

# AN INVISCID DYADIC MODEL OF TURBULENCE: THE GLOBAL ATTRACTOR

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ABSTRACT. Properties of an infinite system of nonlinearly coupled ordinary differential equations are discussed. This system models some properties present in the equations of motion for an inviscid fluid such as the skew symmetry and the 3-dimensional scaling of the quadratic nonlinearity. In a companion paper [6] it is proved that every solution for the system with forcing blows up in finite time in the Sobolev  $H^{5/6}$  norm. In this present paper, it is proved that after the blow-up time all solutions stay in  $H^s$ ,  $s < 5/6$  for almost all time and the energy dissipates.

Moreover, it is proved that the unique equilibrium is an exponential global attractor.

## 1. INTRODUCTION

One of the outstanding open questions in fluid dynamics is existence and uniqueness of solutions to the Cauchy problem for the three-dimensional Euler equations

$$(1.1) \quad \frac{\partial u}{\partial t} = -(u \cdot \nabla)u - \nabla p, \quad \nabla \cdot u = 0.$$

In last few decades simplified models that capture some properties of fluid equations have been proposed and studied. In this article we analyze one of these models, a so called “dyadic” model for the equations of fluid motion. We study the following dyadic model:

$$(1.2) \quad \begin{aligned} \frac{da_j}{dt} &= 2^{\frac{5(j-1)}{2}} a_{j-1}^2 - 2^{\frac{5j}{2}} a_j a_{j+1} + f_j, \quad j > 0 \\ \frac{da_0}{dt} &= -a_0 a_1 + f_0, \end{aligned}$$

where the force  $f$  is chosen so that  $f_0 > 0$  and  $f_j = 0$  for  $j > 0$  for simplicity.

The model (1.2) without forcing is a special case of the infinite dimensional dynamical system

$$(1.3) \quad \frac{da_j}{dt} = \lambda^{j-1} a_{j-1}^2 - \lambda^j a_j a_{j+1}.$$

Such an inviscid model has been studied recently in a number of articles including [12, 15, 17, 25]. Variants of the model that include viscosity are

discussed in [3, 16]. Analysis of more general “shell” models and motivation in terms of turbulence modeling can be found in [2, 8, 9, 14, 20, 21].

In a companion paper [6] we presented a motivation for the model (1.2) from the Fourier space Euler equations (1.1) in 3-dimensions. The coefficient  $a_j^2(t)$  is the total energy in the frequency space shell  $2^j \leq |k| < 2^{j+1}$ . In this context  $l^2$  and  $H^s$ , respectively the energy and Sobolev norms, are defined as

$$(1.4) \quad \|a(t)\|_{l^2} = \left( \sum_{j=0}^{\infty} a_j^2(t) \right)^{1/2}, \quad \|a(t)\|_{H^s} = \left( \sum_{j=0}^{\infty} 2^{2sj} a_j^2(t) \right)^{1/2}.$$

The nonlinear terms on the right hand side of (1.2) retain important features of the advective term in the Euler equation, namely bilinearity and skew-symmetry. The presence of the specific quadratic term  $2^{\frac{5(j-1)}{2}} a_{j-1}^2$  ensures a certain monotonicity (see also, [1] and [23]) in the cascade of energy through the scales  $j$ . We defined a regular solution for the model (1.2) to be a solution with bounded  $H^{5/6}$  norm and such solutions satisfy the energy equality. In [6] we proved that:

- (a) There exists a unique fixed point to (1.2) and the fixed point is not in  $H^{5/6}$ .
- (b) Every regular solution approaches the fixed point in the  $l^2$  norm.
- (c) Every solution blows up in finite time in the  $H^{5/6}$  norm.

As Mattingly et al [19] observe in their recent analysis of an infinite linear dynamical system, the most interesting features of such models belong to solutions *after* the time of blow-up when some norm becomes infinite. This is particularly true in the context of models that illustrate behavior that has been proposed to characterize hydrodynamic turbulence in the works of, for example, Kolmogorov [18], Onsager [22], Frisch [13], Robert [24], Constantin et al [7] and Eyink-Sreenivasan [10].

In our present paper we study the solutions of (1.2) *after* the time of blow-up in  $H^{5/6}$ . We prove:

- (1) The  $H^s$  norms for  $s < 5/6$  are locally square integrable in time.
- (2) The solutions dissipate energy.
- (3) The unique fixed point  $\{a_j\} = \{2^{-5j/6} 2^{5/12} \sqrt{f_0}\}$  is an exponential global attractor.

The existence of a global attractor for an inviscid system is, perhaps, surprising. However it is exactly consistent with the concept of anomalous or turbulent dissipation conjectured by Onsager [22]. As we discuss in Section 6, after the time of blow up in  $H^{5/6}$  the energy spectrum exactly reproduces Kolmogorov’s law:

$$(1.5) \quad E(|k|) = c_0 \bar{\epsilon}^{2/3} |k|^{-5/3},$$

where  $\bar{\epsilon}$  is the average of the energy dissipation rate.

**Notation.** For notational convenience we adopt  $\lambda^j$  as the scaling parameter in the analysis performed in sections 2 - 5. We do this to illustrate that the results are qualitatively independent of the exact choice of  $\lambda$  (which depends on the spatial dimension and the construction of the model). The proofs of results in section 4 require  $\lambda < 2^3$ . As we discussed in [6] the relevant  $\lambda$  for the 3-dimensional model is  $2^{5/2}$ . This exponent determines the values of the exponent of the fixed point and the critical Sobolev space exponent  $H^{5/6}$ . The exponent also reproduces the exponents in the Kolmogorov's law (1.5).

