l_{\mathrm{e}}},$$

where l is a parameter of the boundary layer at the point x_0 (see Section 2). According to the derivation of (12),

$$u(x_0, y) = \frac{u_{\mathrm{e}}}{\beta^2} h' \left(\frac{y}{\beta \sqrt{l_{\mathrm{e}} l}} \right) \tag{42}$$

represents the horizontal component of the averaged velocity at $x = x_0$ for some solution h of

$$m''' + \frac{1}{2} h m'' = 0, \qquad m = h - h'' \tag{43}$$

with $h(0) = h'(0) = 0$, $h''(0) = a > 0$, $h'''(0) = b$, and $h''''(0) = C(a, b)$, where the connecting function $C(a, b)$ is defined in (30). The coefficient β^2 in (42) is the $\lim_{\xi \to \infty} h'(\xi)$ provided by Theorem 2.1, which is numerically determined.

Note that (42) implies

$$a = \frac{1}{2} c_f \beta^3 \sqrt{R_l},$$

i.e. a is a rescaled skin-friction coefficient $c_f = 2 \left(u_\tau / u_{\mathrm{e}} \right)^2$. Writing (42) in wall coordinates, we obtain

$$u^+ = \frac{R_l^{1/4}}{\sqrt{a \beta}} h' \left(\frac{y^+ \sqrt{\beta}}{R_l^{1/4} \sqrt{a}} \right).$$

The parameters a and b correspond to c_f, the skin-friction coefficient, and R_θ, the Reynolds number based on momentum thickness, which can be written in the following way:

$$c_f = \frac{2}{u^+(\infty)^2}, \qquad R_\theta = \int_0^\infty u^+ \left(1 - \frac{u^+}{u^+(\infty)} \right) dy^+, \tag{44}$$

where $u^+(\infty) := \lim_{y^+ \to \infty} u^+$.

In the laminar and transitional cases, for given experimental data R_θ, c_f and R_x, the local Reynolds number, we take $R_l = R_x$ and find numerically a and b so that (44) holds. In the turbulent case it is necessary to determine R_l. Therefore, for given R_θ and c_f we find a and b so that (44) holds. In addition, we find R_l so that the von Karman logarithmic law holds for the middle inflection point in logarithmic coordinates. More precisely, we assume that

$$u^+(y_0^+) = \frac{1}{k} \ln y_0^+ + B$$

for only one point y_0^+, which is the middle inflection point in logarithmic coordinates. There are three inflection points in the turbulent case. The constants are the following: $k \approx 0.4$, $B \approx 5$; however, they have to be adjusted for low Reynolds numbers.

Thus, we have to solve the following equation for R_l:

$$\frac{R_l^{1/4}}{\sqrt{a \beta}} h'(\xi_0) = \frac{1}{k} \ln(\sqrt{a/\beta} R_l^{1/4} \xi_0) + B.$$

Here ξ_0 is the middle inflection point of $h'(\xi)$ in logarithmic coordinates.

Varying R_θ and c_f, we obtain a family of the velocity profiles $\{u^+_{R_\theta,c_f}\}$. This family was compared with experimental data of the Rolls-Royce applied science laboratory, ERCOFTAC t3b test case [32] (see Fig. 5–7). Comparison shows that the case $a+b>0$ corresponds to the laminar boundary layer, whereas $a+b<0$ corresponds to the transitional and turbulent boundary layers.

9. Appendix. Blasius equation

The Blasius equation is written as

$$h''' + \frac{1}{2}hh'' = 0. \tag{45}$$

The physical boundary conditions are $h(0) = h'(0) = 0$, and $h'(\xi) \to 1$ as $\xi \to \infty$.

Theorem 9.1. *There exists a unique solution $h(x)$ to (45) satisfying the physical boundary conditions. In addition, the Blasius profile $h'(x)$ has one inflection point in logarithmic coordinates.*

Proof. The existence and uniqueness is a classical result and can be found, e.g., in [18]. Nevertheless, we present a short proof here for completeness.

