

AN INVISCID DYADIC MODEL OF TURBULENCE: THE FIXED POINT AND ONSAGER'S CONJECTURE

ALEXEY CHESKIDOV, SUSAN FRIEDLANDER, AND NATAŠA PAVLOVIĆ

ABSTRACT. Properties of an infinite system of nonlinearly coupled ordinary differential equations are discussed. This system models some properties present in the equations of motion for an inviscid fluid such as the skew symmetry and the 3-dimensional scaling of the quadratic nonlinearity. It is proved that the system with forcing has a unique equilibrium and that *every* solution blows up in finite time in $H^{5/6}$ -norm. Onsager's conjecture is confirmed for the model system.

1. INTRODUCTION

One of the outstanding questions in fluid dynamics is existence, uniqueness and regularity of solutions to the Cauchy problem for the three-dimensional Euler equations:

$$(1.1) \quad \begin{aligned} \frac{\partial u}{\partial t} &= -(u \cdot \nabla)u - \nabla p, \\ \nabla \cdot u &= 0. \end{aligned}$$

Important features of the nonlinear term $(u \cdot \nabla)u$ are its bilinearity and skew-symmetry. The second property implies conservation of energy for sufficiently regular solution of the Euler equation (and decay of energy in the context of the Navier-Stokes equations). In the past few decades discretized “toy” models that preserve the energy properties of the fluid equations have been proposed and studied both by mathematicians and physicists. These models belong to a general class of “shell” models which simulate the energy cascade in turbulent flow. In all of these models the nonlinearity of the Euler equations is much simplified by considering only local neighbouring interactions between certain scales. However simplifications vary and as a consequence the models differ in the number of conserved quantities and in the presence of a certain “monotonicity” property that we will discuss later. Among the first example of such discretized models is the one introduced by Gledzer [11] in 1970 which was later generalized by Ohkitani and Yamada [17] and is now known as the GOY model. A survey of mathematical developments in connection with shell models can be found in the recent book of Bohr et al [1].

Date: October 3, 2006.

In this article we analyze one of these discretized models, namely

$$(1.2) \quad \begin{aligned} \frac{da_j}{dt} &= 2^{\frac{5}{2}(j-1)} a_{j-1}^2 - 2^{\frac{5}{2}j} a_j a_{j+1} + f_j, \quad j > 0 \\ \frac{da_0}{dt} &= -a_0 a_1 + f_0. \end{aligned}$$

The motivation for this model, and specifically for the scaling appropriate for 3 dimensions is given in [10] and [12] in the case $-\infty < j < \infty$ and $f = 0$. In [10] a shell model is presented, following the Fourier space analysis of Dinaburg and Sinai [8], for the 3-dimensional incompressible Euler equations (1.1) and it is shown that for very specific initial data the vector model reduces to the scalar system analogous to (1.2) without forcing. Furthermore, the wavelet decomposition utilised in [12] and [13] motivates a discretized model for the 3-dimensional Euler equations analogous to (1.2) without forcing.

A derivation of the system (1.2) is presented in Section 2. The coefficient $a_j^2(t)$ is the *total* energy in the frequency space shell $2^j \leq |k| < 2^{j+1}$. In this context l^2 and H^s , respectively energy and Sobolev norms, are defined as:

$$\|a(t)\|_{l^2} = \left(\sum_{j=0}^{\infty} a_j^2(t) \right)^{1/2}, \quad \|a(t)\|_{H^s} = \left(\sum_{j=0}^{\infty} 2^{2sj} a_j^2(t) \right)^{1/2}.$$

Local in time existence of solutions in $H^{5/2}$ was obtained in [10] using the fixed point techniques that produce local existence in $H^{5/2}$ for mild solutions of actual 3 dimensional Euler equations (1.1) (see Cannone [2]). Finite time blow-up of (1.2), without forcing, in an appropriate H^s was initially proved in [12] by exploiting conservation of energy and a monotonicity property present in the quadratic term a_{j-1}^2 . Variants of the model have been studied in [14], [20], and [21]. In particular, it was observed in [20] that (1.2) with $2^{5j/2}$ replaced by 2^j could be derived from a discretized version of the Fourier transform of the 1-dimensional Burger equations under the (unrealistic) restriction that there is no spreading of the support.

Variants of the model including dissipation have been studied [3], [7], [12], [13] and finite time blow up is proved for certain “small” amounts of dissipation. More precisely, in [3] a finite time blow up was proved in the cases where the dissipation degree is such that the model enjoys the same estimate on the nonlinear term as the Navier-Stokes equations in the dimensions larger than 4.

Even though a global existence proof from [3] applied to (1.2) implies that for any initial data in l^2 there exists a global in time classical solution to (1.2), not all the classical solutions satisfy the energy balance equation. For instance, regular solutions, i.e., solutions with bounded $H^{5/6}$ -norm, satisfy the energy equality, but the unique fixed point does not.

In this present paper we study the system (1.2) with forcing as a model for 3-dimensional turbulence. The main results are the following:

- (a) For any force $f = (f_0, 0, \dots)$, $f_0 > 0$, there exists a unique fixed point of the system (1.2). The fixed point is not in $H^{5/6}$ and is given by

$$\{a_j\} = \{2^{-5j/6+5/12} \sqrt{f_0}\}.$$

- (b) The system linearized about the fixed point has exponentially decaying eigenvalues.
 (c) Every regular solution approaches the fixed point in l^2 norm.
 (d) Every solution blows up in finite time in $H^{5/6}$ norm.

In a more technical companion paper [5] we study the solutions of (1.2) *after* the time of blow-up in $H^{5/6}$. Among the properties we prove that the H^s norms for $s < 5/6$ are locally square integrable in time. Moreover, we study the long time behavior of (1.2) and prove that the fixed point is a global attractor. This is a consequence of anomalous or turbulent dissipation that was conjectured by Onsager [18] and related to Kolmogorov's prediction that in fully developed turbulent flow the energy spectrum in the inertial range is given by a power law

$$(1.3) \quad E(|k|) = c_0 \bar{\epsilon}^{2/3} |k|^{-5/3}.$$

In Section 6 of the present paper we observe that the energy spectrum of the fixed point of the forced model (1.2) reproduces Kolmogorov's law (1.3). Furthermore, regular solutions, i.e., solutions with bounded $H^{5/6}$ norm, satisfy the energy equality, whereas the fixed point is not regular and does not satisfy energy equality. Moreover, anomalous dissipation cuts in for all the other solutions as well when the $H^{5/6}$ norm becomes unbounded.

Organization of the paper. In section 2 we present a derivation of the dyadic model based on Littlewood-Paley operators. In section 3 we review existence and finite time blow-up results for the model obtained in the past few years. In sections 4 and 5 we prove the results stated above. In section 6 we discuss how the dyadic model with the 3-dimensional scaling satisfies Onsager's conjecture and Kolmogorov's 5/3 law.