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## 2. FUNCTIONAL SETTING

Let us denote  $H = l^2$  with the usual scalar product and norm:

$$(2.1) \quad (a, b) := \sum_{j=0}^{\infty} a_j b_j, \quad |a| := \sqrt{(a, b)}.$$

The norm  $|a|$  will be called the energy norm. Let

$$(2.2) \quad ((a, b))_s := \sum_{j=0}^{\infty} 2^{2sj} a_j b_j, \quad \|u\|_s := \sqrt{((u, u))_s}.$$

We fix  $\lambda = 2^{5/2}$  and let

$$(2.3) \quad d_s(a, b) := |a - b|, \quad d_w(a, b) := \sum_{j=0}^{\infty} \frac{1}{\lambda^{(j^2)}} \frac{|a_j - b_j|}{1 + |a_j - b_j|}, \quad a, b \in H.$$

Here,  $d_s$  is a strong distance, and  $d_w$  is a weak distance that induces a weak topology on any bounded subset of  $H$ . Hence, a bounded sequence  $\{a^k\} \subset H$  converges to  $a \in H$  weakly, i.e.,

$$(2.4) \quad \lim_{k \rightarrow \infty} (a^k, b) = (a, b), \quad \forall b \in H,$$

if and only if

$$(2.5) \quad d_w(a^k, a) \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty.$$

Let

$$(2.6) \quad C([0, T]; H_w) := \{a(\cdot) : [0, T] \rightarrow H, a_j(t) \text{ is continuous for all } j\}$$

endowed with the distance

$$(2.7) \quad d_{C([0, T]; H_w)}(a, b) = \sup_{t \in [0, T]} d_w(a(t), b(t)).$$

Let also

$$(2.8) \quad C([0, \infty); H_w) := \{a(\cdot) : [0, \infty) \rightarrow H, a_j(t) \text{ is continuous for all } j\}$$

endowed with the distance

$$(2.9) \quad d_{C([0,\infty);H_w)}(a, b) = \sum_{T \in \mathbb{N}} \frac{1}{\lambda^T} \frac{\sup\{d_w(a(t), b(t)) : 0 \leq t \leq T\}}{1 + \sup\{d_w(a(t), b(t)) : 0 \leq t \leq T\}}.$$

### 3. WEAK SOLUTIONS

**Definition 3.1.** *A weak solution on  $[0, T]$  (or  $(0, \infty)$ , if  $T = \infty$ ) of (1.2) is an  $H$ -valued function  $a(t)$  defined for  $t \in [0, T]$ , such that  $a_j \in C^1([0, T])$  and  $a_j(t)$  satisfies (1.2) for all  $j$ .*

Note that since the nonlinear term has a finite number of terms, the notions of a weak solution and a classical solution (of a system of ODEs) coincide. Hence, the weak solutions will be called solutions in the remainder of the paper. Note that if  $a(t)$  is a solution on  $[T, \infty)$ , then automatically  $a_j \in C^\infty([T, \infty))$ .

**Theorem 3.2** (Global existence). *For every  $a^0 \in H$ , there exists a solution of (1.2) on  $[0, \infty)$  with  $a(0) = a^0$ .*

*Proof.* Let  $u^0 \in H$  and  $T > 0$  be arbitrary. We will show the existence of a solution on  $[0, T]$  by taking a limit of the Galerkin approximation  $a^k(t) = (a_0^k(t), \dots, a_k^k(t), 0, 0, \dots)$  with  $a_j^k(0) = a_j^0$  for  $j = 1, 2, \dots, k$ , which satisfies

$$(3.1) \quad \begin{cases} \frac{d}{dt} a_j^k - \lambda^{j-1} (a_{j-1}^k)^2 + \lambda^j a_j^k a_{j+1}^k = f_j, & j \leq k-1, \\ \frac{d}{dt} a_k^k - \lambda^{k-1} (a_{k-1}^k)^2 = f_k, \end{cases}$$

where  $a_{-1}^j = 0$  and  $\lambda = 2^{5/2}$ . From the theory of ordinary differential equations we know that there exists a unique solution  $a^k(t)$  to (3.1) on  $[0, T]$ . We will show that a sequence of the Galerkin approximations  $\{a^k\}$  is weakly equicontinuous. Indeed, it is clear that there exists  $M$ , such that

$$(3.2) \quad a_j^k(t) \leq M, \quad \text{for all } t \in [0, T] \text{ and all } j, k.$$

Therefore,

$$(3.3) \quad |a_j^k(t) - a_j^k(s)| \leq \left| \int_s^t \left( \lambda^{j-1} (a_{j-1}^k(\tau))^2 - \lambda^j a_j^k(\tau) a_{j+1}^k(\tau) + f_j \right) d\tau \right| \\ \leq (\lambda^{j-1} M^2 + \lambda^j M^2 + f_j) |t - s|,$$

for all  $j, k$  and all  $0 \leq t \leq s \leq T$ . Thus,

$$(3.4) \quad d_w(a^k(t), a^k(s)) = \sum_{j=0}^{\infty} \frac{1}{\lambda^{(j^2)}} \frac{|a_j^k(t) - a_j^k(s)|}{1 + |a_j^k(t) - a_j^k(s)|} \\ \leq c|t - s|,$$

for some constant  $c$  independent of  $k$ . Hence,  $\{a^k\}$  is an equicontinuous sequence of functions in  $C([0, T]; H_w)$  with bounded initial data. Therefore, the Ascoli-Arzelà theorem implies that  $\{a^k\}$  is relatively compact in