For a given $a \in \mathbb{R}$, let $h_a(x)$ be the solution of (45) with $h_a(0) = h'_a(0) = 0$, $h''_a(x) = a$. Then

$$h''_a(x) = ae^{-\frac{1}{2}\int_0^x h_a(y)\,dy}. \tag{46}$$

Obviously, if $a \le 0$, then $h_a(x)$ does not satisfy the physical boundary conditions. If $a > 0$, then (46) implies that $h''_a(x) > 0$ for all $x \ge 0$ and, consequently, $\int_0^x h_a(y)\,dy$ increases faster than a linear function. Therefore, using (46) again, we conclude that $h'_a(x) \to \gamma(a)$ for some $\gamma(a) > 0$. Finally, since

$$h_a(x) = \sqrt{a}h_1(\sqrt{a}x),$$

we have that $\gamma(a) = a\gamma(1)$. Hence, there exists a unique solution $h(x)$ to (45) satisfying the physical boundary conditions. In fact, $h(x) = h_a(x)$ for $a = 1/\gamma(1)$. For this $h(x)$, the function $h'(x)$, $x \in [0,\infty)$, will be be called a Blasius profile.

To study the Blasius profile in logarithmic coordinates, let $z = \ln x$ and $F(z) = h'(e^z)$. Then (45) yields

$$F''(z) = ae^{z-\frac{1}{2}\int_0^{e^z} h(y)\,dy}\left(1 - \frac{1}{2}e^z h(e^z)\right).$$

Since $h(0) = 0$ and $h(x)$ is increasing, $F''(z)$ changes sign once, i.e. the Blasius profile has one inflection point in logarithmic coordinates. $\square$

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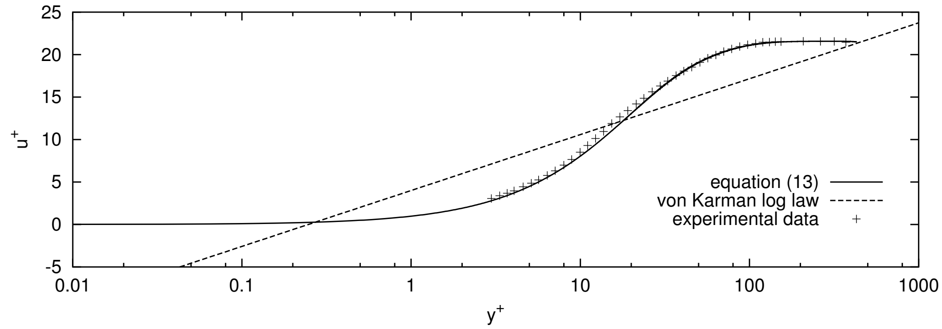

Fig. 5. Comparison with experimental data for $c_f = 0.00432,\ R_\theta = 265$

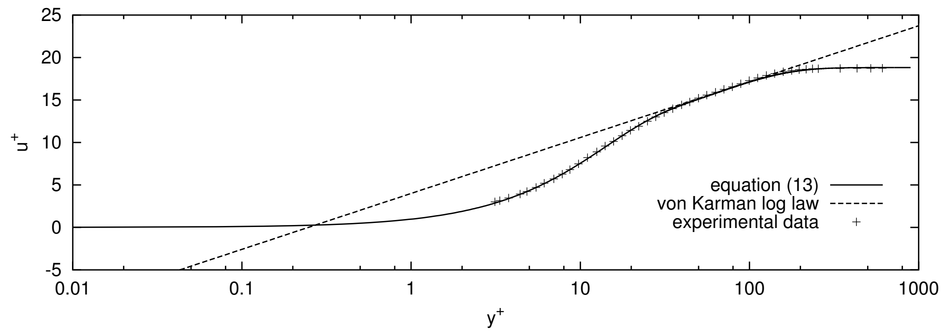

Fig. 6. Comparison with experimental data for $c_f = 0.00569,\ R_\theta = 396$

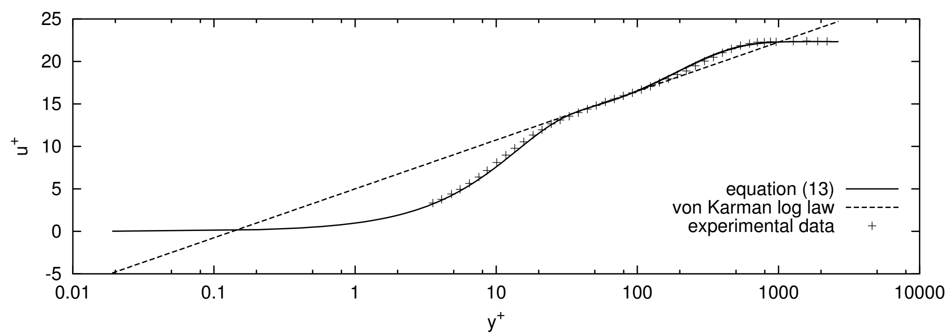

Fig. 7. Comparison with experimental data for $c_f = 0.00401,\ R_\theta = 1436$