Acknowledgements. The authors would like to thank Marie Farge, Jonathan Mattingly, Kai Schneider and Eric Vanden-Eijnden for very helpful discussions. S.F. was partially supported by NSF grant number DMS 0503768. N.P. was partially supported by NSF grant number DMS 0304594.

2. MODEL BASED ON LITTLEWOOD-PALEY DECOMPOSITION

Let us start by considering 3D Euler equations with zero force (for the sake of simplicity):

$$(2.1) \quad \frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = 0.$$

2.1. **Euler equations on \mathbb{R}^3 .** Given a velocity $u(x, t)$, let

$$\hat{u}(\xi, t) = \int_{\mathbb{R}^3} e^{-ix \cdot \xi} u(x, t) dx, \quad \xi \in \mathbb{R}^3,$$

be its Fourier transform. We shall use a standard Littlewood Paley decomposition. More precisely, we consider Fourier multipliers P_j (on $L^2(\mathbb{R}^3)$)

$$\widehat{P_j u}(\xi, t) = p_j(\xi) \hat{u}(\xi, t),$$

so that their symbols $p_j(\xi)$ are smooth and supported in $\frac{2}{3}2^j < |\xi| < 3(2^j)$ and such that $p_j(\xi) = p_0(2^{-j}\xi)$ and $\sum_j p_j(\xi) = 1$. Also we denote by \tilde{P}_j the following multiplier $\tilde{P}_j = \sum_{k=-2}^2 P_{j+k}$.

We apply the Leray projection T onto the divergence free vectors followed by an application of P_j on the equation (2.1) to obtain:

$$(2.2) \quad \frac{\partial P_j u}{\partial t} + P_j(T(u \cdot \nabla u)) = 0.$$

Our goal is to see what we can say about $\|P_j u\|_{L^2}$ because that is the relevant quantity in the definition of the Sobolev norm

$$(2.3) \quad \|f\|_{H^s} = \left(\sum_{j=-\infty}^{\infty} (1 + 2^{2js}) \|P_j f\|_{L^2}^2 \right)^{1/2}.$$

We use Bony's paraproduct formula for the nonlinear term. More precisely,

$$P_j(T(u \cdot \nabla u)) = N_{j, lh} + N_{j, hl} + N_{j, hh} + N_{loc},$$

where the low-high part is given by

$$N_{j, lh} = \sum_{j' < j-4} P_j((P_{j'} u) \cdot \tilde{P}_j \nabla u),$$

the high-low part is given by

$$N_{j, hl} = \sum_{j' < j-4} P_j((\tilde{P}_j u) \cdot P_{j'} \nabla u),$$

the high-high part is given

$$N_{j, hh} = \sum_{j' > j+4} P_j((\tilde{P}_{j'} u) \cdot P_{j'} \nabla u) + \sum_{j' > j+4} P_j((P_{j'} u) \cdot \tilde{P}_{j'} \nabla u),$$

and the local part is given via

$$N_{loc} = \sum_{j-4 \leq j' \leq j+4} P_j((\tilde{P}_{j'} u) \cdot P_{j'} \nabla u) + \sum_{j-4 \leq j' \leq j+4} P_j((P_{j'} u) \cdot \tilde{P}_{j'} \nabla u).$$

Now let us start modeling. We recall that the Fourier multiplier $P_j f(x)$ are given via

$$\widehat{P_j f}(\xi) = p_j(\xi) \hat{f}(\xi)$$

has the symbol $p_j(\xi)$ which is, roughly speaking, supported for $|\xi| \sim 2^j$. This combined with the belief that only local frequency scales are relevant

in a turbulence cascade, motivates us to keep only local interactions in the nonlinear term, i.e. we keep only the equivalent of a modified version of N_{loc} in the model that we propose below. More precisely, we introduce a model that describes the evolution of the coefficients

$$(2.4) \quad a_j(t) = \|P_j u(x, t)\|_{L^2}.$$

Here $a_j^2(t)$ represents the total energy in the shell. In order to introduce in the model an analogue of the N_{loc} expression, we need to see what scaling comes out of $\|P_j u \cdot \nabla P_{j'}(u)\|_{L^2}$ when j and j' are close. By Hölder's inequality combined with the fact that $p_j(\xi)$ are supported for $|\xi| \sim 2^j$ we obtain:

$$(2.5) \quad \|P_j u \cdot \nabla P_{j'} u\|_{L^2} \leq \|P_j u\|_{L^\infty} \|\nabla P_{j'} u\|_{L^2}$$

$$(2.6) \quad \leq 2^{\frac{5}{2}j} \|P_j u\|_{L^2} \cdot 2^{j'} \|P_{j'} u\|_{L^2}$$

$$(2.7) \quad = 2^{\frac{5}{2}j} a_j a_{j'},$$

where to obtain (2.6) we use Bernstein's inequality which can be stated on \mathbb{R}^n in terms of Littlewood-Paley operators as follows

$$\|P_j u\|_{L^q} \leq 2^{(\frac{1}{p} - \frac{1}{q})nj} \|P_j u\|_{L^p}, \text{ for } q > p.$$

Now we are ready to propose the following model for the 3D Euler equations:

$$(2.8) \quad \frac{da_j}{dt} = 2^{\frac{5}{2}(j-1)} a_{j-1}^2 - 2^{\frac{5}{2}j} a_j a_{j+1}, \quad j \in \mathbb{Z}.$$

Note that due to (2.4), the $L^2(\mathbb{R}^3)$ norm of the fluid velocity u is

$$\|u\|_{L^2} = \left(\sum_{j=-\infty}^{\infty} |a_j|^2 \right)^{1/2}.$$

The Sobolev norm (2.3) of u is

$$(2.9) \quad \|u\|_{H^s} = \left(\sum_{j=-\infty}^{\infty} (1 + 2^{2js}) |a_j|^2 \right)^{1/2}.$$

2.2. Space periodic 3D Euler equations. Consider now Euler equations in a periodic box $\Omega = [0, L]^3$. By Galilean change of variables we can assume that the space average of u is zero. Given a velocity $u(x, t)$, let

$$\hat{u}_k(t) = \frac{1}{L^3} \int_{\Omega} e^{-i\frac{2\pi}{L}x \cdot k} u(x, t) dx, \quad k \in \mathbb{Z}^3,$$

be its Fourier coefficients. Define $a_j^2(t)$ to be the total energy in the shell $2^j \leq |k| < 2^{j+1}$

$$(2.10) \quad a_j(t)^2 = \sum_{2^j \leq |k| < 2^{j+1}} |\hat{u}_k(t)|^2, \quad j = 0, 1, 2, \dots$$