$C([0, T]; H_w)$ . Hence, passing to a subsequence, we obtain that there exists a weakly continuous  $H$ -valued function  $a(t)$ , such that

$$(3.5) \quad a^{k_n} \rightarrow a \quad \text{as} \quad k_n \rightarrow \infty \quad \text{in} \quad C([0, T]; H_w).$$

In particular,  $a_j^{k_n}(t) \rightarrow a_j(t)$  as  $k_n \rightarrow \infty$ , for all  $j, t \in [0, T]$ . Thus,  $a(0) = a^0$ . In addition, note that

$$(3.6) \quad a_j^{k_n}(t) = a_j^{k_n}(0) + \int_0^t (\lambda^{j-1} (a_{j-1}^{k_n})^2 - \lambda^j a_j^{k_n} a_{j+1}^{k_n} + f_j) d\tau,$$

for  $j \leq k_n - 1$ . Taking the limit as  $k_n \rightarrow \infty$ , we obtain

$$(3.7) \quad a_j(t) = a_j(0) + \int_0^t (\lambda^{j-1} a_{j-1}^2 - \lambda^j a_j a_{j+1} + f_j) d\tau.$$

Since  $a_j(t)$  is continuous, it follows that  $a_j \in C^1([0, T])$  and satisfies (1.2).  $\square$

**Theorem 3.3** (Energy inequality). *Let  $a(t)$  be a solution of (1.2) with  $a_j(0) \geq 0$ . Then  $a_j(t) > 0$  for all  $t > 0$ , and  $a(t)$  satisfies the energy inequality*

$$(3.8) \quad |a(t)|^2 \leq |a(t_0)|^2 + 2 \int_{t_0}^t (f, a(\tau)) d\tau,$$

for all  $0 \leq t_0 \leq t$ .

*Proof.* A general solution of (1.2) can be written as

$$(3.9) \quad a_j(t) = a_j(0) \exp\left(-\int_0^t \lambda^j a_{j+1}(\tau) d\tau\right) + \int_0^t \exp\left(-\int_s^t \lambda^j a_{j+1}(\tau) d\tau\right) (f_j + \lambda^{j-1} a_{j-1}^2(s)) ds,$$

where  $\lambda = 2^{5/2}$ . Recall that  $f_j \geq 0$  for all  $j$ . Since  $a_j(0) \geq 0$  for all  $j$ , then  $a_j(t) \geq 0$  for all  $j, t > 0$ . Moreover, since  $f_0 > 0$ , we have  $a_0(t) > 0$  for all  $t > 0$  and, consequently,  $a_j(t) > 0$  for all  $j, t > 0$ . Hence, multiplying (1.2) by  $a_j$ , taking a sum from 0 to  $N$ , and integrating between  $t_0$  and  $t$ , we obtain

$$(3.10) \quad \sum_{j=0}^N a_j(t)^2 - \sum_{j=0}^N a_j(t_0)^2 = -2 \int_{t_0}^t \lambda^N a_N^2 a_{N+1} d\tau + 2 \int_{t_0}^t \sum_{j=0}^N f_j a_j d\tau \leq 2 \int_{t_0}^t \sum_{j=0}^N f_j a_j d\tau.$$

Taking the limit as  $N \rightarrow \infty$ , we obtain (3.8).  $\square$

## 4. FIXED POINT

Given a solution  $a(t)$  of (1.2) with arbitrary initial data  $a_j(0) \geq 0$ , let

$$(4.1) \quad b_j(t) := a_j(t) - \lambda^{-j/3}, \quad j \geq 0,$$

where  $\lambda = 2^{5/2}$ . Let also

$$(4.2) \quad \begin{aligned} d_j &= \lambda^{\frac{j-1}{3}} (\lambda^{\frac{1}{6}} b_j - \lambda^{-\frac{1}{6}} b_{j-1}), \quad j \geq 1, \\ d_0 &= \lambda^{-\frac{1}{6}} b_0. \end{aligned}$$

We sum the expressions for  $d_l$ , with  $0 \leq l \leq j$  and thanks to (4.2) obtain:

$$(4.3) \quad b_j = \lambda^{\frac{1}{6} - \frac{j}{3}} (d_0 + d_1 + \cdots + d_j).$$

**Lemma 4.1.** *For every  $t \geq 0$  we have*

$$(4.4) \quad \frac{d}{dt} \sum_{j=0}^k b_j(t)^2 \leq - \sum_{j=0}^k d_j(t)^2 + d_{k+1}(t)^2,$$

and

$$(4.5) \quad \frac{d}{dt} \left( \sum_{j=0}^{k-1} b_j(t)^2 + \frac{1}{2} b_k(t)^2 \right) \leq - \sum_{j=0}^{k-1} d_j(t)^2 + \lambda^{\frac{k-1}{3}} |b(t)|,$$

for all  $k > 1$ .

*Proof.* Note that  $b(t)$  satisfies the following system of equations:

$$(4.6) \quad \begin{aligned} \frac{db_j}{dt} &= \lambda^{\frac{2j}{3}} (2\lambda^{-\frac{2}{3}} b_{j-1} - \lambda^{-\frac{1}{3}} b_j - b_{j+1}) + (\lambda^{j-1} b_{j-1}^2 - \lambda^j b_j b_{j+1}), \quad j \geq 1, \\ \frac{db_0}{dt} &= -\lambda^{-\frac{1}{3}} b_0 - b_1 - b_0 b_1. \end{aligned}$$

Multiplying it by  $b_j$  and taking a sum from  $j = 0$  to  $j = k$  we obtain

$$(4.7) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \sum_{j=0}^k b_j^2 &= -\lambda^{-\frac{1}{3}} b_0^2 - b_0 b_1 - b_0^2 b_1 \\ &+ \sum_{j=1}^k \left( \lambda^{\frac{2j}{3}} (2\lambda^{-\frac{2}{3}} b_{j-1} b_j - \lambda^{-\frac{1}{3}} b_j^2 - b_j b_{j+1}) + (\lambda^{j-1} b_{j-1}^2 b_j - \lambda^j b_j^2 b_{j+1}) \right). \end{aligned}$$