An analysis similar to that given in Subsection 2.1 motivates the following model of the 3D Euler equations:

$$(2.11) \quad \begin{aligned} \frac{da_j}{dt} &= 2^{\frac{5}{2}(j-1)} a_{j-1}^2 - 2^{\frac{5}{2}j} a_j a_{j+1}, & j = 1, 2, \dots, \\ \frac{da_0}{dt} &= -a_0 a_1. \end{aligned}$$

Due to (2.10), the $L^2(\Omega)^3$ and $H^s(\Omega)^3$ norms of the fluid velocity u can be defined as

$$(2.12) \quad \|u\|_{L^2} = \|a\|_{l^2} = \left(\sum_{j=0}^{\infty} |a_j|^2 \right)^{1/2}, \quad \|u\|_{H^s} = \left(\sum_{j=0}^{\infty} 2^{2js} |a_j|^2 \right)^{1/2}.$$

3. GLOBAL EXISTENCE AND BLOW-UP

We write the dyadic model (2.11) as

$$(3.1) \quad \begin{aligned} \frac{da_j}{dt} &= \lambda^{j-1} a_{j-1}^2 - \lambda^j a_j a_{j+1} + f_j, & j > 0, \\ \frac{da_0}{dt} &= -a_0 a_1 + f_0, \end{aligned}$$

with the forcing $f_j \geq 0$ for all j .

For notational simplicity we adopt λ^j as the scaling parameter in the computations in sections 3 and 4. We do this to illustrate that the results are qualitatively independent of the exact choice of lambda (which depends on the spatial dimension and the construction of the model). As we discussed in sections 1 and 2 the relevant lambda for our 3 dimensional dyadic model is $2^{5/2}$. This exponent determines the values of the critical exponents in the Sobolev space results proved below.

We say that $a(t) = (a_0(t), a_1(t), \dots)$ is a solution to (3.1) if it is a classical solution in the usual sense. More precisely, we have the following.

Definition 3.1. A solution on $[0, T]$ (or $[0, \infty)$, if $T = \infty$) of (3.1) is an l^2 -valued function $a(t)$ defined for $t \in [0, T]$, such that $a_j \in C^1([0, T])$ and $a_j(t)$ satisfies (3.1) on $(0, T)$ for all j .

Note that if $a(t)$ is a solution on $[0, T]$, then automatically $a_j \in C^\infty([0, T])$ for all j . The global existence of a classical solution for any initial data $a(0) \in l^2$ was proved in Cheskidov [3] for a closely related system. The idea of the proof is as follows, see Theorem 4.1 in [3] for more details. Given an arbitrary time interval $[0, T]$, consider a sequence of Galerkin approximations to (3.1). It is easy to show that this sequence is weakly equicontinuous

on $[0, T]$. Therefore, thanks to Ascoli-Arzelà theorem, we can pass to a convergent subsequence and obtain the existence of a weak solution to (3.1) on $[0, T]$. Since the nonlinear term has a finite number of terms, this weak solution is also a classical solution. Hence we have the following result.

Theorem 3.2. *For every $a^0 \in l^2$ and $f \in l^2$, there exists a solution $a(t)$ to (3.1) on $[0, \infty)$ with $a(0) = a^0$.*

Define the H^s norm of a as

$$\|a(t)\|_{H^s} = \left(\sum_{j=0}^{\infty} 2^{2sj} a_j^2(t) \right)^{1/2}.$$

Due to (2.12), $\|\cdot\|_{H^s}$ can be viewed as the model analogue of the Sobolev norm of a fluid velocity. For each scale j we define the energy as

$$(3.2) \quad E_j(t) = \frac{1}{2} a_j^2(t),$$

and the energy of the whole system

$$(3.3) \quad E(t) = \frac{1}{2} \|a(t)\|_{l^2}^2 = \sum_{j=0}^{\infty} E_j(t).$$

We define the energy of a “box” B_J as

$$(3.4) \quad E_{B_J}(t) = \sum_{j=J}^{\infty} E_j(t).$$

Formally multiplying (3.1) by a_j and summing over all j gives

$$(3.5) \quad \frac{1}{2} \frac{d}{dt} \sum_{j=0}^{\infty} a_j^2 = \sum_{j=1}^{\infty} \lambda^{j-1} a_{j-1}^2 a_j - \sum_{j=0}^{\infty} \lambda^j a_j^2 a_{j+1} + \sum_{j=0}^{\infty} f_j a_j.$$

We recall that $\lambda = 2^{5/2}$. Hence, if $\|a(t)\|_{H^{5/6}}$ is bounded on some time interval, then the summations on the right hand side of (3.5) are uniformly convergent, and we obtain the following energy balance property:

$$(3.6) \quad \frac{d}{dt} E(t) = \sum_{j=0}^{\infty} f_j a_j.$$

This motivates the following.

Definition 3.3. A solution $a(t)$ of (3.1) is called regular (or strong) on $[T_1, T_2]$ if $\|a(t)\|_{H^{5/6}}$ is bounded on $[T_1, T_2]$.

Hence, the total energy of a regular solution to (3.1) satisfies the energy equality (3.6). A regular solution also satisfies

$$(3.7) \quad \frac{d}{dt} E_{B(J)}(t) = \lambda^{J-1} a_{J-1}^2 a_J, \quad J > 0.$$

Treating the j -th equation in (3.1) as an ODE for $a_j(t)$ gives

$$(3.8) \quad \begin{aligned} a_j(t) &= \frac{1}{\mu(t)} \left[a_j(0) + \int_0^t \lambda^{j-1} a_{j-1}^2(\tau) \mu(\tau) d\tau \right], \quad j > 0, \\ a_0(t) &= \frac{1}{\mu(t)} \left[a_0(0) + f_0 \int_0^t \mu(\tau) d\tau \right], \end{aligned}$$

where

$$(3.9) \quad \mu(t) = \exp \left(\int_0^t \lambda^j a_{j+1}(\tau) d\tau \right).$$

Hence if $a_j(0) \geq 0$ for all j , we conclude that this property is preserved by evolution of (3.1) and

$$a_j(t) \geq 0 \text{ for all } j \geq 0.$$

Thus, (3.7) implies that energy in each box B_J of a regular solution increases monotonically in time, i.e.

$$(3.10) \quad \frac{d}{dt} E_{B_J}(t) \geq 0, \quad J > 0.$$

The inequality (3.10) implies that the system produces a successive cascade of energy into higher and higher scales J .

The basic properties of system (3.1) described above contribute to the proofs of certain results that have appeared in Katz-Pavlović [12], Friedlander-Pavlović [10], Kiselev-Zlatoš [14]. We recall these theorems.

Theorem 3.4. *Let $a^0 \in H^s$ for some $s \geq 5/2$ with $a_j^0 \geq 0$ for all $j \geq 0$. Then there exists a time $T = T(\|a^0\|_{H^s}) > 0$ such that a unique solution $a(t)$ to (3.1) with $a(0) = a^0$ exists and $a(\cdot) \in C([0, T]; H^s)$.*

This theorem is proved in Friedlander-Pavlović [10] using Picard's fixed point argument in the case $f = 0$. In particular, this theorem implies local existence of regular solutions with initial data in $H^{5/2}$. The local existence of regular solutions is not yet known for initial data not in $H^{5/2}$.

Theorem 3.5. *Let $a(t)$ be a solution of (3.1) with $a(t) \in H^1$, $a_j(0) \geq 0$ for all $j \geq 0$, and $a(0) \neq 0$. Then $\|a(t)\|_{H^s}$ becomes infinite in finite time for all $s > 5/6$.*

The first proof of this type of blow-up result in H^s for a system closely related to (3.1) with $f = 0$ was given in Katz-Pavlović [12]. Versions for H^s blow up were given in Friedlander-Pavlović [10], Kiselev-Zlatoš [14] (the above theorem) and for a Navier-Stokes model in [12] and Cheskidov [3].

Crucial to these blow-up proofs is the cancellation of the infinite sums on the right hand side of (3.5) leading to the energy properties (3.6) and (3.7). In the context of the scaling of this present paper, the smallest value of s for which finite time blow-up is proved is $s = 5/6$. We note that the claim by Waleffe [20] to use such a proof to produce blow-up down to $s > 0$ (for a variant of the model with 2^j scaling) is not justified because the telescoping

sums in (3.5) do not converge for $0 < s < s_0$, where s_0 is a relevant critical Sobolev exponent ($s_0 = 1/3$ for 2^j scaling).

Consider now a forcing of the form $f_0 > 0$ and $f_j = 0$ for all $j \geq 1$. Clearly,

$$a_j = \lambda^{-\frac{j}{3} + \frac{1}{6}} \sqrt{f_0}$$

is a fixed point of (3.1). Since $\lambda = 2^{5/2}$, note that the fixed point is not regular, i.e., it is not in $H^{5/6}$. Note also that it does not satisfy the energy balance equation (3.6). Indeed, for a fixed point we have

$$0 = \frac{d}{dt} E(t) < \sum_{j=0}^{\infty} f_j a_j.$$

Moreover, at the end of this paper we will prove that there is no global in time regular solution to (3.1), i.e., every solution $a(t)$ with $a(0) \in l^2$ blows up in finite time in $H^{5/6}$ -norm. This result improves Theorem 3.5.

4. UNIQUENESS OF THE FIXED POINT

We continue studying the dyadic model

$$(4.1) \quad \begin{aligned} \frac{da_j}{dt} &= \lambda^{j-1} a_{j-1}^2 - \lambda^j a_j a_{j+1} + f_j, \quad j > 0 \\ \frac{da_0}{dt} &= -a_0 a_1 + f_0. \end{aligned}$$

For the forcing of the form $f_0 > 0$ and $f_j = 0$ for all $j \geq 1$ we will investigate the spectrum of the system linearized about the fixed point

$$(4.2) \quad \{a_j\} = \{\lambda^{-\frac{j}{3} + \frac{1}{6}} \sqrt{f_0}\}.$$

4.1. Uniqueness of the fixed point.

Theorem 4.1. *For every force $f = (f_0, 0, 0, \dots)$, $f_0 > 0$, there exists a unique fixed point of (4.1)*

$$(4.3) \quad \{a_j\} = \{\lambda^{-\frac{j}{3} + \frac{1}{6}} \sqrt{f_0}\}.$$

Proof. Consider

$$(4.4) \quad A_j = \lambda^{\frac{j}{3} - \frac{1}{6}} f_0^{-\frac{1}{2}} a_j.$$

Then the equations (4.1) for a fixed point become

$$\begin{aligned} A_{j-1}^2 - A_j A_{j+1} &= 0, \quad j > 0, \\ A_0 A_1 &= 1. \end{aligned}$$

Clearly,

$$(4.5) \quad A_j = 1, \quad j \geq 0$$

is a solution to the above system, which corresponds to

$$(4.6) \quad a_j = \lambda^{-\frac{j}{3} + \frac{1}{6}} \sqrt{f_0}.$$

Let us show that this is the only fixed point. Indeed, if $\{a_j\}$ is a fixed point, then

$$(4.7) \quad A_0 A_1 = 1, \quad \frac{A_{j-1}}{A_j} = \frac{A_{j+1}}{A_{j-1}}, \quad j \geq 1.$$

From these equalities we obtain

$$\frac{A_0}{A_1} = \frac{A_2}{A_0}, \quad \frac{A_0}{A_2} = \frac{A_2 A_3}{A_0 A_1}, \quad \frac{A_0}{A_3} = \frac{A_3 A_4}{A_0 A_1}, \quad \frac{A_0}{A_4} = \frac{A_4 A_5}{A_0 A_1}, \quad \dots$$

Thus, we have

$$(4.8) \quad \frac{A_0}{A_j} = A_j A_{j+1}, \quad j \geq 2.$$

Therefore,

$$(4.9) \quad A_{j+1} = A_0 A_j^{-2}, \quad j \geq 2.$$

Hence,

$$(4.10) \quad A_j = \begin{cases} A_0^{(1+2^{j-2})/3} A_2^{-2^{j-2}}, & j \geq 3 \text{ odd}, \\ A_0^{(1-2^{j-2})/3} A_2^{2^{j-2}}, & j \geq 3 \text{ even}. \end{cases}$$

Finally, since $A_2 = A_0^3$, we obtain

$$(4.11) \quad A_j = \begin{cases} A_0^{-3 \cdot 2^{j-2} + (1+2^{j-2})/3}, & j \geq 3 \text{ odd}, \\ A_0^{3 \cdot 2^{j-2} + (1-2^{j-2})/3}, & j \geq 3 \text{ even}. \end{cases}$$

Clearly, A_0 has to be equal to 1 in order for $\{a_j\}$ to be bounded. Then $A_j = 1$ for all j . \square

Note that the above argument can also be applied to a more general force of the form $f = (f_0, f_1, \dots, f_k, 0, 0, \dots)$ with $f_k > 0$.