It now follows that

$$(4.8) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \sum_{j=0}^k b_j^2 &= - \left( \lambda^{-\frac{1}{3}} b_0^2 + b_0 b_1 \right) \\ &+ \sum_{j=1}^k \lambda^{\frac{2j}{3}} \left( 2\lambda^{-\frac{2}{3}} b_{j-1} b_j - \lambda^{-\frac{1}{3}} b_j^2 - b_j b_{j+1} \right) - \lambda^k b_k^2 b_{k+1}. \end{aligned}$$

Also we can rewrite (4.8) as

$$(4.9) \quad \frac{1}{2} \frac{d}{dt} \sum_{j=0}^k b_j^2 = -\lambda^{-\frac{1}{3}} \sum_{j=0}^k \lambda^{\frac{2j}{3}} b_j^2 + \sum_{j=0}^k \lambda^{\frac{2j}{3}} b_j b_{j+1} - 2\lambda^{\frac{2k}{3}} b_k b_{k+1} - \lambda^k b_k^2 b_{k+1}.$$

However,

$$\begin{aligned} & \sum_{j=0}^k \lambda^{\frac{2j}{3}} (\lambda^{-\frac{1}{6}} b_j - \lambda^{\frac{1}{6}} b_{j+1})^2 \\ &= \sum_{j=0}^k \lambda^{\frac{2j}{3}} (\lambda^{-\frac{1}{3}} b_j^2 + \lambda^{\frac{1}{3}} b_{j+1}^2 - 2b_j b_{j+1}) \\ &= -2 \left[ -\lambda^{-\frac{1}{3}} \sum_{j=0}^k \lambda^{\frac{2j}{3}} b_j^2 + \sum_{j=0}^k \lambda^{\frac{2j}{3}} b_j b_{j+1} \right] - \lambda^{-\frac{1}{3}} b_0^2 + \lambda^{\frac{2k}{3} + \frac{1}{3}} b_{k+1}^2. \end{aligned}$$

Hence (4.9) gives

$$(4.10) \quad \frac{1}{2} \frac{d}{dt} \sum_{j=0}^k b_j^2 = -\frac{1}{2} \left[ \sum_{j=0}^k \lambda^{\frac{2j}{3}} (\lambda^{-\frac{1}{6}} b_j - \lambda^{\frac{1}{6}} b_{j+1})^2 + \lambda^{-\frac{1}{3}} b_0^2 \right] + \frac{1}{2} \lambda^{\frac{2k}{3} + \frac{1}{3}} b_{k+1}^2 - 2\lambda^{\frac{2k}{3}} b_k b_{k+1} - \lambda^k b_k^2 b_{k+1}.$$

Now note that since the initial condition of the solution satisfies  $a_j(0) \geq 0$  for all  $j$ , we have that  $a_j(t) > 0$  for all  $j$  and  $t > 0$ . Therefore,

$$(4.11) \quad b_j(t) \geq -\lambda^{-\frac{j}{3}}, \quad \forall j, t \geq 0.$$

Thus,

$$(4.12) \quad -\lambda^k b_k^2 b_{k+1} \leq \lambda^{\frac{2k}{3} - \frac{1}{3}} b_k^2,$$

for all  $k, t \geq 0$ .

Hence,

$$(4.13) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \sum_{j=0}^k b_j^2 &\leq -\frac{1}{2} \sum_{j=0}^{k+1} d_j^2 + \frac{1}{2} \lambda^{\frac{2k}{3} + \frac{1}{3}} b_{k+1}^2 - 2\lambda^{\frac{2k}{3}} b_k b_{k+1} + \lambda^{\frac{2k}{3} - \frac{1}{3}} b_k^2 \\ &= -\frac{1}{2} \sum_{j=0}^{k+1} d_j^2 + d_{k+1}^2 - \frac{1}{2} \lambda^{\frac{2k}{3} + \frac{1}{3}} b_{k+1}^2. \end{aligned}$$

Note that from (4.2) it follows that

$$(4.14) \quad b_{k+1} = \lambda^{-\frac{k}{3} - \frac{1}{6}} d_{k+1} + \lambda^{-\frac{1}{3}} b_k.$$

Now we rewrite (4.13) using (4.14) and (4.2) as follows:

$$\begin{aligned}
(4.15) \quad \frac{d}{dt} \sum_{j=0}^k b_j^2 &\leq - \sum_{j=0}^k d_j^2 - 2\lambda^{\frac{k}{3}-\frac{1}{6}} d_{k+1} b_k - \lambda^{\frac{2k}{3}-\frac{1}{3}} b_k^2 \\
&= - \sum_{j=0}^k d_j^2 - 2\lambda^{\frac{2k}{3}} b_{k+1} b_k + \lambda^{\frac{2k}{3}-\frac{1}{3}} b_k^2 \\
&= - \sum_{j=0}^{k-1} d_j^2 - 2\lambda^{\frac{2k}{3}} b_{k+1} b_k + \lambda^{\frac{2k}{3}-1} b_{k-1}^2 + 2\lambda^{\frac{k}{3}-\frac{1}{2}} d_k b_{k-1} \\
&= - \sum_{j=0}^{k-1} d_j^2 - 2\lambda^{\frac{2k}{3}} b_{k+1} b_k + 2\lambda^{\frac{2k}{3}-\frac{2}{3}} b_k b_{k-1} - \lambda^{\frac{2k}{3}-1} b_{k-1}^2.
\end{aligned}$$