Lemma 4.2. *Let $f = (f_0, f_1, \dots, f_k, 0, 0, \dots)$ with $f_k > 0$. If there exists a fixed point $\{a_j\}$ of (4.1), then*

$$(4.12) \quad a_j = \lambda^{\frac{1}{3}} a_{j+1}, \quad \forall j \geq k.$$

Proof. Let a_j be a fixed point. Since $f_k > 0$, we have that $a_k a_{k+1} > 0$. Consider

$$(4.13) \quad A_j = \frac{\lambda^{(j-k)/3-1/6} a_j}{\sqrt{a_k a_{k+1}}}.$$

Then $A_k A_{k+1} = 1$ and

$$(4.14) \quad A_j A_{j+1} = A_{j-1}^2, \quad j \geq k+1.$$

Therefore, we have

$$(4.15) \quad \frac{A_{j-1}}{A_j} = \frac{A_{j+1}}{A_{j-1}}, \quad j \geq k+1.$$

From these equalities we obtain

$$\frac{A_k}{A_{k+1}} = \frac{A_{k+2}}{A_k}, \quad \frac{A_k}{A_{k+2}} = \frac{A_{k+2}A_{k+3}}{A_kA_{k+1}}, \quad \frac{A_k}{A_{k+3}} = \frac{A_{k+3}A_{k+4}}{A_kA_{k+1}}, \quad \frac{A_k}{A_{k+4}} = \frac{A_{k+4}A_{k+5}}{A_kA_{k+1}}, \dots$$

Hence, it follows that

$$(4.16) \quad \frac{A_k}{A_j} = A_j A_{j+1}, \quad j \geq k+2.$$

Therefore,

$$(4.17) \quad A_{j+1} = A_k A_j^{-2}, \quad j \geq k+2.$$

Thus,

$$(4.18) \quad A_j = \begin{cases} A_k^{(1+2^{j-2})/3} A_{k+2}^{-2(j-2)}, & j \geq k+3, j-k \text{ odd}, \\ A_k^{(1-2^{j-2})/3} A_{k+2}^{2(j-2)}, & j \geq k+3, j-k \text{ even}. \end{cases}$$

Finally, since $A_{k+2} = A_k^3$, we obtain

$$(4.19) \quad A_j = \begin{cases} A_k^{-3 \cdot 2^{(j-2)} + (1+2^{j-2})/3}, & j \geq k+3, j-k \text{ even}, \\ A_k^{3 \cdot 2^{(j-2)} + (1-2^{j-2})/3}, & j \geq k+3, j-k \text{ odd}. \end{cases}$$

Clearly, A_k has to be equal to 1 in order for $\{a_j\}$ to be bounded. Then $A_j = 1$ for all $j \geq k$, which implies that $a_j = \lambda^{1/3} a_{j+1}$ for all $j \geq k$. \square

Using this lemma one can prove the following theorem.

Theorem 4.3. *Let $f = (f_0, f_1, \dots, f_k, 0, 0, \dots)$ with $f_j \geq 0$ for all j . Then there exists a unique fixed point $\{a_j\}$ of (4.1). Moreover,*

$$(4.20) \quad a_j = \lambda^{-\frac{j}{3}} C, \quad \forall j \geq k,$$

for some nonnegative constant C .

4.2. Spectral stability. We consider the fixed point of (3.6) with $f = (f_0, 0, 0, \dots)$, $f_0 > 0$. Rescaling the variables, we can assume that $f_0 = \lambda^{-1/3}$ for simplicity. Then the fixed point is of the form $\{\lambda^{-j/3}\}$. To investigate its spectral stability we write

$$(4.21) \quad a_j(t) = \lambda^{-\frac{j}{3}} + \epsilon b_j.$$

where the perturbation $\{b_j(t)\}$ satisfies

$$(4.22) \quad \frac{db_j}{dt} = \lambda^{\frac{2j}{3}} (2\lambda^{-\frac{2}{3}} b_{j-1} - \lambda^{-\frac{1}{3}} b_j - b_{j+1}) + \epsilon (\lambda^{j-1} b_{j-1}^2 - \lambda^j b_j b_{j+1}) \quad j > 0$$

$$(4.23) \quad \frac{db_0}{dt} = -\lambda^{-\frac{1}{3}} b_0 - b_1 - \epsilon b_0 b_1.$$

The linearized equations for small ϵ are

$$(4.24) \quad \frac{db_j}{dt} = \lambda^{\frac{2j}{3}}(2\lambda^{-\frac{2}{3}}b_{j-1} - \lambda^{-\frac{1}{3}}b_j - b_{j+1}) \quad j > 0$$

$$(4.25) \quad \frac{db_0}{dt} = -\lambda^{-\frac{1}{3}}b_0 - b_1.$$

We seek a solution to (4.24)-(4.25) of the form

$$(4.26) \quad b_j = c_j e^{\mu t}$$

with μ real.

Lemma 4.4. *There are no positive eigenvalues to the system (4.24) - (4.25). Moreover, there exist solutions in H^s , $s < 5/6$ to the system (4.24) - (4.25) of the form (4.26) with the eigenvalues μ negative.*

Proof. Substitution of (4.26) into (4.24)-(4.25) gives

$$(4.27) \quad \lambda^{-\frac{1}{6}}c_{j+1} + \alpha_j c_j - 2\lambda^{-\frac{5}{6}}c_{j-1} = 0, \quad j > 0$$

$$(4.28) \quad \alpha_0 c_0 + \lambda^{-\frac{1}{6}}c_1 = 0,$$

where

$$(4.29) \quad \alpha_j = \lambda^{-\frac{1}{6}}(\lambda^{-\frac{1}{3}} + \lambda^{-\frac{2j}{3}}\mu), \quad j \geq 0.$$

We shall construct a sequence $\{c_j\}$, $c_j \neq 0$, $j \geq 0$ that solves the system (4.27)-(4.28).

Define

$$(4.30) \quad d_j = \lambda^{-\frac{1}{6}} \frac{c_{j+1}}{c_j}, \quad j \geq 0.$$

It follows from (4.27) and (4.30) that

$$(4.31) \quad d_{j-1} = \frac{1}{2^{3/2}(\alpha_j + d_j)} \quad j \geq 1.$$

Hence

$$(4.32) \quad d_n = [\alpha_{n+1}, \alpha_{n+2}, \dots],$$

where $[\alpha_{n+1}, \alpha_{n+2}, \dots]$ denotes the continued fraction

$$[\alpha_{n+1}, \alpha_{n+2}, \dots] = \frac{1}{2^{3/2}\alpha_{n+1} + \frac{2^{3/2}}{2^{3/2}\alpha_{n+2} + \frac{2^{3/2}}{\dots}}}.$$

Observe that from (4.29)

$$(4.33) \quad \lim_{j \rightarrow \infty} \alpha_j = \lambda^{-\frac{1}{2}},$$

for any finite μ . Clearly for sufficiently large n the coefficients α_{n+m} , $m \geq 1$ in (4.32) are positive and the infinite continued fraction (4.32) is convergent to a positive number. Furthermore it follows that

$$(4.34) \quad d_\infty = \lim_{n \rightarrow \infty} d_n = \lambda^{-\frac{1}{2}} < 1.$$

From (4.27) - (4.28) we obtain the characteristic equation

$$(4.35) \quad -\alpha_0 = [\alpha_1, \alpha_2, \dots],$$

where α_j 's are given by (4.29). We observe that (4.35) has no roots $\mu \geq 0$, since in this case all α_j including α_0 are positive.

Suppose there exists μ real and negative such that (4.35) is satisfied. Choose

$$(4.36) \quad \begin{aligned} c_0 &= 1 \\ c_1 &= \lambda^{\frac{1}{6}} d_0 \\ &\dots \\ c_j &= \lambda^{\frac{j}{6}} d_{j-1} d_{j-2} \dots d_0. \end{aligned}$$

Then this sequence satisfies (4.27)-(4.28) by construction and in view of (4.34)

$$(4.37) \quad c_j \sim \lambda^{-\frac{j}{3}}, \text{ for } j \gg 1.$$

Thus the l^2 norm of $\{c_j\}$ is finite. In fact the H^s norm of $\{c_j\}$ is finite for $s < 5/6$.

We are now going to study the characteristic equation (4.35). Let $X(\mu)$ denote the function

$$(4.38) \quad X(\mu) = \alpha_0(\mu) + [\alpha_1(\mu), \alpha_2(\mu), \dots],$$

with $\mu \in (-\infty, 0)$. Then to solve (4.35) we look for zeros of the function $X(\mu)$.

Since $\alpha_j(\mu)$ is given by (4.29) we can write $X(\mu)$ in the form

$$(4.39) \quad X(\mu) = \lambda^{-\frac{1}{6}} (\lambda^{-\frac{1}{3}} + \mu) + \frac{1}{2^{3/2} X(\lambda^{-2/3} \mu)}.$$

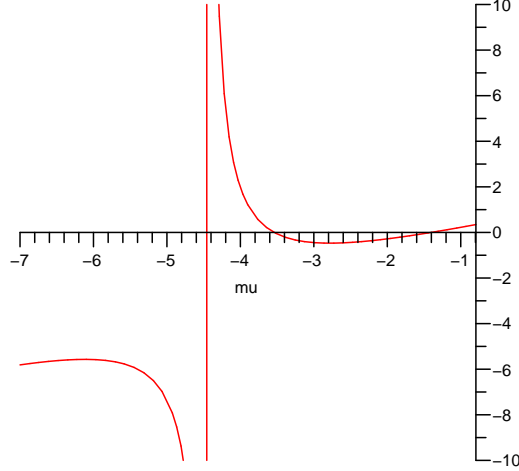
From (4.39) and (4.29) we observe that

$$(4.40) \quad X(0) > 0,$$

and

$$(4.41) \quad \lim_{\mu \rightarrow -\infty} X(\mu) = -\infty.$$

It follows from (4.40) and (4.41) that $X(\mu)$ cannot be continuous and positive for all $\mu \in (-\infty, 0)$. For $\mu \in (-\lambda^{-1/3}, 0)$ each entry of the continuous fraction defining $X(\mu)$ in (4.35) is positive. Hence $X(\mu)$ is continuous in $(-\lambda^{-1/3}, 0)$. Let $(\mu_0, 0)$ be the maximal interval of continuity of $X(\mu)$. Then $X(\mu)$ is continuous at $\lambda^{-2/3} \mu_0$. This implies that $X(\lambda^{-2/3} \mu_0) = 0$. Figure 1 gives

FIGURE 1. Graph of $X(\mu)$

a plot using Maple to graph the function $X(\mu)$ and show that the zeros of $X(\mu)$ are approximately $\mu = -1.4$ and $\mu = -3.6$. \square

5. BEHAVIOUR OF REGULAR SOLUTIONS

In this section we show that every regular solution (i.e. with bounded $H^{5/6}$ norm) approaches the fixed point in the l^2 norm linearly in time. Furthermore, we prove that every solution with initial datum in l^2 blows up in finite time in the $H^{5/6}$ norm.

Theorem 5.1. *Let $a(t)$ be a regular solution of (4.1) on $[T_1, T_2]$ with $f = (\lambda^{-1/3}, 0, 0, \dots)$. We write $a_j(t) = \lambda^{-\frac{j}{3}} + b_j(t)$. Then*

$$\|b(T_2)\|_{l^2} - \|b(T_1)\|_{l^2} \leq -\frac{1}{2} \cdot \lambda^{-\frac{1}{3}}(T_2 - T_1).$$

Proof. Note that $b(t)$ satisfies the following system of equations:

$$(5.1) \quad \begin{aligned} \frac{db_j}{dt} &= \lambda^{\frac{2j}{3}} (2\lambda^{-\frac{2}{3}} b_{j-1} - \lambda^{-\frac{1}{3}} b_j - b_{j+1}) + (\lambda^{j-1} b_{j-1}^2 - \lambda^j b_j b_{j+1}), \quad j \geq 1, \\ \frac{db_0}{dt} &= -\lambda^{-\frac{1}{3}} b_0 - b_1 - b_0 b_1. \end{aligned}$$

Multiplying it by b_j and taking a sum from $j = 0$ to $j = k$ we obtain

$$(5.2) \quad \frac{1}{2} \frac{d}{dt} \sum_{j=0}^k b_j^2 = -\lambda^{-\frac{1}{3}} b_0^2 - b_0 b_1 - b_0^2 b_1 \\ + \sum_{j=1}^k \left(\lambda^{\frac{2j}{3}} (2\lambda^{-\frac{2}{3}} b_{j-1} b_j - \lambda^{-\frac{1}{3}} b_j^2 - b_j b_{j+1}) + (\lambda^{j-1} b_{j-1}^2 b_j - \lambda^j b_j^2 b_{j+1}) \right).$$

It now follows that

$$(5.3) \quad \frac{1}{2} \frac{d}{dt} \sum_{j=0}^k b_j^2 = - \left(\lambda^{-\frac{1}{3}} b_0^2 + b_0 b_1 \right) \\ + \sum_{j=1}^k \lambda^{\frac{2j}{3}} \left(2\lambda^{-\frac{2}{3}} b_{j-1} b_j - \lambda^{-\frac{1}{3}} b_j^2 - b_j b_{j+1} \right) - \lambda^k b_k^2 b_{k+1}.$$

Also we can rewrite (5.3) as

$$(5.4) \quad \frac{1}{2} \frac{d}{dt} \sum_{j=0}^k b_j^2 = -\lambda^{-\frac{1}{3}} \sum_{j=0}^k \lambda^{\frac{2j}{3}} b_j^2 + \sum_{j=0}^k \lambda^{\frac{2j}{3}} b_j b_{j+1} \\ - 2\lambda^{\frac{2k}{3}} b_k b_{k+1} - \lambda^k b_k^2 b_{k+1}.$$