Note that we also have

$$(4.16) \quad \frac{d}{dt} \sum_{j=0}^{k-1} b_j^2 \leq - \sum_{j=0}^{k-1} d_j^2 - 2\lambda^{\frac{2k}{3}-\frac{2}{3}} b_k b_{k-1} + \lambda^{\frac{2k}{3}-1} b_{k-1}^2.$$

Adding equations (4.15) and (4.16) we get

$$\begin{aligned}
(4.17) \quad \frac{d}{dt} \left( \sum_{j=0}^{k-1} b_j^2 + \frac{1}{2} b_k^2 \right) &\leq - \sum_{j=0}^{k-1} d_j^2 - \lambda^{\frac{2k}{3}} b_{k+1} b_k \\
&\leq - \sum_{j=0}^{k-1} d_j^2 + \lambda^{\frac{k}{3}-\frac{1}{3}} |b|.
\end{aligned}$$

□

**Theorem 4.2.** *For every solution  $a(t)$  with the initial data  $a(0) \in l^2$ ,  $a_j(0) \geq 0$ , and every time interval  $[t_1, t_2]$ ,  $0 \leq t_1 \leq t_2$ , we have that*

$$(4.18) \quad |b(t_2)|^2 - |b(t_1)|^2 \leq -\alpha \int_{t_1}^{t_2} |d(t)|^2 dt,$$

where  $\alpha = 2 - \lambda^{3/8}$ .

*Proof.* Since  $\lambda = 2^{5/2}$ , we have that  $\alpha \in (0, 1)$ . First assume that exists  $N > 0$ , such that

$$(4.19) \quad \sum_{j=0}^k b_j(t_2)^2 - \sum_{j=0}^k b_j(t_1)^2 > -\alpha \int_{t_1}^{t_2} \sum_{j=0}^k d_j(t)^2 dt,$$

for all  $k \geq N$ . On the other hand, thanks to Lemma 4.1, we have that

$$(4.20) \quad \sum_{j=0}^k b_j(t_2)^2 - \sum_{j=0}^k b_j(t_1)^2 \leq - \int_{t_1}^{t_2} \sum_{j=0}^k d_j(t)^2 dt + \int_{t_1}^{t_2} d_{k+1}(t)^2 dt.$$



Hence,

$$(4.21) \quad \int_{t_1}^{t_2} d_{k+1}(t)^2 dt > (1 - \alpha) \int_{t_1}^{t_2} \sum_{j=0}^k d_j(t)^2 dt,$$

for all  $k \geq N$ . Let

$$(4.22) \quad I_j := \int_{t_1}^{t_2} d_j(t)^2 dt.$$

From (4.21) it follows that

$$(4.23) \quad I_{k+1} > (1 - \alpha) \sum_{j=0}^k I_j, \quad \forall k \geq N.$$

Hence,

$$(4.24) \quad \begin{aligned} I_{N+1} &> 0, \\ I_{N+2} &> (1 - \alpha)I_{N+1}, \\ I_{N+3} &> (1 - \alpha)(I_{N+2} + I_{N+1}) > (1 - \alpha)(2 - \alpha)I_{N+1}, \\ &\vdots \\ I_{N+j} &> (1 - \alpha)(2 - \alpha)^{j-2}I_{N+1}. \end{aligned}$$

Therefore,

$$(4.25) \quad I_{N+j} > \lambda^{\frac{3j}{8}} \frac{1 - \alpha}{(2 - \alpha)^2} I_{N+1}, \quad \forall j \geq 2.$$

Now let  $M = (t_2 - t_1) \sup_{t \in [t_1, t_2]} \|b(t)\|_{l^2}$ . Lemma 4.1 and (4.25) imply that

$$(4.26) \quad \begin{aligned} &\sum_{j=0}^{N+j} b_j(t_2)^2 - \sum_{j=0}^{N+j} b_j(t_1)^2 + \alpha I_{N+j} \\ &\leq \frac{1}{2} b_{N+j+1}(t_1)^2 - \frac{1}{2} b_{N+j+1}(t_2)^2 - (1 - \alpha)I_{N+j} + \lambda^{\frac{N+j}{3}} M \\ &\leq \frac{1}{2} b_{N+j+1}(t_1)^2 - \frac{1}{2} b_{N+j+1}(t_2)^2 - \lambda^{\frac{3j}{8}} \frac{(1 - \alpha)^2}{(2 - \alpha)^2} I_{N+1} + \lambda^{\frac{N+j}{3}} M, \end{aligned}$$

for all  $j \geq 2$ . Note that the right hand side goes to  $-\infty$  as  $j \rightarrow \infty$ , contradicting (4.19).

Therefore, we have shown that for any  $N > 0$  there exists  $k > N$ , such that

$$(4.27) \quad \sum_{j=0}^k b_j(t_2)^2 - \sum_{j=0}^k b_j(t_1)^2 \leq -\alpha \int_{t_1}^{t_2} \sum_{j=0}^k d_j(t)^2 dt.$$

By the definition of a weak solution,  $b(t) \in l^2$  for all time  $t$ . Therefore, taking a limit as  $N \rightarrow \infty$  and using Levi's convergence theorem, we obtain

that  $|d(t)|^2$  is locally integrable and

$$(4.28) \quad |b(t_2)|^2 - |b(t_1)|^2 \leq -\alpha \int_{t_1}^{t_2} |d(t)|^2 dt,$$

which concludes the proof.  $\square$

**Theorem 4.3.** *Let  $a(t)$  be a solution of (1.2) with the initial data  $a(0) \in l^2$ ,  $a_j(0) \geq 0$ . Then  $\|a(t)\|_s^2$  is locally integrable on  $[0, \infty)$  for all  $s < 5/6$ .*