However,

$$\sum_{j=0}^k \lambda^{\frac{2j}{3}} (\lambda^{-\frac{1}{6}} b_j - \lambda^{\frac{1}{6}} b_{j+1})^2 = \sum_{j=0}^k \lambda^{\frac{2j}{3}} (\lambda^{-\frac{1}{3}} b_j^2 + \lambda^{\frac{1}{3}} b_{j+1}^2 - 2b_j b_{j+1}) \\ = -2 \left[-\lambda^{-\frac{1}{3}} \sum_{j=0}^k \lambda^{\frac{2j}{3}} b_j^2 + \sum_{j=0}^k \lambda^{\frac{2j}{3}} b_j b_{j+1} \right] \\ - \lambda^{-\frac{1}{3}} b_0^2 + \lambda^{\frac{2k}{3} + \frac{1}{3}} b_{k+1}^2.$$

Hence (5.4) gives

$$(5.5) \quad \frac{1}{2} \frac{d}{dt} \sum_{j=0}^k b_j^2 = -\frac{1}{2} \left[\sum_{j=0}^k \lambda^{\frac{2j}{3}} (\lambda^{-\frac{1}{6}} b_j - \lambda^{\frac{1}{6}} b_{j+1})^2 + \lambda^{-\frac{1}{3}} b_0^2 \right] \\ + \frac{1}{2} \lambda^{\frac{2k}{3} + \frac{1}{3}} b_{k+1}^2 - 2\lambda^{\frac{2k}{3}} b_k b_{k+1} - \lambda^k b_k^2 b_{k+1}.$$

Now note that since $a(t)$ is regular on $[T_1, T_2]$, we have that

$$(5.6) \quad \lim_{j \rightarrow \infty} \int_{T_1}^{T_2} \lambda^{\frac{2j}{3}} a_j^2(t) dt = 0.$$

Hence, since $b_j = a_j - \lambda^{-j/3}$, we obtain

$$\begin{aligned}
(5.7) \quad \lim_{j \rightarrow \infty} \int_{T_1}^{T_2} \left(\frac{1}{2} \lambda^{\frac{2k}{3} + \frac{1}{3}} b_{k+1}^2 - 2\lambda^{\frac{2k}{3}} b_k b_{k+1} - \lambda^k b_k^2 b_{k+1} \right) dt \\
= (T_2 - T_1) \left(\frac{1}{2} \lambda^{-\frac{1}{3}} - 2\lambda^{-\frac{1}{3}} + \lambda^{-\frac{1}{3}} \right) \\
= -\frac{1}{2} \cdot \lambda^{-\frac{1}{3}} (T_2 - T_1).
\end{aligned}$$

Now (5.5) and (5.7) imply that

$$(5.8) \quad \|b(T_2)\|_{l^2} - \|b(T_1)\|_{l^2} \leq -\frac{1}{2} \cdot \lambda^{-\frac{1}{3}} (T_2 - T_1).$$

□

5.1. Blow-up in finite time. Using the results of the previous subsection, we prove that there is no global in time regular solution to (4.1) with $f = (\lambda^{-1/3}, 0, 0, \dots)$.

Theorem 5.2. *All the solution to (4.1) with $f = (\lambda^{-1/3}, 0, 0, \dots)$ blow up in finite time in $H^{5/6}$ -norm. More precisely, for every solution $a(t)$ with $a(0) \in l^2$, there exists a time*

$$0 \leq t^* \leq 2\lambda^{\frac{1}{3}} \|b(0)\|_{l^2},$$

such that

$$\limsup_{t \rightarrow t^*} \|a(t)\|_{H^{5/6}} = \infty.$$

Proof. First note that every $a_j(t)$ is continuous, i.e., $a(t)$ is weakly in l^2 continuous. Therefore, if $\|a(t^*)\|_{H^{5/6}} = \infty$, then

$$(5.9) \quad \limsup_{t \rightarrow t^* -} \|a(t)\|_{H^{5/6}} = \limsup_{t \rightarrow t^* +} \|a(t)\|_{H^{5/6}} = \infty.$$

Now assume that the statement of the theorem is not true. Then there exists a solution $a(t)$ to (4.1) which is regular on $[0, T]$ with $T = 2\lambda^{1/3} \|b(0)\|_{l^2}$. Then Theorem 5.1 implies that

$$(5.10) \quad \begin{aligned} \|b(T)\|_{l^2} &\leq \|b(0)\|_{l^2} - 2^{-1} \lambda^{-\frac{1}{3}} T \\ &= 0. \end{aligned}$$

Therefore,

$$(5.11) \quad a_j(T) = \lambda^{-\frac{j}{3}},$$

i.e., $\|a(T)\|_{H^{5/6}} = \infty$. Due to the remark in the beginning of the proof,

$$(5.12) \quad \limsup_{t \rightarrow T -} \|a(t)\|_{H^{5/6}} = \infty,$$

a contradiction. □

Note that there is no global in time regular solution to (4.1) with a more general force $f = (f_0, 0, 0, \dots)$, $f_0 > 0$. This follows by application of Theorem 5.2 to a rescaled system (4.1).

6. ONSAGER'S CONJECTURE AND KOLMOGOROV'S 5/3RD LAW.

In the past 60 years since its original presentation much attention has been given to the statistical theories of turbulence developed by Kolmogorov and Onsager. For recent reviews see, for example Eyink and Sreenivasan [9], Robert [19]. Onsager [18] conjectured a clear distinction between 2 and 3 dimensions: in 2 dimensions the energy of a turbulent flow is conserved, however, in 3 dimensions dissipation of energy persists in the limit of vanishing viscosity. This phenomenon is now referred to as turbulent or anomalous dissipation. It is suggested that an appropriate mathematical description of 3-dimensional turbulent flow is given by weak solutions of the Euler equations which are not regular enough to conserve energy. Kolmogorov's theory [15] predicts that in a fully developed turbulent flow the energy spectrum $E(|k|)$ in the inertial range is given by

$$(6.1) \quad E(|k|) = c_0 \bar{\epsilon}^{2/3} |k|^{-5/3},$$

where $\bar{\epsilon}$ is the average of the energy dissipation rate. This law stated in physical space means that the Holder exponent h of the velocity is $1/3$ in a statistically averaged sense. Onsager conjectured that for exponents $h > 1/3$ the energy is conserved and that this ceases to be true for $h \leq 1/3$. An elegant proof of the conservation of energy of weak solutions of the 3 dimensional Euler equations in Besov spaces $B_3^{h,\infty}$, $h > 1/3$, was given by Constantin et al [6].

The model system that we study in this present paper reproduces the phenomenon described above. Regular solutions, i.e., solutions with bounded $H^{5/6}$ norm, satisfy the energy equality, whereas the fixed point

$$(6.2) \quad \{a_j\} = \{2^{-5j/6+5/12} \sqrt{f_0}\}.$$

does not. Note that a necessary condition for a function in physical space to have Holder exponent h is that the spectral exponent is greater or equal to $1 + 2h$ (i.e., the spectral exponent $5/3$ of the fixed point in our model is exactly the Onsager critical exponent corresponding to $h = 1/3$).

The model (1.2) is derived under the assumption that $a_j^2(t)$ is the *total* energy in the frequency space shell $2^j \leq |k| < 2^{j+1}$. Hence, by (6.2), the energy spectrum $E(|k|)$ for the fixed point is given by the expression

$$(6.3) \quad E(|k|) = 2^{5/6} f_0 |k|^{-5/3}.$$

Since the dissipation rate for the fixed point is equal to the energy input rate, we have

$$\epsilon = a_0 f_0 = 2^{5/12} f_0^{3/2}.$$

Thus the fixed point satisfies Kolmogorov's law (6.1) with $c_0 = 2^{5/9}$.

In Section 5 we proved that every regular solution approaches the fixed point and, furthermore, blows up in finite time in $H^{5/6}$ norm. After this time (as we show in [5]) the solutions are not regular enough to conserve energy and anomalous dissipation for an inviscid system produces Kolmogorov’s turbulent energy spectrum.

Anomalous dissipation and “life after blow-up” has been examined recently in a *linear* 1-dimensional model where the system is “simple” enough to be fully solvable but still produces anomalous dissipation via energy cascade towards higher wave numbers [16]. In our more technical companion paper [5] we examine “life after blow-up” for the system (1.2). In particular, we prove that the fixed point is a global attractor. Since the support of any time-average measure belongs to the global attractor, the average dissipation rate is equal to the dissipation rate of the fixed point. Thus Kolmogorov’s law stated above for the fixed point is valid for the system.

Remark 6.1. It is well known that the 1-dimensional inviscid Burgers equation produces energy dissipation for weak solutions with shocks (as we noted, a rescaled version of (1.2) can be motivated by Burgers equation in Fourier space). In contrast, the Cauchy problem for the 3-dimensional incompressible Euler equations remains open (and a major challenge). The 3-dimensional Euler equations may (or may not) share with the model system (1.2) the turbulent processes conjectured by Onsager for 3-dimensional fluids. The model *per se* could be written with n -dimensional scaling and the system itself has no behavior that is inherently 3-dimensional or incompressible. We note, however, that the reduction of an incompressible 3-dimensional vector model for the fluid equations to the scalar system (1.2) given in [10] cannot be implemented in two dimensions. Furthermore, any such model that does not preserve enstrophy cannot be a suitable model for the 2-dimensional Euler equations.

REFERENCES

- [1] T. Bohr, M. Jensen, G. Paladin and A. Vulpiani: Dynamical Systems Approach to Turbulence, Cambridge University Press, 1998.
- [2] M. Cannone: Harmonic analysis tools for solving the incompressible Navier-Stokes equations, *Handbook of mathematical fluid dynamics* Vol. III, 161–244, North-Holland, Amsterdam, 2004.
- [3] A. Cheskidov, Blow-up in finite time for the dyadic model of the Navier-Stokes equations, *Transactions of AMS*, to appear.
- [4] A. Cheskidov and C. Foias, On global attractors of the 3D Navier-Stokes equations, *J. Diff. eq.*, to appear.
- [5] A. Cheskidov, S. Friedlander, N. Pavlović, An inviscid dyadic model of turbulence: the global attractor, *In preparation*.
- [6] P. Constantin, W. E, and E. S. Titi, Onsager’s conjecture on the energy conservation for solutions of Euler’s equation, *Comm. Math. Phys.* **165** (1994), 207–209.
- [7] P. Constantin, B. Levant and E. Titi, Analytic study of the shell model of turbulence, *Preprint*.
- [8] E.I. Dinaburg and Ya. G. Sinai, A quasilinear approximation for the three-dimensional Navier-Stokes system, *Moscow Mathematical Journal* **1**, No. 3 (2001), 381–388.

- [9] G. L. Eyink and K. R. Sreenivasan, Onsager and the theory of hydrodynamic turbulence, *Rev. Mod. Phys.* **78** (2006).
- [10] S. Friedlander and N. Pavlović, Blow up in a three dimensional vector model for the Euler equations, *Commun. Pure Appl. Math.* **57**, No.6 (2004), 705–725.
- [11] E. B. Gledzer, System of hydrodynamic type admitting two quadratic integrals of motion, *Sov. Phys. Dokl.* **18**, No. 4 (1973), 216–217.
- [12] N. Katz and N. Pavlović, Finite time blow-up for a dyadic model of the Euler equations, *Transactions of AMS* **357**, No. 2 (2005), 695–708.
- [13] N.H. Katz and N. Pavlović, A cheap Caffarelli-Kohn-Nirenberg inequality for the Navier Stokes equation with hyper-dissipation, *GAFSA* **12** (2002), 355–379.
- [14] A. Kiselev and A. Zlatoš, On discrete models of the Euler equation, *IMRN* **38** (2005) No. 38, 2315–2339.
- [15] A. N. Kolmogorov, The local structure of turbulence in incompressible viscous fluids at very large Reynolds numbers, *Dokl. Akad. Nauk. SSSR* **30** (1941), 301–305.
- [16] J. C. Mattingly, T. Suidan, and E. Vanden-Eijnden, Simple systems with anomalous dissipation and energy cascade, *preprint*.
- [17] K. Ohkitani and M. Yamada, Temporal intermittency in the energy cascade process and local Lyapunov analysis in fully developed model turbulence, *Prog. Theor. Phys.* **81**, No. 2 (1989), 329–341.
- [18] L. Onsager, Statistical Hydrodynamics, *Nuovo Cimento (Supplemento)* **6** (1949), 279–287.
- [19] R. Robert, Statistical Hydrodynamics (Onsager revisited), *Handbook of Mathematical Fluid Dynamics*, vol 2 (2003), 1–55. Ed. Friedlander and Serre. Elsevier.
- [20] F. Waleffe On some dyadic models of the Euler equations, *Proc. Amer. Math. Soc.* **134** (2006), 2913–2922.
- [21] V. Zimin and F. Hussain, Wavelet based model for small-scale turbulence, *Phys. Fluids* **7** (1995), 2925–2927.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, ANN ARBOR, MI 48109-1043

E-mail address: acheskid@umich.edu

DEPARTMENT OF MATHEMATICS, STATISTICS, AND COMPUTER SCIENCE, CHICAGO, IL 60607-7045

E-mail address: susan@math.uic.edu

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, PRINCETON, NJ 08544-1000

E-mail address: natasa@math.princeton.edu