*Proof.* Thanks to (4.3), we have that

$$(4.29) \quad \begin{aligned} \sum_{j=0}^{\infty} 2^{2sj} b_j^2 &= \sum_{j=0}^{\infty} 2^{2sj} \lambda^{\frac{1}{3} - \frac{2j}{3}} (d_0 + d_1 + \dots + d_j)^2 \\ &\leq \sum_{j=0}^{\infty} 2^{2sj} \lambda^{\frac{1}{3} - \frac{2j}{3}} (j+1) (d_0^2 + d_1^2 + \dots + d_j^2) \\ &\leq 2^{\frac{5}{6}} |d|^2 \sum_{j=0}^{\infty} 2^{2j(s - \frac{5}{6})} (j+1), \end{aligned}$$

where to obtain the last line we used  $\lambda = 2^{5/2}$ . Due to Theorem 4.2,  $|d(t)|$  is locally integrable. Therefore,  $\|b(t)\|_s^2$  is locally integrable, provided  $s < 5/6$ . Hence,  $\|a(t)\|_s^2$  is locally integrable for  $s < 5/6$ .  $\square$

**Theorem 4.4.** *Let  $a(t)$  be a solution of (1.2) with  $a_j(0) \geq 0$ . Then  $a(t)$  exponentially converges in  $l^2$  to the fixed point as  $t \rightarrow \infty$ . More precisely,*

$$(4.30) \quad |b(t)|^2 \leq |b(0)|^2 e^{-\beta t},$$

for some universal constant  $\beta > 0$ .

*Proof.* Note that inequality (4.29) with  $s = 0$  implies that

$$(4.31) \quad |b|^2 \leq |d|^2 \sum_{j=0}^{\infty} \lambda^{\frac{1}{3} - \frac{2j}{3}} (j+1).$$

Therefore, Theorem 4.2 yields

$$(4.32) \quad |b(t_2)|^2 - |b(t_1)|^2 \leq -\beta \int_{t_1}^{t_2} |b(t)|^2 dt,$$

where

$$(4.33) \quad \beta = \frac{2 - \lambda^{\frac{3}{8}}}{\sum_{j=0}^{\infty} \lambda^{\frac{1}{3} - \frac{2j}{3}} (j+1)}.$$

Using Granwall's inequality, we conclude that

$$(4.34) \quad |b(t)|^2 \leq |b(0)|^2 e^{-\beta t}.$$

$\square$

**Theorem 4.5.** *Let  $a(t)$  be a solution of (1.2) for which  $\|a(t)\|_{5/6}^3$  is integrable on some interval  $[T_1, T_2]$ . Then  $a(t)$  satisfies the energy equality*

$$(4.35) \quad |a(t)|^2 = |a(t_0)|^2 + 2 \int_{t_0}^t (f, a(\tau)) d\tau,$$

for all  $T_1 \leq t_0 \leq t \leq T$ .

*Proof.* Let  $a(t)$  be a solution satisfying the hypothesis of the theorem. First, we recall the property of  $l^p$  spaces which states that if  $p \geq q$  then

$$\|h\|_{l^p} \leq \|h\|_{l^q},$$

for all  $h \in l^q$ . We shall apply this property with  $h_j = \lambda^{\frac{2j}{3}} a_j^2$  and  $p = 3/2$  to obtain

$$(4.36) \quad \int_{t_0}^t \sum_{j=0}^{\infty} \lambda^j a_j^3 d\tau \leq \int_{t_0}^t \left( \sum_{j=0}^{\infty} \lambda^{\frac{2j}{3}} a_j^2 \right)^{3/2} d\tau < \infty,$$

where  $T_1 \leq t_0 \leq t \leq T$ . However the expression (4.36) with  $\lambda = 2^{5/2}$  combined with the assumption of the theorem implies that

$$(4.37) \quad \int_{t_0}^t \lambda^N a_N^3 d\tau \rightarrow 0 \quad \text{as} \quad N \rightarrow \infty.$$

Since

$$(4.38) \quad \int_{t_0}^t \lambda^N a_N^2 a_{N+1} d\tau \leq \int_{t_0}^t \lambda^N a_N^3 d\tau + \int_{t_0}^t \lambda^{N+1} a_{N+1}^3 d\tau,$$

we can take the limit of (3.10) as  $N \rightarrow \infty$  to obtain

$$(4.39) \quad |a(t)|^2 = |a(t_0)|^2 + 2 \int_{t_0}^t (f, a(\tau)) d\tau.$$

□

As a consequence, we can now show that every solution (with any initial data in  $l^2$ ) blows up in finite time in  $H^{5/6}$  norm.

**Corollary 4.6.** *Let  $a(t)$  be a solution of (1.2) with  $a_j(0) \geq 0$ . Then  $\|a(t)\|_{5/6}^3$  is not locally integrable on  $[0, \infty)$ . In particular,  $\|a(t)\|_{5/6}$  blows up in finite time.*

*Proof.* Assume that  $\|a(t)\|_{5/6}^3$  is locally integrable on  $[0, \infty)$ . Note that Theorem 4.4 implies that  $a(t)$  converges to the fixed point in  $l^2$ . Therefore,

$$(4.40) \quad \lim_{t \rightarrow \infty} a_j(t) = \lambda^{-\frac{j}{3}}.$$

In particular, we have that

$$(4.41) \quad \lim_{t \rightarrow \infty} a_0(t) = 1.$$

Hence, there exists  $T > 0$ , such that  $a_0(t) \geq 1/2$  for all  $t \geq T$ . Therefore,

$$(4.42) \quad (f, a(t)) \geq \frac{1}{2} \lambda^{-\frac{1}{3}}, \quad \forall t \geq T.$$

Hence, thanks to Theorem 4.5,

$$(4.43) \quad |a(t)|^2 \geq \frac{1}{2} \lambda^{-\frac{1}{3}} (t - T), \quad \forall t \geq T,$$

which contradicts the fact that  $a(t)$  converges to the fixed point in  $l^2$  as  $t \rightarrow \infty$  (Theorem 4.4).  $\square$

## 5. GLOBAL ATTRACTOR

Since the uniqueness of a solution for given initial data is an open problem, it is not known whether a semigroup of solution operators can be define for the dyadic model. Therefore, we use a more general framework of an evolutionary system  $\mathcal{E}$  from [5, 4].

Let  $X$  be a closed ball in  $H$  centered at the fixed point. Note that  $X$  is weakly compact. In addition,

$$(5.1) \quad a(t) \in X, \quad \forall t \geq 0,$$

for every solution  $a(t)$  to (1.2) with the initial data  $a(0) \in X$ .

Let

$$(5.2) \quad C([0, \infty); X_w) := \{a(\cdot) : [0, \infty) \rightarrow X, a_n(t) \text{ continuous for all } n\}$$

endowed with the distance

$$(5.3) \quad d_{C([0, \infty); X_w)}(a, b) = \sum_{T \in \mathbb{N}} \frac{1}{\lambda^T} \frac{\sup\{d_w(a(t), b(t)) : 0 \leq t \leq T\}}{1 + \sup\{d_w(a(t), b(t)) : 0 \leq t \leq T\}}.$$

Let

$$(5.4) \quad \mathcal{T} := \{I : I = [T, \infty) \subset \mathbb{R}, \text{ or } I = (-\infty, \infty)\},$$

and for each  $I \in \mathcal{T}$  let  $\mathcal{F}(I)$  denote the set of all  $X$ -valued functions on  $I$ . A map  $\mathcal{E}$  that associates to each  $I \in \mathcal{T}$  a subset  $\mathcal{E}(I) \subset \mathcal{F}(I)$  will be called an evolutionary system if the following conditions are satisfied:

- (1)  $\mathcal{E}([0, \infty)) \neq \emptyset$ .
- (2)  $\mathcal{E}(I + s) = \{a(\cdot) : a(\cdot - s) \in \mathcal{E}(I)\}$  for all  $s \in \mathbb{R}$ .
- (3)  $\{a(\cdot)|_{I_2} : a(\cdot) \in \mathcal{E}(I_1)\} \subset \mathcal{E}(I_2)$  for all pairs of  $I_1, I_2 \in \Omega$ , such that  $I_2 \subset I_1$ .
- (4)  $\mathcal{E}((-\infty, \infty)) = \{a(\cdot) : a(\cdot)|_{[T, \infty)} \in \mathcal{E}([T, \infty)) \forall T \in \mathbb{R}\}$ .

Let

$$R(t)A := \{a(t) : a(0) \in A, a \in \mathcal{E}([0, \infty))\},$$

$$\tilde{R}(t)A := \{a(t) : a(0) \in A, a \in \mathcal{E}((-\infty, \infty))\}, \quad A \subset X, t \in \mathbb{R}.$$

A set  $A \subset X$  uniformly attracts a set  $B \subset X$  if for any  $\epsilon > 0$  there exists  $t_0$ , such that

$$(5.5) \quad R(t)B \subset B(A, \epsilon), \quad \forall t \geq t_0.$$

For  $A \subset X$  and  $r > 0$  denote  $B_\bullet(A, r) = \{a : d_\bullet(A, a) < r\}$ , where  $\bullet = s, w$ . Now we define an attracting set and a global attractor as follows.

**Definition 5.1.** A set  $A \subset X$  is  $d_\bullet$ -attracting set ( $\bullet = s, w$ ) if it uniformly attracts  $X$  in  $d_\bullet$ -metric, i.e., for any  $\epsilon > 0$  there exists  $t_0$ , such that

$$(5.6) \quad R(t)X \subset B_\bullet(A, \epsilon), \quad \forall t \geq t_0.$$

A set  $A \subset X$  is invariant if  $\tilde{R}(t)A = A$  for all  $t \geq 0$ . A set  $\mathcal{A}_\bullet \subset X$  is a  $d_\bullet$ -global attractor if  $\mathcal{A}_\bullet$  is a minimal  $d_\bullet$ -closed  $d_\bullet$ -attracting set.

The following result was proved in [5]:

**Theorem 5.2.** *The evolutionary system  $\mathcal{E}$  always possesses a weak global attractor  $\mathcal{A}_w$ . In addition, if  $\mathcal{E}([0, \infty))$  is compact in  $C([0, \infty); X_w)$ , then*

- (a)  $\mathcal{A}_w = \{a^0 : a^0 = u(0) \text{ for some } u \in \mathcal{E}((-\infty, \infty))\}$ .
- (b)  $\mathcal{A}_w$  is the maximal invariant set.

For the dyadic model we define  $\mathcal{E}$  in the following way.

$$\begin{aligned} \mathcal{E}([T, \infty)) &:= \{a : a(\cdot) \text{ is a solution to (1.2) on } [T, \infty), a(0) \in X, a_j(0) \geq 0\}, \\ \mathcal{E}((-\infty, \infty)) &:= \{a : a(\cdot) \text{ is a solution to (1.2) on } (-\infty, \infty), a(0) \in X, a_j(0) \geq 0\}. \end{aligned}$$

Clearly,  $\mathcal{E}$  satisfies properties (1)–(4). Then Theorem 5.2 immediately yields that the weak global attractor  $\mathcal{A}_w$  exists. In order to infer that  $\mathcal{A}_w$  is the maximal invariant set, we need the following result.

**Lemma 5.3.**  $\mathcal{E}([0, \infty))$  is compact in  $C([0, \infty); X_w)$ .

*Proof.* Take any sequence  $a^k \in \mathcal{E}([0, \infty))$ . Thanks to (5.1), there exists  $R > 0$ , such that

$$(5.7) \quad a_j^k(t) \leq R, \quad \forall n, t \geq 0.$$

Therefore,

$$(5.8) \quad |a_j^k(t) - a_j^k(s)| \leq (\lambda^{j-1}R^2 + \lambda^jR^2 + f_j)|t - s|,$$

for all  $j, t \geq 0, s \geq 0$ . Thus,

$$(5.9) \quad d_w(a^k(t), a^k(s)) = \sum_{j=0}^{\infty} \frac{1}{\lambda^{(j^2)}} \frac{|a_j^k(t) - a_j^k(s)|}{1 + |a_j^k(t) - a_j^k(s)|} \leq c|t - s|,$$

for some constant  $c$  independent of  $k$ . Hence,  $\{a^k\}$  is an equicontinuous sequence of functions in  $C([0, \infty); X_w)$  with bounded initial data. Therefore, Ascoli-Arzelà theorem implies that  $\{a^k\}$  is relatively compact in  $C([0, T]; X_w)$ , for all time  $T > 0$ . Using a diagonalization process, we obtain that  $\{a^k\}$  is relatively compact in  $C([0, \infty); X_w)$ . Hence, there exists a weakly continuous  $X$ -valued function  $a(t)$  on  $[0, \infty)$ , such that

$$(5.10) \quad a^{k_n} \rightarrow a \quad \text{as} \quad k_n \rightarrow \infty \quad \text{in} \quad C([0, \infty); X_w),$$

for some subsequence  $k_n$ . In particular, since  $X$  is weakly compact,  $a(0) \in X$ .

In addition, since  $a^{k_n}(t)$  is a solution to (1.2), we have

$$(5.11) \quad a_n^{k_n}(t) = a_j^{k_n}(0) + \int_0^t (\lambda^{j-1}(a_{j-1}^{k_n})^2 - \lambda^j a_j^{k_n} a_{j+1}^{k_n} + f_j) d\tau,$$

for all  $j$ . Taking the limit as  $k_n \rightarrow \infty$ , we obtain

$$(5.12) \quad a_j(t) = a_j(0) + \int_0^t (\lambda^{j-1} a_{j-1}^2 - \lambda^j a_j a_{j+1} + f_j) d\tau,$$

for all  $j$ . Since  $a_j(t)$  is continuous,  $a_j \in C^1([0, \infty))$  and satisfies (1.2).  $\square$

Finally, we show that the weak global attractor is the fixed point.

**Theorem 5.4.** *A strong global attractor  $\mathcal{A}_s$  of (1.2) exists,  $\mathcal{A}_s = \mathcal{A}_w$ , and it is the fixed point:*

$$(5.13) \quad \mathcal{A}_s = \mathcal{A}_w = \{\hat{a}\},$$

where  $\hat{a}_j = \lambda^{-\frac{j}{3}}$ . Moreover,  $\mathcal{A}_s$  is an exponential attractor.

*Proof.* The statement of the theorem immediately follows from Theorem 4.4.  $\square$

## 6. ONSAGER'S CONJECTURE AND KOLMOGOROV'S 5/3 LAW

As we discussed in [6] much attention has been given to the statistical theories of turbulence developed by Kolmogorov [18] and Onsager [22]. It is suggested that an appropriate mathematical description of 3-dimensional turbulent flow is given by weak solutions of the Euler equations which are not regular enough to conserve energy. Onsager conjectured that for the velocity Hölder exponent  $h > 1/3$  the energy is conserved and that this ceases to be true for  $h \leq 1/3$ . This phenomenon is now called turbulent or anomalous dissipation. Kolmogorov's theory predicts that in a fully developed turbulent flow the energy spectrum  $E(|k|)$  in the inertial range is given by

$$(6.1) \quad E(|k|) = c_0 \bar{\epsilon}^{2/3} |k|^{-5/3},$$

where  $\bar{\epsilon}$  is the average of the energy dissipation rate. The model system that we study in this present paper exactly reproduces the phenomena described above. The appropriate choice of  $\lambda$  for the 3-dimensional model is  $2^{5/2}$ . Interpreting the results of sections 4 and 5 with  $\lambda = 2^{5/2}$ , we proved that regular solutions, i.e. solutions with bounded  $H^{5/6}$  norm, satisfy the energy equality whereas solutions after the time of blow-up lose regularity and dissipate energy. The model (1.2) is derived under the assumption that  $a_j^2(t)$  is the total energy in the shell  $2^j \leq |k| < 2^{j+1}$ . Hence the energy spectrum for the fixed point is:

$$(6.2) \quad E(|k|) = 2^{5/6} f_0 |k|^{-5/3}.$$

Since the dissipation rate for the fixed point is equal to the energy input rate, we have

$$\bar{\epsilon} = a_0 f_0 = 2^{5/12} f_0^{3/2}.$$

Furthermore, since the support of any time average measure belongs to the global attractor, the average dissipation rate is equal to the dissipation rate of the fixed point. This result is proved in [11] for the 3D Navier-Stokes equations, but it also holds for the inviscid dyadic model due to the anomalous dissipation. Thus Kolmogorov's law is valid for the system.

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